ESI The Erwin Schrödinger International Institute for Mathematical Physics
Boltzmanngasse 9
A-1090 Wien, Austria

The Filter Dichotomy and Medial Limits

Paul Larson

Vienna, Preprint ESI 2173 (2009)

August 26, 2009

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

The Filter Dichotomy and medial limits

Paul Larson*

September 23, 2009

Abstract

The *Filter Dichotomy* says that every uniform nonmeager filter on the integers is mapped by a finite-to-one function to an ultrafilter. The consistency of this principle was proved by Blass and Laflamme. A function between topological spaces is *universally measurable* if the preimage of every open subset of the codomain is measured by every Borel measure on the domain. A *medial limit* is a universally measurable function from $\mathcal{P}(\omega)$ to the unit interval $[0, 1]$ which is finitely additive for disjoint sets, and maps singletons to 0 and ω to 1. Christensen and Mokobodzki independently showed that the Continuum Hypothesis implies the existence of medial limits. We show that the Filter Dichotomy implies that there are no medial limits.

2000 Mathematics Subject Classification 03E35; 28A20

1 Universally measurable sets

A *measure* on a set X is a function μ whose domain is some σ-algebra of subsets of X, with codomain $[0, \infty]$, such that μ is countably additive for disjoint families (by default, measures are countably additive, but we deal also with *finitely additive measures*, which are finitely additive but not necessarily countably additive). A set is said to be *measurable* with respect to μ if it is in the domain of μ. A *Borel measure* is a measure on a topological space whose domain contains the Borel subsets of the space. A measure is *complete* if all subsets of sets of measure 0 are in the domain of the measure (and thus have measure 0). The *completion* of a measure is the smallest complete measure extending it. If μ is a Borel measure on a topological space X, and μ^* is the completion of μ, then a set $A \subseteq X$ is in the domain of μ^* if and only if there is a set B in the domain of μ such that the symmetric difference $A \triangle B$ is contained in a set of μ-measure 0 (see 212C of [8]). A measure μ on a set X is a *probability* measure if $\mu(X) = 1$, *finite* if $\mu(X)$ is finite, *σ-finite* if X is a countable union of sets of finite measure, and *atomless* if singletons have measure 0.

A subset of a topological space is said to be *universally measurable* if it is measurable with respect to every complete σ-finite Borel measure on the space

*Supported in part by NSF grant DMS-0801009.

(see [15, 19], and 434D of [9], for instance). The collection of universally measurable sets does not change if one replaces "σ-finite" with "finite" or requires the measures to be atomless (see 211X(e) of [8]).

We will make use of the following standard observation.

1.1 Remark. If $f: X \to Y$ is a homeomorphism between topological spaces, then any Borel measure μ on Y induces a measure μ^* on X defined by letting $\mu^*(B) = \mu(f[B])$ for any Borel $B \subseteq X$. It follows that, in this context, for any universally measurable $A \subseteq X$, $f[A]$ is a universally measurable subset of Y.

2 Medial limits

A function between topological spaces is *universally measurable* if all preimages of open sets (equivalently, Borel sets) are universally measurable. For our purposes, a *medial limit* is a universally function from $\mathcal{P}(\omega)$ to $[0, 1]$ which is finitely additive for disjoint sets, and maps singletons to 0 and ω to 1 (i.e., a universally measurable finitely additive measure on $\mathcal{P}(\omega)$ giving ω measure 1). Equivalently, a *medial limit* is a function $f: \mathcal{P}(\omega) \to [0, 1]$, finitely additive for disjoint sets and mapping singletons to 0 and ω to 1, such that for every complete, σ-finite Borel measure μ on $\mathcal{P}(\omega)$ there is a Borel function $b: \mathcal{P}(\omega) \to [0, 1]$ such that $\{x \subseteq \omega \mid f(x) \neq b(x)\}$ is μ-null. Medial limits (in various forms) appear the following publications, among others : [17, 24, 7, 14, 5, 16, 12, 13]. Christensen and Mokobodzki (see [4, 21]) independently showed that medial limits exist under the assumption that the Continuum Hypothesis holds. This assumption was weakened to Martin's Axiom by Normann [20]. As far as we know, the weakest hypothesis known to be sufficient (see 538S of [10]) is the statement that the reals are not a union of fewer than continuum many meager sets (i.e., that the covering number for the meager ideal is the continuum). The term "medial limit" is often used for the corresponding linear functional on ℓ^∞ (see [21]; 538Q of [10]).

