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Finite Rank Bratteli Diagrams and their Invariant Measures

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Abstract

In this paper we study ergodic measures on non-simple Bratteli diagrams of finite rank that are invariant with respect to the cofinal equivalence relation. We describe the structure of finite rank diagrams and prove that every ergodic invariant measure (finite or infinite) is an extension of a finite ergodic measure defined on a simple subdiagram. We find some algebraic criteria in terms of entries of incidence matrices and their norms under which such an extension remains a finite measure. Furthermore, the support of every ergodic measure is explicitly determined. We also give an algebraic condition for a diagram to be uniquely ergodic. It is proved that Vershik maps (not necessarily continuous) on finite rank Bratteli diagrams cannot be strongly mixing and always have zero entropy with respect to any finite ergodic invariant measure. A number of examples illustrating the established results is included.

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Key words: Bratteli diagrams, Vershik maps, zero entropy, mixing, ergodicity, invariant measures.

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1 Introduction

Bratteli diagrams, which originally appeared in the theory of operator algebras, turned out to be a powerful method for study of dynamical systems in ergodic theory and on a Cantor set. The main object of our study in this paper is finite rank Bratteli diagrams, i.e., the diagrams whose vertex set at each level is uniformly bounded or, equivalently (after an easy reduction), with the same number of vertices at each level. We study Borel non-atomic measures on such Bratteli diagrams which are invariant with respect to the tail (cofinal) equivalence relation. Finite rank diagrams have intriguing properties and were considered, in particular, in [DM], [BKM09], [BKMS10], and [BDM10]. It is worth pointing out from the very beginning that, in contrast to most papers on Cantor dynamics and Bratteli diagrams, our interest is focused on general non-simple Bratteli diagrams. In this context, the present paper is a natural continuation of our previous work [BKMS10] devoted to the study of invariant measures and the structure of stationary non-simple Bratteli diagrams.

The main motivation of our study is the application of obtained results to the classification of homeomorphisms of a Cantor set. Namely, the knowledge of supports of invariant measures and the measure values on clopen sets are necessary steps for distinguishing two non-isomorphic or non-orbit equivalent homeomorphisms. We note that every minimal and even aperiodic homeomorphism of a Cantor set can be represented as a Vershik map acting on the path space of a Bratteli diagram ([HPS92], [Med06]). Furthermore, the cofinal equivalence relation and Vershik maps (if they exist) defined on a Bratteli diagram share the same set of invariant measures. This fact allows us to work with the equivalence relation only. On the other hand, all results obtained in the paper are also applicable to the class of transformations that can be represented as Vershik maps acting on finite rank diagrams. We should note that substitution dynamical systems and minimal interval exchange transformations belong to this class [DHS99], [BKM09], [GJ02]. Conversely, every Vershik map on a finite rank diagram is either an odometer or a subshift ([DM], [BKM09]). However, to the best of our knowledge, it is still unknown what kind of subshifts can arise on this way.

The first important observation is that finite rank diagrams have finitely many ergodic invariant (finite or infinite) measures. Thus, the ergodic decomposition of any invariant measure is very simple and completely determined by the supports of ergodic measures. In the paper, we show that every ergodic measure is obtained as an extension of a uniquely ergodic measure from a simple subdiagram (Theorem 3.7). Furthermore, these subdiagrams have no common vertices for distinct ergodic measures. This result gives us a structural description of the support and measure values for any ergodic measure. We will repeatedly use this result throughout the paper to reduce some questions about ergodic measures defined on general Bratteli diagrams to those defined on simple subdiagrams.

The usage of Bratteli diagrams benefits from the opportunity to apply the linear algebra to dynamical problems. This is based on the fact that every Bratteli diagram is completely determined by a sequence of incidence matrices up to an order on the diagram. Thus, if we are interested in properties of the corresponding dynamical system that do not depend on the order of points in its orbits (like invariant measures, minimal components, etc.), then it suffices to study the incidence matrices only. In case of finite rank diagrams, it appears that all properties of invariant measures are dependent on the "growth rate" of coefficients in the products of incidence matrices. For example, it turns out that the simple Bratteli diagram is uniquely ergodic if and only if all the entries of products of incidence matrices grow not very fast and approximately with the same rate (Theorem 4.1).

Thus, the methods we use in the paper lead us with necessity to consideration of products of non-negative matrices of the same size. This subject has been studied extensively in a number of books and research papers. Our primarily sources are [Har02] and [Sen81]. We should point out that the first who clearly used these ideas was A. Fisher [Fis09], [FFT09]. In particular, he showed the crucial importance of Birkhoff contraction coefficient for uniquely ergodic diagrams.

Our main results and the paper organization are the following.

In Section 2 we first give the definition and necessary notation of Bratteli diagrams and discuss the class of homeomorphisms of finite rank. Despite the fact that the main definitions concerning the concept of Bratteli diagrams are considered in almost all papers devoted to Cantor dynamics, we have to include this material in the paper to fix our notation and make the paper more self-contained. In the second part of this section, we give some results related to matrix products and, especially, to the notion of Birkhoff contraction coefficient.

In Section 3 we establish general structural properties of Bratteli diagrams of finite rank and of ergodic measures defined on such diagrams. First of all, we show that every Bratteli diagram of finite rank can be isomorphically transformed to a canonical block-triangular form which is very convenient for solving problems concerning measure extensions. Next, we prove that any finite rank Bratteli diagram admits only a finite number of ergodic measures and every ergodic measure is, in fact, an extension of a finite ergodic measure from a simple subdiagram. In some sense, this result shows that all the dynamical properties of the corresponding Bratteli-Vershik system are mainly determined by its simple subdiagrams and minimal components. Though, in contrast to ergodic theory, the knowledge of minimal (ergodic) components does not provide the complete picture of the underlying dynamics. The results of Section 6 will then show that the contribution of these simple subdiagrams into the number of paths of the entire diagram significantly affect the dynamical properties of the system.

In Section 4 we study simple diagrams of finite rank. We show that the notion of unique ergodicity is equivalent to the fact that the products of incidence matrices tend to column proportionality. As a corollary, assuming that the system is uniquely ergodic, we find the asymptotic growth of tower heights and the measure values on the bases of towers. It turns out that in this case all tower heights are asymptotically the same (up to a constant) and the measure of the tower base is asymptotically the reciprocal of its height (up to a con-

stant). Several easily computable sufficient conditions of unique ergodicity are also presented. This section is partly based on the results of [Fis09].

In Section 5, we prove that any Vershik map on a finite rank Bratteli diagram has zero entropy and cannot be strongly mixing with respect to any finite ergodic invariant measure. To establish zero entropy result, we use the Katok-Stepin idea on the speed of approximation of aperiodic homeomorphisms by periodic ones [KS67]. In particular, we show that the Vershik map can be approximated with the speed θ/n , for some $\theta > 1$. Such a speed is more than sufficient to guarantee zero entropy of a dynamical system (see [KS67]). We also observe that the absence of strong mixing well corresponds to the known results about substitution and linearly recurrent systems.

In section 6 we show that if we have a measure defined on a uniquely ergodic simple subdiagram, then the question about finiteness of measure extension from this subdiagram can be solved purely algebraically by studying relations between the norms of incidence matrices corresponding to subdiagrams. In the last section we also study the diagrams that have only a finite number of distinct incidence matrices (the so called linearly recurrent dynamical systems). For such diagrams we introduce an invariant of orbit equivalence in terms of the speed of growth of number of paths. Furthermore, this invariant is easily computable, which also gives an easy algorithm for describing the simplex of invariant measures.

2 Preliminaries

In this section we collect the notation and basic definitions that are used throughout the paper. We also formulate some technical results needed for the exposition of our results.

2.1 Bratteli diagrams

Since the notion of Bratteli diagrams and the related notion of Vershik transformation have been discussed in numerous recent papers, they might be considered as almost classical nowadays. So, we would prefer to avoid giving detailed definitions. An interested reader may consult the papers [HPS92], [GPS95], [DHS99], [Med06], [BKM09], [BKMS10], and references therein for all details concerning Bratteli diagrams and Vershik maps. We only give here some basic definitions in order to fix our notation.

Definition 2.1. A *Bratteli diagram* is an infinite graph B = (V, E) such that the vertex set $V = \bigcup_{i\geq 0} V_i$ and the edge set $E = \bigcup_{i\geq 1} E_i$ are partitioned into disjoint subsets V_i and E_i such that

(i) $V_0 = \{v_0\}$ is a single point;

(ii) V_i and E_i are finite sets;

(iii) there exist a range map r and a source map s from E to V such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq \emptyset$, $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

The pair (V_i, E_i) or just V_i is called the *i*-th level of the diagram *B*. A finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a *finite* or *infinite path*, respectively. We write e(v, v') to denote a path *e* such that s(e) = v and r(e) = v'. For a Bratteli diagram *B*, we denote by X_B the set of infinite paths starting at the vertex v_0 . We endow X_B with the topology generated by cylinder sets $U(e_1, \ldots, e_n) = \{x \in X_B : x_i = e_i, i = 1, \ldots, n\}$, where (e_1, \ldots, e_n) is a finite path from *B*. Then X_B is a 0-dimensional compact metric space with respect to this topology.

Given a Bratteli diagram B = (V, E), the incidence matrix $F_n = (f_{v,w}^{(n)}), n \ge 1$, is a $|V_{n+1}| \times |V_n|$ matrix whose entries $f_{v,w}^{(n)}$ are equal to the number of edges between the vertices $v \in V_{n+1}$ and $w \in V_n$, i.e.,

$$f_{v,w}^{(n)} = |\{e \in E_{n+1} : r(e) = v, s(e) = w\}|.$$

(Here and thereafter |A| denotes the cardinality of the set A.) We notice that F_0 is a vector. We assume usually that $F_0 = (1, ..., 1)^T$.

Observe that every vertex $v \in V$ is connected to v_0 by a finite path and the set $E(v_0, v)$ of all such paths is finite. Set $h_v^{(n)} = |E(v_0, v)|$ where $v \in V_n$. Then

$$h_v^{(n+1)} = \sum_{w \in V_n} f_{v,w}^{(n)} h_w^{(n)}$$
(2.1)

or

$$h^{(n+1)} = F_n h^{(n)} \tag{2.2}$$

where $h^{(n)} = (h_w^{(n)})_{w \in V_n}$.

Together with the sequence of incidence matrices $\{F_n\}$ we will use the sequence of matrices $\{Q_n\}$ where the entries $q_{v,w}^{(n)}$ of Q_n are defined by the formula:

$$q_{v,w}^{(n)} = f_{v,w}^{(n)} \frac{h_w^{(n)}}{h_v^{(n+1)}}, \quad n \ge 1.$$
(2.3)

It follows from (2.1) that every Q_n is a stochastic matrix.

It is not hard to show that for a given sequence of non-negative rational stochastic $d \times d$ matrices $\{Q_n\}$ there exists a Bratteli diagram B with incidence matrices $\{F_n\}$ whose entries satisfy (2.3). The sequence $\{F_n\}$ is not uniquely determined: matrices F_n and pF_n , $p \in \mathbb{N}$, correspond to the same stochastic matrix Q_n .

For $w \in V_n$, the set $E(v_0, w)$ defines the clopen subset

$$X_w^{(n)} = \{ x = (x_i) \in X_B : r(x_n) = w \}.$$

The sets $\{X_w^{(n)} : w \in V_n\}$ form a clopen partition of X_B , $n \ge 1$. Analogously, each finite path $\overline{e} = (e_1, \ldots, e_n) \in E(v_0, w)$ determines the clopen subset

$$X_w^{(n)}(\overline{e}) = \{x = (x_i) \in X_B : x_i = e_i, i = 1, \dots, n\}.$$

These sets form a clopen partition of $X_w^{(n)}$. We will use also the notation $[\overline{e}]$ for the clopen set $X_w^{(n)}(\overline{e})$ when it does not lead to a confusion. The base of the tower $X_w^{(n)}$ is denoted by $B_n(w)$. (In fact, this means that an order is specified on $E(v_0, w)$. But since, in the most cases, an order is inessential for us, the subset $B_n(w)$ may be represented by any finite path from $E(v_0, w)$).

