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# The Local Structure of some Complex Poisson Brackets

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# THE LOCAL STRUCTURE OF SOME COMPLEX POISSON BRACKETS

Andriy Panasyuk

Abstract. We consider structures of complex Poisson brackets on the space of $C^\infty$-functions on complex $2n$-dimensional manifold generated by $\partial$-closed non-degenerate $(2,0)$-form (with non-holomorphic coefficients). Is the Darboux theorem valid for such structures? We show that the local structure of brackets depends on equivalence class of corresponding $(2,0)$-form under biholomorphic maps.

## 0.Introduction

This work takes its origin in the papers [1], [2]. It deals with some examples of Poisson algebras (Hamiltonian in terminology of [1]) over the field of complex numbers $\mathbb{C}$. These algebras appear naturally if one tries to generalize the real results of [2] to the complex case (see [3], [4]). The above mentioned examples of Poisson algebras are built as follows.

Let $M$ be a complex manifold of even dimension, $A(M)=C^\infty(M,\mathbb{C})$ the space of smooth complex valued functions on $M$, $\omega$ a nondegenerate $\partial$-closed $(2,0)$-form of class $C^\infty$ on $M$. If $c$ is an inverse $(2,0)$-vector field on $M$ it satisfies the equation $[c,c]=0$, where $[\,,]$ is a Schouten bracket (see [5]). The validity of this equation is a necessary and suficient condition for $c$ to generate a Lie algebra structure on $A(M)$ by

$$\{f,g\}=c(f)g, \tag{1}$$

where $f,g\in A(M), c(f)$ is a $(1,0)$-vector field obtained by convolution on the first index of bivector field $c$ and $(1,0)$-form $\partial f$ ($c(f)$ is called the Hamiltonian vector field). In holomorphic local coordinates $z^1,\dots,z^{2n}$ $\omega$ is written as $\omega=\omega_{ij}(z,\bar z)dz^i\wedge dz^j$ (the summation convention is used, $z$ and $\bar z$ are $2n$-tuples), $c$ is equal to $c^{kl}(z,\bar z)\frac{\partial}{\partial z^k}\wedge\frac{\partial}{\partial z^l}$, where $c^{kl}(z,\bar z)\cdot\omega_{sl}(z,\bar z)=\delta^k_s$ (Kronecker symbol), and the corresponding Lie bracket is given by

$$\{f,g\}=c^{kl}(z,\bar z)\frac{\partial f}{\partial z^k}\frac{\partial g}{\partial z^l},\quad f,g\in A(M).$$

If the coefficients $\omega_{ij}$ are holomorphic (do not depend on $\bar z$-coordinates), we obtain a holomorphic symplectic structure on $M$. To distinguish our more general situation from the latter we call our $(2,0)$-forms and corresponding brackets $\partial$-symplectic. To obtain a natural generalization of our construction one should take a $CR$-manifold $M$, $dim_{CR}M=2n$, and nondegenerate $\partial$-closed $(2,0)$-form $\omega$ (again called $\partial$-symplectic) on $M$. Formula (1) gives a structure of Lie algebra on $A(M)$ once more

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although the corresponding local coordinate formulas are not valid for general $CR$-manifold (inverse bivector field $c$ can be defined in an arbitrary noncoordinate basis of vector fields).

The main question which we try to answer in this paper is the following: is the Darboux theorem valid for $\partial$-symplectic brackets on complex manifold? More precisely we can formulate two different questions.

a) Let $\omega_0$ be a standard symplectic form on $\mathbb{C}^{2n}$ which can be written in coordinates $z^1, \dots, z^{2n}$ as $\omega_0 = dz^1 \wedge dz^{n+1} + \cdots + dz^n \wedge dz^{n+n}$. Can we find for arbitrary $\partial$-symplectic form $\omega$ on complex manifold $M$ a local $C^\infty$- diffeomorphism $\psi$ such that $\omega = \psi^*\omega_0$? Let us note that for general $\omega$ the map $\psi$ can not be biholomorhic and must be very specific: $(1,1)$- and $(0,2)$-components of the form $\psi^*\omega_0$ must be equal to zero.