An ideal I on ω is said to be *c.c.c. over Fin* if all almost disjoint families (i.e., families of infinite subsets of ω which pairwise have finite intersection) disjoint from I are countable (this is Definition 3.3.1 of [6]). We say that an ideal on ω is *uniform* if it contains all finite subsets of ω, and similarly that a filter is uniform if its corresponding ideal is.

Theorem 2.1. *If f is a medial limit and $I = \{x \subseteq \omega \mid f(x) = 0\}$, then I is a universally measurable uniform c.c.c. over Fin ideal.*

Proof. That I is a universally measurable ideal follows from the definition of medial limit. To see that I is c.c.c. over Fin, let A be an almost disjoint family disjoint from I, and suppose that A is uncountable. Then there is a positive integer n such that the set of $x \in A$ such that $f(x) \geq 1/n$ is uncountable. Since f is finitely additive for almost disjoint sets, and $f(\omega) = 1$, any such set can have size at most n. $\square$

The following is Lemma 3.3.2(c) of [6].

Theorem 2.2. *A uniform c.c.c. over Fin proper ideal on ω cannot have the property of Baire.*

Proof. Let I be a uniform c.c.c. over Fin proper ideal on ω. If I is somewhere comeager, than there are two members of I whose union is cofinite. To see this, suppose that $s \subseteq \omega$ is finite and D_n $(n \in \omega)$ are dense open subsets of $\mathcal{P}(\omega)$ such that all $x \in [s] \cap \bigcap_{n\in\omega} D_n$ are in I, where $[s] = \{x \subseteq \omega \mid x \cap (max(s) + 1) = s\}$. It is relatively straightforward to build sets x, y in $[s] \cap \bigcap_{n\in\omega} D_n$ whose union is $\omega \setminus s$. Similarly, if I is meager, then there is a perfect set of subsets of ω which are almost disjoint and all not in I. The construction of such a perfect set is similar, using a collection of dense open sets D_n $(n \in \omega)$ such that $\bigcap_{n\in\omega} D_n$ is disjoint from I. □

Godefroy and Talagrand [11] proved in 1977 that if f is a medial limit, then the filter $\{x \subseteq \omega \mid f(x) = 1\}$ does not have the property of Baire.

3 The Filter Dichotomy

3.1 Definition. The *Filter Dichotomy* is the statement that for each nonmeager filter F on ω, there is a finite-to-one function $h \colon \omega \to \omega$ such that $\{h[x] \mid x \in F\}$ is an ultrafilter.

Blass and Laflamme showed [1] that Filter Dichotomy holds in models previously considered by Miller [18] and Blass and Shelah [2, 3]. Note that a filter on ω is comeager if and only if its corresponding ideal is.

Theorem 3.2. *The Filter Dichotomy implies that universally measurable uniform filters on ω are meager.*

Proof. Let F be a nonmeager universally measurable uniform filter on ω, and let $h \colon \omega \to \omega$ be finite-to-1 such that $\{h[x] \mid x \in F\}$ is an ultrafilter. Let

$$S = \{\bigcup_{n\in Z} h^{-1}[n] \mid Z \subseteq \omega\},$$

and let $G \colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be defined by $G(x) = h[x]$. Then:

- S is a perfect subset of $\mathcal{P}(\omega)$.
- $F \cap S$ is a universally measurable subset of S.
- $G \restriction S \colon S \to \mathcal{P}(\omega)$ is a homeomorphism.
- $G[F \cap S] = G[F]$ is not Lebesgue measurable.

This gives a contradiction, by Remark 1.1.

The first and third items above are easy, and the fourth follows from the fact that nonprincipal ultrafilters are not Lebesgue measurable ([22]; to see that $G[F]$ has to be nonprincipal, note that F is uniform and h is finite-to-1). To see the second item above, suppose that μ is a finite Borel measure on S. Define a measure μ^* on $\mathcal{P}(\omega)$ by letting $\mu^*(A) = \mu(A \cap S)$ for all Borel $A \subseteq \mathcal{P}(\omega)$. Since F is universally measurable, there exist Borel sets B and N such that $F \triangle B \subseteq N$ and $\mu^*(N) = 0$. Then $\mu(N \cap S) = 0$, and $(F \cap S) \triangle (B \cap S) \subseteq (N \cap S)$. □

Putting together the Blass-Laflamme result with Lemmas 2.1 and 2.2 and Theorem 3.2, we have the following.