Remark 2.2. Given a dynamical system (X, T), a Bratteli diagram is constructed by a sequence of Kakutani-Rokhlin partitions generated by (X, T) (see [HPS92] and [Med06]). The *n*-th level of the diagram corresponds to the *n*-th Kakutani-Rokhlin partition and the number $h_w^{(n)}$ is the height of the *T*-tower labeled by the symbol *w* from that partition.

Throughout the paper we will constantly use the telescoping procedure for a Bratteli diagram. Roughly speaking, in order to telescope a Bratteli diagram, one takes a subsequence of levels $\{n_k\}$ and considers the set of all finite paths as new edges between the consecutive levels $\{n_k\}$ and $\{n_{k+1}\}$. A rigorous definition of telescoping can be found in many papers on Bratteli diagrams, for example, in [GPS95].

Let B be a Bratteli diagram. By a finite measure on B we always mean a Borel non-atomic (not necessarily probability) measure on X_B . For an infinite σ -finite measure μ on X_B , we assume that μ takes finite values on some clopen sets.

Definition 2.3. Given a Bratteli diagram B = (V, E), a measure μ on X_B is called *invariant* if $\mu([\overline{e}]) = \mu([\overline{e'}])$ for any two finite paths \overline{e} and $\overline{e'}$ with the same range. In other words, $\mu(X_w^{(n)}(\overline{e})) = \mu(X_w^{(n)}(\overline{e'}))$ for any $n \ge 1$ and $w \in V_n$.

Remark 2.4. For a Bratteli diagram B, consider the cofinal (tail) equivalence relation \mathcal{E} on the path space X_B such that $x\mathcal{E}y$ if $x_n = y_n$ for all n sufficiently large. Then the measure μ is invariant on B if and only if it is invariant with respect to the equivalence relation \mathcal{E} .

Definition 2.5. An invariant measure μ is *ergodic* for the diagram *B* (or *B*-*ergodic*) if it is ergodic with respect to the cofinal equivalence relation \mathcal{E} .

If a Bratteli diagram B admits a unique invariant probability measure, then B is called *uniquely ergodic*.

Definition 2.6. A Bratteli diagram that has a uniformly bounded number of vertices at each level is called a diagram of *finite type*.

By definition, a Bratteli diagram B = (V, E) is ordered if every set $r^{-1}(v)$, $v \in \bigcup_{n \ge 1} V_n$, is linearly ordered, see [HPS92]. Thus, any two paths from $E(v_0, v)$ are comparable with respect to the lexicographical order. We call a finite or infinite path $e = (e_i)$ maximal (minimal) if every e_i is maximal (minimal) amongst the edges from $r^{-1}(r(e_i))$. Notice that for $v \in V_i$, $i \ge 1$, the minimal and maximal (finite) paths in $E(v_0, v)$ are unique. Denote by X_{\max} and X_{\min} the sets of all maximal and minimal infinite paths from X_B , respectively. It is not hard to see that X_{\max} and X_{\min} are finite sets for finite rank diagrams, see

the proof of Proposition 6.2 in [BKM09]. Let X_B^* be the set of all infinite paths which are cofinal neither to a maximal path nor to a minimal one. Then the set $X_B \setminus X_B^*$ is at most countable for finite rank diagrams.

Definition 2.7. Define a map $T : X_B^* \to X_B^*$ by setting $T(x_1, x_2, ...) = (x_1^0, \ldots, x_{k-1}^0, \overline{x_k}, x_{k+1}, x_{k+2}, \ldots)$, where $k = \min\{n \ge 1 : x_n \text{ is not maximal}\}$, $\overline{x_k}$ is the successor of x_k in $r^{-1}(r(x_k))$, and $(x_1^0, \ldots, x_{k-1}^0)$ is the minimal path in $E(v_0, s(\overline{x_k}))$. In this paper we will refer to the map T as the Vershik map on the ordered diagram B.

Remark 2.8. (i) It is still unknown when the Vershik map can be extended to a homeomorphism of X_B for non-simple Bratteli diagrams. We note only that it is not always possible (see [Med06]).

(ii) Since all orbits of T coincide with classes of \mathcal{E} , maybe except for at most countable collection of orbits, any \mathcal{E} -invariant measure is also T-invariant and vice versa.

To avoid trivialities, throughout the paper we will always assume that each Bratteli diagram of finite rank meets the following conditions:

- (i) The path space X_B has no isolated points, i.e., X_B is a Cantor set.
- (ii) The diagram has the same number of vertices at each level, say d. So, each incidence matrix is a $d \times d$ matrix. (This can always be achieved by telescoping.)
- (iii) The diagram has simple edges between the top vertex v_0 and the vertices of the first level, i.e., the vector F_0 consists of 1's. (This assumption is not restrictive because any diagram can be isomorphically transformed into a diagram with simple edges on the first level, as in [DHS99, Lemma 9].)
- (iv) The cofinal equivalence relation is aperiodic, i.e. it has no finite classes.

2.2 Homeomorphisms of finite rank

We recall here the concept of the rank of a homeomorphism of a Cantor set that was defined in [BDK06]. It is worth to mention that the rank of an automorphism of a standard measure space is an important invariant in ergodic theory, studied in many papers.

Definition 2.9. Let T be a homeomorphism of a Cantor set X and let $\{\xi_n\}, n \in \mathbb{N}$, be a sequence of Kakutani-Rokhlin partition of X into r T-towers:

$$X = \bigcup_{j=1}^{r} \bigcup_{i=0}^{h_n(i)-1} T^i F_n(j)$$

We say that T has rank at most r if ξ_{n+1} refines ξ_n and all ξ_n 's generate the clopen topology on X. We say that T has rank r if T has rank at most r but does not have rank at most r-1. The homeomorphism T has infinite rank if it does not have rank r for any finite r.

We denote the set of all homeomorphisms having rank at most r by $\mathcal{R}(\leq r)$ and the set of homeomorphisms of rank r by $\mathcal{R}(r)$.

Obviously, $T \in \mathcal{R}(1)$ if and only if T is topologically conjugate to an odometer.

We note that if $T \in \mathcal{R}(r)$ then the Bratteli diagram constructed from the sequence $\{\xi_n\}$ in Definition 2.9 has exactly r vertices at each level, and this number cannot be reduced. This means that there is a correspondence between finite rank Bratteli diagrams and homeomorphisms of finite rank. The converse is also true for finite rank diagrams that admit Vershik maps.

Let τ_w be the topology of uniform convergence on the group H(X) of all homeomorphisms of X. The following result was proved in [BDK06].

Proposition 2.10. For every finite r, the set $\mathcal{R}(\leq r)$ is a G_{δ} -set in τ_w -topology.

We do not know how one can describe the closure of $\mathcal{R}(\leq r)$ and $\mathcal{R}(r)$ in τ_w . Note only that $\mathcal{R}(1)$ is dense in the closure of all minimal homeomorphisms [BDK06].

2.3 Birkhoff Contraction Coefficient

Each Bratteli diagram is defined by a sequence $\{F_n\}$ of non-negative integer matrices. The vector of heights $h^{(n)} = \{h_v^{(n)}\}$ is equal to the product $F_{n-1} \cdots F_1 \overline{1}$, where $\overline{1} = (1, \ldots, 1)^T$. In the following sections we will show that dynamical properties of the diagram depend on the asymptotic growth of the difference between the minimal and maximal entries of $h^{(n)}$.

In matrix theory and theory of Markov chains there is a method, known as the Birkhoff contraction coefficient method, which allows one to study the asymptotical behavior of nonnegative matrix products by looking at how they "squeeze" the orthant of positive vectors. The first results in the area appeared in Birkhoff's fundamental works [Bir57] and [Bir67]. We refer the reader to the books [Har02] and [Sen81] where a detailed exposition of the material as well as extensive reference list are presented. For the reader's convenience we include some results from [Har02].

Recall that we deal here with Bratteli diagrams of finite rank. From now on, a matrix will always mean a square $d \times d$ matrix.

Definition 2.11. For two positive vectors $x, y \in \mathbb{R}^d$ define the *projective metric* (*Hilbert metric*) as

$$D(x,y) = \ln \max_{i,j} \frac{x_i y_j}{x_j y_i} = \ln \frac{\max_i \frac{x_i}{y_i}}{\min_j \frac{x_j}{y_i}}$$

where (x_i) and (y_i) are entries of the vectors x and y.

Denote by Δ the set of all positive probability vectors of \mathbb{R}^d . Note that (Δ, D) is a complete metric space (Theorem 2.5 in [Har02]).

The next theorem says that all non-negative matrices act as (weak) contractions on the orthant of positive vectors. For the proof, see Lemma 2.1 in [Har02]. **Proposition 2.12.** Let A be a non-negative $d \times d$ matrix. Then for any positive vectors $x, y \in \mathbb{R}^d$ we have that $D(Ax, Ay) \leq D(x, y)$.

Definition 2.13. For a non-negative matrix A, we set

$$\tau(A) = \sup_{x,y>0} \frac{D(Ax, Ay)}{D(x, y)}$$

The coefficient $\tau(A)$ is called the *Birkhoff contraction coefficient*.

It follows from the definition that $D(Ax, Ay) \leq \tau(A)D(x, y)$. Proposition 2.12 implies that $0 \leq \tau(A) \leq 1$. Note that the Birkhoff contraction coefficient has the property $\tau(AB) \leq \tau(A)\tau(B)$.

For a positive matrix $A = (a_{i,j})$, set

$$\phi(A) = \min_{i,j,r,s} \frac{a_{i,j}a_{r,s}}{a_{r,j}a_{i,s}}.$$

If A has a zero entry, then, by definition, we put $\phi(A) = 0$. The next theorem gives the formula for computing the Birkhoff contraction coefficient.

Proposition 2.14 (Theorem 2.6, [Har02]). Suppose that a matrix A has a nonzero entry in each row. Then

$$\tau(A) = \frac{1 - \sqrt{\phi(A)}}{1 + \sqrt{\phi(A)}}$$

In particular, if A is positive, then $\tau(A) < 1$.

Let $\{A_k\}_{k\geq 1}$ be a sequence of $d \times d$ matrices. Denote by P_m^n the forward product $A_m A_{m+1} \cdots A_n$, n > m.

Definition 2.15. The products $P_m^n = (p_{i,j}^{(m,n)})$ are said to tend to row proportionality if for all k, s the sequence $\frac{p_{k,i}^{(m,n)}}{p_{s,i}^{(m,n)}}$ converges (as $n \to \infty$) to some constant a = a(k, s, m) > 0 which does not depend on the column index i.

Similarly, changing column indexes to row indexes, we can define the notion of *column proportionality* (see [Har02, Chapter 5] for details).

Remark 2.16. We note that if P_m^n tends to row proportionality as $n \to \infty$, then its transpose, which is the backward product of $\{A_n^T\}$, tends to column proportionality. Proposition 2.14 also implies that $\tau(A_1 \cdots A_n) = \tau(F_n \cdots F_1)$ where $F_i = A_i^T$.

Lemma 2.17 (Lemma 3.4, [Sen81]). If $\{A_k\}$ is a sequence of positive matrices, then $\tau(P_m^n) \to 0$ as $n \to \infty$ if and only if the products $\{P_m^n\}$ tend to row proportionality.