b) Let $\{\,,\}_0$ be a Lie bracket associated to standard form $\omega_0$ on $\mathbb{C}^{2n}$. Can we find for an arbitrary $\partial$-symplectic bracket $\{\,,\}$ which corresponds to $\partial$-symplectic form $\omega$ on complex manifold $M$ open sets $U_1 \subset \mathbb{C}^{2n}, U_2 \subset M$ and a map $\varphi : A(U_1) \longrightarrow A(U_2)$ such that $\varphi$ is an isomorphism of Poisson algebras $(A(U_1), \{\,,\}_0, \cdot)$ and $(A(U_2), \{\,,\}, \cdot)$, where the dot denotes multiplication of functions?

It follows from our main theorem (Theorem 1) that if we have an isomorphism of Poisson algebras such as mentioned above it coincides with a map $\psi^* : A(U_1) \longrightarrow A(U_2)$, where $\psi : U_2 \longrightarrow U_1$ is a biholomorphic map such that $\omega_0 = \psi^*\omega$. Thus we obtain the negative answer for both questions (see Corollary 1, p.7) and it is clear that the study of the local structure of $\partial$-symplectic brackets is reduced to the study of the classes of equivalence of $\partial$-symplectic forms under local biholomorphic transformations.

One can consider this paper also as a continuation of a series of papers in which the following problem is stated: for two manifolds $M_1$ and $M_2$ (with some additional structures) and Lie isomorphism $\varphi : L_1 \longrightarrow L_2$, where $L_1$ and $L_2$ are certain Lie algebras of vector fields on $M_1$ and $M_2$ respectively, can we find a diffeomorfism $\psi : M_1 \longrightarrow M_2$ (preserving additional structures) which induces $\varphi$? This problem was solved in papers [7], [8], [9] respectively for Lie algebras of all smooth vector fields, Lie algebras of smooth $(1,0)$-vector fields on complex manifold and Lie algebras associated to symplectic structure (Lie algebras of local and global Hamiltonian vector fields e.t.c.). We consider Lie algebras of Hamiltonian vector fields associated with $\partial$-symplectic bracket and solve the above mentioned problem for them. Our proof is modelled on the proof of the same theorem for real and complex analytic cases in [9] but differs from the latter in some points. We have kept the most of the notations of [9].

In the last section of the paper we list some open questions.

## 1.LOCAL ISOMORPHISMS OF $\partial$-SYMPLECTIC BRACKETS

Since we are interested in local questions all our manifolds $M_i$ will be open connected subsets in $\mathbb{C}^{2n}$. We shall denote the space of smooth complex valued functions on $M_i$ by $A(M_i)$, the space of smooth $(1,0)$-vector fields on $M_i$ by $?(M_i)$ and the space of smooth $(1,0)$-forms on $M_i$ by $\Omega^1(M_i)$ ("smooth" is the synonym to "$C^\infty$-smooth" in this paper). If $\omega_i$ is a $\partial$-symplectic form of class $C^\infty$ on $M_i$ let $L^*(\omega_i)$ be the space of all (globally) Hamiltonian vector fields associated with

$\omega_i$ (see Introduction for definitions). The map $f \mapsto c(f) : A(M_i) \longrightarrow ?(M_i)$ is a Lie homomorphism with image $L^*(\omega_i)$ and kernel $C(M_i)$ which coincides with the space $\overline{\mathcal{O}}(M_i)$ of antiholomorphic functions on $M_i$. Let $\mu_i$ denote a $C^\infty$-isomorphism between the holomorphic tangent bundle $T'M_i$ and the holomorphic cotangent bundle $(T'M_i)^*$ which is extended to the isomorphism between $?(M_i)$ and $\Omega^1(M_i)$ and is given by $\mu_i(v)(\cdot) = \omega_i(v, \cdot), \quad v \in ?(M_i)$.

