

Small Families of Complex Lines for Testing Holomorphic Extendibility

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**SMALL FAMILIES OF COMPLEX LINES FOR TESTING
HOLOMORPHIC EXTENDIBILITY**

Josip Globevnik

To Urban, Manja and Lidija

ABSTRACT Let B be the open unit ball in \mathbb{C}^2 . The paper deals with the analog of Hartogs separate analyticity theorem for CR functions on the sphere bB . We prove such a theorem for functions in $C^\infty(bB)$: If $a, b \in \overline{B}$, $a \neq b$ and if $f \in C^\infty(bB)$ extends holomorphically into B along any complex line passing through either a or b , then f extends holomorphically through B . On the other hand, for each $k \in \mathbb{N}$ there is a function $f \in C^k(bB)$ which extends holomorphically into B along any complex line passing through either a or b yet f does not extend holomorphically through B . More generally, in the paper we obtain a fairly complete description of pairs of points $a, b \in \mathbb{C}^2, a \neq b$, such that if $f \in C^\infty(bB)$ extends holomorphically into B along every complex line passing through either a or b that meets B , then f extends holomorphically through B .

1. Introduction and the main result

Denote by Δ the open unit disc in \mathbb{C} and by B the open unit ball in \mathbb{C}^2 . If f is a continuous function on bB and L is a complex line in \mathbb{C}^2 that meets B then we say that f extends holomorphically along L if $f|_{L \cap bB}$ has a continuous extension through $L \cap \overline{B}$ which is holomorphic on $L \cap B$. Given $a \in \mathbb{C}^2$ we denote by $\mathcal{L}(a)$ the family of all complex lines passing through a .

QUESTION 1.1 *Let $a, b \in B$, $a \neq b$. Assume that $f \in C(bB)$ extends holomorphically along every complex line in $\mathcal{L}(a) \cup \mathcal{L}(b)$. Must f extend holomorphically through B ?*

EXAMPLE 1.1 Let $k \in \mathbb{N}$. For each $(z, w) \in bB$ let

$$f(z, w) = \begin{cases} z^{k+2}/\overline{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

The function f is of class C^k on bB . Let L be any complex line that meets the disc $\{0\} \times \Delta$. Then, if L is not equal to the w -axis, we have $L \cap bB = \{(p_1 + \zeta q_1, p_2 + \zeta q_2) : \zeta \in b\Delta\}$ where $|p_1| < |q_1|$ so that the circle $\{p_1 + \zeta q_1 : \zeta \in b\Delta\}$ surrounds the origin. It is easy to see that in this case the function $\zeta \mapsto (p_1 + \zeta q_1)^{k+2}/(p_1 + \zeta q_1)$ ($\zeta \in b\Delta$) extends holomorphically

through Δ . Thus, f extends holomorphically along each complex line that meets $\{0\} \times \Delta$, yet f does not extend holomorphically through B . Thus, for each $k \in \mathbb{N}$ there is a function $f \in C^k(bB)$ which provides a negative answer to the question above. In the present paper we show that there are no such functions of class C^∞ :

THEOREM 1.1 *Let $a, b \in \overline{B}$, $a \neq b$. If a function $f \in C^\infty(bB)$ extends holomorphically along each complex line in $\mathcal{L}(a) \cup \mathcal{L}(b)$ that meets B then f extends holomorphically through B .*

In other words, for each pair of points $a, b \in \overline{B}$, $\mathcal{L}(a) \cup \mathcal{L}(b)$ is a test family for holomorphic extendibility for $C^\infty(bB)$.

In the present paper we consider more general pairs of points a, b . Given $a, b \in \mathbb{C}^2$, $a \neq b$, denote by $\Lambda(a, b)$ the complex line passing through a and b . Our main result is the following

THEOREM 1.2 *Let $a, b \in \mathbb{C}^2$, $a \neq b$.*

(A) *Suppose that one of the points a, b is contained in B .*

(A1) *If $\langle a|b \rangle \neq 1$ then $\mathcal{L}(a) \cup \mathcal{L}(b)$ is a test family for holomorphic extendibility for $C^\infty(bB)$.*

(A2) *If $\langle a|b \rangle = 1$ then*

$$\left. \begin{array}{l} \text{there is a function } f \in C^\infty(bB) \text{ which extends holomorphically} \\ \text{along every complex line in } \mathcal{L}(a) \cup \mathcal{L}(b) \text{ that meets } B, \\ \text{yet } f \text{ does not extend holomorphically through } B. \end{array} \right\} \quad (1.1)$$

(B) *Suppose now that both points a, b are contained in $\mathbb{C}^2 \setminus B$*

(B1) *If $\Lambda(a, b)$ meets B then $\mathcal{L}(a) \cup \mathcal{L}(b)$ is a test family for holomorphic extendibility for $C^\infty(bB)$*

(B2) *If $\Lambda(a, b)$ misses \overline{B} then (1.1) holds.*

REMARK Note that Theorem 1.1 follows from Theorem 1.2. The methods used in the present paper do not apply to the case when $\Lambda(a, b)$ is tangent to bB and it will remain an open question for which pairs of points a, b of this sort $\mathcal{L}(a) \cup \mathcal{L}(b)$ is a test family for holomorphic extendibility for $C^\infty(bB)$.

If φ is a continuous function on a circle Γ then we say that φ extends holomorphically from Γ if it extends holomorphically through the disc bounded by Γ .

Given $\alpha \in \Delta$ denote by C_α the family of the circles obtained as the images under the Moebius map $z \mapsto (\alpha - z)/(1 - \overline{\alpha}z)$ of all circles in $\overline{\Delta}$ centered at the origin. To prove Theorem 1.2 we will have to prove the following new, one variable result:

THEOREM 1.3 *Let $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, and let φ be a continuous function on $\overline{\Delta}$ which extends holomorphically from each circle in $C_\alpha \cup C_\beta$. Then φ is holomorphic on Δ .*

2. Simplifying the geometry by using automorphisms of B

Let $a \in B$. Write $P_0 = 0$ and

$$P_a(z) = \frac{\langle z|a \rangle}{\langle a|a \rangle} a \quad ((z \in \mathbb{C}^2) \text{ if } a \neq 0,$$

$$s_a = \sqrt{1 - |a|^2}, \quad Q_a = I - P_a.$$

The map φ_a defined by

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z|a \rangle} \quad (z \in \mathbb{C}^2, \langle z|a \rangle \neq 1). \quad (2.1)$$

is an automorphism of B , a homeomorphism of \overline{B} and a C^ω -diffeomorphism of bB . It is a fractional linear map of \mathbb{C}^2 that maps $\{z \in \mathbb{C}^2: \langle z|a \rangle \neq 1\}$ onto itself and satisfies $\varphi_a(a) = 0$, $\varphi_a(0) = a$, $\varphi \circ \varphi = id$. It maps complex lines to complex lines [Ru]. In particular, if $b \in \mathbb{C}^2$ and $\langle b|a \rangle \neq 1$ then it maps $\mathcal{L}(b)$ to $\mathcal{L}(\varphi_a(b))$ where $\varphi_a(b)$ is a point in \mathbb{C}^2 and if $\langle b|a \rangle = 1$ then it maps $\mathcal{L}(b)$ to a family of parallel complex lines. In the special case when $a = (\lambda, 0)$ where $\lambda \in \Delta$, we have

$$\varphi_a(z, w) = \left(\frac{\lambda - z}{1 - \overline{\lambda}z}, -\frac{\sqrt{1 - |\lambda|^2}}{1 - \overline{\lambda}z} w \right). \quad (2.2)$$

It is known that every automorphism of B is of the form $U \circ \varphi_a$ for some $a \in B$ and for some unitary map U [Ru].