Definition 2.18. For any positive $d \times d$ matrix A denote by $\Theta(A)$ the D-diameter (in the projective metric) of the image of \mathbb{R}^d_+ under the action of A.

The next lemma, which was proved by A. Fisher (see Proposition 6.13 and Corollary 6.4 of [Fis09]), is important for our study. If B is a Bratteli diagram with incidence matrices $\{F_k\}_{k\geq 1}$, then the vectors of heights are determined by the backward products of the incidence matrices $\{F_k\}_{k\geq 1}$, whereas the values of invariant measures are specified by the forward products of the transposes $\{F_k^T\}_{k\geq 1}$. This lemma will allow us to show that for uniquely ergodic Bratteli diagrams the measure value on the base of each tower is asymptotically the reciprocal of the tower height (see Propositions 4.8 and 4.10).

Lemma 2.19. Let $A = (a_{i,j})$ be a positive matrix. Then $\Theta(A) = \Theta(A^T)$. Furthermore,

$$\Theta(A) = \max_{i,j,k,l} \log \frac{a_{i,k}a_{j,l}}{a_{j,k}a_{i,l}}.$$

As a corollary of this result we deduce the following simple fact saying that the image of the cone of positive vectors under P_m^n has sufficiently small diameter in the projective metric D when n is large enough if and only if the Birkhoff contraction coefficient of P_m^n tends to zero, as $n \to \infty$.

Lemma 2.20. Suppose that all matrices $\{A_k\}_{k\geq 1}$ are positive. Then $\tau(P_m^n) \to 0$ as $n \to \infty$ if and only if for given $\varepsilon > 0$, $m \in \mathbb{N}$, and any non-negative vectors x, y there exists $N \in \mathbb{N}$ such that $D(P_m^n x, P_m^n y) < \varepsilon$ for $n \geq N$.

Proof. Set $F_k = A_k^T$. Suppose that $\tau(P_m^n) \to 0$ as $n \to \infty$. It follows from Remark 2.16 that $\tau(F_n \cdots F_m) \to 0$. Hence the backward product $(P_m^n)^T = F_n \cdots F_m$ tends to column proportionality.

Denote by e_i the *i*-th column vector from the standard basis. Consider $x = \sum x_i e_i$ and $y = \sum y_j e_j$ where the summation is over indices with $x_i > 0$ and $y_j > 0$, respectively. Then we get that

$$D((P_m^n)^T x, (P_m^n)^T y) \le \sum_{i,j=1}^d D((P_m^n)^T e_i, (P_m^n)^T e_j).$$

Thus it suffices to estimate the distance between the images of basis vectors. Set $v_n = (P_m^n)^T e_i$ and $w_n = (P_m^n)^T e_j$. Then v_n and w_n are exactly the *i*-th and *j*-th columns of the matrix $(P_m^n)^T$. Using the definition of projective metric D and the property of column proportionality of $(P_m^n)^T$, we get that $D(v_n, w_n) \to 0$ as $n \to \infty$. Thus, we obtain that $\Theta(P_m^n) = \Theta((P_m^n)^T) \to 0$.

Conversely, using the equality $\Theta(P_m^n) = \Theta((P_m^n)^T)$, we get that $D(v_n, w_n) \to 0$ where v_n and w_n are the columns of $(P_m^n)^T$. It follows from the definition of the metric D that

$$\frac{v_n(i)}{w_n(i)} \cdot \frac{w_n(j)}{v_n(j)} \to 1 \text{ for all } i, j.$$

This implies precisely that the matrices $\{(P_m^n)^T\}$ tend to column proportionality as $n \to \infty$.

Appropriate matrix norms may serve as numerical characteristics of growth rate for matrix products. For a vector $v \in \mathbb{R}^d$ denote by $||v||_1$ the norm given by

$$||v||_1 = \sum_i |v_i|$$

Similarly, for a square matrix $A = (a_{i,j})_{i,j}$ we denote by $||A||_1$ the entrywise 1-norm

$$||A||_1 = \sum_{i,j} |a_{i,j}|.$$
(2.4)

Note that this is not the operator norm arising from the vector 1-norm. However, it is easy to check that

$$||AB||_1 \le ||A||_1 ||B||_1$$
 and $||Ax||_1 \le ||A||_1 ||x||_1$,

whenever the products are defined. Note also that

$$||A||_{1} = ||A\overline{1}||_{1} = ||\overline{1}^{T}A||_{1}$$
(2.5)

for any non-negative matrix A, where $\overline{1} = (1, \dots, 1)^T$.

For any two sequences of real numbers $\{x_n\}$ and $\{y_n\}$, we will write $x_n \sim y_n$ as $n \to \infty$ to indicate that $\lim_{n\to\infty} x_n/y_n = 1$.

3 Measure Extension for Finite Rank Bratteli Diagrams

In this section we first find the structure of any Bratteli diagram of finite rank and then we show that any ergodic measure on such a diagram arises as an extension of a finite ergodic measure from a "vertical" simple subdiagram. These results reduce the study of dynamical measure-theoretical properties of Vershik maps to those defined by dynamics on simple Bratteli diagrams.

We begin this section by describing the process of measure extension from a subdiagram which is central for the paper. Consider a Bratteli diagram B = (V, E) where the vertex set $V = \bigcup_n V_n$ and the edge set $E = \bigcup E_n$ are as in Definition 2.1.

Definition 3.1. By a subdiargam of B, we mean a Bratteli diagram S = (W, R) with $W \subseteq V$ and $R \subseteq E$ such that the set of vertices W_n at level n of the diagram S is precisely the set $W \cap V_n$ and the set of edges between levels n - 1 and n of the diagram S is formed by the edges from $R \cap E_n$. In other words, a subdiagram S is constructed by taking a subset of vertices at each level n of the diagram B and then considering all the edges of B that connect these vertices.

Let S = (W, R) be a subdiagram of B. Consider the set $Y = Y_S$ of all infinite paths of the subdiagram S. Then the set Y is naturally seen as a subset of X_B . Let μ be a finite invariant (with respect to the tail equivalence relation \mathcal{E}) measure on Y. Let X_S be the saturation of Y with respect to \mathcal{E} . In other words, a path $x \in X_B$ belongs to X_S if it is \mathcal{E} -equivalent to a path $y \in Y$. Then X_S is \mathcal{E} -invariant and Y is a complete section for \mathcal{E} on X_S . By the *extension of measure* μ to X_S we mean an \mathcal{E} -invariant measure $\hat{\mu}$ on X_S (finite or infinite) such that $\hat{\mu}$ induced on Y coincides with μ .

Although the procedure of the measure extension with respect to an equivalence relation is well-known, the geometrical nature of the tail equivalence relation makes this construction more illuminating.

Specifically, take a finite path $\overline{e} \in E_S(v_0, v)$ from the top vertex to a vertex v of level n that belongs to the subdiagram S. Let $[\overline{e}]_S$ be the set of all paths in Y that coincide with \overline{e} in the first n edges. Then $[\overline{e}]_S$ is a cylinder subset of Y. For any finite path \overline{e}' from the diagram B with the same range v we set $\hat{\mu}([\overline{e'}]) = \mu([\overline{e}]_S)$. In such a way, the measure $\hat{\mu}$ is extended onto a σ -algebra of Borel subsets of X_B generated by all clopen sets of the form $[\overline{x}]$ where a finite path \overline{x} has the range in a vertex from S. Using the properties of tail equivalence relations, one can show that such an extension is well-defined. Furthermore, the support of $\hat{\mu}$ is, by definition, the set X_S of all paths which are cofinal to paths from Y. We observe that $\hat{\mu}(X_S)$ may be either finite or infinite. In fact, one can use the following formula for computing $\hat{\mu}(X_S)$. Let $W_n = W \cap V_n$ and set $X_S(n) = \{x = (x_i) \in X_B : r(x_i) \in W_i, \forall i \geq n\}$. Clearly, $X_S(n) \subset X_S(n+1)$. Then we have the following formula for the measure $\hat{\mu}(X_S)$:

$$\widehat{\mu}(X_S) = \lim_{n \to \infty} \widehat{\mu}(X_S(n)) = \lim_{n \to \infty} \sum_{w \in W_n} \widehat{h}_w^{(n)} \mu([e_S(v_0, w)])$$
(3.1)

where $\hat{h}_w^{(n)}$ is the height of the tower $X_w^{(n)}$ in the diagram B and $e_S(v_0, w)$ is a finite path from v_0 to w that belongs to S.

The next theorem shows that each Bratteli diagram can be isomorphically transformed into a canonical block-triangular form, which immediately gives a natural decomposition of X_B into a finite number of tail-invariant subsets (discussed below).

Theorem 3.2. Any Bratteli diagram of finite rank is isomorphic to a diagram whose incidence matrices $\{F_n\}_n$ are as follows:

$$F_{n} = \begin{pmatrix} F_{1}^{(n)} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & F_{2}^{(n)} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & F_{s}^{(n)} & 0 & \cdots & 0 \\ X_{s+1,1}^{(n)} & X_{s+1,2}^{(n)} & \cdots & X_{s+1,s}^{(n)} & F_{s+1}^{(n)} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ X_{m,1}^{(n)} & X_{m,2}^{(n)} & \cdots & X_{m,s}^{(n)} & X_{m,s+1}^{(n)} & \cdots & F_{m}^{(n)} \end{pmatrix}.$$
(3.2)

For every $n \ge 1$, the matrices $F_i^{(n)}$, i = 1, ..., s, have strictly positive entries and matrices $F_i^{(n)}$, i = s + 1, ..., m, have either strictly positive or zero entries. For every fixed j = s + 1, ..., m, there is at least one non-zero matrix $X_{i,k}^{(n)}$.

Proof. Let B be a finite rank Bratteli diagram. By telescoping, we obtain that $|V_n| = d$ for all $n \ge 1$. It follows from Proposition 4.6 of [BKM09] that B has finitely many minimal components with respect to the tail equivalence relation, say they are $Z_1, ..., Z_s$. Denote

$$W_n(i) = \{r(x_n) \in V_n : x = (x_n) \in Z_i\}, i = 1, ..., s.$$

Claim: For any $i_1 \neq i_2$ there exists N such that for all $n \geq N$

$$W_n(i_1) \cap W_n(i_2) = \emptyset, \ i_1, i_2 = 1, ..., s.$$

To prove the claim, we fix Z_i and consider the subdiagram B_i of B which is formed by the vertex set $W(i) = \bigcup_{n \ge 1} W_n(i)$ and the edges induced by all paths from Z_i . Then B_i is a simple Bratteli diagram.

Suppose now that the contrary holds, i.e., there exist distinct i_1 and i_2 and a sequence $\{n_k\}$ such that $W_{n_k}(i_1) \cap W_{n_k}(i_2) \neq \emptyset$. Let $\{v_{n_k}\}$ be a sequence of vertices which is chosen from $W_{n_k}(i_1) \cap W_{n_k}(i_2)$. Without loss of generality, we may assume that $n_{k+1} - n_k > 2$. By simplicity of subdiagrams B_{i_1} and B_{i_2} , there are finite paths $\overline{e}_k(1)$ and $\overline{e}_k(2)$ connecting the vertices v_{n_k} and $v_{n_{k+1}}$ and such that $\overline{e}_k(1)$ and $\overline{e}_k(2)$ belong to B_{i_1} and B_{i_2} , respectively. Therefore, there exist infinite paths $x \in Z_{i_1}$ (obtained as a concatenation of $\overline{e}_1(k)$) and $y \in Z_{i_2}$ (obtained as a concatenation of $\overline{e}_2(k)$) which go through the vertices v_{n_k} for every $k \ge 1$. Thus, for every $k \ge 1$, there exists a path $x_k \in Z_{i_1}$ cofinal to xwhich coincides with the first n_k edges of y. This implies that $\operatorname{dist}(x_k, y) \to 0$ as $k \to \infty$. Hence $\operatorname{dist}(Z_{i_1}, Z_{i_2}) = 0$, which is impossible. To complete the proof of the claim, we use a standard argument based on finiteness of the set of minimal components.

