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Abstract. If CH fails, then there exist $2^{2^{\aleph_0}}$ universal sofic groups up to isomorphism.

1. Introduction

Let $\mathcal{U}$ be a nonprincipal ultrafilter over ω and let $G_{\mathcal{U}} = \prod_{\mathcal{U}} \mathrm{Sym}(n)$ be the corresponding ultraproduct of the finite symmetric groups. Then Allsup-Kaye [1] and Elek-Szabó [2] have independently shown that $G_{\mathcal{U}}$ has a unique maximal proper normal subgroup; namely,

$$M_{\mathcal{U}} = \left\{ (\pi_n)_{\mathcal{U}} \in G_{\mathcal{U}} : \lim_{\mathcal{U}} \frac{|\operatorname{supp}(\pi_n)|}{n} = 0 \right\},$$

where $\operatorname{supp}(\pi_n) = \{\ell \in n : \pi_n(\ell) \neq \ell\}$. Let $S_{\mathcal{U}} = G_{\mathcal{U}}/M_{\mathcal{U}}$. Then by Elek-Szabó [2], if Γ is a finitely generated group, the following statements are equivalent:

- Γ is a sofic group.
- Γ embeds into $S_{\mathcal{U}}$ for some (equivalently every) nonprincipal ultrafilter $\mathcal{U}$.

For this reason, $S_{\mathcal{U}}$ is said to be a *universal sofic group*.[1] In this paper, we will consider the problem of computing the number of universal sofic groups up to isomorphism. Perhaps surprisingly, this problem turns out to be much easier to handle under the assumption that the Continuum Hypothesis CH fails.

Theorem 1.1. *If CH fails, then there exist $2^{2^{\aleph_0}}$ universal sofic groups $S_{\mathcal{U}}$ up to isomorphism.*

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[1] A clear account of the basic theory of sofic groups can be found in Pestov [8]. It is an important open problem whether *every* finitely generated group is sofic.

On the other hand, suppose that CH holds. Then each ultraproduct $G_{\mathcal{U}} = \prod_{\mathcal{U}} \mathrm{Sym}(n)$ is saturated and hence is determined up to isomorphism by its first order theory. Thus there are at most $2^{\aleph_0}$ such ultraproducts up to isomorphism and hence also at most $2^{\aleph_0}$ universal sofic groups up to isomorphism. It is easily shown that (as expected) there are $2^{\aleph_0}$ such ultraproducts up to elementary equivalence. However, it is currently not even known whether there exist two nonisomorphic universal sofic groups if CH holds.

Conjecture 1.2. If CH holds, then there exist $2^{\aleph_0}$ universal sofic groups $S_{\mathcal{U}}$ up to isomorphism.

Question 1.3. Are all universal sofic groups $S_{\mathcal{U}}$ elementarily equivalent?

This paper is organized as follows. In Section 2, we will use a basic property of expander graphs to show that certain ultraproducts $\prod_{\mathcal{D}} G_n$ of finite groups can be realized as centralizers of finitely generated subgroups in suitably chosen universal sofic groups. Then in Section 3, using recent results of Kassabov [4] and Ellis-Hachtman-Schneider-Thomas [3], we will complete the proof of Theorem 1.1.

2. Centralizers and Expander Families

In this section, we will use a basic property of expander graphs to show that certain ultraproducts $\prod_{\mathcal{D}} G_n$ of finite groups can be realized as centralizers of finitely generated subgroups in suitably chosen universal sofic groups. We will begin by defining the notion of an expander family of finite graphs. Suppose that $\Gamma = (V, E)$ is a finite connected graph with vertex set V and edge set E. Then for each subset $A \subseteq V$, the corresponding *edge boundary* is defined to be

$$\partial A = \{ e \in E : |e \cap A| = 1 \};$$

and the *expansion constant* of Γ is defined to be

$$h(\Gamma) = \min \left\{ \frac{|\partial A|}{|A|} : A \subset V \text{ with } 1 \leq |A| \leq \frac{|V|}{2} \right\}.$$

From now on, we will identify each finite graph $\Gamma = (V, E)$ with its vertex set V and we will write $h(V)$ instead of $h(\Gamma)$. A finite graph V is said to be *k-regular* if each vertex $v \in V$ has degree k.

Definition 2.1. Let $(V_n : n \in \omega)$ be a family of finite connected k-regular graphs such that $|V_m| < |V_n|$ for all $m < n \in \omega$. Then $(V_n : n \in \omega)$ is said to be an *expander family* if there exists $\tau > 0$ such that $h(V_n) \geq \tau$ for all $n \in \omega$.