Let φ be an automorphism of B . If $f \in C^\infty(bB)$ then $f \circ \varphi^{-1} \in C^\infty(bB)$ and f extends holomorphically along each complex line in $\mathcal{L}(a)$ that meets B if and only if $f \circ \varphi^{-1}$ extends holomorphically along each complex line $\varphi(L)$, $L \in \mathcal{L}(a)$, that meets B , that is, along each complex line in $\mathcal{L}(\varphi(a))$ that meets B . Since $f \circ \varphi^{-1}$ extends holomorphically through B if and only if f does it follows that $\mathcal{L}(a) \cup \mathcal{L}(b)$ is a test family for holomorphy for $C^\infty(bB)$ if and only if $\mathcal{L}(\varphi(a)) \cup \mathcal{L}(\varphi(b))$ is a test family for holomorphy for $C^\infty(bB)$. This observation simplifies the geometry by applying automorphisms before beginning the proof of Theorem 1.2.

Consider (A1). Let $a \in B$ and $\langle a|b \rangle \neq 1$. Then $\varphi_a(a) = 0$ and $\varphi_a(b)$ is a point in \mathbb{C}^2 . Hence, after a composition with a unitary map, we may, with no loss of generality assume that $a = (0, 0)$ and $b = (t, 0)$ where $t > 0$. Now, consider (B1). Suppose that $|a| \geq 1$, $|b| \geq 1$ and assume that $\Lambda(a, b)$ meets B . Then we may, after composition by a unitary map, assume that $\Lambda(a, b) = \{(z, w) \in \mathbb{C}^2: z = \lambda\}$ where $0 \leq \lambda < 1$. The transform (2.2) maps $\Lambda(a, b)$ to the w -axis. So, with no loss of generality assume that a, b are both on one coordinate axis, say on z -axis, that is $a = (\alpha, 0)$, $b = (\beta, 0)$, $\alpha \neq \beta$, where $|\alpha| \geq 1$, $|\beta| \geq 1$. One possibility is $|\alpha| = |\beta| = 1$. If one of α, β , say α , satisfies $|\alpha| > 1$ then the map (2.2) with $\lambda = 1/\overline{\alpha}$ maps the complex lines through $(\alpha, 0)$ to the complex lines parallel to the z -axis and the point $(\beta, 0)$ to a point on z -axis, which, after a suitable rotation $(z, w) \mapsto (e^{i\omega}z, w)$ we may assume that is of the form $(t, 0)$ where $t \geq 1$. Thus, (A1) and (B1) will be proved once we have proved that each of the following families of lines (2.3), (2.4) and (2.5) is a test family for holomorphic extendibility for $C^\infty(bB)$:

$$\left. \begin{array}{l} \text{the complex lines passing through the origin and the complex} \\ \text{lines passing through a point } (t, 0) \text{ where } t > 0, \end{array} \right\} \quad (2.3)$$

$$\left. \begin{array}{l} \text{the complex lines passing through the point } (\alpha, 0) \text{ and the complex} \\ \text{lines passing through the point } (\beta, 0) \text{ where } |\alpha| = |\beta| = 1, \alpha \neq \beta, \end{array} \right\} \quad (2.4)$$

the complex lines parallel to the z – axis and the complex lines passing through a point $(t, 0)$ where $t \geq 1$. } (2.5)

Consider now (A2), so let $a \in B$ and $\langle a|b \rangle = 1$. The map φ_a maps $\mathcal{L}(a)$ to $\mathcal{L}(0)$ and $\mathcal{L}(b)$ to a family of parallel lines for which, after a composition with a unitary map, we may assume that are parallel to the z -axis. Now, consider (B2). After composition by a unitary map we may assume that $\Lambda(a, b) = \{(z, w) \in \mathbb{C}^2: z = \mu\}$ where $\mu > 1$ so that $a = (\mu, \alpha)$, $(b = (\mu, \beta)$. An easy computation shows that the map (2.2) with $\lambda = 1/\mu$ maps $\mathcal{L}(a)$ to the complex lines parallel to $(1, \eta_\alpha)$ and $\mathcal{L}(b)$ to the complex lines parallel to $(1, \eta_\beta)$ where

$$\eta_\alpha = \frac{\alpha}{\sqrt{\mu^2 - 1}}, \quad \eta_\beta = \frac{\beta}{\sqrt{\mu^2 - 1}}.$$

It follows that (A2) will be proved once we have proved that

there is an $f \in C^\infty(bB)$ that extends holomorphically along every complex line passing through the origin and every complex line parallel to z – axis which meets B yet f does not extend holomorphically through B , } (2.6)

and (B2) will be proved once we have proved that

given $p, q \in \mathbb{C}$, $p \neq q$, there is an $f \in C^\infty(bB)$ which extends holomorphically along every complex line parallel to $(1, p)$ which meets B , and along every complex line parallel to $(1, q)$ which meets B , yet f does not extend holomorphically through B . } (2.7)

It is easy to see that the function $f(z, w) = |w|^2$ satisfies (2.6). To get an example of a function f that satisfies (2.7) we follow A. M. Kytmanov and S. G. Myslivets [KM] and put

$$f(z, w) = \bar{z}[z(1 + |p|^2) + \bar{p}(w - pz)] \cdot [z(1 + |q|^2) + \bar{q}(w - qz)].$$

It is easy to check that f satisfies (2.7): If a complex line is parallel to $(1, p)$ then it has the form $\{(\zeta, c + \zeta p): \zeta \in \mathbb{C}\}$ for some $c \in \mathbb{C}$. Note that $|\zeta|^2 + |c + \zeta p|^2 = 1$ implies that $\bar{\zeta}[\zeta(|p|^2 + 1) + \zeta \bar{p}] = 1 - |c|^2 - \zeta p \bar{c}$ so for such ζ , $f(\zeta, c + \zeta p) = [1 - |c|^2 - \zeta p \bar{c}] \cdot [\zeta(1 + |q|^2) + \bar{q}(c + \zeta p - \zeta q)]$ depends holomorphically on ζ . Repeating the reasoning for $(1, q)$ we see that f satisfies (2.7).

Note that both examples above are real analytic.

It remains to prove that the families (2.3), (2.4) and (2.5) are test families for holomorphic extendibility for $C^\infty(bB)$.