**Theorem 1.** *Let $M_i$, $i = 1, 2$, be an open connected set in $\mathbb{C}^{2n}$, $n \geq 1$, and let $\omega_i$ be a $\partial$-symplectic form of class $C^\infty$ on $M_i$ with the corresponding Poisson bracket $\{\,,\}_i$ on $A(M_i)$. Let $\varphi : (A(M_1), \{\,,\}_1, \cdot) \longrightarrow (A(M_2), \{\,,\}_2, \cdot)$ be an isomorphism of Poisson algebras (the dot denotes the multiplication of functions). Then there exists a biholomorphic map $\psi : M_2 \longrightarrow M_1$ such that $\varphi = \psi^*$ and $\omega_2 = \psi^*\omega_1$, where $\psi^*$ denotes the pull-back map.*

The scheme of the proof of the theorem is the following. The isomorphism $\varphi$ induces an isomorphism of Lie algebras $\bar{\varphi} : L^*(\omega_1) \longrightarrow L^*(\omega_2)$ since $L^*(\omega_i) \cong A(M_i)/C(M_i)$. At first we shall build a one to one correspondence between $M_i$ and $\Sigma(L(\omega_i))$ - Lie spectrum (certain set of Lie subalgebras) of $L(\omega_i)$ (Theorem 2). Then we shall show that the map $\psi : M_2 \longrightarrow M_1$ which is generated by $\bar{\varphi}$ and this correspondence satisfies some conditions (Theorems 3 and 4) which imply the needed properties of $\psi$.

For arbitrary Poisson algebra $A$ over $\mathbb{C}$ let $\Sigma(A)$ denote the family of all self-normalizing maximal proper finite-codimensional Lie subalgebras of $A$ and $\mathcal{M}(A)$ denote the family of all maximal proper finite-codimensional associative ideals of $A$. The following result is proved in [10] and [11].

**Lemma 1.** *Suppose the Poisson algebra $A$ satisfies:*
*a) $A^2 = A$;*
*b) for any $J \in \mathcal{M}(A)$ the Lie normalizer*

$$\mathcal{N}_A(J) = \{f \in A : [f, J] \subset J\}$$

*is a proper finite-codimensional linear subspace of $A$.*

*Then the mapping $J \mapsto \mathcal{N}_A(J)$ establishes a one-one correspondence between $\mathcal{M}(A)$ and $\Sigma(A)$.*

Now again let $M_i$ be an open connected subset in $\mathbb{C}^{2n}$, $\omega_i$ - $\partial$-symplectic form of class $C^\infty$. Given $p \in M$, define

$$p^* = \{f \in A(M_i) : f(p) = 0\}$$

$$N(M_i, p) = \{f \in A(M_i) : \partial f(p) = 0\}$$

$$L^*(\omega_i)_p = \{v \in L^*(\omega_i) : v(p) = 0\}.$$

**Lemma 2.** *The map $p \mapsto p^*$ furnishes a bijection between $M_i$ and $\mathcal{M}(A(M_i))$.*

The proof of this fact is the same as for associative algebra $\mathcal{O}(M_i)$ of holomorphic functions on $M_i$ (see [12],p.17). Similarly the proof of the next two lemmas and theorem is exactly the same as in [9], p.328-329, modulo substitution of $d$ by $\partial$.

**Lemma 3.** $\mathcal{N}_{A(M_i)}(p^*) = N(M_i, p)$.

**Lemma 4.** *The map* $p \mapsto N(M_i, p)$ *constitutes a bijection of* $M_i$ *with* $\Sigma(A(M_i))$.

**Theorem 2.** *The map* $s_i : p \mapsto L^*(\omega_i)_p$ *constitutes a bijection of* $M_i$ *with* $\Sigma(L^*(\omega_i))$.

The next two lemmas are needed for the proof of Theorem 3.

