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SPECTRAL INSTABILITY OF SEMICLASSICAL OPERATORS

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ABSTRACT. We give a short review of the spectral instability of non-normal semiclassical differential operators, both for scalar operators and systems.

1. INTRODUCTION

The spectral instability of non-normal operators is a phenomenon which has recently attracted interest in applications. It gives an obstruction to the accurate computation of eigenvalues of large non-normal matrices and has applications in a wide field, from random matrix theory, the stability of flows to things as mundane as brake squeal (see [72] for more examples). The standard example of the spectral instability of non-normal matrices is the following perturbation of the $N \times N$ Jordan matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \varepsilon & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \qquad \varepsilon > 0$$

which has characteristic equation $z^N + \varepsilon = 0$ and eigenvalues

$$\lambda_k = \varepsilon^{1/N} e^{i\pi(1+2k)/N} \qquad k = 1, \dots, N$$

We find for large N that $|\lambda_k| = \varepsilon^{1/N} \approx 1$, thus a small perturbation can give a large change of the eigenvalues. This holds in general, Lidskiĭ, in a pioneering work [54, 58], showed that small perturbations $A + \varepsilon B$ of an $N \times N$ matrix A, could produce "Lidskii circles" of eigenvalues. Davies and Hager [16] has gone further, proving that for large N, most eigenvalues of random perturbations of the $N \times N$ Jordan matrix will be very close to the unit circle. The spectral instability of the Jordan matrices is the worst case, since in general one has that the minimal distance (under permutations) between the spectra of two $N \times N$ matrices is bounded by

$$(\|A\| + \|B\|)^{1-1/N} \|A - B\|^{1/N}$$

where $\|A\|$ is the standard matrix norm, see [26]. This spectral instability complicates the mathematical modelling of non-symmetric problems, since more accurate models usually

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use larger matrices giving less stability under computational errors. The spectral instability for non-normal matrices seems to have been discovered and rediscovered at least five times according to [72], see the references there. A simple way of measuring the spectral stability of matrices is the *ε-pseudospectrum*, given by the following definition.

Definition 1.1. Let A be an $N \times N$ matrix, then the ε-pseudospectrum $\operatorname{Spec}_\varepsilon(A)$ is the set of $z \in \mathbf{C}$ such that

$$\|(z \operatorname{Id}_N - A)^{-1}\| > \varepsilon^{-1}$$

Here $(z \operatorname{Id}_N - A)^{-1}$ is the resolvent of A and we define $\|(z \operatorname{Id}_N - A)^{-1}\| = \infty$ if $z \in \operatorname{Spec}(A)$, the spectrum of A, thus

$$\operatorname{Spec}(A) = \bigcap_{\varepsilon>0} \operatorname{Spec}_\varepsilon(A)$$

An equivalent definition is that $\operatorname{Spec}_\varepsilon(A)$ the set of $z \in \mathbf{C}$ such that $z \in \operatorname{Spec}(A + B)$ for some B with $\|B\| < \varepsilon$ (see [72] for other equivalent definitions). These eigenvalues are called *ε-pseudoeigenvalues* and the corresponding eigenvectors are called *ε-pseudoeigenvectors* or *ε-pseudomodes*.

For normal matrices we have that

$$\|(z \operatorname{Id}_N - A)^{-1}\| = \operatorname{dist}(z, \operatorname{Spec}(A))^{-1}$$

so that the ε-pseudospectrum is contained in an ε neighborhood of the spectrum. But for non-normal matrices, the ε-pseudospectrum could be a much larger set, as seen above for the Jordan matrices. Observe that when $|z| \geq \|A\|$ we have for any matrix A the estimate

$$\|(z \operatorname{Id}_N - A)^{-1}\| \leq \cot\left(\frac{\pi}{4N}\right) \operatorname{dist}(z, \operatorname{Spec}(A))^{-1}$$

by Davies and Simon [17]. The use of the resolvent norm has given rise to the pseudospectral method in numerical analysis. There are several other ways of measuring spectral stability, for example the *structured ε-pseudospectrum* [24] and the *second-order relative spectrum* [41]. In the present review article, it will not be possible to make a more thorough treatment of the spectral instability of matrices. For more results, examples and references we refer the reader to [72].

Spectral instability also occurs for non-selfadjoint partial differential operators. Lidskiĭ, in a series of papers [43]–[53], studied the completeness and summability of eigenfunction expansions of certain types of non-selfadjoint differential operators. The first to study the stability of the spectrum for non-selfadjoint differential operators seems to be Reddy, Schmid and Henningsen [66] who studied the complex Airy and the Orr-Sommerfeld equations. When studying the spectral instability of differential operators, it is illuminating (and physically relevant) to study semiclassical operators $P(x, hD_x)$, where the parameter $0 < h \leq 1$ usually is called "Planck's constant". Observe that this parameter could in applications be, for example, the inverse of the Reynolds number, the Péclet number or the frequency. Then, the spectral instability can be defined as a function of h, see Definition 2.7. Davies studied the semiclassical complex harmonic oscillator [5], and the general semiclassical Schrödinger equation with complex potential in one dimension [6]. He proved that spectral instability is generic, in the sense that the norm of the resolvent blows up as any power of semiclassical parameter h almost everywhere in the numerical range of the semiclassical principal symbol (see Theorem 2.1 below). Zworski [74, 75] made the important observation that the spectral instability of semiclassical differential operators is directly connected with the bracket condition and the solvability question. In these infinite dimensional problems, the condition numbers of the eigenvectors could grow exponentially, see [1]. There also occurs "spectral pollution", in the sense that the spectra of the finite dimensional projections do not converge to the spectrum of the operator, see [14] for a numerical example. Another important problem is the behaviour of evolution semigroups for non-normal operators and the relation to pseudospectrum, see for example [9].

In this paper, we shall review the spectral instability of non-normal semiclassical differential operators in several variables, in particular for systems.

2. Semiclassical operators

In this section, we shall consider the spectral instability of semiclassical partial differential operators, following [22]. Let us start with a typical (and physically relevant) example, the semiclassical Schrödinger operator:

$$P(h) = h^2\Delta + V(x) \qquad V \in C^\infty(\mathbf{R}^n) \tag{2.1}$$

where $\Delta = -\sum_{j=1}^{n} \partial_{x_j}^2$ is the positive Laplacean. In [22] the *semiclassical pseudospectrum* of $P(h)$ was defined as

$$\Lambda(p) = \overline{\{\xi^2 + V(x) :\ (x,\xi) \in \mathbf{R}^{2n}\ ,\ \operatorname{Im}\langle \xi, V'(x)\rangle \neq 0\}}\,, \tag{2.2}$$

Of course, in the analytic case $\Lambda(p)$ is either empty or the closure of the set of all values of $p = \xi^2 + V(x)$. The following result shows that the resolvent blows up almost everywhere inside the semiclassical pseudospectrum.

Theorem 2.1. *Suppose that $P(h) = -h^2\Delta + V(x)$, with $V \in \mathcal{C}^\infty(\mathbf{R}^n)$. Then, for any $z \in \{\xi^2 + V(x) :\ (x,\xi) \in \mathbf{R}^{2n},\ \operatorname{Im}\langle \xi, V'(x)\rangle \neq 0\}$ there exists $u(h) \in L^2(\mathbf{R}^n)$ with the property that $\|u(h)\| = 1$ and*

$$\|(P(h) - z)u(h)\| \le C_N h^N \quad \forall\, N \qquad h \to 0 \tag{2.3}$$

In addition, $u(h)$ is localized to a point in phase space, (x,ξ), with $p(x,\xi) = z$, in the sense that $WF_h(u) = \{(x,\xi)\}$. Here the semiclassical wave front set, $WF_h(u)$, is given by Definition 2.2. If the potential V is real analytic then we can replace h^∞ by $\exp(-1/Ch)$ in (2.3).

These "almost eigenvectors" are called pseudomodes. Theorem 2.1 was proved by Davies [6] for Schrödinger operators in one dimension, but as was pointed out by Zworski [74, 75], it follows in general from a simple adaptation of the now classical results of Hörmander [34, 35] and Duistermaat-Sjöstrand [25], and is connected with the solvability problem by the bracket condition. It follows from (2.3) that, unlike the case of normal operators, the resolvent blows up as any power of h:

$$\|(P(h)-z)^{-1}\| \geq C_N h^{-N} \qquad \forall\, N \quad h \to 0$$

for z in an open set. That is particularly striking when $P(h)$ only has discrete spectrum $\mathrm{Spec}(P(h))$. The Schrödinger equation (2.1) has discrete spectrum if, for example,

$$\begin{aligned} &|\partial_x^\alpha V(x)| \leq C_\alpha (1+|x|)^{m-|\alpha|} \\ &(1+|x|^m+|\xi|^2)/C \leq |\xi^2+V(x)| \quad |(x,\xi)| \geq C \end{aligned} \tag{2.4}$$

where $m > 0$. In the analytic case, it suffices that (2.4) holds for $|\alpha| = 0$ as $|x| \to \infty$ and $|\operatorname{Im} x| < c_0$. The symbol $p = \xi^2 + V(x)$ will then avoid all sufficiently negative values and Fredholm theory guarantees that $P(h)$ has discrete spectrum for h small enough (see Proposition 3.12 below).