3. Reduction to a sequence of one variable problems

As in [G1, G2] we shall use the Fourier series decomposition and averaging. Suppose that $f \in C^\infty(bB)$. Given $n \in \mathbb{Z}$ and $z \in \Delta$ let

$$c_n(z) = \left(\frac{1}{\sqrt{1 - |z|^2}} \right)^n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(z, e^{i\theta} \sqrt{1 - |z|^2}) d\theta.$$

so that, since $e^{i\theta} \mapsto f(z, \sqrt{1-|z|^2}e^{i\theta})$ is smooth, we have

$$f(z, e^{i\theta} \sqrt{1-|z|^2}) = \sum_{n=-\infty}^{\infty} (\sqrt{1-|z|^2})^n c_n(z) e^{in\theta} = \sum_{n=-\infty}^{\infty} c_n(z) (\sqrt{1-|z|^2} e^{i\theta})^n.$$

If $(z, w) \in bB$ then writing $w = e^{i\theta} \sqrt{1-|z|^2}$ the preceding discussion implies that

$$f(z, w) = \sum_{n=-\infty}^{\infty} c_n(z) w^n \quad ((z, w) \in bB, w \neq 0).$$

The coefficients c_n are continuous on Δ and from the definition it follows that if $n \leq 0$ they also continuously extend to \overline{D} . We shall show that when $f \in C^\infty(bB)$ the same holds for $n > 0$:

LEMMA 3.1 *Suppose that $f \in C^\infty(bB)$. Then for each $n \in \mathbb{Z}$ the function $z \mapsto c_n(z)$ ($z \in \Delta$) extends continuously to $\overline{\Delta}$.*

Let $z_0 \in \mathbb{C}$ and assume that $f \in C(bB)$ extends holomorphically along every complex line passing through $(z_0, 0)$ which meets B . If L is such a complex line then so is $\omega_\theta(L)$ where, for $\theta \in \mathbb{R}$, $\omega_\theta(z, w) = (z, e^{-i\theta}w)$. It follows that f extends holomorphically along every complex line $\omega_\theta(L)$ which is the same to say that for each $\theta \in \mathbb{R}$, $(z, w) \mapsto f(z, e^{i\theta}w)$ extends holomorphically along L and hence for each $n \in \mathbb{Z}$, the same holds for

$$\Psi_n(z, w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(z, we^{i\theta}) d\theta,$$

a continuous function on bB . Note that

$$\Psi_n(z, w) = w^n c_n(z) \quad ((z, w) \in bB, w \neq 0).$$

This proves

PROPOSITION 3.1 *If a function $f \in C(bB)$ extends holomorphically along each complex line that passes through $(z_0, 0)$ and meets B then for each $n \in \mathbb{Z}$ the same holds for the continuous extension of $(z, w) \mapsto w^n c_n(z)$ ($(z, w) \in bB, w \neq 0$) to bB .*

Analogous statement holds for complex lines parallel to z -axis. For such complex lines $w = \text{const}$ so we have

PROPOSITION 3.2 *If a function $f \in C(bB)$ extends holomorphically along each complex line that is parallel to the z -axis and meets B then for each $n \in \mathbb{Z}$ the same holds for the function $(z, w) \mapsto c_n(z)$ ($(z, w) \in bB, w \neq 0$).*

Note that this is the same as to say that c_n extends holomorphically from each circle in Δ centered at the origin.

Denote by π_1 the projection onto z -axis, $\pi_1(z, w) = z$.

Assume now that for given $z_0 \in \mathbb{C}$ a function $f \in C(bD)$ extends holomorphically along each complex line passing through $(z_0, 0)$. By Proposition 3.1 we know that for

each $n \in Z$ the continuous extension to bB of the function $(z, w) \mapsto w^n c_n(z)$ has the same property. So let L be such a complex line that is not the z -axis and is not parallel to the w -axis so that $\pi_1(L \cap bB) = \{p + \zeta q: \zeta \in b\Delta\}$ is not a point, and let $n \in Z$. Write $L \cap bB = \{(p + \zeta q, r + \zeta s): \zeta \in b\Delta\}$, note that $q \neq 0$, $s \neq 0$ and note that $c_n(p + \zeta q)$ is not defined for at most one point $\zeta \in b\Delta$ for which we have $r + \zeta s = 0$ and that this can happen only in the case when $|z_0| = 1$ when $p + \zeta q = z_0$. Thus,

$$\left. \begin{array}{l} \text{the continuous extension of } \zeta \mapsto (r + \zeta s)^n c_n(p + \zeta q) \text{ to } b\Delta \\ \text{extends holomorphically through } \Delta. \end{array} \right\} \quad (3.1)$$

Since L passes through $(z_0, 0)$ there is a $\zeta_0 \in \mathbb{C}$ such that $p + \zeta_0 q = z_0$ and $r + \zeta_0 s = 0$. Clearly $\zeta_0 = (z_0 - p)/q$ so that $r + (z_0 - p)s/q = 0$. Writing $p + \zeta q = z$ we get $\zeta = (z - p)/q$ and it follows that $r + \zeta s = (z - z_0)s/q$, so (3.1) implies that the continuous extension of $z \mapsto (z - z_0)^n c_n(z)$ to the circle $\{p + \zeta q: \zeta \in b\Delta\}$ extends holomorphically from this circle. This proves

PROPOSITION 3.3 *If $f \in C(bB)$ extends holomorphically along each complex line passing through $(z_0, 0)$ that meets B then for each such complex line L and for each $n \in Z$ the function $z \mapsto (z - z_0)^n c_n(z)$ extends holomorphically from $\pi_1(L \cap bB)$.*

REMARK In the case when $z_0 \in b\Delta$ the circle $\pi_1(L \cap bB)$ contains z_0 and in this case we have to be more precise and say that the continuous extension of $z \mapsto (z - z_0)^n c_n(z)$ to the circle $\pi_1(L \cap bB)$ extends holomorphically from this circle.

We shall prove the following

LEMMA 3.2 *Suppose that φ is a continuous function on $\overline{\Delta}$. Suppose that \mathcal{F} is one of the families (2.3), (2.4), (2.5) of complex lines. Suppose that for each complex line belonging to \mathcal{F} which meets the ball, φ extends holomorphically from the circle $\pi_1(L \cap bB)$. Then φ is holomorphic on Δ .*

Assuming Lemma 3.1 and Lemma 3.2 for a moment we can now complete the proof of Theorem 1.2 as follows:

Let $f \in C^\infty(bB)$. By Lemma 3.1 for each $n \in Z$ the function c_n extends continuously to $\overline{\Delta}$.