Most of the known expander families consist of suitably chosen Cayley graphs of finite groups. (For example, see Lubotzky [7] and Kassabov-Lubotzky-Nikolov [5].) Here if G is a finite group and $S \subseteq G \smallsetminus 1$ is a generating set, then the corresponding *Cayley graph* $\mathrm{Cay}(G, S)$ is the graph with vertex set G and edge set

$$E = \{\{x, y\} \mid y = s\,x \text{ for some } s \in S \cup S^{-1}\}.$$

As we will see in Section 3, Theorem 1.1 is an easy consequence of the following theorem, together with recent results of Kassabov [4] and Ellis-Hachtman-Schneider-Thomas [3].

Theorem 2.2. *For each $n \in \omega$, let G_n be a finite group and let $S_n \subseteq G_n$ be a generating set of fixed size d. If $(\mathrm{Cay}(G_n, S_n) : n \in \omega)$ is an expander family, then for each nonprincipal ultrafilter $\mathcal{D}$ over ω, there exists a nonprincipal ultrafilter $\mathcal{U}$ over ω and a finitely generated subgroup $\Gamma \leqslant S_{\mathcal{U}}$ such that $C_{S_{\mathcal{U}}}(\Gamma) \cong \prod_{\mathcal{D}} G_n$.*

The proof of Theorem 2.2 makes use of the following observation.

Proposition 2.3. *Suppose that V is a finite connected k-regular graph and that $h(V) \geq \tau$. Suppose that $\varepsilon > 0$ and let $\delta = \varepsilon\tau/(\tau + k)$. Then whenever $Y \subseteq V$ is a subgraph with $|Y| \geq (1 - \delta)|V|$, there exists a connected subgraph $Z \subseteq Y$ with $|Z| \geq (1 - \varepsilon)|V|$.*

Proof. Let $Y \subseteq V$ be a subgraph with $|Y| \geq (1-\delta)|V|$ and suppose that $C_1, \cdots, C_t$ are connected components of Y with $|C_i| \leq \frac{1}{2}|V|$ for each $1 \leq i \leq t$. Consider the set

$$P = \{ (v, e) : e \in \bigcup_{i=1}^{t} \partial C_i \text{ and } v \in e \smallsetminus Y \}.$$

Notice that if $e \in \bigcup_{i=1}^{t} \partial C_i$, then $|e \cap Y| = 1$. Thus

$$|P| = \sum_{i=1}^{t} |\partial C_i| \geq \tau \sum_{i=1}^{t} |C_i|.$$

Clearly we also have that

$$|P| \leq k\,|V \smallsetminus Y| \leq k\delta|V|.$$

Hence we have that

$$\tau \sum_{i=1}^{t} |C_i| \leq k\delta |V|$$

and so

$$|V \smallsetminus Y| + \sum_{i=1}^{t} |C_i| \leq \delta |V| + \frac{k\delta}{\tau} |V| = \varepsilon |V|.$$

It follows that Y has a connected component Z with $|Z| \geq (1-\varepsilon)|V|$. □

The proof of Theorem 1.1 also makes use of the notions of the left regular and right regular permutation representations of a finite group G. Here the *left regular permutation representation* of G is the embedding $\lambda : G \to \mathrm{Sym}(G)$ defined by $\lambda(g)(x) = g\,x$; and the *right regular permutation representation* of G is the embedding $\rho : G \to \mathrm{Sym}(G)$ defined by $\rho(g) = x\,g^{-1}$. It is well-known that

$$C_{\mathrm{Sym}(G)}(\,\lambda[\,G\,]\,) = \rho[\,G\,].$$

(For example, see Tsuzuku [9, Theorem 3.2.10].)