By telescoping the diagram B, we may assume that $W_n(i_1) \cap W_n(i_2) = \emptyset$ $(i_1 \neq i_2)$ for all $n \geq 1$. One can also regroup the vertices at each level so that the sets $W_n(1), ..., W_n(s)$ are enumerated from left to right.

Choose a positive constant δ so that $\operatorname{dist}(Z_i, Z_j) \geq \delta, i \neq j$. Again using the method of telescoping, we can easily reduce the general case to that when no edges between vertices from different minimal components exist. Hence we have constructed the collection of simple subdiagrams B_i with incidence matrices $\{F_i^{(n)}\}, i = 1, \ldots, s$. Further telescoping the diagram we may ensure that each matrix $F_i^{(n)}$ has strictly positive entries.

Next, we consider the subdiagram B' of B whose vertex set V' is $\bigcup_n V'_n$ where $V'_n = V_n \setminus \bigcup_{i=1}^s W_n(i)$ and the edge set E' consists of the edges that connect vertices from V' only. In other words, we temporarily ignore the set of edges that link vertices from B' and those from B_i , $i = 1, \ldots, s$. Then B' is a finite rank Bratteli diagram whose rank is strictly less than the rank of B. We can apply the described above procedure to find all minimal components of B'. In a finite number of such steps, we obtain all simple subdiagrams of B that correspond to non-zero matrices from the set $\{F_j^{(n)}\}$, $j = s + 1, \ldots, m$. It may happen that there will be some vertices at infinitely many levels that do not belong to the constructed simple subdiagrams. This means that after appropriate telescoping the corresponding incidence matrices $F_j^{(n)}$ must be either zero or strictly positive.

To finish the proof, we return to B and restore all edges that have been temporarily removed. They will now connect some vertices from different subdiagrams $B'_{j}, j = s + 1, \ldots, m$ and also connect them with some vertices from $B_i, i = 1, \ldots, s$. This set of edges determines the matrices $X_{i,j}^{(n)}$. Certainly, some of these matrices may be zero. But if one fixes a row $i \in \{s + 1, \ldots, m\}$, then at least one matrix from the collection $\{X_{i,j}^{(n)}\}$ is non-zero.

The next theorem which was proved in [BKMS10] shows that the simplex of invariant measures is completely determined by the sequence of incidence matrices of the diagram. To state the theorem, we will need to introduce the following notation.

For $x = (x_1, \ldots, x_N)^T \in \mathbb{R}^N$, we will write $x \ge 0$ if $x_i \ge 0$ for all i, and consider the positive cone $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x \ge 0\}$. Let

$$C_k^{(n)} := F_k^T \cdots F_n^T \left(\mathbb{R}_+^{|V_{n+1}|} \right), \quad 1 \le k \le n.$$

Clearly, $\mathbb{R}^{|V_k|}_+ \supset C^{(n)}_k \supset C^{(n+1)}_k$ for all $n \geq 1.$ Let

$$C_k^{\infty} = \bigcap_{n \ge k} C_k^{(n)}, \ k \ge 1.$$

Observe that C_k^{∞} is a closed non-empty convex subcone of $\mathbb{R}_+^{|V_k|}$. It also follows from these definitions that

$$F_k^T C_{k+1}^\infty = C_k^\infty. \tag{3.3}$$

Theorem 3.3. [BKMS10, Theorem 2.9] Let B = (V, E) be a Bratteli diagram such that the tail equivalence relation \mathcal{E} on X_B is aperiodic. If μ is an invariant measure with respect to the tail equivalence relation \mathcal{E} , then the vectors $p^{(n)} =$ $(\mu(X_w^{(n)}(\overline{e})))_{w \in V_n}, \overline{e} \in E(v_0, w), \text{ satisfy the following conditions for } n \geq 1$:

(i)
$$p^{(n)} \in C_n^{\infty}$$
,

(*ii*)
$$F_n^T p^{(n+1)} = p^{(n)}$$
.

Conversely, if a sequence of vectors $\{p^{(n)}\}$ from $\mathbb{R}^{|V_n|}_+$ satisfies condition (ii), then there exists a non-atomic finite Borel \mathcal{E} -invariant measure μ on X_B with $p_w^{(n)} = \mu(X_w^{(n)}(\overline{e}))$ for all $n \ge 1$ and $w \in V_n$.

The \mathcal{E} -invariant measure μ is a probability measure if and only if (iii) $\sum_{w \in V_n} h_w^{(n)} p_w^{(n)} = 1$ for n = 1, in which case this equality holds for all $n \ge 1$.

Remark 3.4. It was also proved in [BKMS10, Theorem 3.8] that for stationary Bratteli diagrams the sequence of vectors $\{p^{(n)}\}$ which determines an invariant measure can be completely restored by the initial distribution vector $p^{(1)}$. One can construct an example when this result fails for general diagrams. However, for diagrams of finite rank we can still telescope the diagram in such a way that any two different invariant measures μ and ν can already be distinguished on the first level, i.e. the corresponding vectors $p^{(1)}$ are distinct.

Indeed, it follows from the proof of Proposition 3.5 (see below) that the number of extreme rays of C_k^{∞} stabilizes to the number of ergodic measures as $k \to \infty$. By telescoping we may assume that this already holds for every k. By (3.3), we see that the linear map F_k^T sends extreme rays onto extreme rays. Thus, F_k^T is a bijection of the cones C_{k+1}^{∞} and C_k^{∞} for all k proving the claim.

In the next result we apply Theorem 3.3 to a finite rank Bratteli diagram to show that any such a diagram has a finite number of ergodic measures. We observe that this result was already mentioned in [BDM10] for simple diagrams without a proof.

Proposition 3.5. Let B be a Bratteli diagram of finite rank. Suppose that the number of vertices at each level is bounded by d. Then B has no more than d invariant ergodic probability measures.

Proof. We will use Theorem 2.1 from [Pul71]:

Let $\{C_n\}$ be a sequence of finitely generated cones such that $C_n \supset C_{n+1}$ for all $n \ge 1$. If for all sufficiently large n the cone C_n is finitely generated by at most d rays, then $C = \bigcap_n C_n$ is also a finitely generated cone by at most d rays (the number of generating rays is called the size of the cone).

We can apply Pullman's theorem to the sequence of cones $\{C_k^{(n)}\}_n$ for all $k \geq 1$ and conclude that the cones C_k^{∞} are finitely generated of size not greater than d. It follows from (3.3) that $\operatorname{size}(C_k^{\infty}) \leq \operatorname{size}(C_{k+1}^{\infty})$. Hence the sizes must stabilize: $\operatorname{size}(C_k^{\infty}) = m$ for all $k \geq N_0$. Then B has m ergodic invariant probability measures. In fact, it easily follows from Theorem 3.3 that there is a 1-1 correspondence between \mathcal{E} -invariant measures and $C_{N_0}^{\infty}$ such that the extreme rays correspond to the ergodic measures.

Remark 3.6. We note that minimal dynamical systems have no infinite invariant measures that take a finite value on a clopen set. For an aperiodic dynamical system (and, in particular, for finite rank non-simple diagrams) such measures can occur, see [BKMS10].

From now on, we may assume that a finite rank Bratteli diagram is reduced by Theorem 3.2 to the form (3.2) when it is convenient for us. Denote by Λ the subset of $\{1, \ldots, m\}$ such that the corresponding incidence matrices are nonzero in (3.2). For $\alpha \in \Lambda$, denote by B_{α} the subdiagram of B whose incidence matrices are $\{F_{\alpha}^{(n)}\}$. The fact that the matrix $F_{\alpha}^{(n)}$ is strictly positive implies that the subdiagram B_{α} is simple. Let Y_{α} be the path space of the Bratteli diagram B_{α} , $\alpha \in \Lambda$. Denote by $X_{\alpha} = \mathcal{E}(Y_{\alpha})$ the saturation of Y_{α} with respect to the tail equivalence relation. It is clear that $\{X_{\alpha} : \alpha \in \Lambda\}$ is a partition of X_B into Borel invariant subsets.

In the next theorem, we describe the structure of the supports of ergodic invariant measures. The support of each ergodic measure turns out to be the set of all paths that stabilize in some subdiagram, which geometrically can be seen as "vertical". Furthermore, these subdiagrams are pairwise disjoint for different ergodic measures. Everywhere below the term "measure" stands for an \mathcal{E} -invariant measure. Recall that by an infinite measure we mean any σ -finite non-atomic measure which is finite on some clopen set.

Theorem 3.7. Let B be a Bratteli diagram of finite rank.

(1) Each finite ergodic measure on Y_{α} extends to an ergodic measure on X_{α} . The extension can be a finite or an infinite measure.

(2) Each ergodic measure (both finite and infinite) on X_B is obtained as an extension of a finite ergodic measure from some Y_{α} .

(3) The number of finite and infinite ergodic measures (up to scalar multiple) is not greater than d.

(4) We may telescope the diagram B in such a way that for every probability ergodic measure μ there exists a subset W_{μ} of vertices from $\{1, \ldots, d\}$ such that the support of μ consists of all infinite paths that eventually go along the vertices of W_{μ} only. Furthermore,

(4-i) $W_{\mu} \cap W_{\nu} = \emptyset$ for different ergodic measures μ and ν ;

(4-ii) given a probability ergodic measure μ , there exists a constant $\delta > 0$ such that for any $v \in W_{\mu}$ and any level n

$$\mu(X_v^{(n)}) \ge \delta$$

where $X_v^{(n)}$ is the set of all paths that go through the vertex v at level n;

(4-iii) the subdiagram generated by W_{μ} is simple and uniquely ergodic. The only ergodic measure on the path space of the subdiagram is the restriction of measure μ .

Proof. (I) Statements (1), (2), and (3) are similar to Lemma 4.2 from [BKMS10] so that we give a sketch of the proof only.

Let μ be a finite or infinite ergodic measure on the path-space X_B . Then there exists α such that μ is supported on X_{α} . As Y_{α} is a complete section of X_{α} , the restriction of μ to Y_{α} determines an ergodic measure μ_0 on Y_{α} . Thus, to define a measure on X_{α} we need to take any ergodic measure on Y_{α} (due to Proposition 3.5 we have finitely many of them up to a normalization) and extend it by invariance to X_{α} . This process was described at the beginning of this section, see equation (3.1). We note that if the extended measure μ is infinite, but finite on a clopen set, then the minimality of the tail equivalence relation on Y_{α} implies that the restriction μ_0 is a finite measure. This proves (1), (2), and (3).