Suppose that f extends holomorphically along each complex line belonging to one of the families (2.3), (2.4), (2.5) that meets B . Suppose that $n \geq 0$. By Proposition 3.3 and Proposition 3.2 either

$$\left. \begin{array}{l} \text{for each } L \text{ belonging to the family (2.3) the function } z \mapsto \varphi(z) = \\ z^n (z - t)^n c_n(z) \text{ extends holomorphically from } \pi_1(L \cap bB) \end{array} \right\} \quad (3.2)$$

or

$$\left. \begin{array}{l} \text{for each } L \text{ belonging to the family (2.4) the function } z \mapsto \varphi(z) = \\ (z - \alpha)^n (z - \beta)^n c_n(z) \text{ extends holomorphically from } \pi_1(L \cap bB) \end{array} \right\} \quad (3.3)$$

or

$$\left. \begin{array}{l} \text{for each } L \text{ belonging to the family (2.5) the function } z \mapsto \varphi(z) = \\ (z - t)^n c_n(z) \text{ extends holomorphically from } \pi_1(L \cap bB.) \end{array} \right\} \quad (3.4)$$

Here we used the fact that if a function ψ extends holomorphically from a circle Γ and if $p \in \mathbb{C}$ then the function $z \mapsto (z - p)^n \psi(z)$ extends holomorphically from Γ . Since in each of the three cases the function φ is continuous on $\overline{\Delta}$ it follows by Lemma 3.2 that in each of the three cases the function φ is holomorphic on Δ . It follows that c_n is holomorphic on Δ except perhaps at two poles, 0 and t , which, by the continuity of c_n , are removable singularities, hence c_n is holomorphic on Δ .

Now let $n < 0$. Then, if for a function ψ the function $z \mapsto (z - p)^n \psi(z)$ extends holomorphically from a circle Γ it follows that the function ψ extends holomorphically from Γ . Together with Proposition 3.3 and Proposition 3.2 this implies that the function c_n extends holomorphically from $\pi_1(L \cap bB)$ either for each L belonging to the family (2.3), or for each L belonging to the family (2.4) or for each L belonging to the family (2.5). Since c_n is continuous on Δ it follows by Lemma 3.2 that in each of the three cases the function c_n is holomorphic on Δ .

It follows that

$$f(z, w) = \sum_{n=0}^{\infty} w^n c_n(z) \quad ((z, w) \in bB)$$

where each c_n is continuous on $\overline{\Delta}$ and holomorphic on Δ . Since this is a Fourier decomposition of a smooth function, the series converges uniformly on bB and so by the maximum principle, uniformly on \overline{B} . So f extends holomorphically through B . This completes the proof of Theorem 1.2. It remains to prove Lemma 3.1 and Lemma 3.2.

4. Proof of Lemma 3.1

Assume that $f \in C^\infty(bB)$. We have to show that for each $n \in \mathbb{N}$ the function

$$z \mapsto c_n(z) = \left\{ \frac{1}{\sqrt{1 - |z|^2}} \right\}^n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(z, e^{i\theta} \sqrt{1 - |z|^2}) d\theta,$$

which is defined and smooth on Δ , extends continuously to $\overline{\Delta}$.

Write $z = \sqrt{1 - R^2} e^{i\varphi}$. We have $\sqrt{1 - |z|^2} = R$ so

$$c_n(z) = \frac{1}{R^n} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\sqrt{1 - R^2} e^{i\varphi}, R e^{i\theta}) d\theta. \quad (4.1)$$

We will prove the continuous extendibility of c_n to $\overline{\Delta}$ by showing that as $R \searrow 0$, the functions $e^{i\varphi} \mapsto c_n(\sqrt{1 - R^2} e^{i\varphi})$ converge to a function $e^{i\varphi} \mapsto c_n(e^{i\varphi})$, uniformly in φ , $-\pi \leq \varphi \leq \pi$. With no loss of generality assume that $f \in C^\infty(C^2)$. To prove the continuous extendibility notice first that whenever Φ is a smooth function on \mathbb{C}^2 and

$j \in \mathbb{N}$ then the integration by parts gives

$$\begin{aligned}
& \int_{-\pi}^{\pi} \Phi(\sqrt{1-R^2}e^{i\varphi}, Re^{i\theta})e^{-ij\theta}d\theta = \\
&= \frac{1}{ij} \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta} \left[\Phi(\sqrt{1-R^2}e^{i\varphi}, Re^{i\theta}) \right] e^{-ij\theta} d\theta = \\
&= \frac{1}{ij} \int_{-\pi}^{\pi} \left[\frac{\partial \Phi}{\partial w}(\sqrt{1-R^2}e^{i\varphi}, Re^{i\theta})iRe^{i\theta} - \right. \\
&\quad \left. - \frac{\partial \Phi}{\partial \bar{w}}(\sqrt{1-R^2}e^{i\varphi}, Re^{i\theta})(-iRe^{-i\theta}) \right] e^{-ij\theta} d\theta = \\
&= \frac{R}{j} \int_{-\pi}^{\pi} \frac{\partial \Phi}{\partial w}(\sqrt{1-R^2}e^{i\varphi}, Re^{i\theta})e^{-i(j-1)\theta} d\theta - \\
&\quad - \frac{R}{j} \int_{-\pi}^{\pi} \frac{\partial \Phi}{\partial \bar{w}}(\sqrt{1-R^2}e^{i\varphi}, Re^{i\theta})e^{-i(j+1)\theta} d\theta
\end{aligned}$$

where both integrands have the same form as the integrand at the beginning.

Given $n \in \mathbb{N}$ we apply the preceding reasoning n times to see that

$$R^n c_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\sqrt{1-R^2}e^{i\varphi}, Re^{i\theta}) d\theta$$

is a finite sum of terms of the form

$$R^n \gamma_{j,k} \int_{-\pi}^{\pi} \frac{\partial^n f}{\partial w^j \partial \bar{w}^k}(\sqrt{1-R^2}e^{i\varphi}, Re^{i\theta}) e^{-in\theta} e^{ij\theta} e^{-ik\theta} d\theta$$

where $j+k=n$ and where $\gamma_{j,k}$ are constants, so $c_n(z)$ is a finite sum of terms of the form

$$\gamma_{j,k} \int_{-\pi}^{\pi} \frac{\partial^n f}{\partial w^j \partial \bar{w}^k}(\sqrt{1-R^2}e^{i\varphi}, Re^{i\theta}) e^{-i(n-j+k)\theta} d\theta.$$

As $R \searrow 0$ each of these integrals converges, uniformly in φ , $-\pi \leq \varphi \leq \pi$, to

$$\gamma_{j,k} \int_{-\pi}^{\pi} \frac{\partial^n f}{\partial w^j \partial \bar{w}^k}(\sqrt{1-R^2}e^{i\varphi}, 0) e^{-i(n-j+k)\theta} d\theta$$

which equals 0 if $n-j+k > 0$ and

$$\gamma_{n,0} \int_{-\pi}^{\pi} \frac{\partial^n f}{\partial w^n}(e^{i\theta}, 0) d\theta$$

if $n-j+k=0$ which happens when $j=n$, $k=0$. Thus, as $R \searrow 0$, $c_n(\sqrt{1-R^2}e^{i\theta})$ converges, uniformly for $-\pi \leq \varphi \leq \pi$ to

$$2\pi \gamma_{n,0} \frac{\partial^n f}{\partial w^n}(e^{i\varphi}, 0).$$

This completes the proof.