If $P(h) - z_1 \,\mathrm{Id}_N$ is invertible for some z_1, we may consider the following operator having bounded symbol:

$$(P(h)-z_1)^{-1}(P(h)-z_2) \qquad z_2 \neq z_1$$

see Remark 3.13. Thus is sufficient to consider the quantization of functions which have uniform bounds on any derivative:

$$p \in C_{\mathrm{b}}^\infty(T^*\mathbf{R}^n) = \{u \in C^\infty(T^*\mathbf{R}^n) : \ \partial^\alpha u \in L^\infty(T^*\mathbf{R}^n) \ \ \forall\, \alpha\}$$

In the analytic case we will assume that $p(x,\xi)$ is bounded and holomorphic in a tubular neighbourhood of $T^*\mathbf{R}^n \simeq \mathbf{R}^{2n} \subset \mathbf{C}^{2n}$.

We shall use the Weyl quantization

$$p^w(x,hD_x)u = \frac{1}{(2\pi)^n} \iint p\left(\frac{x+y}{2}, h\xi\right) e^{i\langle x-y,\xi\rangle} u(y)\,dy\,d\xi\,. \tag{2.5}$$

which for bounded symbols $p \in C_{\mathrm{b}}^\infty(T^*\mathbf{R}^n)$ gives operators bounded on $L^2(\mathbf{R}^n)$ – see [23, Chapter 7]. The advantage of the Weyl quantization is that real symbols gives symmetric operators. We shall also consider more general operators,

$$P(h) \sim \sum_{j=0}^\infty h^j p_j^w(x,hD)$$

in which case we call p_0 the principal symbol of $P(h)$. Since the results only depend on the principal symbol and different quantizations only differ in the lower order terms, we could as well have used the Kohn-Nirenberg or other quantizations.

In the case of analytic symbols we assume that p_j are bounded holomorphic functions in a tubular neighbourhood of $T^*\mathbf{R}^n$, and that

$$|p_j(z,\zeta)| \leq C^j j^j \qquad |\operatorname{Im}(z,\zeta)| \leq 1/C \tag{2.6}$$

This gives exponentially small errors in the expansions.

Let $\Sigma(p) = \overline{p(T^*\mathbf{R}^n)}$, then it is well-known that the spectrum $\mathrm{Spec}(P(h)) \subseteq \Sigma(p)$ (see Proposition 3.11) and spectral instability can only occur in $\Sigma(p)$. Now, [22] generalized the definition (2.2) of the *semiclassical pseudospectrum* to:

$$\Lambda(p) \stackrel{\text{def}}{=} \overline{p(\{(x,\xi) \in T^*\mathbf{R}^n : \ \{p, \bar{p}\}(x,\xi) \neq 0\})} \subseteq \Sigma(p) \tag{2.7}$$

where we have used the Poisson bracket:

$$\{f, g\} = H_f g = \sum_{j=1}^{n} \partial_{\xi_j} f \partial_{x_j} g - \partial_{x_j} f \partial_{\xi_j} g$$

The non-vanishing of $\{p, \bar{p}\}$ is a classical equivalent of the operator not being normal, since this is the principal symbol of the commutator $[P(h), P^*(h)]$. Hörmander showed in [34] that the vanishing of $\{p, \bar{p}\}$ at $p^{-1}(0)$ is a necessary condition for the solvability of $P(h)$, thus almost all partial differential equations are non-solvable. Note that in the analytic case we have

$$\Lambda(p) = \emptyset \quad \text{or} \quad \Lambda(p) = \Sigma(p)$$

Actually, it is the sign of the Poisson bracket that is important. Therefore, we define

$$\begin{aligned} &\Lambda_\pm(p) = \{p(x,\xi) : \ \pm\{\mathrm{Re}\, p, \mathrm{Im}\, p\}(x,\xi) > 0\} \subseteq \Sigma(p) \\ &\Sigma_\infty(p) = \{z : \ \exists\ (x_j, \xi_j) \to \infty \quad \lim_{j\to\infty} p(x_j, \xi_j) = z\}\,, \end{aligned} \tag{2.8}$$

then $\Lambda(p) = \overline{\Lambda_-(p) \bigcup \Lambda_+(p)}$ and $\Sigma_\infty(p)$ is the set of limits of p at infinity. In order to measure the singularities of the solutions, we shall recall the semiclassical wave front sets.

Definition 2.2. For $u \in L^2(\mathbf{R}^n)$ we say that $(x,\xi) \notin \mathrm{WF}_h(u)$ if there exists $a \in C_0^\infty(T^*\mathbf{R}^n)$ such that $a(x,\xi) \neq 0$ and the L^2 norm

$$\|a^w(x, hD)u\| \leq C_k h^k \quad \forall\, k \qquad h \to 0 \tag{2.9}$$

We call $\mathrm{WF}_h(u)$ the semiclassical wave front set of u.

Observe that this definition is equivalent to the definition (2.5) in [22] which use the FBI transform T: $(x_0, \xi_0) \notin \mathrm{WF}_h(u)$ if $\|Tu(x,\xi)\| = \mathcal{O}(h^\infty)$ when $|(x,\xi) - (x_0,\xi_0)| \ll 1$. We may also define the *analytic* semiclassical wave front set by the condition that $\|Tu(x,\xi)\| = \mathcal{O}(e^{-c/h})$ in a neighborhood of (x_0, ξ_0) for some $c > 0$, see (2.6) in [22]. In this more general setting, Theorem 1.2 in [22] gives

Theorem 2.3. *Suppose that $n \geq 2$, $p \in C_{\mathrm{b}}^\infty(T^*\mathbf{R}^n)$ and that $p^{-1}(z)$ is compact for a dense set of values $z \in \mathbf{C}$. If $P(h)$ has the principal symbol $p(x,\xi)$ then*

$$\Lambda(p) \setminus \Sigma_\infty \subseteq \overline{\Lambda_-(p)}$$

and for every $z \in \Lambda_-(p)$ there exists $u(h) \in L^2(\mathbf{R}^n)$ with the property that $\|u(h)\| = 1$ and

$$\|(P(h) - z)u(h)\| \leq C_N h^N \quad \forall\, N \qquad h \to 0 \tag{2.10}$$

In addition, $u(h)$ is localized to a point in phase space, (x,ξ), with $p(x,\xi) = z$, so that $WF_h(u) = \{(x,\xi)\}$. If in addition p has a bounded holomorphic continuation to $\{(x,\xi) \in$

$\mathbf{C}^{2n}$, $|\operatorname{Im}(x,\xi)| \le 1/C\}$ *then we obtain that* (2.10) *holds with* h^∞ *replaced by* $\exp(-c/h)$, $c > 0$. *If* $n = 1$ *then the same conclusions hold provided that each component of* $\mathbf{C} \setminus \Sigma_\infty(p)$ *has a non-empty intersection with* $\complement\Lambda(p)$.

Theorem 2.3 gives that norm of the resolvent $\|(P(h) - z_0)^{-1}\| \ge C_N h^{-N}$ as $h \to 0$ for any N. In general, one cannot construct an almost solution (or quasimode) $u(h)$ for an arbitrary interior point of $\Lambda(p) \setminus \Sigma_\infty(p)$, see the example in [22, Section 3]. However, for many points $\Lambda(p) \setminus \Lambda_-(p)$ quasimodes can exist, since that the vanishing of the Poisson bracket $\{\operatorname{Re} p, \operatorname{Im} p\}$ is not enough to guarantee the absence of a quasimode. In fact, as proved by Pravda-Starov [59], a violation of the condition $(\overline{\Psi})$ (see [37, Section 26.4]) can produce quasimodes. Condition $(\overline{\Psi})$ is directly connected to the Nirenberg-Treves conjecture, which says that a pseudodifferential operator $P(x, D)$ of principal type is locally solvable if and only if the principal symbol p satisfies condition (Ψ) or, equivalently, $\overline{p}$ satifies condition $(\overline{\Psi})$. Here principal type means that the principal symbol satisfies (2.12) for $z_0 = 0$. The Nirenberg-Treves conjecture was recently proved in [20].

In the case of dimension one the topological condition is necessary, according to the following example from [22].

Example 2.4. Let

$$p(x,\xi) = \frac{(\xi + ix)^2}{1 + x^2 + \xi^2}$$

then $\{\operatorname{Re} p, \operatorname{Im} p\}(x,\xi) > 0$ for $(x,\xi) \ne (0,0)$.

For principal symbols arising from differential equations of *even order* in one variable, like the Schrödinger equation, we always have

$$\sum_{(x,\xi)\in p^{-1}(z)} \operatorname{sgn} \{\operatorname{Re} p, \operatorname{Im} p\}(x,\xi) = 0 \tag{2.11}$$

for a dense set of values z. In fact, $p(x,\xi) = p(x,-\xi) = z$,

$$\{\operatorname{Re} p, \operatorname{Im} p\}(x,-\xi) = -\{\operatorname{Re} p, \operatorname{Im} p\}(x,\xi)$$

and the set of values z corresponding to $\xi \ne 0$ is dense in the set of values for which the bracket is non-zero.