Proof of Theorem 2.2. To simplify notation, suppose that $d = 2$ and let $S_n = \{\,a_n, b_n\,\}$. Let $\mathcal{U}$ be the nonprincipal ultrafilter over ω such that for each $X \subseteq \omega$,

$$\{\,|G_n| : n \in X\,\} \in \mathcal{U} \iff X \in \mathcal{D}.$$

Then we can define a natural isomorphism

$$\sigma : \prod\nolimits_{\mathcal{D}} \mathrm{Sym}(|G_n|) \to \prod\nolimits_{\mathcal{U}} \mathrm{Sym}(n)$$
$$(\,\theta_n\,)_{\mathcal{D}} \mapsto (\,\psi_n\,)_{\mathcal{U}}$$

by setting

$$\psi_n = \begin{cases} \theta_m & \text{if } n = |G_m|; \\ 1 & \text{otherwise.} \end{cases}$$

Clearly there also exists a natural isomorphism $\iota : \prod_{\mathcal{D}} \mathrm{Sym}(G_n) \to \prod_{\mathcal{D}} \mathrm{Sym}(|G_n|)$. Let $\pi : \prod_{\mathcal{D}} \mathrm{Sym}(G_n) \to S_{\mathcal{U}}$ be the surjective homomorphism obtained by composing the following maps:

$$\prod\nolimits_{\mathcal{D}} \mathrm{Sym}(G_n) \xrightarrow{\iota} \prod\nolimits_{\mathcal{D}} \mathrm{Sym}(|G_n|) \xrightarrow{\sigma} \prod\nolimits_{\mathcal{U}} \mathrm{Sym}(n) \to \Big(\prod\nolimits_{\mathcal{U}} \mathrm{Sym}(n)\Big)/M_{\mathcal{U}} = S_{\mathcal{U}}.$$

For each $n \in \omega$, let $\lambda_n : G_n \to \mathrm{Sym}(G_n)$ and $\rho_n : G_n \to \mathrm{Sym}(G_n)$ be the left regular and right regular permutation representations. Let α, $\beta \in S_{\mathcal{U}}$ be the elements defined by

$$\alpha = \pi(\, (\lambda_n(a_n))_{\mathcal{D}}\,) \quad \text{and} \quad \beta = \pi(\, (\lambda_n(b_n))_{\mathcal{D}}\,).$$

Then we claim that the subgroup $\Gamma = \langle \alpha, \beta \rangle$ of $S_{\mathcal{U}}$ satisfies our requirements. Let $G = \prod_{\mathcal{D}} \mathrm{Sym}(G_n)$. Then clearly we have that

$$\prod\nolimits_{\mathcal{D}} \rho_n[\, G_n\,] \leqslant C_G(\{\, (\lambda_n(a_n))_{\mathcal{D}}, (\lambda_n(b_n))_{\mathcal{D}}\,\}).$$

(In fact, using Tsuzuku [9, Theorem 3.2.10], it is easily seen that the two groups in the above inclusion are actually equal.) It is also clear that π maps $\prod_{\mathcal{D}} \rho_n[\, G_n\,]$ injectively into $C_{S_{\mathcal{U}}}(\Gamma)$. Thus it is enough to show that if $\gamma \in C_{S_{\mathcal{U}}}(\Gamma)$, then there exists $g \in \prod_{\mathcal{D}} \rho_n[\, G_n\,]$ such that $\pi(g) = \gamma$. To see this, let $\varphi = (\varphi_n)_{\mathcal{D}}$ be an element such that $\pi(\varphi) = \gamma$ and fix some $0 < \varepsilon < 1/3$. Since $(\, \mathrm{Cay}(G_n, S_n) : n \in \omega\,)$ is an expander family, Proposition 2.3 implies that there exists $\delta > 0$ such that for all $n \in \omega$, if $Y \subseteq G_n$ is a subgraph of $\mathrm{Cay}(G_n, S_n)$ with $|Y| \geq (1-\delta)|G_n|$, then there exists a connected subgraph $Z \subseteq Y$ with $|Z| \geq (1-\varepsilon)|G_n|$. For each $n \in \omega$, let $Y_n \subseteq G_n$ be the set of elements $y \in G_n$ such that

$$s\,\varphi_n(y) = \varphi_n(s\,y) \quad \text{for all } s \in S_n \cup S_n^{-1}. \tag{2.1}$$