5. Projections of the intersection of bB with complex lines

In this section we give a precise description of the circles $\pi_1(L \cap bB)$ where L is a complex line passing through $(t, 0)$ where $t \geq 0$. Given $z \in \mathbb{C}$ and $r > 0$ write $\Delta(z, r) = \{\zeta \in \mathbb{C}: |\zeta - z| < r\}$.

The intersections of bB with complex lines through the origin are

$$\{(R\zeta, e^{i\omega}\sqrt{1-R^2}\zeta): \zeta \in b\Delta\}, \quad 0 \leq R \leq 1, \quad \omega \in \mathbb{R} \quad (5.1)$$

and the intersections of bB with the complex lines through $(t, 0)$, $0 < t < 1$, are the images of (5.1) under the Moebius map

$$\varphi(z, w) = \left(\frac{t-z}{1-tz}, -\sqrt{1-t^2} \frac{w}{1-tz} \right)$$

which takes the origin to the point $(t, 0)$. Fix t , $0 < t < 1$. Then $\pi_1(L \cap bB)$ for complex lines passing through $(t, 0)$ are the circles

$$\left\{ \frac{t-R\zeta}{1-tR\zeta}, \zeta \in b\Delta \right\} = \left\{ \frac{R\zeta+t}{1+tR\zeta}, \zeta \in b\Delta \right\}, \quad 0 < R \leq 1.$$

Given R , the diameter of such a circle is the closed interval $[(-R+t)/(1-tR), (R+t)/(1+tR)]$ on the real axis so the center is

$$T = \frac{1}{2} \left(\frac{-R+t}{1-tR} + \frac{R+t}{1+tR} \right) = \frac{t(1-R^2)}{1-t^2R^2}$$

and the radius is

$$\rho = \frac{1}{2} \left(\frac{R+t}{1+tR} - \frac{-R+t}{1-tR} \right) = \frac{R(1-t^2)}{1-t^2R^2}.$$

Notice that when R increases from 0 to 1, T decreases from t to 0. Since $R^2 = (T-t)/(t(Tt-1))$ it follows that $\rho^2 = (T-t)(T-1/t)$. This shows that if $0 < t < 1$ then for the complex lines L passing through the point $(t, 0)$, the circles $\pi_1(L \cap bB)$ are

$$b\Delta(T, \sqrt{(T-t)(T-1/t)}), \quad 0 \leq T \leq t. \quad (5.2)$$

If $t = 1$ then for complex lines L passing through the point $(1, 0)$, $\pi_1(L \cap bB)$ are the circles contained in $\overline{\Delta}$ which pass through the point 1.

If $t > 1$ then for complex lines L , passing through the point $(t, 0)$ which meet B , the circles $\pi_1(L \cap B)$ are, similarly to (5.2), the circles

$$b\Delta(T, \sqrt{(T-t)(T-1/t)}), \quad 0 \leq T < 1/t. \quad (5.3)$$

To verify this, one writes $z = x + iy$, $w = u + iv$ and looks at the (x, u) -plane $E = \{(x, u) \in \mathbb{C}^2: x \in \mathbb{R}, y \in \mathbb{R}\}$. Let ℓ be a line in E passing through the point $(t, 0)$, $t > 1$, which intersects the open unit disc in E ; denote the intersection by J . The segment J is the diameter of the disc $L \cap B$ where L is the complex line in \mathbb{C}^2 that contains ℓ and the projection of J in E to the x -axis, the real axis in the z -axis, is the diameter of $\pi_1(L \cap B)$. If T is its midpoint then a simple calculation in \mathbb{R}^2 shows that the length of $\pi_1(J)$ is $2\sqrt{(T-t)(T-1/t)}$.

This shows that to prove Lemma 3.2 we have to consider three different families (actually pairs of families) of circles in $\overline{\Delta}$:

(\mathcal{C}_1) for $\alpha, \beta \in b\Delta$, $\alpha \neq \beta$, the family of all circles in $\overline{\Delta}$ passing through α and the family of all circles in $\overline{\Delta}$ passing through β

(\mathcal{C}_2) the family of all circles centered at the origin and the family of all circles in $\overline{\Delta}$ passing through 1

(\mathcal{C}_3) the family of all circles centered at the origin and, for t , $0 < t < 1$, the family

$$b\Delta(T, \sqrt{(T-t)(T-1/t)}), \quad 0 \leq T < t,$$

that is, the family of all circles obtained from the circles centered at the origin by the Moebius transform $z \mapsto (t-z)/(1-tz)$.

Lemma 3.2 will be proved once we have proved, for each j , $1 \leq j \leq 3$, that if φ is a continuous function on $\overline{\Delta}$ which extends holomorphically from each circle belonging to \mathcal{C}_j , then φ is holomorphic on Δ . For \mathcal{C}_2 this is the main result of [G6]. As mentioned in [G6] it is proved in the same way for \mathcal{C}_1 . It remains to prove it for \mathcal{C}_3 :

LEMMA 5.1 *Let $0 < t < 1$. Suppose that $\varphi \in C(\overline{\Delta})$ extends holomorphically from each circle in $\overline{\Delta}$ centered at the origin and from each circle $b\Delta(T, \sqrt{(T-t)(T-1/t)})$, $0 \leq T < t$. Then φ is holomorphic on Δ .*

It is clear that by proving Lemma 5.1 we also prove Theorem 1.3. In fact, the statements of Lemma 5.1 and Theorem 1.3 are equivalent.

Lemma 3.2 will be proved once we have proved Lemma 5.1. To prove Lemma 5.1 we will first use semiquadratics as in [AG,G3] to formulate the problem in \mathbb{C}^2 and then show that the idea of A.Tumanov [T2] to use an argument of H.Lewy [L] together with the Liouville theorem still applies in our situation. The proof will be similar to the proof of the main result of [G6] but more complicated.

6. Proof of Lemma 5.1, Part 1

We introduce semiquadratics to pass to an associated problem in \mathbb{C}^2 . Given $a \in \mathbb{C}$ and $r > 0$ let

$$\Lambda(a, r) = \{(z, w) \in \mathbb{C}^2: (z-a)(w-\bar{a}) = r^2, \quad 0 < |z-a| < r\}.$$

be the semiquadratic associated with the circle $b\Delta(a, r)$. Write $\Sigma = \{(\zeta, \bar{\zeta}): \zeta \in \mathbb{C}\}$. $\Lambda(a, r)$ is a closed complex submanifold of $\mathbb{C}^2 \setminus \Sigma$ which is attached to Σ along $b\Lambda(a, r) = \{(\zeta, \bar{\zeta}): \zeta \in b\Delta(a, r)\}$. A continuous function g extends holomorphically from the circle $b\Delta(a, r)$ if and only if the function G , defined on $b\Lambda(a, r)$ by $G(\zeta, \bar{\zeta}) = g(\zeta)$ ($\zeta \in b\Delta(a, r)$)