**Lemma 5.** *If* $M$ *is a complex manifold, then for arbitrary* $p \in M$
*a)* $\partial$*-exact smooth* $(1,0)$*-forms taken in* $p$ *generate* $(T'_pM)^*$*;*
*even more precisely*
*b)* $\partial$*-exact holomorphic* 1*-forms taken in* $p$ *generate* $(T'_pM)^*$*.*

The proof is obvious.

**Lemma 6.** *Let* $M_i$*,* $i = 1, 2$*, be an open connected set in* $\mathbb{C}^{2n}$*,* $n \geq 1$*, write* $A_i$ *for* $A(M_i)$*. Suppose* $\hat{\varphi} : B^1(M_1) \longrightarrow B^1(M_2)$ *is a* $\mathbb{C}$*-linear surjective map, where* $B^1(M_i)$ *denotes the space of* $\partial$*-exact smooth* $(1,0)$*-forms on* $M_i$*. Suppose also that there exists a map* $\psi : M_2 \longrightarrow M_1$ *which is not constant such that, for any* $p \in M_2$ *and* $f \in A_1$*,*

$$[\hat{\varphi}(\partial f)](p) = 0 \iff \partial f(\psi(p)) = 0. \tag{2}$$

*Then: 1)* $\psi$ *is holomorphic; 2) there exits* $e \in \overline{\mathcal{O}}(M_2)$ *such that, for all* $f \in A_1$*,*

$$e \cdot \hat{\varphi}(\partial f) = \psi^*(\partial f).$$

*Proof.* The same considerations as in [9], p.331 (in which $d$ is substituted by $\partial$) show that $\psi$ is of class $C^\infty$. Let us show that $\psi$ is holomorphic. Obviously

$$\{\partial(fg) - f(\psi(p))\partial g - g(\psi(p))\partial f\}(\psi(p)) = 0$$

for all $f, g \in A_1$. This equation is equivalent by (2) to

$$\hat{\varphi}(\partial(fg)) = (f \circ \psi)\hat{\varphi}(\partial g) + (g \circ \psi)\hat{\varphi}(\partial f). \tag{3}$$

Now substitute arbitrary holomorphic $f$ and arbitrary antiholomorphic $g$ in (3):

$$\hat{\varphi}(g\partial f) = (g \circ \psi)\hat{\varphi}(\partial f). \tag{4}$$

Differentiate (4) with respect to $\partial$ remembering that $\hat{\varphi}$ is a map into $\partial$-closed forms:

$$\partial(g \circ \psi) \wedge \hat{\varphi}(\partial f) = 0. \tag{5}$$

Since $f$ is an arbitrary holomorphic function and $\hat{\varphi}$ is surjective, Lemma 5 b) implies that the dimension of the image of $\hat{\varphi}$ in every point is greater than 1 and (5) shows that

$$\partial(g \circ \psi) = 0 \tag{6}$$

for every $g \in \overline{\mathcal{O}}(M_1)$.

Let $z^1, \dots, z^{2n}$ and $w^1, \dots, w^{2n}$ be local holomorphic coordinates in $M_2$ and $M_1$ respectively and let $w^i = \psi^i(z^1, \dots, z^{2n}, \bar{z}^1, \dots, \bar{z}^{2n})$, $i = 1, \dots, 2n$, be a local expression of $\psi$. Then

$$\partial(g \circ \psi) = \frac{\partial g(\psi(z,\bar{z}), \overline{\psi(z,\bar{z})})}{\partial \bar{z}^i} \cdot \frac{\partial \overline{\psi^i(z,\bar{z})}}{\partial z^k} dz^k = 0, \tag{7}$$

where $z, \bar{z}, \psi(z, \bar{z})$ denote the corresponding vector-functions. Now recalling that $g \in \overline{\mathcal{O}}(M_1)$ is arbitrary we can use the conjugated version of Lemma 5 b) and conclude that

$$\frac{\partial \psi^i(z,\bar{z})}{\partial \bar{z}^k} = 0$$

for all $i, k = 1, \dots, 2n$.

