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Local Analytic Hypoellipticity for $D_x^2 + D_y^2 + (x^{2m} + y^{2n})D_t^2$

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Abstract

We give a new and very simple proof of the local analytic hypoellipticity of some partial differential operators studied by Matzusawa in [3] and their generalizations. These include the operator

$$\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + (x^{2m} + y^{2n})\left(\frac{\partial}{\partial t}\right)^2.$$

These operators represent a simple class which had not yielded to the methods of Derridj and Tartakoff to date.

1 Introduction

We are concerned here with the operators

$$P_1 = \left(\frac{\partial}{\partial x}\right)^2 + x^{2m}\left(\frac{\partial}{\partial t}\right)^2, \tag{1}$$

$$P_2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + (x^{2m} + y^{2n})\left(\frac{\partial}{\partial t}\right)^2, \tag{2}$$

. The first type is that studied by Matzusawa in [3], the second occurs as an analogue of the first in more variables.

While there are easy generalizations of this method to higher dimensions, we content ourselves for the moment with these low dimensions, which suffice to demonstrate the technique.

2 Notation and Statement of Results

The notation being clear, we proceed to the statement of the theorem:

Theorem 1 *Both P_1 and P_2 are microlocally analytic hypoelliptic.*

Remark For P_1 the theorem is due to Matzusawa in a local form, although our proof will be entirely different.

3 Technical Reductions

Our first observation is that the operators are subelliptic, as can be seen from the Hörmander condition applied to the vector fields

$$X_1 = \frac{\partial}{\partial x} \text{ and } Y_1 = x^{2m}\frac{\partial}{\partial t} \text{ for } P_1,$$

and

$$X_2 = \frac{\partial}{\partial x}, Y_2 = \frac{\partial}{\partial y}, Z_2 = x^{2m}\frac{\partial}{\partial t} \text{ and } W_2 = y^{2n}\frac{\partial}{\partial t} \text{ for} P_2.$$

It seems to be better in this context, however, *not* to think of these operators in terms of sums of squares of vector fields. The rôle played by vector fields *per se* seems much less important here than elsewhere.

If we denote by $(\xi(,\eta),\tau)$ the variables dual to $(x,(y,)t)$ in the cotangent space, in both cases the operator is clearly elliptic for $(x(,y)) \neq (0(,0))$ and at $(0,(0,)t;\xi(,\eta)) \neq (0,(0,)t;0(,0))$, the characteristic variety of both operators is symplectic and hence the methods and results of [4], and [1] together with classical results on elliptic operators yield (microlocal) analytic hypoellipticity of the P_i. Note that at symplectic points for these opertors one does have maximal estimates and hence indeed the results of [1] do apply.

If one is only interested in local results, and not micro-local ones, what this amounts to saying is that if $P_i u = f$ with

$$\|\left(\frac{\partial}{\partial t}\right)^r f\|_{L^2} \leq C_f^{r+1} r!$$

near the origin uniformly in r, then the same is true of u instead of f, perhaps in a smaller neighborhood of the origin, with a new constant C. Furthermore, this 'neighborhood' may be chosen by merely localizing in t; while of course a localization of the form $\phi_1(x(,y))\phi_2(t)$ should be used, whenever a derivative falls on ϕ_1 the support will be away from $(x(,y)) = (0(,0))$, and hence at a symplectic point where the result is known by [1].

Thus technically speaking it suffices to estimate, in L^2 norm, derivatives of the solution u of the form $\phi(t)(\frac{\partial}{\partial t})^r u$.