Corollary 3.3. *If ZFC is consistent, then so is ZFC + "there exist no medial limits."*

Talagrand [23] proved that a filter on ω is meager if and only if there exists a finite-to-one function $h\colon \omega \to \omega$ such that $\{h[x] \mid x \in F\}$ is the set of cofinite subsets of ω (filters can be replaced in his result with sets closed under supersets and finite changes, and one can use this to generalize our result as well). The Filter Dichotomy can then be restated as: for every uniform filter F on ω there is a finite-to-one function $h\colon \omega \to \omega$ such that $\{h[x] \mid x \in F\}$ is either the cofinite filter or a nonprincipal ultrafilter. The condition that h is finite-to-one is used in the proof of Theorem 3.2 only to get the image ultrafilter to be nonprincipal. Blass has pointed out to us that the finite-to-one condition can be relaxed in the other (meager) case as well, via the following argument.

Theorem 3.4. *Suppose that $f\colon \mathcal{P}(\omega) \to [0,1]$ is finitely additive for disjoint sets, sends singletons to 0 and ω to 1. Let $F = \{x \subseteq \omega \mid f(x) = 1\}$, and let $h\colon \omega \to \omega$ be such that $f(h^{-1}[n]) = 0$ for all $n \in \omega$. Then $\{h[x] \mid x \in F\}$ is not the cofinite filter on ω.*

Proof. First note that $y \mapsto f(h^{-1}[y])$ defines a function from $\mathcal{P}(\omega)$ to $[0,1]$ which is finitely additive for disjoint sets, sends singletons to 0 and ω to 1. Furthermore, $\{h[x] \mid x \in F\} = \{y \subseteq \omega \mid f(h^{-1}[y]) = 1\}$, so it suffices to consider the case where h is the identity function, and to show that F is not the cofinite filter. Split ω into two infinite pieces, and let A_0 be the one with smaller measure with respect to f, or either piece in case of a tie. Let B_0 be the other piece minus its first element. Then $f(A_0) \leq 1/2$, $f(B_0) \geq 1/2$ and $|\omega \setminus (A_0 \cup B_0)| = 1$ (note that finite sets have f-measure 0). Split A_0 into two infinite pieces, let A_1 be the smaller one with respect to f, and let B_1 be the larger one minus its first element. Then $f(A_1) \leq 1/4$, $f(B_0 \cup B_1) \geq 3/4$ and $|\omega \setminus (A_1 \cup B_0 \cup B_1)| = 2$. Continue in this way, defining infinite sets A_n and B_n for each $n \in \omega$ such that $f(A_n) \leq 2^{-n-1}$, $f(B_0 \cup \ldots \cup B_n) \geq 1 - 2^{-n-1}$ and $|\omega \setminus (A_n \cup B_0 \cup \ldots \cup B_n)| = n + 1$. Then $\bigcup_{n \in \omega} B_n$ has f-measure 1, but is not cofinite. □

From this it follows that the nonexistence of medial limits follows from the following weak form of the Filter Dichotomy: for every uniform filter F on ω there exists a function $h\colon \omega \to \omega$ such that $h^{-1}[\omega \setminus \{n\}] \in F$ for all $n \in \omega$ and $\{h[x] \mid x \in F\}$ is either the cofinite filter or a nonprincipal ultrafilter. Since our proof uses only the universally measurability of the filter of measure 1 sets, we have that this weak form of the Filter Dichotomy implies that whenever f is a finitely additive measure on $\mathcal{P}(\omega)$ such that $f(\omega) = 1$, the filter $\{x \subseteq \omega \mid f(x) = 1\}$ is not universally measurable.

3.5 Definition. Given two filters G and F on ω, G is said to be *Rudin-Keisler* below F ($G \leq_{RK} F$) if there is a a function $h\colon \omega \to \omega$ such that $\{h[x] \mid x \in F\}$ generates the filter G.

The following is Definition 538Ah of [10]. We refer the reader there or to [15] for the definition of *probability space*.