In simple one dimensional examples we can already see that the spectrum, $\sigma(P(h))$, typically lies deep inside the pseudospectrum $\Lambda(p)$ — see [5, 6, 71] for numerical examples of this phenomenon. Consider for example the following non-selfadjoint operator $P(h) = (hD_x)^2 + i(hD_x) + x^2$. A formal conjugation

$$e^{-x/2h} P(h) e^{x/2h} = (hD_x)^2 + x^2 + \frac{1}{4}$$

shows that the spectrum of $P(h)$ is given by $(2n + 1)h + 1/4$, while

$$\Lambda(p) = \{z : \ \operatorname{Re} z \ge (\operatorname{Im} z)^2\}$$

since $p = \xi^2 + i\xi + x^2$.

To obtain that the spectrum is inside the pseudospectrum for general operators we have to make assumptions on $z_0 \in \partial\Sigma(p)$. One is the principal type condition:

$$p(x,\xi) = z_0 \implies dp(x,\xi) \neq 0\,, \quad (x,\xi) \in T^*\mathbf{R}^n \tag{2.12}$$

then p is of principal type if (2.12) holds for $z_0 = 0$. We also assume a dynamical condition:

$$\exists\, \lambda \in \mathbf{C} \text{ so that } no \text{ complete trajectory of } H_{\mathrm{Re}(\lambda p)} \text{ is contained in } p^{-1}(z_0) \tag{2.13}$$

Recall that the trajectories of $H_{\mathrm{Re}(\lambda p)}$ on $\mathrm{Re}(\lambda p) = 0$ are called bicharacteristics of $\mathrm{Re}(\lambda p)$ and semibicharacteristics of p. Under these conditions we obtain from Theorem 1.3 in [22]:

Theorem 2.5. *Suppose that $p \in \mathcal{C}_{\mathrm{b}}^\infty(T^*\mathbf{R}^n)$ and that the principal symbol of $P(h)$ is given by $p(x,\xi)$. If $z_0 \in \partial\Sigma(p) \setminus \Sigma_\infty(p)$ satisfies* (2.12) *and* (2.13)*, then for any $M > 0$ there exists $h_M > 0$ such that*

$$\{z:\ |z - z_0| < Mh\log(1/h)\} \cap \sigma(P(h)) = \emptyset \qquad 0 < h < h_M$$

and $\|(P(h) - z_0)^{-1}\| \leq C/h$ for $0 < h \ll 1$. If in addition p is a bounded holomorphic function in a complex tubular neighbourhood of $\mathbf{R}^n$, then there exists $C_0 > 0$ such that

$$\{z:\ |z - z_0| < C_0\} \cap \sigma(P(h)) = \emptyset \qquad 0 < h \ll 1$$

Observe that we can replace $\Sigma(p)$ in Theorem 2.5 by $\Lambda(p)$. In fact, condition (2.13) gives that $z_0 \in \partial\Lambda(p)$, since the bracket cannot vanish identically on all level sets close to $p^{-1}(z_0)$. If (2.13) is violated, then for a large class of *dissipative operators*, the spectrum lies arbitrarily close (as $h \to 0$) to the boundary of the pseudospectrum, see [22, Section 6]. The example in [22, Section 4] shows that the $\mathcal{C}^\infty$ result of Theorem 2.5 is optimal.

At the boundary of the pseudospectrum we may expect an improved bound on the resolvent when some additional non-degeneracy is assumed. We shall borrow our notation from [37, Chapter 27]. If $p = p_1 + ip_2 \in \mathcal{C}^\infty$ with real valued p_j then we define the repeated Poisson brackets

$$p_I = H_{p_{i_1}} H_{p_{i_2}} \dots H_{p_{i_{k-1}}} p_{i_k}$$

where $I = (i_1, i_2, \dots, i_k) \in \{\,1,2\,\}^k$ and $|I| = k > 1$ is the order of the bracket.

We say that $z_0 \in \Sigma(p) \setminus \Sigma_\infty(p)$ is of *finite type* for p if for any $(x,\xi) \in p^{-1}(z_0)$ there exists $k > 1$ and $I \in \{\,1,2\,\}^k$ such that

$$p_I(x,\xi) \neq 0 \tag{2.14}$$

which implies (2.12). The *order* of p at (x,ξ) is

$$k(w) = \max\{\, j \in \mathbf{Z}:\ p_I(x,\xi) = 0 \ \text{ for } 1 < |I| \leq j\,\}\,. \tag{2.15}$$

The order of z_0 is the maximum of the order of p at (x,ξ) for $(x,\xi) \in p^{-1}(z_0)$.

Theorem 2.6. *Assume that $p \in \mathcal{C}_{\mathrm{b}}^\infty(T^*\mathbf{R}^n)$, and that the principal symbol of $P(h)$ is $p(x,\xi)$. If $z_0 \in \partial\Sigma(p) \setminus \Sigma_\infty(P)$ is of finite type for p of order $k \geq 1$, then k is* even *and $\exists\ h_0,\, C > 0$ so that*

$$\|(P(h) - z_0)^{-1}\| \leq Ch^{-\frac{k}{k+1}} \qquad 0 < h \leq h_0 \tag{2.16}$$

Also, there exists h_1, $c_0 > 0$ such that

$$\Big\{ z : \ |z - z_0| \le c_0 (h \log h^{-1})^{\frac{k}{k+1}} \Big\} \cap \operatorname{Spec}(P(h)) = \emptyset \qquad 0 < h \le h_1 \tag{2.17}$$

We obtain (2.17) from [22, Theorem 1.4] and (2.17) from [68]. In one dimension, the resolvent estimate was proved in [73], and in some special cases by Boulton [3] who also showed that the bounds are optimal. As was demonstrated by Trefethen [71] this is also easy to see numerically.

We have the following simple higher dimensional example from [22] to which we can apply Theorem 2.6. Let $W \in \mathcal{C}_{\mathrm{b}}^{\infty}(\mathbf{R}^2)$ be a non-negative function, vanishing on the circle $x_1^2 + x_2^2 = 1$ and consider

$$P(h) = -h^2\Delta + iW(x) + i(x_1^2 + x_2^2 - 1)^m \quad \text{with } m \text{ even.}$$

Then the estimate (2.16) holds for $z_0 > 0$ uniformly on compact subsets of $(0, \infty)$, with $k = 2m$. This is due to the (simple) tangency of some bicharacteristics of the real part to the set where the imaginary part vanishes.

One could also define the semiclassical pseudospectrum of $P(h)$ as the closure of the set of values z for which (2.10) holds, actually this was an alternative definition in [22]. Pravda-Starov [61] introduced the following refined definitions.

Definition 2.7. Let $P(h)$, $0 < h \le 1$, be a semiclassical family of operators on $L^2(\mathbf{R}^n)$ with domain D. For $\mu > 0$ we define the *pseudospectrum of index* μ as the set

$$\begin{aligned} \Lambda_\mu^{\mathrm{sc}}(P(h)) = \{ z \in \mathbf{C} :\ & \forall\, C > 0,\ \forall\, h_0 > 0, \\ & \exists\, 0 < h < h_0,\ \|(P(h) - z\operatorname{Id}_N)^{-1}\| \ge C h^{-\mu} \} \end{aligned}$$

and the *injectivity pseudospectrum of index* μ as

$$\begin{aligned} \lambda_\mu^{\mathrm{sc}}(P(h)) = \{ z \in \mathbf{C} :\ & \forall\, C > 0,\ \forall\, h_0 > 0, \\ & \exists\, 0 < h < h_0,\ \ \exists\, u \in D,\ \|u\| = 1,\ \|(P(h) - z\operatorname{Id}_N)u\| \le C h^{\mu} \} \end{aligned}$$

We define the *pseudospectrum of infinite index* as $\Lambda_\infty^{\mathrm{sc}}(P(h)) = \bigcap_\mu \Lambda_\mu^{\mathrm{sc}}(P(h))$ and correspondingly the *injectivity pseudospectrum of infinite index.*

With these definitions, we find that the semiclassical harmonic oscillator has pseudospectrum $\lambda_\infty^{\mathrm{sc}} = \Lambda_\infty^{\mathrm{sc}} = \overline{\mathbf{R}_+}$, since the eigenvalues are $(2n+1)h$, $n \in \mathbf{N}$. One way of avoiding this, is to have $\exists\, h_0 > 0\ \ \forall\, 0 < h < h_0$ instead in the definitions.

Observe that we have the obvious inclusion $\lambda_\mu^{\mathrm{sc}}(P(h)) \subseteq \Lambda_\mu^{\mathrm{sc}}(P(h))$, $\forall\, \mu$. We get equality if, for example, $P(h)$ is Fredholm of index ≥ 0. Theorem 2.3 and Proposition 3.11 show that

$$\Lambda_-(p) \subseteq \lambda_\infty^{\mathrm{sc}}(P(h)) \subseteq \Lambda_\infty^{\mathrm{sc}}(P(h)) \subseteq \Sigma(p)$$

The interplay between classical properties of symbols and the existence of localized quasimodes can be also be observed in the Berezin-Toeplitz quantization of compact symplectic Kähler manifolds. See Chapman and Trefethen [4] for the case of the torus, and Borthwick and Uribe [2] for the general C^∞ case.

For non-principal type scalar semiclassical operators, there are still many open questions about the pseudospectrum. Pravda-Starov has studied the pseudospectrum for non-principal type operators in one dimension [62], in the case when the Hessian of the principal symbol is elliptic and non-normal. He has also studied the pseudospectrum for general elliptic non-normal quadratic Weyl operators [63].