(II) To prove (4), we enumerate probability ergodic measures on X_B as μ_1, \ldots, μ_p . In view of (I), we may assume, without loss of generality, that each

measure μ_i is restricted to a simple subdiagram B_{α_i} . We start with the measure μ_1 . Then

$$\sum_{v} \limsup_{n \to \infty} \mu_1(X_v^{(n)}) \ge \limsup_{n \to \infty} \sum_{v} \mu_1(X_v^{(n)}) = 1.$$

Therefore, there exists a vertex v_1 with

$$\limsup_{n \to \infty} \mu_1(X_{v_1}^{(n)}) = \delta_1 > 0.$$

This means that we can telescope the diagram so that $\mu_1(X_{v_1}^{(n)}) > \delta_1/2$ for all levels n. Considering the set of vertices $\{1, \ldots, d\} \setminus \{v_1\}$, choose a vertex v_2 (if possible) such that for some positive number δ_2

$$\limsup_{n \to \infty} \mu_1(X_{v_2}^{(n)}) = \delta_2 > 0.$$

Telescope the diagram so that $\mu_1(X_{v_2}^{(n)}) > \delta_2/2$ for all levels n. Repeating this procedure finitely many times, we will end up with a set of vertices W_1 such that

$$\mu_1(X_v^{(n)}) > \delta > 0$$

for all levels n and any vertex $v \in W_1$ (here $\delta = \frac{1}{2} \min_i \delta_i$) and such that

$$\limsup_{v \to 0} \mu_1(X_v^{(n)}) = 0 \text{ for all } v \notin W_1.$$

We will further telescope the diagram to ensure that

$$\sum_{k=n}^{\infty} \mu_1(\bigsqcup_{v \notin W_1} X_v^{(k)}) < \frac{1}{n} \text{ for any } n.$$

Consider the set S_1 of all paths that eventually go only through the vertices from W_1 . We claim that the measure μ_1 is supported on S_1 . Indeed, consider the set

$$R_1 = X_B \setminus S_1 = \bigcap_{n \ge 1} \bigcup_{k \ge n} \bigcup_{v \notin W_1} X_v^{(k)}.$$

Then

$$\mu_1(R_1) = \lim_{n \to \infty} \mu_1(\bigcup_{k \ge n} \bigcup_{v \notin W_1} X_v^{(k)}) \le \lim_{n \to \infty} \sum_{k=n}^\infty \mu_1(\bigcup_{v \notin W_1} X_v^{(k)}) = 0,$$

which proves the claim.

As soon as W_1 is constructed, we may repeat the arguments above to find the corresponding sets W_2, \ldots, W_p for the rest of the ergodic measures. We claim that $W_i \cap W_j = \emptyset$ for all $i \neq j$. Assume the converse, i.e. that

We claim that $W_i \cap W_j = \emptyset$ for all $i \neq j$. Assume the converse, i.e. that there are two probability ergodic measures μ and ν and a vertex w such that

$$\mu(X_w^{(n)}) \ge \gamma \text{ and } \nu(X_w^{(n)}) \ge \gamma$$

for all n, where $\gamma = \frac{1}{2} \min(\delta(\mu), \delta(\nu)) > 0$.

Set $C = \bigcap_{k \ge 1} \bigcup_{n \ge k} X_w^{(k)}$. It follows that $\mu(C) \ge \gamma$ and $\nu(C) \ge \gamma$. Note that C is exactly the set of all paths that visit the vertex w infinitely many times, which is an \mathcal{E} -invariant set. By ergodicity of μ and ν , we see that $\mu(C) = \nu(C) = 1$.

Since μ and ν are mutually singular as distinct ergodic measures, the Radon-Nikodym derivative satisfies

$$\frac{d\mu}{d(\mu+\nu)}(x) \equiv 0$$

for ν -a.e. $x \in X_B$.

For every $x \in X_B$, let $v_n(x)$ denote the vertex of level *n* the path *x* goes through. Set $[x]_n = \{y \in X_B : y_j = x_j, j = 1, ..., n\}$. We observe that $h_{v_n(x)}^{(n)} \mu([x]_n) = \mu(X_{v_n(x)}^{(n)})$, where $h_{v_n(x)}^{(n)}$ is the number of paths from the vertex $v_n(x)$ to the top vertex.

As $\nu(C) = 1$, we have that for ν -a.e. $x \in C$

$$0 = \frac{d\mu}{d(\mu+\nu)}(x)$$

= $\lim_{n \to \infty} \frac{\mu([x]_n)}{(\mu+\nu)([x]_n)}$
= $\lim_{n \to \infty} \frac{h_{v_n(x)}^{(n)}\mu([x]_n)}{h_{v_n(x)}^{(n)}(\mu+\nu)([x]_n)}$
= $\lim_{\{n:v_n(x)=w\}} \frac{\mu(X_w^{(n)})}{\mu(X_w^{(n)}) + \nu(X_w^{(n)})}$
 $\geq \frac{\gamma}{2} > 0,$

which is a contradiction. Thus, statements (4-i) and (4-ii) are proved.

(III) For each ergodic measure $\mu \in {\mu_1, \ldots, \mu_p}$, denote by B_{μ} the subdiagram generated by the vertices W_{μ} . We note that the diagram B_{μ} is a subdiagram of the corresponding simple diagram B_{α_i} . Thus, we can telescope the original diagram in such a way that there is at least one edge between any pair of vertices of W_{μ} at consecutive levels. This will ensure that B_{μ} is a simple subdiagram.

Assume now that the diagram B_{μ} admits another probability ergodic measure, say ν . Denote by Y_{μ} the path space of B_{μ} and by μ_0 the normalized restriction of μ onto Y_{μ} .

We may apply the arguments of part (II) to the measures ν and μ_0 and find the set of vertices $W_{\nu}, W_{\mu_0} \subseteq W_{\mu}$ that define the supports of ν and μ_0 within B_{μ} , respectively. Observe that $W_{\nu} \cap W_{\mu_0} = \emptyset$. However, we claim that $W_{\mu_0} = W_{\mu}$, which will lead to a contradiction. Denote by Y_{μ_0} the set of all paths that go though the vertices from W_{μ_0} only. Since Y_{μ_0} is a cross-section of the support of μ_0 , we get that $\mu(Y_{\mu_0}) > 0$. Set $X_{\mu_0} = \mathcal{E}(Y_{\mu_0})$ and $\gamma = \mu(Y_{\mu})$.

Since $\mu(X_{\mu_0}) > \gamma \mu_0(Y_{\mu_0}) > 0$, the ergodicity of μ implies that $\mu(X_{\mu_0}) = 1$. On the other hand, if $W_{\mu_0} \subsetneq W_{\mu}$, then we get from the statement (4-ii) and the relation (3.1) that

$$1 = \mu(X_{\mu_0}) = \lim_{n \to \infty} \sum_{v \in W_{\mu_0}} \mu(X_v^{(n)} \cap X_{\mu_0}) < 1 - \delta,$$

which is impossible. Thus, the statement (4-iii) is proved.

The proved results clarify the structure of supports of finite ergodic invariant measures on a finite rank diagram. The next statements address the following problem: under what conditions a probability measure defined on a subdiagram can be extended to a finite measure on its support.

Let $\overline{W} = \{W_n\}$ be a sequence of finite subsets of V_n . We will consider the non-trivial case when W_n is a proper subset of V_n for all n. Denote $W'_n = V_n \setminus W_n$. Thus, the sequence \overline{W} determines a proper Bratteli subdiagram $B(\overline{W})$ which is formed by the vertices from \overline{W} and the edges that connect them. Let $Y = Y_{B(\overline{W})}$ be the path space of $B(\overline{W})$. Suppose μ is a probability invariant measure defined on Y.

Proposition 3.8. Let B be a finite rank diagram with incidence matrices $\{F_n = (f_{v,w}^{(n)})\}$ and $B(\overline{W}), \mu$ as above.

(1) Suppose the extension $\hat{\mu}$ of μ on the support $X = X_{B(\overline{W})}$ is finite. Then

$$\sum_{n=1}^{\infty} \sum_{v \in W_{n+1}} \sum_{w \in W'_n} q_{v,w}^{(n)} \mu(X_v^{(n+1)}(\overline{W})) < \infty$$
(3.4)

where $q_{v,w}^{(n)}$ are the entries of the stochastic matrix Q_n (see (2.3)) and $X_v^{(n+1)}(\overline{W})$ is the tower in the subdiagram $B(\overline{W})$ corresponding to the vertex $v \in V_{n+1}$. (2) If

$$\sum_{n=1}^{\infty} \sum_{v \in W_{n+1}} \sum_{w \in W'_n} q_{v,w}^{(n)} < \infty,$$
(3.5)

then any probability measure μ defined on the path space Y of the subdiagram $B(\overline{W})$ extends to a finite measure $\hat{\mu}$ on X.

Proof. (1) Let $X_w^{(n)}(\overline{W})$ be the tower in $B(\overline{W})$ corresponding to a vertex $w \in W_n$ and \overline{e} a finite path from $X_w^{(n)}(\overline{W})$ (we can assume that this path corresponds to the base of the tower $\overline{B}_n(w)$). Denote by $h_w^{(n)}(\overline{W})$ the height of $X_w^{(n)}(\overline{W})$, then $\mu(X_w^{(n)}(\overline{W})) = h_w^{(n)}(\overline{W})\mu(\overline{B}_n(w))$. Let $\widehat{h}_w^{(n)}$ be the number of all finite paths from v_0 to w contained in B, i.e., $\widehat{h}_w^{(n)}$ is the height of $X_w^{(n)}$. Denote

$$I_n = \sum_{w \in W_n} \widehat{h}_w^{(n)} \mu(\overline{B}_n(w)).$$

Compute

$$I_{n+1} = \sum_{v \in W_{n+1}} \widehat{h}_{v}^{(n+1)} \mu(\overline{B}_{n+1}(v))$$

$$= \sum_{v \in W_{n+1}} \sum_{w \in V_{n}} f_{v,w}^{(n)} \widehat{h}_{w}^{(n)} \mu(\overline{B}_{n+1}(v))$$

$$= \sum_{v \in W_{n+1}} \sum_{w \in W_{n}} f_{v,w}^{(n)} \widehat{h}_{w}^{(n)} \mu(\overline{B}_{n+1}(v))$$

$$+ \sum_{v \in W_{n+1}} \sum_{w \in W_{n}'} f_{v,w}^{(n)} \widehat{h}_{w}^{(n)} \mu(\overline{B}_{n+1}(v))$$

$$= \sum_{w \in W_{n}} \widehat{h}_{w}^{(n)} \sum_{v \in W_{n+1}} f_{v,w}^{(n)} \mu(\overline{B}_{n+1}(v)) + J_{n}$$

$$= I_{n} + J_{n}$$

where we used the relation

$$\sum_{v \in W_{n+1}} f_{v,w}^{(n)} \mu(\overline{B}_{n+1}(v)) = \mu(\overline{B}_n(w))$$

and the notation

$$J_n = \sum_{v \in W_{n+1}} \sum_{w \in W'_n} f_{v,w}^{(n)} \widehat{h}_w^{(n)} \mu(\overline{B}_{n+1}(v)).$$

It follows that $I_{n+1} = 1 + J_1 + \ldots + J_n$ because $I_1 = 1$. By the assumption of proposition, the measure extension $\hat{\mu}(X_{\overline{W}})$ is finite and equals $\lim_n I_n$. Therefore, we obtain that $\sum_n J_n < \infty$.

Next,

$$J_{n} = \sum_{v \in W_{n+1}} \sum_{w \in W'_{n}} f_{v,w}^{(n)} \frac{\widehat{h}_{w}^{(n)}}{\widehat{h}_{v}^{(n+1)}} \widehat{h}_{v}^{(n+1)} \mu(\overline{B}_{n+1}(v))$$

$$= \sum_{v \in W_{n+1}} \sum_{w \in W'_{n}} q_{v,w}^{(n)} \widehat{h}_{v}^{(n+1)} \mu(\overline{B}_{n+1}(v))$$

$$\geq \sum_{v \in W_{n+1}} \sum_{w \in W'_{n}} q_{v,w}^{(n)} h_{v}^{(n+1)}(\overline{W}) \mu(\overline{B}_{n+1}(v))$$

$$= \sum_{v \in W_{n+1}} \sum_{w \in W'_{n}} q_{v,w}^{(n)} \mu(X_{v}^{(n+1)}(\overline{W})).$$

Thus, (3.4) follows from convergence of the series $\sum_{n} J_n$.