To complete the proof one can use once more the reasoning from [9], p. 332-333, and show that there exists an antiholomorphic function $e \not\equiv 0$ on $M_2$ such that for any $f \in A_1$ and $p \in M_2$

$$\psi^*(\partial f)(p) = e(p)[\hat{\varphi}(\partial f)](p).$$

□

**Theorem 3.** *Let $M_i$, $i = 1, 2$, be an open connected subset in $\mathbb{C}^{2n}$, $n \geq 1$. Let $\omega_i$ be a $\partial$-symplectic form of class $C^\infty$ on $M_i$ and $\varphi : L^*(\omega_1) \longrightarrow L^*(\omega_2)$ be a Lie algebra isomorphism. Then there exists a biholomorphic map*

$$\psi : M_2 \longrightarrow M_1$$

*and a nowhere vanishing function $l \in C(M_2)$ such that*

$$\omega_2 = l \cdot \psi^*(\omega_1).$$

*Proof.* Write $A_i$ for $A(M_i)$ and $L_i$ for $L^*(\omega_i)$. Since $\varphi$ is a Lie algebra isomorphism $\varphi^{-1}(K) \in \Sigma(L_1)$ iff $K \in \Sigma(L_2)$. Define a map $\psi : M_2 \longrightarrow M_1$ by

$$\psi = s_1^{-1} \varphi^{-1} s_2,$$

where $s_i$ is from Theorem 2. It satisfies the following condition: for any $X \in L_1$ and $p \in M_2$

$$\varphi(X)(p) = 0 \iff X(\psi(p)) = 0. \tag{8}$$

This implies that the map $\hat{\varphi} : B^1(M_1) \longrightarrow B^1(M_2)$ defined by $\hat{\varphi} = \mu_2 \varphi \mu_1^{-1}$ and the map $\psi$ satisfy conditions of Lemma 6. Thus $\psi$ is holomorphic and there exists $e_1 \in \overline{\mathcal{O}}(M_2)$ such that for all $f \in A_1$

$$e_1 \cdot \hat{\varphi}(\partial f) = \psi^*(\partial f).$$

The above construction can also be applied to the inverse map $\varphi^{-1}$ and we shall find that the maps $\widehat{\varphi^{-1}} = \mu_1^{-1} \varphi^{-1} \mu_2 : B^1(M_2) \longrightarrow B^1(M_1)$ and $\psi^{-1} = s_2^{-1} \varphi s_1$ :

$M_1 \longrightarrow M_2$ also satisfy conditions of Lemma 6. Thus $\psi$ is biholomorphic and there exists $e_2 \in \overline{\mathcal{O}}(M_1)$ such that for all $f \in A_2$

$$e_2 \cdot \widehat{\varphi^{-1}}(\partial f) = (\psi^{-1})^*(\partial f).$$

Therefore $\psi^* : B^1(M_1) \longrightarrow B^1(M_2)$ is an isomorphism and we can conclude that $e_1$ does not have zeroes on $M_2$ and that $e_2 = 1/(e_1 \circ \psi^{-1})$. Now put $l = 1/e_1$ and recall that

$$\hat{\varphi} = l \cdot \psi^*.$$

The same considerations as in [9], p.336, show that the function on $M_2$

$$F(X, Y) = l\{\omega_1(X, Y) \circ \psi\} - \omega_2(\varphi(X), \varphi(Y)),$$

where $X, Y \in L_1$, is antiholomorphic. We shall show that $F(X, Y)$ is identically equal to zero for all $X, Y$.