3.6 Definition. A filter $\mathcal{F}$ on ω is said to satisfy the *Fatou property* if for any probability space (X, Σ, μ), if $\langle E_n : n \in \omega \rangle$ is a sequence in Σ, and $X = \bigcup_{A \in \mathcal{F}} \bigcap_{n \in A} E_n$, then $\lim_{n \to \mathcal{F}} \mu(E_n)$ is defined and equal to 1.

The nonexistence of medial limits under the Filter Dichotomy also follows from the following facts, where f is a supposed medial limit and F is the filter $\{x \subseteq \omega \mid f(x) = 1\}$.

- The filter F is uniform and nonmeager. ([11])
- The Filter Dichotomy implies that every uniform nonmeager filter on ω is Rudin-Keisler above a nonprincipal ultrafilter on ω.
- The filter F has the Fatou property. (Proposition 538Rd of [10])
- If G is a filter such that $F \geq_{RK} G$, and F has the Fatou property, then so does G. (Proposition 538Ob of [10])
- No nonprincipal ultrafilter on ω has the Fatou property (Exercise 538Xn of [10])

References

[1] A. Blass, C. Laflamme, *Consistency results about filters and the number of inequivalent growth types*, J. Symbolic Logic 54 (1989), no. 1, 50–56

[2] A. Blass, S. Shelah, *There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed*, Ann. Pure Appl. Logic 33 (1987), no. 3, 213–243

[3] A. Blass, S. Shelah, *Near coherence of filters. III. A simplified consistency proof*, Notre Dame J. Formal Logic 30 (1989), no. 4, 530–538

[4] J.P.R. Christensen, **Topology and Borel structure**, North Holland, Amsterdam, 1974

[5] A. Connes, **Noncommutative Geometry**, Academic Press, 1994

[6] I. Farah, *Analytic Quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc. 148 (2000), no. 702

[7] A. Fisher, *Convex-invariant means and a pathwise central limit theorem*, Adv. in Math. 63 (1987) 3, 213–246

[8] D.H. Fremlin, **Measure Theory**, Volume Two, Torres Fremlin, 2001

[9] D.H. Fremlin, **Measure Theory**, Volume Four, Part I, Torres Fremlin, 2003

[10] D.H. Fremlin, **Measure Theory**, Volume Five, Torres Fremlin, 2008

[11] G. Godefroy, M. Talagrand, *Filtres et mesures simplement additives sur N*, Bull. Sci. Math. (2) 101 (1977) 3, 283–286

[12] S. Jackson, A.S. Kechris, A. Louveau, *Countable Borel equivalence relations*, J. Math. Log. 2 (2002) 1, 1–80

[13] V.A. Kaimanovich, *Amenability and the Liouville property*, Israel J. Math. 149 (2005), 45–85

[14] A. Kechris, *Amenable equivalence relations and Turing degrees*, J. Symb. Logic 56 (1991) 1, 182–194

[15] A. Kechris, **Classical Descriptive Set Theory**, Springer-Verlag, 1995

[16] L. Lauwers, *Intertemporal objective functions: strong Pareto versus anonymity*, Math. Social Sci. 35 (1998) 1, 37–55

[17] R.D. Mauldin, D. Preiss, H. v. Weizsäcker, *Orthogonal transistion kernels*, Ann. Prob. 11 (1983) 4, 970-988

[18] A. Miller, *Rational perfect set forcing*, Contemporary Mathematics 31 (1984), 143-159

[19] T. Nishiura, **Absolute Measurable Spaces**, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2008

[20] D. Normann, *Martin's axiom and medial functions*, Math. Scand. 38 (1976), no. 1, 167–176

[21] P.A. Meyer, *Limites médiales, d'après Mokobodzki*, Séminaire de Probabilités, VII (Univ. Strasbourg, année universitaire 1971–1972), pp. 198–204. Lecture Notes in Math., Vol. 321, Springer, Berlin, 1973

[22] W. Sierpiński, *Fonctions additives non complètement additives et fonctions non mesurables*, Fund. Math. 30 (1938), 96–99

[23] M. Talagrand, *Compacts de fonctions mesurables et filtres non mesurables*, Studia Mathematicae 67 (1980), 13 43

[24] M. Talagrand, *Invariant means on an ideal*, Trans. Amer. Math. Soc. 288 (1985) 1, 257–272

Department of Mathematics and Statistics
Miami University
Oxford, Ohio 45056
United States
larsonpb@@muohio.edu