4 Proof of the Theorem

We seek to show in both cases that we may bound

$$||\phi_1(x(,y))\phi_2(t)\left(\frac{\partial}{\partial t}\right)^r u||_{L^2(\omega)} \leq C_u^{r+1} r! \tag{3}$$

where ω is a small neighborhood of the origin. Here the localizing functions $\phi_1(x(,y))$ and $\phi_2(t)$ are identically one near $(0(,0))$ and 0, respectively, and of compact support. We shall omit reference to $\phi_1(x(,y))$, since if we take all norms and inner products over a neighborhood of the origin and a derivative in x (or y) falls on $\phi_1(x(,y))$, then in the support of this differentiated function, the result is known. Thus we shall proceed as though the $(x(,y))$ support were compact and omit reference to $\phi_1(x(,y))$. Thus we need to bound

$$\|\phi_2(t)\left(\frac{\partial}{\partial t}\right)^r u\|_{L^2} \leq C_u^{r+1} r! \tag{4}$$

4.1 The Proof for $D_x^2 + x^{2m}D_t^2$

We write $D_z = \frac{\partial}{\partial z}$ and start with the *a priori* inequality,

$$\|D_x v\|_{L^2}^2 + \|x^m D_t v\|_{L^2}^2 + \|v\|_{\frac{1}{m+1}}^2 \leq C|\Re(P_1 v, v)_{L^2}| \tag{5}$$

for $v \in C_0^\infty$, and replace v by $\phi(t)D_t^r u$. We have

$$\|D_x\phi(t)D_t^r u\|_{L^2}^2 + \|x^m D_t\phi(t)D_t^r u\|_{L^2}^2 + \|\phi(t)D_t^r u\|_{\frac{1}{m+1}}^2 \leq$$

$$\begin{aligned}
&\leq C|\Re(P_1\phi(t)D_t^r u, \phi(t)D_t^r u)_{L^2}| \\
&= C|\Re(\phi(t)D_t^r P_1 u, \phi(t)D_t^r u)_{L^2} + E_r| \\
&= C|\Re(\phi(t)D_t^r f, \phi(t)D_t^r u)_{L^2} + E_r| \\
&\leq C'\left(\|\phi(t)D_t^r f\|_{L^2}^2 + |E_r|\right),
\end{aligned}$$

where

$$E_r = \Re([P_1, \phi(t)D_t^r]u, \phi(t)D_t^r u)_{L^2}.$$

Now $[P_1, \phi(t)D_t^r]u$ may be written:

$$[P_1, \phi(t)D_t^r]u = [D_x^2, \phi(t)D_t^r]u + [x^{2m}D_t^2, \phi(t)D_t^r]u.$$

But the first term is zero, and for the second term,

$$[x^{2m}D_t^2, \phi(t)D_t^r]u = x^{2m}(2D_t\phi'(t)D_t^r u + \phi''(t)D_t^r)u$$

$$= \ldots\ldots$$

$$= \sum_{j=1}^{r} x^{2m}(j+1)D_t^2\phi^{(j)}(t)D_t^{r-j}u + (r+2)x^{2m}D_t\phi^{(r+1)}(t)u + (r+3)x^{2m}\phi^{(r+2)}(t)u,$$

and hence, with some integration by parts,

$$|E_r| = |\Re([P_1, \phi(t)D_t^r]u, \phi(t)D_t^r u)_{L^2}|$$

$$\begin{aligned}
&\leq C\{\sum_{j=1}^{r} |((j+1)x^m D_t\phi^{(j)}(t)D_t^{r-j}u, x^m D_t\phi(t)D_t^r u)_{L^2}| + \\
&\quad + |(x^m(r+2)\phi^{(r+1)}(t)u, x^m D_t\phi(t)D_t^r u)_{L^2}| \\
&\quad + |(x^m(r+3)\phi^{(r+2)}(t)u, x^m\phi(t)D_t^r u)_{L^2}|\}.
\end{aligned} \tag{6}$$

With the help of the weighted Schwarz inequality, we have, then,

$$|\Re([P_1, \phi(t)D_t^r]u, \phi(t)D_t^r u)_{L^2}| \leq C\sum_1^r \|(j+1)x^m D_t\phi^{(j)}(t)D_t^{r-j}u\|_{L^2}^2 \tag{7}$$

$$+C\|(r+2)x^m\phi^{(r+1)}(t)u\|_{L^2}^2 + C\|(r+3)x^m\phi^{(r+2)}(t)u\|_{L^2}^2\}.$$