At first note that $F(X_1, Y) \equiv F(X_2, Y)$ if $X_1 \equiv X_2$ on some open set $U \subset M_1$ by the uniqueness property of antiholomorphic functions since the function $F(X_1 - X_2, Y)$ is the identical zero on $\psi^{-1}$ (see (8)). Now for arbitrary $X = \mu_1^{-1}\partial f \in L_1,\ f \in A_1$, find two open sets $U_1, U_2 \subset M_1,\ U_1 \cap U_2 = \varnothing$, and a smooth function $g \in A_1$ such that $g \equiv f$ on $U_1$ and $g \equiv 0$ on $U_2$. Then $F(X, Y) \equiv F(Z, Y) \equiv 0$, where $Z = \mu_1^{-1}\partial g$. We obtain

$$\omega_2(\varphi(X), \varphi(Y)) = l\{\omega_1(X, Y) \circ \psi\} \tag{9}$$

for all $Z, Y \in L_1$.

Continueing the reasoning from [9], p.337, we deduce that

$$(\forall X \in L_1)(\forall p \in M_2) \ \ \psi_{*p}(\varphi(X)(p)) = X(\psi(p))$$

and substituting this in (9) we find that $(\forall X, Y \in L_1)(\forall p \in M_2)$

$$l(p)\omega_1(X(\psi(p)), Y(\psi(p))) = l(p)\{\psi^*\omega_1\}(\varphi(X)(p), \varphi(Y)(p))$$

$$= \omega_2(\varphi(X)(p), \varphi(Y)(p)). \tag{10}$$

Now recall that the vectors $\varphi(X)(p),\ X \in L_1$, span $T'_p M_2$ by Lemma 5 a). Thus (10) shows that $\omega_2 = l \cdot \psi^*(\omega_1)$ and Theorem 3 is proved. □

The following theorem is obtained as a consequence of Theorem 3 and is proved exactly as Theorem 8.7 in [9].

**Theorem 4.** *Let $M_i,\ i = 1, 2$, be an open connected subset in $\mathbb{C}^{2n},\ n \geq 1$, and let $\omega_i$ be a smooth $\partial$-symplectic form on $M_i$. Let $\varphi : A(M_1) \longrightarrow A(M_2)$ be a Lie algebra isomorphism. Then there exists a biholomorphic map $\psi : M_2 \longrightarrow M_1$, an everywhere non-zero function $l \in C(M_2)$ and a $\mathbb{C}$-linear map $\Phi : A(M_1) \longrightarrow C(M_2)$ such that $\omega_2 = l \cdot \psi^*(\omega_1)$ and $\varphi = l \cdot \psi^* + \Phi$.*

Now we are ready to complete the proof of Theorem 1. The only thing we need is to show that if $\varphi$ is multiplicative, then $l \equiv 1$ and $\Phi \equiv 0$. We have for multiplicative $\varphi$

$$l\psi^*(f^2) + \Phi(f^2) = (l\psi^*(f) + \Phi(f))^2 = l^2(\psi^*(f))^2 + 2l\psi^*(f)\Phi(f) + \Phi(d)^2,$$

where $f \in A(M_1)$ is arbitrary. Since $\psi^*(f^2) = (\psi^*(f))^2$ we obtain

$$(l - l^2)(\psi^*(f))^2 - 2l\Phi(g)\psi^*(f) + \Phi(f^2) - \Phi^2(f) = 0. \tag{10}$$

Apply $\partial$ to (10) using the fact that $l, \Phi(f), \Phi(f^2)$ are antiholomorphic

$$2\partial\psi^*(f)[(l - l^2)\psi^*(f) - l\Phi(f)] = 0.$$

Thus

$$(l - l^2)\psi^*(f) = l\Phi(f) \tag{11}$$

and since $\psi^*(f)$ is an arbitrary smooth function and the r.h.s. of (11) is antiholomorphic, $l = l^2$ and $\Phi(f) = 0$. Now recall that $l$ is everywhere non-zero and deduce that $l = 1$. Theorem 1 is proved. $\square$