has a bounded continuous extension to $\overline{\Lambda(a, r)} = \Lambda(a, r) \cup b\Lambda(a, r)$ which is holomorphic on $\Lambda(a, r)$. In fact, if we denote by the same letter g the holomorphic extension of g through $\Delta(a, r)$ we have

$$G\left(z, \bar{a} + \frac{r^2}{z-a}\right) = g(z) \quad (z \in \overline{\Delta(a, r)} \setminus \{a\})$$

and if we define $G(a, \infty) = g(a)$ we get a continuous function G on $\overline{\Lambda(a, r)} \cup \{(a, \infty)\}$, the closure of $\Lambda(a, r)$ in $\mathbb{C} \times \overline{\mathbb{C}}$. It is known that if $(a, r) \neq (b, \rho)$ then $\Lambda(a, r)$ meets $\Lambda(b, \rho)$ if and only if $a \neq b$ and one of the circles $b\Delta(a, r)$, $b\Delta(b, \rho)$ surrounds the other. If this happens then $\Lambda(a, r)$ and $\Lambda(b, \rho)$ meet at precisely one point [G3].

We begin with the proof of Lemma 5.1. Let φ and t be as in Lemma 5.1. By our assumption, φ extends holomorphically from two families of circles: $\{b\Delta(0, R): 0 < R \leq 1\}$ and $\{b\Delta(T, \sqrt{(T-t)(T-1/t)}): 0 \leq T < t\}$. Accordingly, there are two families of semiquadratics: $\{\Lambda(0, R), 0 < R \leq 1\}$ and $\{\Lambda(T, \sqrt{(T-t)(T-1/t)}): 0 \leq T < t\}$, and the function $\Phi(\zeta, \bar{\zeta}) = \varphi(\zeta)$, defined on $\{(\zeta, \bar{\zeta}): \zeta \in \overline{\Delta}\}$ has a bounded holomorphic extension through each of these semiquadratics.

Consider the first family. In this family the semiquadratics are pairwise disjoint. Let L be the closure of their union in $[\mathbb{C} \setminus \{0\}] \times \mathbb{C}$:

$$L = \bigcup_{0 < r \leq 1} [\Lambda(0, R) \cup b\Lambda(0, R)].$$

The continuity of φ together with the maximum principle implies that our function Φ extends from $L \cap \Sigma = \{(\zeta, \bar{\zeta}): \zeta \in \overline{\Delta} \setminus \{0\}\}$ to a bounded continuous function Φ on L so that the extension Φ is holomorphic on each fiber $\Lambda(0, R)$. Note that L is a CR-manifold in $[\mathbb{C} \setminus \{0\}] \times \mathbb{C}$ with piecewise smooth boundary consisting of two pieces, $\Lambda(0, 1)$ and $\{(\zeta, \bar{\zeta}): \zeta \in \overline{\Delta} \setminus \{0\}\}$ and the function Φ , being holomorphic on fibers, is CR on its interior $L_0 = \bigcup_{0 < R < 1} \Lambda(0, R)$, that is

$$\int_{L_0} \Phi \bar{\partial} \omega = 0$$

for each smooth, (2,0)-form ω on \mathbb{C}^2 whose support meets L_0 in a compact set.

Now, look at the second family of semiquadratics

$$\Lambda\left(T, \sqrt{(T-t)(T-1/t)}\right) = \{(z, w) \in \mathbb{C}^2: (z-T)(w-T) = (T-t)(T-1/t), \\ 0 < |z-T| < \sqrt{(T-t)(T-1/t)}\}, \quad 0 \leq T < t.$$

Observe that these semiquadratics are not pairwise disjoint since they all contain the point $(t, 1/t)$. Since two semiquadratics can meet at at most one point it follows that the sets

$$\Lambda\left(T, \sqrt{(T-t)(T-1/t)}\right) \setminus \{(t, 1/t)\}, \quad 0 < T < t,$$

are pairwise disjoint and so their union

$$N_0 = \left[\bigcup_{0 < T < t} \Lambda\left(T, \sqrt{(T-t)(T-1/t)}\right) \right] \setminus \{(t, 1/t)\}$$

is a CR manifold. Let N be the closure of N_0 in $[\mathbb{C} \setminus \{t\}] \times \overline{\mathbb{C}}$, that is,

$$N = \bigcup_{0 \leq T < t} \left[\Lambda(T, \sqrt{(T-t)(T-1/t)}) \cup b\Lambda(T, \sqrt{(T-t)(T-1/t)}) \cup \{(T, \infty)\} \right].$$

Again, the continuity of φ together with the maximum principle implies that our function Φ extends from $N \cap \Sigma = \{(\zeta, \bar{\zeta}) : \zeta \in \overline{\Delta} \setminus \{t\}\}$ continuously to N so that our extension Φ is holomorphic on each fiber $\Lambda(T, \sqrt{(T-t)(T-1/t)})$, $0 \leq T < t$. The part of N contained in $\mathbb{C} \times \mathbb{C}$ is a smooth CR manifold in $[\mathbb{C} \setminus \{t\}] \times \mathbb{C}$ with boundary consisting of two pieces, $\Lambda(0, 1)$ and $N \cap \Sigma$ and the function Φ is CR in the interior N_0 .

7. Proof of Lemma 5.1, Part 2

Let φ and t be as in Lemma 5.1. We have shown that the function Φ extends continuously from $\{(\zeta, \bar{\zeta}) : \zeta \in \Delta \setminus \{0\}\}$ to L and from $\{(\zeta, \bar{\zeta}) : \zeta \in \Delta \setminus \{t\}\}$ to N so that the extensions are holomorphic on fibers of L and N . We would like that these extensions define a function Φ on $L \cup N$. However, this is not possible since L and N intersect. There is one piece of $L \cap N$, namely $\Lambda(0, 1)$ on which both extensions coincide. There are other points of $L \cap N$. Obviously, all such points of intersection are of the form (x, y) where $x \in \mathbb{R}$, $y \in \mathbb{R}$. We now show that there is an $\eta > 0$ such that there are no such intersections with $0 < x < \eta$.

For each R , $0 < R \leq 1$, write $\mathcal{T}_R = \Lambda(0, R)$ and for each T , $0 \leq T < t$, write $\mathcal{S}_T = \Lambda(T, \sqrt{(T-t)(T-1/t)})$. Clearly $\mathcal{S}_0 = \mathcal{T}_1$. Choose η so that

$$0 < \eta < \frac{1}{2(t+1/t)} \quad (7.1)$$

and notice that $\eta < 1/2$ and $\eta < t$.