Then $A_\varepsilon = \{\, n \in \omega : |Y_n| \geq (1-\delta)|G_n|\,\} \in \mathcal{D}$. Fix some $n \in A_\varepsilon$. Then regarding Y_n as a subgraph of the Cayley graph $\mathrm{Cay}(G_n, S_n)$, there exists a connected subgraph $Z_n \subseteq Y_n$ such that $|Z_n| \geq (1-\varepsilon)|G_n|$. Fix some $z_n \in Z_n$ and let $\varphi_n(z_n) = z_n g_n$. Then applying (2.1) repeatedly, we obtain that $\varphi_n(z) = z g_n$ for all $z \in Z_n$. Note that if $g_n' \in G_n$ with $g_n' \neq g_n$, then $x g_n' \neq x g_n$ for all $x \in G_n$. Hence if $0 < \varepsilon' < 1/3$ and $n \in A_{\varepsilon'} \cap A_\varepsilon$, then the above argument will yield precisely the same element $g_n \in G_n$. Hence, letting $g_n = 1$ for $n \notin A_\varepsilon$, it follows that

$$\pi(\, (\rho_n(g_n))_{\mathcal{D}}\,) = \pi(\, (\varphi_n)_{\mathcal{D}}\,) = \gamma,$$

as required. □

3. Ultraproducts of finite alternating groups

In this section, we will complete the proof of Theorem 1.1. We will make use of the following recent result of Kassabov [4].

Theorem 3.1. *For each $n \geq 5$, there exists a generating subset $S_n \subseteq \mathrm{Alt}(n)$ with $|S_n| = 20$ such that $(\,\mathrm{Cay}(\mathrm{Alt}(n), S_n) : n \geq 5\,)$ is an expander family.*

The following result is an immediate consequence of Theorems 2.2 and 3.1.

Theorem 3.2. *For each nonprincipal ultrafilter $\mathcal{D}$ over ω, there exists a nonprincipal ultrafilter $\mathcal{U}$ over ω and a finitely generated subgroup $\Gamma \leqslant S_{\mathcal{U}}$ such that $C_{S_{\mathcal{U}}}(\Gamma) \cong \prod_{\mathcal{D}} \mathrm{Alt}(n)$.*

We will also make use of the following recent result of Ellis-Hachtman-Schneider-Thomas [3].

Theorem 3.3. *If CH fails, then there exist $2^{2^{\aleph_0}}$ ultraproducts $\prod_{\mathcal{U}} \mathrm{Alt}(n)$ up to isomorphism.*

Sketch Proof. By Allsup-Kaye [1], if $\mathcal{U}$ is a nonprincipal ultrafilter over ω, then there is an inclusion-preserving bijection between the collection of proper normal subgroups of $\prod_{\mathcal{U}} \mathrm{Alt}(n)$ and the linearly ordered set

$$L_{\mathcal{U}} = \left\{ I \subseteq \prod\nolimits_{\mathcal{U}} n : I \text{ is an additive cut of } \prod\nolimits_{\mathcal{U}} \mathbb{N} \right\},$$

where an *additive cut* is a nonempty initial segment of the nonstandard model of arithmetic $\mathcal{M} = \prod_{\mathcal{U}} \mathbb{N}$ which is closed under addition in $\mathcal{M}$. A routine modification of the proof of Kramer-Shelah-Tent-Thomas [6, Theorem 3.3] shows that if CH fails, then there exist $2^{2^{\aleph_0}}$ such linearly ordered sets $L_{\mathcal{U}}$ up to isomorphism. □

Proof of Theorem 1.1. Let $\{\, \mathcal{D}_\alpha : \alpha < 2^{2^{\aleph_0}} \,\}$ be a collection of nonprincipal ultrafilters over ω such that the corresponding ultraproducts $\prod_{\mathcal{D}_\alpha} \mathrm{Alt}(n)$ are pairwise nonisomorphic. Then for each $\alpha < 2^{2^{\aleph_0}}$, there exists a nonprincipal ultrafilter $\mathcal{U}_\alpha$ over ω and a finitely generated subgroup $\Gamma_\alpha \leqslant S_{\mathcal{U}_\alpha}$ such that $C_{S_{\mathcal{U}_\alpha}}(\Gamma_\alpha) \cong \prod_{\mathcal{D}_\alpha} \mathrm{Alt}(n)$. Fix some $\alpha < 2^{2^{\aleph_0}}$. Since $|S_{\mathcal{U}_\alpha}| = 2^{\aleph_0}$, it follows that $S_{\mathcal{U}_\alpha}$ has only $2^{\aleph_0}$ finitely generated subgroups and hence there exist at most $2^{\aleph_0}$ ordinals $\beta < 2^{2^{\aleph_0}}$ such that $S_{\mathcal{U}_\alpha} \cong S_{\mathcal{U}_\beta}$. It follows that $\{\, S_{\mathcal{U}_\alpha} : \alpha < 2^{2^{\aleph_0}} \,\}$ includes a collection of $2^{2^{\aleph_0}}$ pairwise nonisomorphic groups. □

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