(2) Suppose (3.5) holds. To prove the finiteness of $\hat{\mu}(X)$, it suffices to show

that the sequence $\{I_n\}$ is bounded. We have that

$$I_n = \sum_{w \in W_n} \frac{\widehat{h}_w^{(n)}}{h_w^{(n)}(\overline{W})} h_w^{(n)}(\overline{W}) \mu(\overline{B}_n(w))$$
$$= \sum_{w \in W_n} \frac{\widehat{h}_w^{(n)}}{h_w^{(n)}(\overline{W})} \mu(X_w^{(n)}(\overline{B}))$$

Next, if we show that there exists M such that for all n and $w \in W_n$

$$\frac{\widehat{h}_{w}^{(n)}}{h_{w}^{(n)}(\overline{W})} \le M, \tag{3.6}$$

then we obtain that

$$I_n \le M \sum_{w \in W_n} \mu(X_w^{(n)}(\overline{B})) \le M.$$

Let

$$M_n = \max\{\frac{\widehat{h}_w^{(n)}}{h_w^{(n)}(\overline{W})} : w \in W_n\}.$$

Fix a vertex $v \in W_{n+1}$ and consider

$$\begin{aligned} \frac{\hat{h}_{v}^{(n+1)}}{h_{v}^{(n+1)}(\overline{W})} &= \frac{1}{h_{v}^{(n+1)}(\overline{W})} \left(\sum_{w \in W_{n}} f_{v,w}^{(n)} \hat{h}_{w}^{(n+1)} + \sum_{w \in W_{n}'} f_{v,w}^{(n)} \hat{h}_{w}^{(n+1)} \right) \\ &\leq \frac{M_{n}}{h_{v}^{(n+1)}(\overline{W})} \sum_{w \in W_{n}} f_{v,w}^{(n)} h_{w}^{(n+1)}(\overline{W}) + \frac{1}{h_{v}^{(n+1)}(\overline{W})} \sum_{w \in W_{n}'} f_{v,w}^{(n)} \hat{h}_{w}^{(n+1)} \\ &= M_{n} + \frac{\hat{h}_{v}^{(n+1)}}{h_{v}^{(n+1)}(\overline{W})} \sum_{w \in W_{n}'} f_{v,w}^{(n)} \frac{\hat{h}_{w}^{(n+1)}}{\hat{h}_{v}^{(n+1)}} \\ &= M_{n} + \frac{\hat{h}_{v}^{(n+1)}}{h_{v}^{(n+1)}(\overline{W})} \sum_{w \in W_{n}'} g_{v,w}^{(n)} \\ &\leq M_{n} + \frac{\hat{h}_{v}^{(n+1)}}{h_{v}^{(n+1)}(\overline{W})} \varepsilon_{n} \end{aligned}$$

where

$$\varepsilon_n = \sum_{v \in W_{n+1}} \sum_{w \in W'_n} q_{v,w}^{(n)}.$$

It follows from the above inequalities that

$$\frac{\widehat{h}_{v}^{(n+1)}}{h_{v}^{(n+1)}(\overline{W})}(1-\varepsilon_{n}) \leq M_{n}$$

$$M_{n+1} \le \frac{M_n}{1 - \varepsilon_n}$$

Finally,

$$M_{n+1} \le \frac{M_1}{\prod_{k=1}^{\infty} (1 - \varepsilon_n)}$$

where the product is convergent in view of (3.5).

It immediately follows from Theorem 3.7 and Proposition 3.8 that the following corollary holds:

Corollary 3.9. Let B be a finite rank diagram and \overline{W} determines a simple uniquely ergodic subdiagram $B(\overline{W})$. If μ is an ergodic probability invariant measure on $B(\overline{W})$, then the measure extension $\hat{\mu}$ is finite if and only if

$$\sum_{n=1}^{\infty} \sum_{v \in W_{n+1}} \sum_{w \in W'_n} q_{v,w}^{(n)} < \infty,$$
(3.7)

Remark 3.10. It is clear that for a given sequence \overline{W} of finite subsets we can telescope the corresponding Bratteli subdiagram $B(\overline{W})$ so that it is transformed into the standard "vertical" subdiagram.

We consider an important particular case of Proposition 3.8. Let B be a finite rank Bratteli diagram with incidence matrices F_n . Take a sequence $\overline{v} = (v_0, v_1, ...)$ of vertices in B such that $v_i \in V_i$ and denote by $Y_{\overline{v}}$ the corresponding "odometer", i.e., $Y_{\overline{v}}$ is the set of paths $x = (x_i)$ such that $r(x_i) = v_i$ for all i. Let $\mu_{\overline{v}}$ be the ergodic measure on $Y_{\overline{v}}$ such that

$$\mu_{\overline{v}}([e(v_0, v_n])) = \left(\prod_{i=1}^{n-1} f_{v_{i+1}, v_i}^{(i)}\right)^{-1}.$$

Let $\hat{\mu}_{\overline{v}}$ be the extension of $\mu_{\overline{v}}$. It follows immediately from (3.7) that

$$\widehat{\mu}_{\overline{v}}(X_B) < \infty \Longleftrightarrow \sum_{i=1}^{\infty} (1 - q_{v_{i+1}, v_i}^{(i)}) < \infty$$
(3.8)

where $q_{v_{i+1},v_i}^{(i)}$ are the entries of the corresponding stochastic matrix (2.3) taken along the sequence \overline{v} .

Corollary 3.11. Let $\overline{v} = (v_0, v_1, ...)$ and $\overline{w} = (w_0, w_1, ...)$ be two sequences of vertices of a finite rank diagram B such that the corresponding measures $\widehat{\mu}_{\overline{w}}$ and $\widehat{\mu}_{\overline{v}}$ are finite. Then there exists a level n_0 such that for all $n \ge n_0$ either $w_n = v_n$ or $w_n \neq v_n$.

Indeed, it follows from (3.8) that, without loss of generality, one can assume that for all n the inequality $q_{v_{n+1},v_n}^{(n)} > 1/2$ holds. Since the vector $(q_{v,w}^{(n)})_w$

and

is probability, there exists at most one vertex $w \in V_n$ such that, for a given $v \in V_{n+1}$, the entry $q_{v,w}^{(n)}$ is greater than 1/2.

Now we consider several examples which illustrate different cases of the proved theorems. In particular, one of the examples shows that if a component Y_{α} of a Bratteli diagram B supports several ergodic probability measures, then some of them might give rise to finite measures and some to infinite ones on $\mathcal{E}(Y_{\alpha})$. We observe that our examples have some similarities with the examples constructed in [FFT09], but we use a completely different approach here. In all the examples below we extend ergodic measures from subdiagrams which have the simplest form possible, i.e. they have only one vertex at each level. In other words, we extend the measure from "odometers". We should note that not every measure can be obtained as an extension from such an elementary subdiagram.

Example 3.12. Let B be the Bratteli diagram with incidence matrices

$$F_n = \begin{pmatrix} b_n & 1\\ 1 & c_n \end{pmatrix}, \quad n \ge 1.$$

Then B contains two natural subdiagrams B_1 and B_2 defined by odometers $\{b_n\}$ and $\{c_n\}$ "sitting" on left and right vertices v_1 and v_2 , respectively. Let μ_1 and μ_2 be the two invariant probability measures on B_1 and B_2 , respectively. Find the extensions $\hat{\mu}_1$ and $\hat{\mu}_2$ of measures μ_1 and μ_2 on $X_1 = \mathcal{E}(Y_1)$ and $X_2 = \mathcal{E}(Y_2)$. To compute $\hat{\mu}_1(X_1)$, we use the relation (for $\hat{\mu}_2(X_2)$ we have similar formulas)

$$\widehat{\mu}_1(X_1) = \lim_{n \to \infty} \widehat{\mu}_1(X_1(n))$$

where $X_1(n) = \{x = (x_i) \in X_B : r(x_i) = v_1, i \ge n\}$. Notice that for $n \ge 1$

$$h_1^{(n)} = b_{n-1}h_1^{(n-1)} + h_2^{(n-1)},$$

$$h_2^{(n)} = c_{n-1}h_2^{(n-1)} + h_1^{(n-1)}.$$

Then

$$\begin{aligned} \widehat{\mu}_{1}(X_{1}(n)) &= \widehat{\mu}_{1}(X_{1}(1)) + \sum_{i=2}^{n} (\widehat{\mu}_{1}(X_{1}(i)) - \widehat{\mu}_{1}(X_{1}(i-1))) \\ &= 1 + \sum_{i=2}^{n} (\frac{h_{1}^{(i)}}{b_{i-1} \cdots b_{1}} - \frac{h_{1}^{(i-1)}}{b_{i-2} \cdots b_{1}}) \\ &= 1 + \sum_{i=2}^{n} (\frac{b_{i-1}h_{1}^{(i-1)} + h_{2}^{(i-1)}}{b_{i-1} \cdots b_{1}} - \frac{h_{1}^{(i-1)}}{b_{i-2} \cdots b_{1}}) \\ &= 1 + \sum_{i=2}^{n} \frac{h_{2}^{(i-1)}}{b_{i-1} \cdots b_{1}} \end{aligned}$$

Finally,

$$\widehat{\mu}_1(X_1) = 1 + \sum_{i=1}^{\infty} \frac{h_2^{(i)}}{b_i \cdots b_1}.$$
(3.9)

Thus,

$$\widehat{\mu}_1(X_1) < \infty \iff \sum_{i=1}^{\infty} \frac{h_2^{(i)}}{b_i \cdots b_1} < \infty.$$

We note that the function $h_2^{(i)}$ depends on $b_1, ..., b_{i-2}$ and $c_1, ..., c_{i-1}$. Based on this observation, we can easily show that the following statement holds:

For any sequence $\{c_n\}$, there exists a sequence $\{b_n\}$ such that $\hat{\mu}_1(X_1) < \infty$. Similarly, given a sequence $\{b_n\}$, one can find a sequence $\{c_n\}$ such that $\hat{\mu}_2(X_1) < \infty$. Moreover, one can construct sequences $\{b_n\}$ and $\{c_n\}$ to obtain both measures $\hat{\mu}_1$ and $\hat{\mu}_2$ simultaneously either finite or infinite.

Indeed, formula (3.9) says that, independently of $h_2^{(i)}$, we can always choose b_i to ensure the convergence of the series $\sum_{i=1}^{\infty} h_2^{(i)} (b_i \cdots b_1)^{-1}$. This is possible because b_i is not involved in the formula for $h_2^{(i)}$. Clearly, this kind of argument proves the claim above.

Now we consider the following Bratteli diagram \overline{B} :



The incidence matrices of \overline{B} have the form:

$$F_n = \left(\begin{array}{ccc} 2 & 0 & 0 \\ x_n & b_n & 1 \\ 1 & 1 & c_n \end{array} \right).$$

We have proved above that there are sequences $\{b_n\}$ and $\{c_n\}$ such that the subdiagram B of \overline{B} has two finite ergodic measures $\hat{\mu}_1$ and $\hat{\mu}_2$. Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be extensions of $\hat{\mu}_1$ and $\hat{\mu}_2$ from B to \overline{B} . In other words, we extend these measures to path spaces $\mathcal{E}(X_i)$, i = 1, 2, in the diagram \overline{B} .

Direct computations, similar to those above, show that one can choose sequences $\{x_n\}$, $\{b_n\}$, and $\{c_n\}$ such that the measure $\overline{\mu}_1$ is infinite and the measure $\overline{\mu}_2$ is finite. First we take $\{b_n\}$, and $\{c_n\}$ to guarantee the finiteness of $\hat{\mu}_1$ and $\hat{\mu}_2$. It is clear that if additionally $c_n > 3$, then $\overline{\mu}_2(\mathcal{E}(X_2)) < \infty$. On the other hand, we can, for any given $\{b_n\}$ and $\{c_n\}$, choose x_n large enough to obtain the divergent series

$$\sum_{n=1}^{\infty} \frac{2^{n-1}x_n}{b_1 \dots b_n} = \infty$$

It follows then that $\overline{\mu}_1(\mathcal{E}(X_1)) = \infty$.