PROPOSITION 7.1 *Let η satisfy (7.1) and assume that $0 < x < \eta$. Then $\{x\} \times \mathbb{C}$ contains no point of $\mathcal{S}_T \cap \mathcal{T}_R$ with $0 < T < t$ and $0 < R < 1$.*

Proof. We shall prove the proposition by proving that

$$\text{if } 0 < R < 1 \text{ and if } (x, y) \in \mathcal{T}_R \text{ then } x < y < 1/x, \quad (7.2)$$

$$\text{if } 0 < T < t \text{ and if } (x, y) \in \mathcal{S}_T \text{ then either } 1/x < y < \infty \text{ or } -\infty < y < x. \quad (7.3)$$

A simple picture shows that there is no $(x, y) \in \mathcal{T}_R$ if $R \leq x$ and when $x < R < 1$ than there is precisely one y such that $(x, y) \in \mathcal{T}_R$ and

$$\text{as } R \text{ moves from } x \text{ to } 1, y \text{ moves from } x \text{ to } 1/x. \quad (7.4).$$

This takes care of (7.2). We now turn to (7.3). We first determine the interval $(0, T_0)$ of those T for which x is contained in the disc $\Delta(T, \sqrt{(T-t)(T-1/t)})$, that is, of all those T for which there is a point $(x, y) \in \mathcal{S}_T \cup \{(T, \infty)\}$. Clearly T_0 is determined by the condition that $(x, x) \in b\mathcal{S}_{T_0}$, that is, $(x - T_0)^2 = (T_0 - t)(T_0 - 1/t)$ which gives

$$T_0 = \frac{1 - x^2}{(t + 1/t) - 2x}.$$

Since $x < \eta < 1/2$, (7.1) implies that $x(t + 1/t) < 1/2$ hence

$$x^2 - x(t + 1/t) + 1 > 0. \quad (7.5)$$

Obviously $x < T_0$. One can see this also by rearranging (7.5) to $1 - x^2 > x(t + 1/t) - 2x^2$ which gives $T_0 > x$.

Now, for each T , $0 \leq T \leq T_0$ we compute the unique $y = y(T)$ such that $(x, y) \in \mathcal{S}_T \cup \{(T, \infty)\}$. It is clear that $y(0) = 1/x$ and $y(x) = \pm\infty$. We now show that

$$\left. \begin{array}{l} \text{as } T \text{ increases from } 0 \text{ to } x, y(T) \text{ increases from } 1/x \text{ to } +\infty \\ \text{and as } T \text{ increases from } x \text{ to } T_0, y(T) \text{ increases from } -\infty \text{ to } x. \end{array} \right\} \quad (7.6)$$

To show this it suffices to show that $\frac{dy}{dx} > 0$ for all T , $0 < T < T_0$, $T \neq x$. We have

$$y(T) = T + \frac{1 - (t + 1/t)T + T^2}{x - T}$$

and a short computation shows that

$$\frac{dy}{dT} = \frac{x^2 - (t + 1/t)x + 1}{(x - T)^2}$$

which, by (7.5) is always positive. This completes the proof of (7.3) and so completes the proof of Proposition 7.1.

We have extended Φ to both L and N and so this common extension is well defined on $(L \cup N) \setminus (L \cap N)$. Thus, by Proposition 7.1, the extension is well defined on

$$M = (L \cup N) \setminus [([-1, 0] \cup [\eta, 1]) \times \overline{\mathbb{C}}].$$

It is continuous and holomorphic on fibers, that is, it is CR in the interior of the part of M in \mathbb{C}^2 . It is on this M where we will apply the idea of Tumanov.

8. Completion of the proof of Lemma 5.1

Let $\mathcal{S} = \Delta \setminus ((-1, 0] \cup [\eta, 1))$. Given $z \in \mathcal{S}$, let $M_z = \{\zeta \in \mathbb{C}: (z, \zeta) \in M\}$.

Given $z \in \mathcal{S}$, $z \notin \mathbb{R}$, let C_z be the circle passing through $t, 1/t$ and \bar{z} . Note that $1/z$ is on the same circle. Denote by λ_z the arc on C_z with endpoints \bar{z} and $1/z$ which does not contain t and $1/t$. We shall show that

$$M_z \text{ consists of } \lambda_z \text{ and of the segment joining } \bar{z} \text{ and } 1/z. \quad (8.1)$$

We have $L \cap (\{z\} \times \mathbb{C}) = \{(z, R^2/z): |z| \leq R \leq 1\}$ which is the segment joining (z, \bar{z}) and $(z, 1/z)$. To find what $N \cap (\{z\} \times \mathbb{C})$ is, we recall first that

$$\Lambda(T, \sqrt{(T-t)(T-1/t)}) = \left\{ (z, w): w = T + \frac{(T-t)(T-1/t)}{z-T}, \right. \\ \left. |z-T| < \sqrt{(T-t)(T-1/t)} \right\}.$$

So we must determine $\{w(T): 0 \leq T < T(z)\}$ where

$$w(T) = T + \frac{(T-t)(T-1/t)}{z-T} = \frac{Tz - tT - T/t + 1}{z-T}$$

and where $T(z)$ is such that $w(T(z)) = \bar{z}$. The set $\{w(T): T \in \mathbb{R}\}$ is clearly a circle. We have $w(t) = t$, $w(1/t) = 1/t$ and $w(0) = 1/z$. So this circle is C_z described above and it follows that when $0 \leq T \leq T(z)$ then $y(T)$ is on the arc λ_z . This proves (8.1).

We now look at M_x where $x \in \mathcal{S}$ is real, By (7.4) we know that if $(x, y) \in \mathcal{T}_R$ then, as R increases from x to 1, y increases from x to $1/x$. By (7.6) we know that if $(x, y) \in \mathcal{S}_T \cup \{(T, \infty)\}$ then, as T increases from 0 to x , y increases from $1/x$ to ∞ and as T increases from x to T_0 , y increases from $-\infty$ to x . This shows that

$$\text{if } x \in \mathcal{S} \text{ is real then } M_x \text{ is the real axis .} \quad (8.2)$$

For each $z \in \mathcal{S}$, let D_z be the domain bounded by M_z , The domains D_z change continuously with $z \in \mathcal{S} \setminus \mathbb{R}$ and as z approaches a point $a \in b\Delta \setminus \mathbb{R}$ they shrink to the point \bar{a} .

At this point we are precisely in the situation described in Section 4 of [G6]. Repeating word by word the part of the proof there we use an argument of H.Lewy [L] as generalized by H.Rossi [R], to prove that for each $z \in \mathcal{S} \setminus \mathbb{R}$ the function $w \in \Phi(z, w)$, defined on M_z , extends holomorphically through D_z .