Sjöstrand and Hager has proved that for certain random perturbations of pseudodifferential operators, the spectrum will satisfy a asymptotic Weyl law, see [31]. It is interesting to compare this to the recent proof by Tao and Vu [70] of the *circular law* for random matrices, for which the spectrum is uniformly distributed in a disk.

3. Systems of semiclassical operators

In this section, we will show how the results for semiclassical scalar operators generalizes to systems, following [21]. We shall consider $N \times N$ systems of semiclassical pseudodifferential operators on the form:

$$P(h) \sim \sum_{j=0}^{\infty} h^j P_j^w(x, hD) \tag{3.1}$$

with $P_j(x,\xi) \in C_{\mathrm{b}}^\infty(T^*\mathbf{R}^n, \mathcal{L}(\mathbf{C}^N, \mathbf{C}^N))$ using the Weyl quantization given by (2.5). As before, C_{b}^∞ is the set of C^∞ functions having all derivatives in L^∞ and $P_0 = \sigma(P(h))$ is the principal symbol of $P(h)$. The operator is said to be elliptic if the principal symbol P_0 is invertible, and of principal type if P_0 vanishes of first order on the kernel, see Definition 4.1. We shall also consider operators with analytic symbols, then we shall assume that $P_j(x,\xi)$ are bounded and holomorphic in a tubular neighborhood of $T^*\mathbf{R}^n$ satisfying (2.6), which gives exponentially small errors in the calculus. In the following, we shall use the notation $w = (x,\xi) \in T^*\mathbf{R}^n$.

We shall consider the spectrum $\operatorname{Spec} P(h)$ which is the set of values $z \in \mathbf{C}$ such that the resolvent $(P(h) - z\,\mathrm{Id}_N)^{-1}$ is *not* a bounded operator, here Id_N is the identity in $\mathbf{C}^N$. The spectrum of $P(h)$ is essentially contained in the spectrum of the principal symbol $\operatorname{Spec}(P(w))$, which is given by

$$|P(w) - z\,\mathrm{Id}_N| = 0$$

where $|A|$ is the determinant of the matrix A. For example, if $P(w) = \sigma(P(h))$ is bounded and z is not an eigenvalue of $P(w)$ for any $w = (x,\xi)$ (or a limit of eigenvalues at infinity) then $P(h) - z\,\mathrm{Id}_N$ is invertible by Proposition 3.11 below. When $P(w)$ is an unbounded symbol one needs additional conditions, see for example Proposition 3.12. As before, we shall restrict our study to bounded symbols, but we can reduce to this case if $P(h) - z_1\,\mathrm{Id}_N$ is invertible for some z_1 by considering

$$(P(h) - z_1\,\mathrm{Id}_N)^{-1}(P(h) - z_2\,\mathrm{Id}_N) \qquad z_2 \neq z_1$$

see Remark 3.13. But unless we have conditions on the eigenvalues at infinity, this does not always give a bounded operator.

Example 3.1. Let

$$P(\xi) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \qquad \xi \in \mathbf{R}$$

then 0 is the only eigenvalue of $P(\xi)$ but

$$(P(\xi) - z\,\mathrm{Id}_N)^{-1} = -1/z \begin{pmatrix} 1 & \xi/z \\ 0 & 1 \end{pmatrix} \tag{3.2}$$

and $(P^w - z\,\mathrm{Id}_N)^{-1}P^w = -z^{-1}P^w$ is unbounded for any $z \neq 0$.

Definition 3.2. Let $P \in C^\infty(T^*\mathbf{R}^n)$ be an $N \times N$ system. We denote the closure of the set of eigenvalues of P by

$$\Sigma(P) = \overline{\{\, \lambda \in \mathbf{C} : \exists\, w \in T^*\mathbf{R}^n,\ |P(w) - \lambda\,\mathrm{Id}_N\,| = 0 \,\}} \tag{3.3}$$

and the eigenvalues at infinity:

$$\begin{aligned} \Sigma_\infty(P) = \big\{ \lambda \in \mathbf{C} :\ \exists\, w_j \to \infty,\ \exists\, u_j \in \mathbf{C}^N,\ |u_j| = 1, \\ |P(w_j)u_j - \lambda u_j| \to 0,\ j \to \infty \,\big\} \end{aligned} \tag{3.4}$$

which is closed in $\mathbf{C}$.

In fact, that $\Sigma_\infty(P)$ is closed follows by taking a suitable diagonal sequence. Observe that as in the scalar case, we could have $\Sigma_\infty(P) = \Sigma(P)$, for example if $P(w)$ is constant in one direction. It follows from the definition that $\lambda \notin \Sigma_\infty(P)$ if and only if the resolvent is defined and bounded when $|w|$ is large enough:

$$\|(P(w) - \lambda\,\mathrm{Id}_N)^{-1}\| \le C \qquad |w| \gg 1 \tag{3.5}$$

where as before $\|A\|$ is the norm of the matrix A.

It is clear from the definition that $\Sigma_\infty(P)$ contains all finite limits of eigenvalues of P at infinity. In fact, if $P(w_j)u_j = \lambda_j u_j$, $|u_j| = 1$, $w_j \to \infty$ and $\lambda_j \to \lambda$ then

$$P(w_j)u_j - \lambda u_j = (\lambda_j - \lambda)u_j \to 0$$

Example 3.1 shows that in general $\Sigma_\infty(P)$ could be a larger set, in fact, then $\Sigma(P) = \{\, 0 \,\}$ but $\Sigma_\infty(P) = \mathbf{C}$ by (3.2) and (3.5). But for bounded $N \times N$ symbols $P \in C_{\mathrm{b}}^\infty(T^*\mathbf{R}^n)$ we obtain from Proposition 2.4 in [21] that $\Sigma_\infty(P)$ is the set of all limits of the eigenvalues of P at infinity.

One problem with studying systems $P(w)$, is that the eigenvalues are not regular in the parameter w, in general they depend only continuously on w when the multiplicity is not constant.

Definition 3.3. For an $N \times N$ system $P \in C^\infty(T^*\mathbf{R}^n)$ we define

$$\kappa_P(w, \lambda) = \operatorname{Dim} \operatorname{Ker}(P(w) - \lambda\,\mathrm{Id}_N)$$

and

$$K_P(w, \lambda) = \max \{\, k :\ \partial_\lambda^j p(w, \lambda) = 0 \text{ for } j < k \,\}$$

where $p(w,\lambda) = |P(w) - \lambda \operatorname{Id}_N|$ is the characteristic polynomial. Let

$$\Omega_k(P) = \{ (w,\lambda) \in T^*\mathbf{R}^n \times \mathbf{C} : \ K_P(w,\lambda) \geq k \} \qquad k \geq 1$$

then $\emptyset = \Omega_{N+1}(P) \subseteq \Omega_N(P) \subseteq \cdots \subseteq \Omega_1(P)$ and we may define

$$\Xi(P) = \bigcup_{j>1} \partial\Omega_j(P) \tag{3.6}$$

where $\partial\Omega_j(P)$ is the boundary of $\Omega_j(P)$ in the relative topology of $\Omega_1(P)$.

Clearly, $\Omega_j(P)$ is a closed set for any $j \geq 1$. We have $\kappa_P \leq K_P$ with equality for symmetric systems but in general we need not have equality, see Example 3.1 where $\kappa_P < K_P = 2$ when $\lambda = 0$ and $\xi \neq 0$. By the definition we find that the multiplicity $K_P(w,\lambda)$ of the zeros of $|P(w) - \lambda \operatorname{Id}_N|$ is locally constant on $\Omega_1(P) \setminus \Xi(P)$. If $P(w)$ is symmetric then κ_P is also constant on $\Omega_1(P) \setminus \Xi(P)$ but in general this is not true, see Example 3.1 where κ_P is discontinuous when $\xi = 0$.

Remark 3.4. *We find that $\Xi(P)$ is closed and nowhere dense in the relative topology of $\Omega_1(P)$ since it is the union of boundaries of closed sets. We also find that*

$$(w,\lambda) \in \Xi(P) \Leftrightarrow (w,\overline{\lambda}) \in \Xi(P^*)$$

since $|P^* - \overline{\lambda} \operatorname{Id}_N| = \overline{|P - \lambda \operatorname{Id}_N|}$.

Example 3.5. Let

$$P(t) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \qquad t \in \mathbf{R}$$

then $P(t)$ has the eigenvalues $\pm\sqrt{t}$, $\kappa_P \equiv 1$ on $\Omega_1(P)$ and $\Omega_2(P) = \{ 0 \}$.

Example 3.6. Let

$$P = \begin{pmatrix} w_1 + w_2 & w_3 \\ w_3 & w_1 - w_2 \end{pmatrix}$$

then

$$\Omega_1(P) = \left\{ (w;\lambda_j) : \ \lambda_j = w_1 + (-1)^j \sqrt{w_2^2 + w_3^2}, \ j = 1, \ 2 \right\}$$

and $\Omega_2(P) = \{ (w_1, 0, 0; w_1) : \ w_1 \in \mathbf{R} \}$.