Remark 3.13. (1) One can slightly modify Example 3.12 and consider the sequence of incidence matrices

$$F_n = \begin{pmatrix} b_n & s_n \\ t_n & c_n \end{pmatrix}, \quad n \ge 1$$

such that the additional condition $b_n + s_n = t_n + c_n = h_n$ holds. Then the corresponding stochastic matrix Q_n has the form

$$Q_n = \begin{pmatrix} \frac{b_n}{h_n} & 1 - \frac{b_n}{h_n} \\ 1 - \frac{c_n}{h_n} & \frac{c_n}{h_n} \end{pmatrix} = \begin{pmatrix} 1 - \varepsilon_n & \varepsilon_n \\ \eta_n & 1 - \eta_n \end{pmatrix}$$

because $h_v^{(n+1)} = h_n h_v^{(n)}$ for any vertex v. It is not hard to show that if $\sum_n (\varepsilon_n + \eta_n) < \infty$, then there are two finite ergodic invariant measures and if $\sum_n (\varepsilon_n + \eta_n) = \infty$, then the diagram constructed by $\{F_n\}$ is uniquely ergodic.

(2) We also note that the method of Example 3.12 can be applied to construct a simple diagram with d vertices at each level, having exactly k finite ergodic measures, $k \leq d$.

4 Unique Ergodicity of Simple Diagrams

We will assume in our investigation of finite rank Bratteli diagrams that they have the form (3.2), and, moreover, all matrices $F_i^{(n)}$, i = 1, ..., m, are non-zero. In view of Theorem 3.7, the study of invariant measures on Bratteli diagrams of finite rank can be reduced to the study of invariant ergodic measures on simple subdiagrams. In this section we answer the question when a simple diagram is uniquely ergodic. We prove an analogue of Boshernitzan's theorem for symbolic systems [Bos92]. It is shown that unique ergodicity is equivalent to the fact that the measures of all towers at all levels are uniformly bounded from below by a positive constant.

The next theorem gives a criterion of unique ergodicity for a simple Bratteli diagram in terms of Birkhoff contraction coefficients. A version of this result was earlier established by Fisher [Fis09, Theorem 1.3], but with somewhat different terminology and approach.

Theorem 4.1. Let B be a simple Bratteli diagram of finite rank with incidence matrices $\{F_n\}_{n\geq 1}$. Let $A_n = F_n^T$. Then the diagram B is uniquely ergodic if and only if

$$\lim_{n \to \infty} \tau(A_1 \dots A_n) = 0$$

Proof. Set $P_m^n = A_m \dots A_n$. Denote the cone $\bigcap_{n \ge m} P_m^n \mathbb{R}^d_+$ by C_m . By the compactness argument, $C_m \neq \emptyset$. Furthermore, $A_m \overline{C}_{m+1} = C_m$. Therefore, for any vector $p^{(1)} \in C_1$ there exists a sequence of nonnegative vectors $\{p^{(m)}\}_{m \ge 1}$ such that $A_{m-1}p^{(m)} = p^{(m-1)}$. Such a sequence of vectors defines a finite invariant measure. The converse is also true. It follows from Remark 3.4 that in

order to establish the unique ergodicity, it is necessary and sufficient to show that C_1 is a single ray. Now the result follows immediately from Lemma 2.20.

In the next proposition we collect a number of conditions yielding unique ergodicity that can be easily checked in practice. For the proof, see Corollary 5.1 in [Har02] and Theorem 3.2 in [Sen81].

Proposition 4.2. Let $\{A_n\}_{n\geq 1}$ be transposes of primitive incidence matrices of a finite rank diagram B.

(1) The diagram B admits a unique invariant probability measure on X_B if and only if there exists a strictly increasing sequence $\{n_s\}$ such that

$$\sum_{s=1}^{\infty} \sqrt{\phi(P_{n_s}^{n_{s+1}})} = \infty$$

where $P_{n_s}^{n_{s+1}} = A_{n_s} \cdots A_{n_{s+1}}$. In particular, if

$$\sum_{n=1}^{\infty} \sqrt{\phi(A_n)} = \infty,$$

then B admits a unique invariant probability measure.
(2) If

$$\sum_{n=1}^{\infty} \left(\frac{m_n}{M_n} \right) = \infty$$

where m_n and M_n are the smallest and the largest entry of A_n respectively, then *B* admits a unique invariant probability measure.

Example 4.3. Let B be a simple Bratteli diagram with incidence matrices

$$F_n = \begin{pmatrix} f_1^{(n)} & 1 & \cdots & 1 \\ 1 & f_2^{(n)} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & f_d^{(n)} \end{pmatrix}.$$

Let $q_n = \max\{f_i^{(n)}f_j^{(n)}: i \neq j\}$. Compute $\phi(F_n) = q_n^{-1}$. For $A_n = F_n^T$, we observe that if

$$\sum_{n=1}^{\infty} \sqrt{\phi(A_n)} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{q_n}} = \infty,$$

then there is a unique invariant probability measure on B. This example generalizes an example considered in [FFT09] for the case of 2×2 matrices.

As a corollary of Proposition 4.2 we immediately obtain that if the incidence matrices do not grow too fast, then the diagram admits a unique invariant measure.

Corollary 4.4. If a simple Bratteli diagram with incidence matrices $\{F_n\}_{n\geq 1}$ satisfies the condition $||F_n||_1 \leq Cn$ for some C > 0 and all sufficiently large n, then the diagram admits a unique invariant probability measure.

In particular, this result holds if the diagram has only finitely many different incidence matrices.

Proof. Denote by m_n and M_n the smallest and the largest entry of F_n respectively. Using the simplicity of the diagram and an appropriate telescoping, we may assume that $m_n \geq 1$ for all n. By the definition of the entrywise matrix 1-norm, we get that

$$\frac{m_n}{M_n} \ge \frac{1}{||F_n||_1} \ge \frac{1}{Cn}$$

for all n large enough. The result follows from Proposition 4.2.

Remark 4.5. (1) This corollary gives another proof of the fact that linearly recurrent systems are uniquely ergodic, which was originally established in Proposition 5 of [CDHM03].

(2) It is mentioned in [Haj76, p. 528] that the products of the following sets of positive matrices tend to column proportionality and, in particular, give rise to uniquely ergodic systems:

- (i) Any set of primitive incidence matrices which pairwise commute.
- (ii) The set Σ of primitive incidence matrices such that if $A \in \Sigma$ and F is primitive, then AF and FA are primitive.

The next results show that unique ergodicity of the Bratteli diagram ensures the property that all towers in Kakutani-Rokhlin partitions grow with the same rate. Furthermore, it allows us to use the norm of the product of incidence matrices as a measure of the growth rate for tower heights.

Lemma 4.6. Let B be a simple uniquely ergodic Bratteli diagram with incidence matrices $\{F_n\}_{n\geq 1}$. Then there exist strictly positive vectors ξ and η such that for any n > 0 and any vector $x \in \mathbb{R}^d_+$ we have that

$$\lim_{m \to \infty} \frac{F_m \cdots F_n x}{||F_m \cdots F_n x||_1} = \xi$$

and

$$\lim_{m \to \infty} \frac{x^T F_m \cdots F_n}{||x^T F_m \cdots F_n||_1} = \eta^T.$$

Furthermore, $\eta^T \cdot \xi = 1$ where the dot denotes the inner product of vectors.

Proof. Since the diagram is uniquely ergodic, we have, by Theorem 4.1, that $\tau(A_n \cdots A_m) \to 0$ as $m \to \infty$. Applying Lemmas 2.19 and 2.20, we obtain that the *D*-diameter of the cone $C_m = F_m \cdots F_n \mathbb{R}^d_+$ tends to zero as $m \to \infty$. Hence, by compactness argument, there exists a strictly positive probability

vector ξ_n with $||\xi_n||_1 = 1$ such that $\bigcap_{m \ge n} C_m = ray\{\xi_n\}$. It follows that for any non-negative vector x

$$\frac{F_m \cdots F_n x}{||F_m \cdots F_n x||_1} \to \xi_n \text{ as } m \to \infty.$$

Setting $x = F_{n-1}y$ for some non-negative vector y, we get that

$$\frac{F_m \cdots F_n F_{n-1} y}{\|F_m \cdots F_n F_{n-1} y\|_1} \to \xi_{n-1} \text{ as } m \to \infty.$$

Hence $\xi_n = \xi_{n-1} = \ldots = \xi_1 = \xi$.

The existence of the probability vector η that satisfies the second equality of the lemma can be proved in a similar way. We note only that Lemma 2.19 implies that the *D*-diameter of the cone $C_m = F_n^T \dots F_m^T \mathbb{R}^d_+$ tends to zero as $m \to \infty$.

Now we show that $\eta^T \cdot \xi = 1$. Since $||\eta||_1 = ||\xi||_1 = 1$, we get that $\eta^T \cdot \xi \leq 1$. Thus, it is sufficient to prove only that $\eta^T \cdot \xi \geq 1$.

For fixed n and m > n, define the column vectors $x^{(n)}$ and $y_n(m)$ by

$$x^{(n)} = \frac{F_{n-1} \cdots F_1 \overline{1}}{||F_{n-1} \cdots F_1||_1}$$
 and $y_n(m)^T = \frac{\overline{1}^T F_m \cdots F_n}{||F_m \cdots F_n||_1}$

where $\overline{1} = (1, \ldots, 1)^T$. We proved that $x^{(n)} \to \xi$ as $n \to \infty$ and $y_n(m)^T \to \eta^T$ as $m \to \infty$ for any fixed n, in view of (2.5). Thus, for every fixed n, we have that

$$y_n(m)^T \cdot x^{(n)} = \frac{\overline{1}^T F_m \cdots F_n}{||F_m \cdots F_n||_1} \cdot \frac{F_{n-1} \cdots F_1 \overline{1}}{||F_{n-1} \cdots F_1||_1}$$
$$= \overline{1}^T \cdot \frac{F_m \cdots F_1 \overline{1}}{||F_m \cdots F_n||_1 ||F_{n-1} \cdots F_1||_1}$$
$$\geq \overline{1}^T \cdot \frac{F_m \cdots F_1 \overline{1}}{||F_m \cdots F_n F_{n-1} \cdots F_1||_1}$$
$$\to \overline{1}^T \cdot \xi = 1$$

as $m \to \infty$. On the other hand, $y_n(m)^T \cdot x^{(n)} \to \eta^T \cdot x^{(n)}$ as $m \to \infty$. Hence $\eta^T \cdot x^{(n)} \ge 1$ for every *n*. Therefore,

$$\eta^T \cdot x^{(n)} \to \eta^T \cdot \xi \ge 1.$$

Remark 4.7. We note that if a Bratteli diagram B is stationary, i.e. $F_n = F$ for all n, then ξ and η are the normalized right and left Perron-Frobenius eigenvectors of F, respectively.

As a corollary of Lemma 4.6 we immediately obtain the following proposition. Recall that we write $x_n \sim y_n$ as $n \to \infty$ to denote that $\lim_{n\to\infty} x_n/y_n = 1$.