All but the last of these terms have the same form as the second term on the left, with powers of D_t converted into derivatives on ϕ. The last term will be treated as a final error term, with ϕ suitably chosen. Thus absorbing the first term on the right into the left hand side of the above,

$$\|D_x\phi(t)D_t^r u\|_{L^2}^2 + \|x^m D_t\phi(t)D_t^r u\|_{L^2}^2 + \|\phi(t)D_t^r u\|_{\frac{1}{m+1}}^2 \leq \tag{8}$$

$$\leq C\|\phi(t)D_t^r f\|_{L^2}^2 + C\{\sum_1^r \|(j+1)x^m D_t\phi^{(j)}(t)D_t^{r-j}u\|_{L^2}^2$$

$$+\|(r+2)x^m\phi^{(r+1)}(t)u\|_{L^2}^2 + \|(r+3)x^m\phi^{(r+2)}(t)u\|_{L^2}^2\}.$$

Now the inductive hypothesis that

$$\|x^m D_t\phi^{(j)}(t)D_t^{r-j}u\|_{L^2}^2 \leq C_0 C_1^j C_2^{N-j} N^r$$

for $j > j_0$, (for suitable constants $C_l, l = 0,1,2$), true for $j = j_0$ provided the localizing function $\phi = \phi_N$ satisfies $|D^k\phi| \leq K^k N^k, k \leq 2N$ (for instance, see [2], [4] for the existence of such 'analytic families of localizing functions'), together with the above inequality, yields

$$\|x^m D_t\phi^{(j_0)}(t)D_t^{r-j_0}u\|_{L^2}^2 \leq$$

$$C\|\phi^{(j_0)}(t)D_t^{r-j_0}f\|_{L^2}^2 + C\{\sum_1^{r-j_0}\|(j+1)x^m D_t\phi^{(j_0+j)}(t)D_t^{r-j_0-j}u\|_{L^2}^2$$

$$+\|(r-j_0+2)x^m\phi^{(r+1)}(t)u\|_{L^2}^2 + \|(r-j_0+3)x^m\phi^{(r+2)}(t)u\|_{L^2}^2\} \leq$$

$$\leq CC_f^{r+1}r! + C\sum_1^{r-j_0}(j+1)C_0C_1^{j_0+j}C_2^{N-j_0-j}N^r + 2(r-j_0+3)C_0C_1^{r+2}N^r \leq$$

$$\leq C_0C_1^{j_0}C_2^{N-j_0}N^r$$

provided

$$CC_f^{r+1}r! \leq \frac{1}{3}C_0C_1^{j_0}C_2^{N-j_0}N^r,$$

$$C\sum_1^{r-j_0}(j+1)C_0C_1^{j_0+j}C_2^{N-j_0-j}N^r \leq \frac{1}{3}C_0C_1^{j_0}C_2^{N-j_0}N^r,$$

and

$$2(r-j_0+3)C_0C_1^{r+2}N^r \leq \frac{1}{3}C_0C_1^{j_0}C_2^{N-j_0}N^r.$$

But these are easily achieved with proper ratios of the constants C_j.

4.2 The Proof for $P_2 = D_x^2 + D_y^2 + x^{2m} D_t^2 + y^{2n} D_t^2$

The proof for P_2 is no different from that for P_1, since there is no interaction between the two operators

$$P_{1,1} = P_1 = D_x^2 + x^{2m} D_t^2$$

and

$$P_{1,2} = D_y^2 + y^{2n} D_t^2$$

that make up $P_2 = P_{1,1} + P_{1,2}$, and the subellipticity employed is only used to capture a large multiple of the L^2 norm and the norm $\|\phi(t) D_t^r u\|^2_{\frac{1}{m+1}}$ may be replaced by $\|\phi(t) D_t^r u\|^2_{\frac{1}{\max(m,n)+1}}$.

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