Definition 3.7. Let π_j be the projections

$$\pi_1(w,\lambda) = w \qquad \pi_2(w,\lambda) = \lambda$$

then we define for $\lambda \in \mathbf{C}$ the closed sets

$$\Sigma_\lambda(P) = \pi_1\left(\Omega_1(P) \bigcap \pi_2^{-1}(\lambda)\right) = \{ w : \ |P(w) - \lambda \operatorname{Id}_N| = 0 \}$$

and

$$X(P) = \pi_1\left(\Xi(P)\right) \subseteq T^*\mathbf{R}^n$$

Observe that $X(P)$ is nowhere dense in $T^*\mathbf{R}^n$ and $P(w)$ has constant characteristics near $w_0 \notin X(P)$. This means that $|P(w) - \lambda \operatorname{Id}_N| = 0$ if and only if $\lambda = \lambda_j(w)$ for $j = 1, \dots k$, where the eigenvalues $\lambda_j(w) \neq \lambda_k(w)$ for $j \neq k$ when $|w - w_0| \ll 1$.

Definition 3.8. For an $N \times N$ system $P \in C^\infty(T^*\mathbf{R}^n)$ we define the *weakly singular eigenvalue set*

$$\Sigma_{ws}(P) = \pi_2(\Xi(P)) \subseteq \mathbf{C} \tag{3.7}$$

and the *strongly singular eigenvalue set*

$$\Sigma_{ss}(P) = \left\{ \lambda : \ \pi_2^{-1}(\lambda) \bigcap \Omega_1(P) \subseteq \Xi(P) \right\}. \tag{3.8}$$

It follows from the definition that $\Sigma_{ss}(P) \subseteq \Sigma_{ws}(P)$. Clearly $\Sigma_{ws}(P) \bigcup \Sigma_\infty(P)$ and $\Sigma_{ss}(P) \bigcup \Sigma_\infty(P)$ are closed, and $\Sigma_{ss}(P)$ is nowhere dense. On the other hand, it is possible that $\Sigma_{ws}(P) = \Sigma(P)$, for example in Example 3.6 we have $\Sigma_{ws}(P) = \Sigma(P) = \mathbf{R}$ and $\Sigma_{ss}(P) = \emptyset$. In fact, the eigenvalues then coincide only when $w_2 = w_3 = 0$ but the eigenvalue $\lambda = w_1$ is also attained at some point where $w_2 \neq 0$. When we have constant characteristics, the Implicit Function Theorem immediately gives the following result.

Remark 3.9. *Let $P \in C^\infty(T^*\mathbf{R}^n)$ be an $N \times N$ system. If $(w_0, \lambda_0) \in \Omega_1(P) \setminus \Xi(P)$ then there exists a unique C^∞ function $\lambda(w)$ so that $(w, \lambda) \in \Omega_1(P)$ if and only if $\lambda = \lambda(w)$ in a neighborhood of (w_0, λ_0). If $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{ws}(P) \bigcup \Sigma_\infty(P))$ then $\exists\ \lambda(w) \in C^\infty$ such that $(w, \lambda) \in \Omega_1(P)$ if and only if $\lambda = \lambda(w)$ in a neighborhood of $\Sigma_{\lambda_0}(P)$.*

By Remark 3.9 we find that $\Omega_1(P) \setminus \Xi(P)$ is locally given as a C^∞ manifold over $T^*\mathbf{R}^n$, and that the eigenvalues $\lambda_j(w) \in C^\infty$ outside $X(P)$. This is not true if we instead assume that κ_P is constant on $\Omega_1(P)$, see Example 3.5.

Definition 3.10. A C^∞ function $\lambda(w)$ is called a *germ of eigenvalues* at w_0 for the $N \times N$ system $P \in C^\infty(T^*\mathbf{R}^n)$ if

$$|P(w) - \lambda(w) \operatorname{Id}_N| \equiv 0 \qquad \text{in a neighborhood of } w_0 \tag{3.9}$$

If this holds in a neighborhood of every point in $\omega \Subset T^*\mathbf{R}^n$ then we say that $\lambda(w)$ is a germ of eigenvalues for P on ω.

If $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{ss}(P) \bigcup \Sigma_\infty(P))$ then there exists $w_0 \in \Sigma_{\lambda_0}(P)$ so that $(w_0, \lambda_0) \in \Omega_1(P) \setminus \Xi(P)$. By Remark 3.9 there exists a C^∞ germ $\lambda(w)$ of eigenvalues at w_0 for P such that $\lambda(w_0) = \lambda_0$. If $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{ws}(P) \bigcup \Sigma_\infty(P))$ then there exists a C^∞ germ $\lambda(w)$ of eigenvalues on $\Sigma_{\lambda_0}(P)$.

As in the scalar case we obtain that the spectrum is essentially discrete outside $\Sigma_\infty(P)$ by the following result, which is Proposition 2.19 in [21].

Proposition 3.11. *Assume that the $N \times N$ system $P(h)$ is given by* (3.1) *with principal symbol $P \in C_{\mathrm{b}}^\infty(T^*\mathbf{R}^n)$. Let Ω be an open connected set, satisfying*

$$\overline{\Omega} \bigcap \Sigma_\infty(P) = \emptyset \quad \textit{and} \quad \Omega \bigcap \complement\Sigma(P) \neq \emptyset$$

Then $(P(h)-\lambda\,\mathrm{Id}_N)^{-1}$, $0<h\ll 1$, $\lambda\in\Omega$, *is a meromorphic family of operators with poles of finite rank. In particular, for h sufficiently small, the spectrum of* $P(h)$ *is discrete in any such set. When* $\Omega\bigcap\Sigma(P)=\emptyset$ *we find that* Ω *contains no spectrum of* $P(h)$.

Proposition 3.11 shows that $\Lambda^{\mathrm{sc}}_{\mu}(P(h))\subseteq\Sigma(P)$ for any $\mu>0$. We shall show how the reduction to the case of bounded operator can be done in the systems case, following [21, 22]. Let $m(w)$ be a positive function on $T^*\mathbf{R}^n$ satisfying

$$1\le m(w)\le C(1+|w-w_0|)^M m(w_0)\qquad\forall\ w,\ w_0\in T^*\mathbf{R}^n$$

for some C and M. Then m is an admissible weight function and we can define the symbol classes $P\in S(m)$ by

$$\|\partial_w^\alpha P(w)\|\le C_\alpha m(w)\qquad\forall\alpha$$

Following [23] we can then define the semiclassical operator $P(h)=P^w(x,hD)$. In the analytic case we require that the symbol estimates hold in a tubular neighborhood of $T^*\mathbf{R}^n$:

$$\|\partial_w^\alpha P(w)\|\le C_\alpha m(\operatorname{Re}w)\qquad\text{for}\quad|\operatorname{Im}w|\le 1/C\qquad\forall\alpha\tag{3.10}$$

One typical example of an admissible weight function is $m(x,\xi)=(\langle\xi\rangle^2+\langle x\rangle^p)$. We shall make the ellipticity assumption

$$\|P^{-1}(w)\|\le C_0m^{-1}(w)\qquad|w|\gg 1\tag{3.11}$$

and in the analytic case we assume this in a tubular neighborhood of $T^*\mathbf{R}^n$ as in (3.10). We then obtain the following result from Proposition 2.20 in [21].

Proposition 3.12. *Assume that* $P\in S(m)$ *is an* $N\times N$ *system satisfying* (3.11) *and that* $z\notin\Sigma(P)\bigcup\Sigma_\infty(P)$. *Then we find that* $P^w(x,hD)-z\,\mathrm{Id}_N$ *is invertible for small enough h.*

When m is bounded, (3.11) can be replaced by $z\notin\Sigma_\infty(P)$ and then the result follows from Proposition 3.11. In the reduction to operators with bounded symbols we may use the following result.

Remark 3.13. *If* $z_1\notin\operatorname{Spec}(P)$ *we may define the operator*

$$Q=(P-z_1\,\mathrm{Id}_N)^{-1}(P-z_2\,\mathrm{Id}_N)\qquad z_2\ne z_1$$

then the resolvents of Q *and* P *are related by*

$$(Q-\zeta\,\mathrm{Id}_N)^{-1}=(1-\zeta)^{-1}(P-z_1\,\mathrm{Id}_N)\left(P-\frac{\zeta z_1-z_2}{\zeta-1}\,\mathrm{Id}_N\right)^{-1}\qquad\zeta\ne 1$$

when $\frac{\zeta z_1-z_2}{\zeta-1}\notin\operatorname{Spec}(P)$.

Example 3.14. Let

$$P(x,\xi)=|\xi|^2\,\mathrm{Id}_N+iK(x)$$

where $0\le K(x)\in C_{\mathrm{b}}^\infty$, then we find that $P\in S(m)$ with $m(x,\xi)=|\xi|^2+1$. If $0\notin\Sigma_\infty(K)$ then $K(x)$ is invertible for $|x|\gg 1$, so $P^{-1}\in S(m^{-1})$ at infinity. Since $\operatorname{Re}z\ge 0$ in

$\Sigma(P)$ we find from Proposition 3.12 that $P^w(x,hD)+\mathrm{Id}_N$ is invertible for small enough h and $P^w(x,hD)(P^w(x,hD)+\mathrm{Id}_N)^{-1}$ is bounded in L^2 with principal symbol $P(w)(P(w)+\mathrm{Id}_N)^{-1} \in C_{\mathrm{b}}^\infty$.