Recall that $\{(\lambda, \infty): 0 < \lambda < \eta\} \subset M$ and that Φ is continuous on M . Given τ , $0 < \tau < \eta$, we show that Φ is constant on $\{\tau\} \times M_\tau$. To see this, recall that $\eta < t$ and fix τ , $0 < \tau < \eta$. Observe that for small $\omega > 0$, $M_{\tau+i\omega}$ are simple closed curves bounding $D_{\tau+i\omega}$ which depend continuously on ω and, as domains in $\overline{\mathbb{C}}$ converge to the halfplane $\Im\zeta < 0$ as ω tends to 0. Since for each small ω the function $\zeta \mapsto \Phi(\tau + i\omega, \zeta)$ extends from $M_{\tau+i\omega}$ holomorphically through $D_{\tau+i\omega}$, the continuity of Φ on M implies that $s \mapsto \Phi(\tau, s)$ has a bounded continuous extension from \mathbb{R} to the halfplane $\Im\zeta \leq 0$ which is holomorphic on $\Im\zeta < 0$. Repeating the reasoning with $\omega < 0$ we see that $s \mapsto \Phi(\tau, s)$ has a bounded continuous extension from \mathbb{R} to the halfplane $\Im\zeta \geq 0$ which is holomorphic on $\Im\zeta > 0$. Thus, $s \mapsto \Phi(\tau, s)$ has a bounded holomorphic extension to \mathbb{C} , which, by the Liouville theorem, must be constant. Thus, for each τ , $0 < \tau < \eta$, the holomorphic extensions of φ from all circles in our families that surround τ , coincide at τ . This implies that φ is holomorphic in a neighbourhood of the segment $(0, \eta)$ and it is then easy to see that the analyticity propagates along circles so φ is holomorphic on Δ . This completes the proof of Lemma 5.1. and also proves Theorem 1.3. The proof of Lemma 3.2 is thus complete. This completes the proof of Theorem 1.2.

9. Higher dimensions

After the results presented above were obtained Mark Agranovsky told the author that he has proved Theorem 1.1. for real analytic functions and its generalization to higher dimensions and then put his results on the web [A1, A2]. In a later version of [A1] he showed that it is very easy to generalize his result to higher dimensions and that,

surprisingly, only complex lines through two points in \overline{B} suffice in any dimension. We want to repeat the elegant simple argument that he used and apply it in our case.

COROLLARY 9.1 *Let B be the open unit ball in \mathbb{C}^N , $N \geq 2$, and assume that $a, b \in \mathbb{C}^N$, $a \neq b$, are such that the complex line containing a and b meets B and such that $\langle a|b \rangle \neq 1$. Suppose that a function $f \in C^\infty(bB)$ extends holomorphically along every complex line in $\mathcal{L}(a) \cup \mathcal{L}(b)$. Then f extends holomorphically through B .*

Proof. Using Moebius transforms we may, with no loss of generality assume that $\Lambda(a, b)$, the complex line through a and b , contains the origin. Choose $c \in \Lambda(a, b) \cap B$, $c \neq 0$. By Theorem 1.2 the function $f|_{(\Sigma \cap bB)}$ extends holomorphically through $\Sigma \cap B$ for every complex two-plane Σ containing $\Lambda(a, b)$. So, if a complex line $L \in \mathcal{L}(0)$ meets a complex line $E \in \mathcal{L}(c)$ then both L and E are contained in such a two-plane which implies that f extends holomorphically along both E and L and that the extensions are the same at $E \cap L$. Thus, all such holomorphic extensions along complex lines in $\mathcal{L}(0) \cup \mathcal{L}(c)$ define an extension \tilde{f} of f to \overline{B} which is holomorphic on $L \cap B$ for each $L \in \mathcal{L}(0)$ and holomorphic on $E \cap B$ for each $E \in \mathcal{L}(c)$. Expressing $\tilde{f}|_{E \cap B}$ as the Cauchy integral of $f|_{E \cap bB}$ for each $E \in \mathcal{L}(c)$, the fact that $f \in C^\infty(bB)$ implies that \tilde{f} is of class C^∞ in a neighbourhood of the origin which, by a theorem of F. Forelli [Ru] implies that \tilde{f} is holomorphic on B , which completes the proof.

Very recently L. Baracco [B] showed that for real analytic functions on bB only one point in the boundary suffices, that is, if $a \in bB$ then $\mathcal{L}(a)$ is a test family for holomorphic extendibility for real analytic functions on bB .

10. An open question

Let us conclude by formulating our initial question for general domains:

QUESTION 10.1 *Let $a, b \in \mathbb{C}^2$, $a \neq b$, and let $D \subset \mathbb{C}^2$ be a bounded domain with smooth boundary. Assume that a continuous function f on bD extends holomorphically along every complex line $L \in \mathcal{L}(a) \cup \mathcal{L}(b)$ that meets D . Must f extend holomorphically through D ?*

We have seen that if D is a ball the answer is no even in the case when $a, b \in D$ and to get a positive answer one must assume that f is infinitely smooth. The author believes that one must require smoothness only in very special cases:

CONJECTURE *For a generic domain D the answer to Question 10.1 is positive.*

M. Lawrence has proved a result of this kind for arbitrary small perturbations of the ball [La].

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REFERENCES

- [A1] M. L. Agranovsky: Holomorphic extension from the unit sphere in C^n into complex lines passing through a finite set.
<http://arxiv.org/abs/0910.3592>
- [A2] M. L. Agranovsky: Characterization of polyanalytic functions by meromorphic extensions into chains of circles.
<http://arxiv.org/abs/0910.3578>
- [AG] M. L. Agranovsky and J. Globevnik: Analyticity on circles for rational and real-analytic functions of two real variables.
J. d'Analyse Math. 91 (2003) 31-65
- [B] L. Baracco: Holomorphic extension from the sphere to the ball.
<http://arxiv.org/abs/0911.2560>
- [G1] J. Globevnik; On holomorphic extensions from spheres in C^2
Proc. Roy. Soc. Edinb. 94A (1983) 113-120
- [G2] J. Globevnik: A family of lines for testing holomorphy in the ball of C^2 .
Indiana Univ. Math. J. 36 (1987) 639-644
- [G3] J. Globevnik: Holomorphic extensions from open families of circles.
Trans. Amer. Math. Soc. 355 (2003) 1921-1931
- [G4] J. Globevnik: Analyticity on families of circles.
Israel J. Math. 142 (2004) 29-45
- [G5] J. Globevnik: Analyticity on translates of a Jordan curve.
Trans. Amer. Math. Soc. 359 (2007) 5555-5565
- [G6] J. Globevnik: Analyticity of functions analytic on circles
Journ. Math. Anal. Appl. 360 (2009) 363-368
- [KM] A. M. Kytmanov, S. G. Myslivets: On families of complex lines sufficient for holomorphic extension.
Mathematical Notes 83 (2008) 500-505 (translated from *Mat.Zametki* 83 (2008) 545-551
- [La] M. G. Lawrence: Hartogs' separate analyticity theorem for CR functions.
Internat. J. Math. 18 (2007) 219-229.
- [L] H. Lewy: On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex

variables.

Ann. of Math. 64 (1956) 514-522

[R] H. Rossi: A generalization of a theorem of Hans Lewy.
Proc. Amer. Math. Soc. 19 (1968) 436-440

[Ru] W. Rudin: *Function theory in the unit ball of C^n* .
Springer, Berlin-Heidelberg-New York 1980

[T1] A. Tumanov: A Morera type theorem in the strip.
Math. Res. Lett. 11 (2004) 23-29

[T2] A. Tumanov: Testing analyticity on circles.
Amer. J. Math. 129 (2007) 785-790

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