**Corollary 1.** *Let $M_i$ be an open connected set in $\mathbb{C}^{2n}$, $n \geq 1$, and let $\omega_i$ be a $\partial$-symplectic form of class $C^\infty$ on $M_i$, $i = 1, 2$. Let $\varphi : M_2 \longrightarrow M_1$ be a $C^\infty$-diffeomorphism such that $\omega_2 = \varphi^*\omega_1$. Then $\varphi$ is biholomorphic.*

*Proof.* The diffeomorphism $\varphi$ induces an isomophism of Poisson algebras $\varphi^* : (A(M_1), \{\,,\}_1, \cdot) \longrightarrow (A(M_2), \{\,,\}_2, \cdot)$, where $\{\,,\}_i$ denotes the corresponding Poisson bracket on $A(M_i)$, $i = 1, 2$. It follows from Theorem 1 that $\varphi^*$ coincides with $\psi^*$, where $\psi : M_2 \longrightarrow M_1$ is some biholomorphic map, whence $\varphi = \psi$. $\square$

## 2. OPEN QUESTIONS

1. Is Theorem 1 valid in $C^\omega$-category? (We have used a flexibility of $C^\infty$-functions in the proof of Theorem 3, see p.6.)

2. Is Theorem 1 valid for $\partial$-symplectic structures on CR-manifolds? (See Introduction for definition.)

3. One can consider the problem of local equivalence of $\partial$-symplectic forms under biholomorphic maps as a problem of local equivalence of certain $G$-structures (see [13] for definitions).

Namely, let $\omega$ be a $C^\infty$-smooth nondegenerate $(2,0)$-form on a complex manifold $M$, $dim_{\mathbb{C}}M = 2n$, let $\omega_0 = dz^1 \wedge dz^{n+1} + \cdots + dz^n \wedge dz^{2n}$ be a standard symplectic form on $\mathbb{C}^{2n}$ with coordinates $z^j = x^j + iy^j$, $j = 1, \ldots, 2n$. Let $G \subset GL(4n, \mathbb{R})$ be a group of all $\mathbb{C}$-linear preserving $\omega_0$ transformations of $\mathbb{C}^{2n} \cong \mathbb{R}^{4n}$. One can build a $C^\infty$-smooth $G$-structure $S_\omega$ on $M$ considering the form $\omega$ as a tensor on the real tangent bundle $TM$. $G$-structure $S_\omega$ satifies the following condition.

(∗) If $u_p$, $p \in M$, is a smooth field of tangent repers which is a section of $S_\omega$ over some local chart $U \subset M$ with coordinates $z^j = x^j + iy^j$, $j = 1, \ldots, 2n$, then there exists a smooth field of endomorphisms $E_p : T_pM \longrightarrow T_pM$ over $U$ such that $E_p$ is $\mathbb{C}$-linear with respect to the natural complex structure on $T_pM$ and takes $u_p$ to $u_p^0 = ((\frac{\partial}{\partial x^1})_p, \ldots, (\frac{\partial}{\partial x^{2n}})_p, (\frac{\partial}{\partial y^1})_p, \ldots, (\frac{\partial}{\partial y^{2n}})_p)$.

Conversely, every $C^\infty$-smooth $G$-structure $S$ on a complex manifold $M$, $dim_{\mathbb{C}}M = 2n$, which satisfies condition $(*)$ induces some nondegenerate $C^\infty$-smooth $(2,0)$-form $\omega_S$ on $M$.

Two $C^\infty$-smooth satisfying $(*)$ $G$-structures $S_1,\ S_2$ on $M$ are equivalent if and only if there exists a $C^\infty$-diffeomorphism $f : M \longrightarrow M$ such that $\omega_{S_1} = f^*\omega_{S_2}$. In the case of $\partial$-symplectic structures this diffeomorphism is biholomorphic by Corollary 1. Thus the problem of local equivalence of $\partial$-symplectic forms under biholomorphic maps can be considered as the problem of local ($C^\infty$-smooth) equivalence of $G$-structures of a special kind. One can try to use the various approaches of the theory of $G$-structures to solve this problem.

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