Observe that if $u=(u_1,\dots,u_N)\in L^2(\mathbf{R}^n,\mathbf{C}^N)$ we may define $\mathrm{WF}_h(u)=\bigcup_j \mathrm{WF}_h(u_j)$ but this gives no information about which components of u that are singular. Therefore we shall define the corresponding vector valued semiclassical polarization sets.

Definition 3.15. For $u\in L^2(\mathbf{R}^n,\mathbf{C}^N)$, we say that $(w_0,z_0)\notin \mathrm{WF}_h^{pol}(u)\subseteq T^*\mathbf{R}^n\times\mathbf{C}^N$ if there exists $A(h)$ given by (3.1) with principal symbol $A(w)$ such that $A(w_0)z_0\neq 0$ and $\|a^w(x,hD)u\|\le C_kh^k$, $\forall\, k$. We call $\mathrm{WF}_h^{pol}(u)$ the semiclassical polarization set of u.

We could similarly define the *analytic* semiclassical polarization set by using the FBI transform and analytic pseudodifferential operators, see (2.6) in [22].

Remark 3.16. *The semiclassical polarization sets are closed, linear in the fiber and has the functorial properties of the C^∞ polarization sets in* [18]. *In particular, we find that*

$$\pi(\mathrm{WF}_h^{pol}(u)\setminus 0)=\mathrm{WF}_h(u)=\bigcup_j \mathrm{WF}_h(u_j)$$

if π is the projection along the fiber variables: $\pi: T^*\mathbf{R}^n\times\mathbf{C}^N\mapsto T^*\mathbf{R}^n$. We also find that

$$A(\mathrm{WF}_h^{pol}(u))=\Big\{\,(w,A(w)z):\ (w,z)\in\mathrm{WF}_h^{pol}(u)\,\Big\}\subseteq \mathrm{WF}_h^{pol}(A(h)u)$$

if $A(w)$ is the principal symbol of $A(h)$, which gives that $\mathrm{WF}_h^{pol}(Au)=A(\mathrm{WF}_h^{pol}(u))$ when $A(h)$ is elliptic.

Remark 3.16 follows from the proofs of Propositions 2.5 and 2.7 in [18].

Example 3.17. Let $u=(u_1,\dots,u_N)\in L^2(\mathbf{R}^n,\mathbf{C}^N)$ where $\mathrm{WF}_h(u_1)=\{\,w_0\,\}$ and $\mathrm{WF}_h(u_j)=\emptyset$ for $j>1$. Then

$$\mathrm{WF}_h^{pol}(u)=\{\,(w_0,(z,0,\dots)):\ z\in\mathbf{C}\,\}$$

since $w_0\in\mathrm{WF}_h(u)$ and $\|A^w(x,hD)u\|=\mathcal{O}(h^\infty)$ if $A^wu=\sum_{j>1}A_j^wu_j$. By taking a suitable invertible E we obtain

$$\mathrm{WF}_h^{pol}(Eu)=\{\,(w_0,zv):\ z\in\mathbf{C}\,\}$$

for any $0\neq v\in\mathbf{C}^N$.

4. Systems of principal type

Recall that the scalar symbol $p(x,\xi)\in C^\infty(T^*\mathbf{R}^n)$ is of *principal type* if $dp\neq 0$ when $p=0$ by (2.12). In the following we let $\partial_\nu P(w)=\langle \nu, dP(w)\rangle$ for $P\in C^1(T^*\mathbf{R}^n)$ and $\nu\in T^*\mathbf{R}^n$. We shall use the following definition of systems of principal type, in fact, most of the systems we consider will be of this type. We shall denote by $\mathrm{Ker}\,P$ and $\mathrm{Ran}\,P$ the kernel and range of the matrix P.

Definition 4.1. The $N \times N$ system $P(w) \in C^\infty(T^*\mathbf{R}^n)$ is of *principal type* at w_0 if

$$\operatorname{Ker} P(w_0) \ni u \mapsto \partial_\nu P(w_0)u \in \operatorname{Coker} P(w_0) = \mathbf{C}^N / \operatorname{Ran} P(w_0) \tag{4.1}$$

is bijective for some $\nu \in T_{w_0}(T^*\mathbf{R}^n)$. The operator $P(h)$ given by (3.1) is of principal type if the principal symbol $P = \sigma(P(h))$ is of principal type.

Remark 4.2. *If $P(w) \in C^\infty$ is of principal type and $A(w)$, $B(w) \in C^\infty$ are invertible then APB is of principal type. We have that $P(w)$ is of principal type if and only if the adjoint P^* is of principal type.*

In fact, by Leibniz' rule we have

$$\partial(APB) = (\partial A)PB + A(\partial P)B + AP\partial B \tag{4.2}$$

and $\operatorname{Ran}(APB) = A(\operatorname{Ran} P)$ and $\operatorname{Ker}(APB) = B^{-1}(\operatorname{Ker} P)$ when A and B are invertible, which gives the invariance under left and right multiplication. Since $\operatorname{Ker} P^*(w_0) = \operatorname{Ran} P(w_0)^\perp$ we find that P satisfies (4.1) if and only if

$$\operatorname{Ker} P(w_0) \times \operatorname{Ker} P^*(w_0) \ni (u, v) \mapsto \langle \partial_\nu P(w_0)u, v\rangle \tag{4.3}$$

is a non-degenerate bilinear form. Since $\langle \partial_\nu P^* v, u\rangle = \overline{\langle \partial_\nu P u, v\rangle}$ we find that P^* is of principal type if and only if P is.

Observe that if P only has one vanishing eigenvalue λ (with multiplicity one) then the condition that P is of principal type reduces to the condition in the scalar case: $d\lambda \neq 0$. In fact, by using the spectral projection one can find invertible systems A and B so that

$$APB = \begin{pmatrix} \lambda & 0 \\ 0 & E \end{pmatrix}$$

with E invertible $(N-1)\times(N-1)$ system, which is of principal type if and only if $d\lambda \neq 0$.

Example 4.3. Consider the system

$$P(w) = \begin{pmatrix} \lambda_1(w) & 1 \\ 0 & \lambda_2(w) \end{pmatrix}$$

where $\lambda_j(w) \in C^\infty$, $j = 1, 2$. We find that $P(w) - \lambda \operatorname{Id}_2$ is not of principal type when $\lambda = \lambda_1(w) = \lambda_2(w)$ since $\operatorname{Ker}(P(w) - \lambda \operatorname{Id}_2) = \operatorname{Ran}(P(w) - \lambda \operatorname{Id}_2) = \mathbf{C} \times \{0\}$ is invariant under ∂P.

Observe that the property of being of principal type is not stable under C^1 perturbation, not even when $P = P^*$ is symmetric, by the following example.

Example 4.4. The system

$$P(w) = \begin{pmatrix} w_1 - w_2 & w_2 \\ w_2 & -w_1 - w_2 \end{pmatrix} = P^*(w) \qquad w = (w_1, w_2)$$

is of principal type when $w_1 = w_2 = 0$, but *not* of principal type when $w_2 \neq 0$ and $w_1 = 0$. In fact,

$$\partial_{w_1} P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is invertible, and when $w_2 \neq 0$ we have that

$$\operatorname{Ker} P(0, w_2) = \operatorname{Ker} \partial_{w_2} P(0, w_2) = \{\, z(1,1) : z \in \mathbf{C} \,\}$$

which is mapped to $\operatorname{Ran} P(0, w_2) = \{\, z(1,-1) : z \in \mathbf{C} \,\}$ by $\partial_{w_1} P$.

We obtain from Proposition 3.5 in [21] a simple characterization of systems of principal type. For that we recall κ_P, K_P and $\Xi(P)$ from Definition 3.3.

Proposition 4.5. *Assume $P(w) \in C^\infty$ is an $N \times N$ system and that $(w_0, \lambda_0) \in \Omega_1(P) \setminus \Xi(P)$, then $P(w) - \lambda_0 \operatorname{Id}_N$ is of principal type at w_0 if and only if $\kappa_P \equiv K_P$ at (w_0, λ_0) and $d\lambda(w_0) \neq 0$ for the C^∞ germ of eigenvalues for P at w_0 satisfying $\lambda(w_0) = \lambda_0$.*

Here, the germs of eigenvalues are given by Definition 3.10. Now, for symmetric systems we have $\kappa_P \equiv K_P$ and the differential $d\lambda \neq 0$ almost everywhere on $\Omega_1(P) \setminus \Xi(P)$. Proposition 4.5 shows that for a *symmetric* system the property to be of principal type is stable outside $\Xi(P)$: if the symmetric system $P(w) - \lambda \operatorname{Id}_N$ is of principal type at a point $(w_0, \lambda_0) \notin \Xi(P)$ then it is in a neighborhood. It follows from the Sard Theorem that symmetric systems $P(w) - \lambda \operatorname{Id}_N$ are of principal type almost everywhere on $\Omega_1(P)$. For eigenvalues $\lambda_0 \notin \Sigma_{ss}(P)$ we can define the following bracket condition.

Definition 4.6. Let $P \in C^\infty(T^*\mathbf{R}^n)$ be an $N \times N$ system, then we define

$$\Lambda(P) = \overline{\Lambda_-(P) \bigcup \Lambda_+(P)}$$

where $\Lambda_\pm(P)$ is the set of $\lambda_0 \in \Sigma(P)$ such that there exists $w_0 \in \Sigma_{\lambda_0}(P)$ so that $(w_0, \lambda_0) \notin \Xi(P)$ and

$$\pm \{\, \operatorname{Re} \lambda, \operatorname{Im} \lambda \,\}(w_0) > 0 \tag{4.4}$$

for the unique C^∞ germ $\lambda(w)$ of eigenvalues at w_0 for P such that $\lambda(w_0) = \lambda_0$.

Observe that $\Lambda_\pm(P) \bigcap \Sigma_{ss}(P) = \emptyset$, and it follows from Proposition 4.5 that $P(w) - \lambda_0 \operatorname{Id}_N$ is of principal type at $w_0 \in \Lambda_\pm(P)$ if and only if $\kappa_P = K_P$ at (w_0, λ_0), since $d\lambda(w_0) \neq 0$ when (4.4) holds. Because of the bracket condition (4.4) we find that $\Lambda_\pm(P)$ is contained in the interior of $\Sigma(P)$. The following result generalizes Theorem 2.3 to systems, for a proof see Theorem 3.10 in [21].

Theorem 4.7. *Let $P \in C^\infty(T^*\mathbf{R}^n)$ be an $N \times N$ system, then we have that*

$$\Lambda(P) \setminus \Big(\Sigma_{ws}(P) \bigcup \Sigma_\infty(P) \Big) \subseteq \overline{\Lambda_-(P)} \tag{4.5}$$

when $n \geq 2$. Assume that $P(h)$ is given by (3.1) with principal symbol $P \in C_{\mathrm{b}}^\infty(T^*\mathbf{R}^n)$, and that $\lambda_0 \in \Lambda_-(P)$, $0 \neq u_0 \in \operatorname{Ker}(P(w_0) - \lambda_0 \operatorname{Id}_N)$ and $P(w) - \lambda \operatorname{Id}_N$ is of principal type on $\Sigma_\lambda(P)$ near w_0 for $|\lambda - \lambda_0| \ll 1$, for the $w_0 \in \Sigma_{\lambda_0}(P)$ in Definition 4.6. Then there exists $u(h) \in L^2(\mathbf{R}^n)$ so that $\|u(h)\| \leq 1$

$$\|(P(h) - \lambda_0 \operatorname{Id}_N) u(h)\| \leq C_M h^M \quad \forall\, M \qquad h \to 0 \tag{4.6}$$

and $\mathrm{WF}_h^{pol}(u(h)) = \{\,(w_0,u_0)\,\}$. *There also exists a dense subset of values* $\lambda_0 \in \Lambda(P)$ *so that*

$$\|(P(h)-\lambda_0\,\mathrm{Id}_N)^{-1}\| \ge C'_M h^{-M} \quad \forall\, M \qquad h\to 0 \tag{4.7}$$

If all the terms P_j *in the expansion* (3.1) *are analytic satisfying* (2.6) *then* $h^{\pm M}$ *may be replaced by* $\exp(\mp c/h)$ *in* (4.6)–(4.7).

Theorem 4.7 together with Proposition 3.11 give that

$$\Lambda_-(P) \subseteq \lambda_\infty^{\mathrm{sc}}(P(h)) \subseteq \Lambda_\infty^{\mathrm{sc}}(P(h)) \subseteq \Sigma(P)$$

as in the scalar case.

Example 4.8. Let

$$P(x,\xi) = |\xi|^2\,\mathrm{Id} + iK(x) \qquad (x,\xi)\in T^*\mathbf{R}^n$$

where $K(x)\in C^\infty(\mathbf{R}^n)$ is symmetric for all x. Then we find from Sard's Theorem that

$$\overline{\Lambda_-(P)} = \Lambda(P) = \left\{\, \mathrm{Re}\, z \ge 0 \wedge \mathrm{Im}\, z \in \overline{\Sigma(K)\setminus\Big(\Sigma_{ss}(K)\bigcup\Sigma_\infty(K)\Big)}\,\right\}$$

As in the scalar case, when the dimension is equal to one we have to add some conditions in order to get the inclusion (4.5).

Lemma 4.9. *Let* $P(w)\in C^\infty(T^*\mathbf{R})$ *be an* $N\times N$ *system, then for every component* Ω *of* $\mathbf{C}\setminus(\Sigma_{ws}(P)\bigcup\Sigma_\infty(P))$ *which has non-empty intersection with* $\complement\Lambda(P)$ *we have*

$$\Lambda(P)\setminus\Big(\Sigma_{ws}(P)\bigcup\Sigma_\infty(P)\Big) \subseteq \overline{\Lambda_-(P)} \tag{4.8}$$

For a proof, see the proof of [21, Lemma 3.15] (which has an error in the formulation). Recall that the topological condition in Lemma 4.9 is necessary even in the scalar case by Example 2.4.

5. Quasi-symmetrizable systems

Next, we shall study the behavior at the boundary $\partial\Sigma(P)$ of the eigenvalues. First we note that if the system $P(w) - z\,\mathrm{Id}_N$ is of principal type near $\Sigma_z(P)$ for z close to $\lambda\in\partial\Sigma(P)\setminus(\Sigma_{ws}(P)\bigcup\Sigma_\infty(P))$ and $\Sigma_\lambda(P)$ has no closed bicharacteristics, then one can generalize Theorem 1.3 in [22] to obtain

$$\|(P(h)-\lambda\,\mathrm{Id}_N)^{-1}\| \le C/h \qquad h\to 0 \tag{5.1}$$

In fact, by using the reduction in the proof of [21, Theorem 3.10] this follows from the scalar case, see Example 5.7. But in this case the eigenvalues close to λ have constant multiplicity.

Generically, we have that the eigenvalues of the principal symbol P have constant multiplicity almost everywhere since $\Xi(P)$ is nowhere dense. But at the boundary $\partial\Sigma(P)$ this needs not be the case. For example, if

$$P(t,\tau) = \tau\,\mathrm{Id} + iK(t)$$

where $C^\infty \ni K \geq 0$ is unbounded and $0 \in \Sigma_{ss}(K)$, then $\mathbf{R} = \partial\Sigma(P) \subseteq \Sigma_{ss}(P)$.

When the multiplicity of the eigenvalues of the principal symbol is not constant the situation is more complicated. Then the following example shows that it is not sufficient to have conditions only on the eigenvalues in order to obtain the estimate (5.1), not even in the principal type case.

Example 5.1. Let $a(t) \in C^\infty(\mathbf{R})$ be real valued and let

$$P^w(t, hD_t) = \begin{pmatrix} hD_t + a(t) & t - ia(t) \\ t + ia(t) & -hD_t + a(t) \end{pmatrix} = P^w(t, hD_t)^*$$

Then the eigenvalues of $P(t, \tau)$ are

$$\lambda = a(t) \pm \sqrt{\tau^2 + t^2 + a^2(t)} \in \mathbf{R}$$

which coincide if and only if $\tau = t = a(t) = 0$. We have that

$$\frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} P^w \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} hD_t + it & 0 \\ 2a(t) & hD_t - it \end{pmatrix} = \widetilde{P}(h)$$

Thus we can construct pseudomodes $u_h(t) = {}^t(0, u_2(t))$ so that $\|u_h\| = 1$ and $\widetilde{P}(h)u_h = \mathcal{O}(\exp(-c/h))$ by Theorem 2.3. By the invariance, we see that P is of principal type at $t = \tau = 0$ if and only if $a(0) = 0$. If $a(0) = 0$ then $\Sigma_{ws}(P) = \{\, 0 \,\}$ and when $a \neq 0$ we have that P^w is a selfadjoint diagonalizable system. In the case $a \equiv 0$ the eigenvalues of P^w are $\pm\sqrt{2kh}$, $k \in \mathbf{N}$, see the proof of Proposition 3.6.1 in [33].

Of course, the problem is that the eigenvalues are not invariant under multiplication with elliptic systems. To obtain the estimate (5.1) for operators that are *not* of principal type, it is not even sufficient that the principal symbol is C^∞ diagonalizable with real eigenvalues with constant multiplicity one, according to the following example.

Example 5.2. Let $a(t) \in C^\infty(\mathbf{R})$, $a(0) = 0$, $a'(0) > 0$ and

$$P(h) = \begin{pmatrix} 1 & hD_t \\ h & iha(t) \end{pmatrix}$$

with principal symbol $\begin{pmatrix} 1 & \tau \\ 0 & 0 \end{pmatrix}$ having eigenvalues 0 and 1, thus $\Xi(P) = \emptyset$. Since

$$\begin{pmatrix} 1 & 0 \\ -h & 1 \end{pmatrix} P(h) \begin{pmatrix} 1 & -hD_t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -h \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & hD_t - ia(t) \end{pmatrix}$$

we obtain from by Theorem 2.3 that $\|P(h)^{-1}\| \geq C_M h^{-M}$ when $h \to 0$, $\forall\, M$, and for analytic $a(t)$ we obtain $\|P(h)^{-1}\| \geq Ce^{c/h}$, $h \to 0$. Now $\partial_\tau P$ maps $\operatorname{Ker} P(0)$ into $\operatorname{Ran} P(0)$ so the system is not of principal type. Observe that this property is not preserved under the multiplications above, since not all the systems are elliptic.

Instead of using properties of the eigenvalues of the principal symbol, we shall use properties that are invariant under multiplication with invertible systems. We recall the following normal form for scalar symbols of principal type near the boundary $\partial\Sigma(P)$.

Example 5.3. Assume that $p(x, \xi) \in C^\infty(T^*\mathbf{R}^n)$ is scalar of principal type and $0 \in \partial\Sigma(p) \setminus \Sigma_\infty(p)$. Then we find from the proof of Lemma 4.1 in [22] that we can choose symplectic coordinates so that

$$p(x, \xi) = e(x, \xi)(\xi_1 + if(x, \xi')) \qquad \xi = (\xi_1, \xi') \tag{5.2}$$

in a neighborhood of $w_0 \in \Sigma_0(p)$, where $e \neq 0$ and $f \geq 0$. If there are no closed semibicharacteristics of p, we obtain (5.2) in a neighborhood of $\Sigma_0(P)$.

Recall that a *semibicharacteristic* of p is a non-trivial bicharacteristic of $\operatorname{Re} qp$, for some $q \neq 0$. The normal form (5.2) in the scalar case motivates the following definition, which is Definition 4.5 in [21].

Definition 5.4. We say that the $N \times N$ system $P(w) \in C^\infty(T^*\mathbf{R}^n)$ is *quasi-symmetrizable* with respect to the real C^∞ vector field V in $\Omega \subseteq T^*\mathbf{R}^n$ if $\exists$ $N \times N$ system $M(w) \in C^\infty(T^*\mathbf{R}^n)$ and c, $C > 0$ so that

$$\begin{cases} \operatorname{Re}\langle M(VP)u, u\rangle \geq c\|u\|^2 - C\|Pu\|^2 \\ \operatorname{Im}\langle MPu, u\rangle \geq -C\|Pu\|^2 \end{cases} \qquad \text{in } \Omega \tag{5.3}$$

for any $u \in \mathbf{C}^N$. The system M is called a *symmetrizer* for P.

The definition is clearly independent of the choice of coordinates in $T^*\mathbf{R}^n$ and choice of base in $\mathbf{C}^N$. When P is elliptic, we may take $M = iP^*$ as multiplier, then P is quasi-symmetrizable with respect to any vector field because $\|Pu\| \cong \|u\|$. We see from Example 5.3 that the scalar symbol p of principal type is quasi-symmetrizable in neighborhood of any point at $\partial\Sigma(p) \setminus \Sigma_\infty(p)$. We obtain the following result from Propositions 4.7 and 4.10 in [21].

Proposition 5.5. *If $P(w) \in C^\infty(T^*\mathbf{R}^n)$ be an quasi-symmetrizable $N \times N$ system, then P is of principal type and P^* is quasi-symmetrizable. If $A(w)$ and $B(w) \in C^\infty(T^*\mathbf{R}^n)$ are invertible $N \times N$ systems then BPA is quasi-symmetrizable.*

Example 5.6. The $N \times N$ system

$$P(w) = w_1 \operatorname{Id}_N + iF(w) \qquad F \geq 0$$

is quasi-symmetrizable with respect to ∂_{w_1}.

Example 5.7. Assume $P(w) \in C^\infty$ is $N \times N$ and $z \in \Sigma(P) \setminus (\Sigma_{ws}(P) \bigcap \Sigma_\infty(P))$ such that $P(w) - \lambda \operatorname{Id}_N$ is of principal type when $|\lambda - z| \ll 1$. Then we can make a base change $B(w) \in C^\infty$ so that

$$P(w) = B^{-1}(w) \begin{pmatrix} \lambda(w) \operatorname{Id}_K & 0 \\ 0 & P_{22}(w) \end{pmatrix} B(w) \tag{5.4}$$

in a neighborhood of $\Sigma_z(P)$, where $|P_{22} - \lambda(w) \operatorname{Id}| \neq 0$. We find from Proposition 4.5 that $d\lambda \neq 0$ when $\lambda = z$, so $\lambda - z$ is of principal type. Proposition 5.5 and Example 5.3 give that $P - z \operatorname{Id}_N$ is quasi-symmetrizable near any $w_0 \in \Sigma_z(P)$ if $z \in \partial\Sigma(\lambda)$. If there are no closed semibicharacteristics of $\lambda - z$ then we also find from Example 5.3 that $P - z\,\mathrm{Id}_N$ is quasi-symmetrizable in a neighborhood of $\Sigma_z(P)$.

Example 5.8. Let

$$P(x,\xi) = |\xi|^2\,\mathrm{Id}_N + iK(x)$$

where $0 \le K(x) \in C^\infty$. When $\lambda > 0$ we find that $P - \lambda\,\mathrm{Id}_N$ is quasi-symmetrizable in a neighborhood of $\Sigma_\lambda(P)$ with respect to the exterior normal $\langle \xi, \partial_\xi \rangle$ to $\Sigma_\lambda(P) = \{\, |\xi|^2 = \lambda \,\}$.

It is interesting to note that the operator in Example 5.8 is of the type considered by Lidskiĭ in [45, 46, 48]. For scalar symbols, we find that $0 \in \partial\Sigma(p)$ if and only if p is quasi-symmetrizable, see Example 5.3. But in the system case, this needs not be the case according to the following example from [21].

Example 5.9. Let

$$P(w) = \begin{pmatrix} w_2 + iw_3 & w_1 \\ w_1 & w_2 - iw_3 \end{pmatrix} \qquad w = (w_1, w_2, w_3)$$

then P is quasi-symmetrizable with respect to ∂_{w_1} with symmetrizer $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In fact, $\partial_{w_1} MP = \mathrm{Id}_2$ and

$$MP(w) = \begin{pmatrix} w_1 & w_2 - iw_3 \\ w_2 + iw_3 & w_1 \end{pmatrix} = (MP(w))^*$$

Since eigenvalues of $P(w)$ are $w_2 \pm \sqrt{w_1^2 - w_3^2}$ we find that $\Sigma(P) = \mathbf{C}$, thus $0 \in \overset{\circ}{\Sigma}(P)$ is an interior point of the eigenvalues.

For quasi-symmetrizable systems we obtain the following result from Theorem 4.15 in [21], which generalizes Theorem 2.5 to systems.

Theorem 5.10. *Let the $N \times N$ system $P(h)$ be given by* (3.1) *with principal symbol $P \in C_{\mathrm{b}}^\infty(T^*\mathbf{R}^n)$. Assume that $z \notin \Sigma_\infty(P)$ and there exists a real valued function $T(w) \in C^\infty$ such that $P(w) - z\,\mathrm{Id}_N$ is quasi-symmetrizable with respect to the Hamilton vector field $H_T(w)$ in a neighborhood of $\Sigma_z(P)$. Then for any $K > 0$ we have*

$$\big\{\zeta \in \mathbf{C} :\ |\zeta - z| < Kh\log(1/h)\big\} \bigcap \mathrm{Spec}(P(h)) = \emptyset \qquad 0 < h \ll 1 \tag{5.5}$$

and

$$\big\|(P(h) - z)^{-1}\big\| \le C/h \qquad 0 < h \ll 1 \tag{5.6}$$

If P is analytic in a tubular neighborhood of $T^*\mathbf{R}^n$ then $\exists\, c_0 > 0$ such that

$$\big\{\zeta \in \mathbf{C} :\ |\zeta - z| < c_0\big\} \bigcap \mathrm{Spec}(P(h)) = \emptyset \qquad 0 < h \ll 1 \tag{5.7}$$

Condition (5.6) means that $z \notin \Lambda_1^{\mathrm{sc}}(P)$, which is the pseudospectrum of index 1 by Definition 2.7. The conditions in Theorem 5.10 give some geometrical information on the bicharacteristic flow of the eigenvalues according to the following remark.

Remark 5.11. *The conditions in Theorem 5.10 imply that the limit set at $\Sigma_\lambda(P)$ of the non-trivial semibicharacteristics of the eigenvalues close to zero of $Q = M(P - \lambda \operatorname{Id}_N)$ is a union of compact curves on which T is strictly monotone, thus they cannot form closed orbits.*

Example 5.12. Consider the system in Example 5.8

$$P(x,\xi) = |\xi|^2 \operatorname{Id}_N + iK(x)$$

where $0 \le K(x) \in C^\infty$, then for $\lambda > 0$ we find that $P - \lambda \operatorname{Id}_N$ is quasi-symmetric in a neighborhood of $\Sigma_\lambda(P)$ with respect to $V = H_T$, for $T(x,\xi) = -\langle \xi, x\rangle$. If $K(x) \in C_{\mathrm{b}}^\infty$ with $0 \notin \Sigma_\infty(K)$ then we obtain from Proposition 3.12, Remark 3.13, Example 3.14 and Theorem 5.10 that

$$\|(P^w(x,hD) - \lambda)^{-1}\| \le C/h \qquad 0 < h \ll 1$$

since $0 < \lambda \notin \Sigma_\infty(P)$.

One can also generalize the improved bounds of finite type boundary points given by Theorem 2.6 to systems. The situation is more complicated in the systems case, see Definition 3.15 and Theorem 5.20 in [21].

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