Proposition 4.8. Let B be a uniquely ergodic simple Bratteli diagram with incidence matrices $\{F_n\}_{n\geq 1}$. Let $h^{(n)} = F_{n-1}\cdots F_1\overline{1}$ be the vector representing the tower heights. Then

$$h_w^{(n)} \sim \xi_w ||F_{n-1} \cdots F_1||_1.$$

In particular, there exist constants $k_1 > 0$ and $k_2 > 0$ such that for all $w = 1, \ldots, d$ and $n \ge 1$

$$||F_{n-1}\cdots F_1||_1 \le h_w^{(n)} \le k_2 ||F_{n-1}\cdots F_1||_1.$$

Remark 4.9. We note that the converse statement is not true. As a counterexample, consider the Bratteli diagram B with incidence matrices

$$F_n = \left(\begin{array}{cc} n^2 & 1\\ 1 & n^2 \end{array}\right).$$

Setting $h^{(n)} = F_{n-1} \cdots F_1 \overline{1}$, we note that $h_1^{(n)} = h_2^{(n)} = 2^{-1} ||F_{n-1} \cdots F_1||_1$. However, it was shown in [FFT09, Proposition 3.1] that the diagram *B* is not uniquely ergodic. We may also apply the methods of Section 3 to show that each of these measures is obtained as an extension of a unique invariant measure from the left (right) vertical subdiagram.

The next proposition shows that the measure of the tower base asymptotically behaves as the reciprocal of the tower height in uniquely ergodic systems.

Proposition 4.10. Let B be a uniquely ergodic simple Bratteli diagram. Let μ be the unique invariant measure. Then

$$\mu(B_n(w)) \sim \frac{\eta_w}{||F_{n-1}\cdots F_1||_1}$$

where $B_n(w)$ is the base of the tower $X_w^{(n)}$.

Proof. Enumerate the edges of the Bratteli diagram B from left to right. Denote by T the resulting Vershik map. Then T is defined everywhere on the set $X \setminus \mathcal{O}$ where \mathcal{O} is the union of orbits of maximal and minimal paths. Note that \mathcal{O} is a countable set.

Fix an integer m > 0. For each infinite path $x \in X_B \setminus \mathcal{O}$, denote by $v_m(x)$ the vertex of level m the path x goes through. Denote also by $e(v_0, v_m(x))$ the finite path of x between the vertices v_0 and $v_m(x)$. Let $i_m(x)$ be the least integer such that $T^{-i_m(x)}$ maps $e(v_0, v_m(x))$ to the minimal finite path from the set $E(v_0, v_m(x))$. Similarly, let $j_m(x)$ be the least integer such that $T^{j_m(x)}$ maps $e(v_0, v_m(x))$. Notice that $T^{j_m(x)}$ maps $e(v_0, v_m(x))$ to the maximal path from $E(v_0, v_m(x))$. Notice that $h_{v_m(x)}^{(m)} = j_m(x) + i_m(x)$.

Fix an arbitrarily $\varepsilon > 0$. By Proposition 4.8, choose n large enough so that

$$\left|\frac{h_r^{(n)}}{||F_{n-1}\cdots F_1||_1} - \xi_r\right| < \varepsilon$$

for every r. Then by the pointwise ergodic theorem, for μ -a.e $x \in X_B$, we get that

$$\mu(B_n(w)) = \lim_{m \to \infty} \frac{1}{i_m(x) + j_m(x)} \sum_{i = -i_m(x)}^{j_m(x)} \mathbb{1}_{B_n(w)}(T^i(x)).$$

The sum in the right hand side of the equality is equal to the number of paths that connect the range of $B_n(w)$, the vertex w of the level n, to the vertex $v_m(x)$ of the level m. Hence, we get that

$$\mu(B_n(w)) = \lim_{m \to \infty} \frac{(F_{m-1} \cdots F_n)_{v_m(x),w}}{h_{v_m(x)}^{(m)}}.$$

We should note that such an interpretation of the pointwise ergodic theorem apparently appeared first in [VK81, Theorem 2], see also [Mel06, Lemma 3.4].

Telescoping the diagram or, in other words, passing to a subsequence, we may assume, without loss of generality, that $v_m(x) = v$ for all m > n. Denote by e_v the v-th basis vector. Using Proposition 4.8, we get that

$$\mu(B_n(w)) = \lim_{m \to \infty} \frac{(F_{m-1} \cdots F_n)_{v,w}}{h_v^{(m)}}$$

$$= \lim_{m \to \infty} \frac{(F_{m-1} \cdots F_n)_{v,w}}{\sum_r (F_{m-1} \cdots F_n)_{v,r} h_r^{(n)}}$$

$$\geq \lim_{m \to \infty} \frac{(F_{m-1} \cdots F_n)_{v,r}}{||F_{n-1} \cdots F_1||_1 \sum_r (F_{m-1} \cdots F_n)_{v,r} (\xi_r + \varepsilon)}$$

$$= \lim_{m \to \infty} \frac{e_v^T F_{m-1} \cdots F_n e_w}{||F_{n-1} \cdots F_1||_1 e_v^T F_{m-1} \cdots F_n \cdot (\xi + \varepsilon \overline{1})}$$

$$= \frac{\eta_w}{||F_{n-1} \cdots F_1||_1 \eta^T \cdot (\xi + \varepsilon \overline{1})}$$

$$= \frac{\eta_w}{||F_{n-1} \cdots F_1||_1 (1 + \varepsilon)}.$$

Analogously, we can prove that

$$\mu(B_n(w)) \le \frac{\eta_w}{||F_{n-1}\cdots F_1||_1(1-\varepsilon)}.$$

Since ε can be chosen arbitrarily small, we get that

$$\mu(B_n(w)) \sim \frac{\eta_w}{||F_{n-1}\cdots F_1||_1}.$$

Now we are ready to establish a topological analogue of Boshernitzan's theorem [Bos92].

Theorem 4.11. Let B be a simple Bratteli diagram of a finite rank. Let μ be an ergodic probability invariant measure. Then μ is a unique invariant measure if and only if there exists a constant δ such that

$$h_w^{(n)}\mu(B_n(w)) \ge \delta > 0$$

for any level n and any vertex $w \in V_n$, where $B_n(w)$ is the base of the tower.

Proof. The "only if" part follows immediately from Propositions 4.8 and 4.10. The "if" part is a corollary of Theorem 3.7(4).

We complete this section by extending Proposition 4.8 to arbitrary diagrams of finite rank. Amongst other things, this result shows that the extension of a measure from a subdiagram remains finite if and only if the heights of the towers in the entire diagram are asymptotically equivalent (up to a factor) to the heights of the subdiagram. From the dynamical point of view, this can be interpreted as a fact that each point spends significantly greater amount of time in the subdiagram than in its complement.

Proposition 4.12. Let μ be a measure on the path space of a Bratteli diagram B (not necessarily simple). Let B_{μ} be a simple subdiagram constructed in Theorem 3.7 that determines the support of μ . Denote by W the set of vertices at each level of B_{μ} .

(1) The measure μ is finite if and only if

$$\limsup_{n \to \infty} \frac{h_v^{(n)}}{h_v^{(n)}(W)} < \infty \text{ for any } v, w \in W,$$

where $\{h_w^{(n)}\}_w$ is the vector of tower heights in the diagram B and $\{h_v^{(n)}(W)\}_v$ is the vector of heights in the subdiagram B_{μ} . Moreover, the finiteness of μ is equivalent to the fact that there exist some positive constants r_1 and r_2 such that

$$r_1 \leq \frac{h_v^{(n)}}{||G_{n-1}\cdots G_1||_1} < r_2 \text{ for any } v \in W \text{ and } n \geq 1,$$

where $\{G_n\}_{n\geq 1}$ is the sequence of incidence matrices for B_{μ} .

(2) If the measure μ is finite, then

$$\limsup_{n \to \infty} \frac{h_v^{(n)}}{h_w^{(n)}} < \infty \text{ for any } v, w \in W.$$

Proof. Let Y_{μ} stand for the path space of B_{μ} . Assume that the measure μ is finite. Then for the base $\overline{B}_n(w)$ of the tower $X_v^{(n)} \cap Y_{\mu}$, we have that

$$\limsup_{n \to \infty} \mu(\overline{B}_n(w)) h_w^{(n)} < \infty \text{ for every } w \in W.$$
(4.1)

Applying Propositions 4.10 and 4.8 to the uniquely ergodic simple diagram B_{μ} and to the measure $\mu|_{Y_{\mu}}$, we get that for $w \in W$

$$\mu(\overline{B}_n(w)) \sim \eta_w / ||G_{n-1} \cdots G_1||_1 \text{ and } h_w^{(n)}(W) \sim \xi_w ||G_{n-1} \cdots G_1||_1.$$

Observe that the norms of vectors (ξ_w) and (η_w) are equal to $\mu(Y_{\mu})$. Clearly, $h_w^{(n)} \ge h_w^{(n)}(W)$ for all $w \in W$. Thus, it follows from (4.1) that

$$0 < \xi_w \eta_w = \limsup_{n \to \infty} h_w^{(n)}(W) \mu(\overline{B}_n(w)) \le \eta_w \limsup_{n \to \infty} \frac{h_w^{(n)}}{||G_{n-1} \cdots G_1||_1} < \infty.$$

Thus, by telescoping, we may assume that there exist positive constants r_1 and r_2 such that

$$r_1 < \frac{h_w^{(n)}}{||G_{n-1}\cdots G_1||_1} < r_2$$

for all $n \ge 1$ and all $w \in W$. This yields the statement (2) and "only if" part of (1).

Conversely, if $\limsup_n h_v^{(n)}/h_v^{(n)}(W) < \infty$ for all $v \in W$, then the relation (3.1) (Section 3) implies that

$$\mu(X_B) = \lim_{n \to \infty} \sum_{v \in W} \mu(Y_\mu \cap X_v^{(n)}) \frac{h_v^{(n)}}{h_v^{(n)}(W)}$$
$$\leq \limsup_{n \to \infty} \sum_{v \in W} \frac{h_v^{(n)}}{h_v^{(n)}(W)} < \infty.$$

Hence, the measure μ is finite.

The second part of the statement (1) follows directly from the argument above and asymptotic of $h_w^{(n)}(W)$ obtained from the application of Proposition 4.8 to the subdiagram B_{μ} .

5 Zero Entropy and Absence of Strong Mixing

In this section we first show that Vershik maps of finite rank Bratteli diagrams have zero measure entropy. Our proof of this assertion is based on ideas from [KS67]. Katok and Stepin defined there the notion of "speed of approximation of a dynamical system by periodic transformations" and discovered that dynamical properties of automorphisms of a measure space crucially depend on the speed of approximation. For a Vershik map we have a natural sequence of Kakutani-Rokhlin partitions and periodic transformations that imitates the original dynamics. This property of Bratteli diagrams allows us to "read" the speed of approximation directly from the diagrams and even find its quantitative estimate.

In the second part of the section we prove that Bratteli-Vershik systems of finite rank cannot be strongly mixing. The absence of strong mixing has been earlier established for substitution systems and linearly recurrent systems (see [DK78], [BKMS10], and [CDHM03]). We also mention the Ph.D thesis of Wargan [War02] devoted to the study of some generalizations of linearly recurrent systems where he proved the absence of strong mixing for such systems. In our work, we study the more general case of finite rank Bratteli diagrams and also prove that Bratteli-Vershik systems for such diagrams are not strongly mixing.

We recall the definition of the speed of approximation following [KS67].

Definition 5.1. Let (X, μ, T) be an ergodic probability measure preserving system. Let f(n) be a monotonically decreasing function with $f(n) \to 0$ as $n \to \infty$. It is said that the transformation T admits a periodic approximation of type Iwith the speed f(n) if there exists a sequence of measurable partitions $\{\Xi_n\}_{n\geq 1}$ with $\Xi_n = \{C_{n,1}, \ldots, C_{n,q_n}\}$ and measure-preserving periodic automorphisms P_n of the measure space (X, μ) such that

- (i) $\Xi_n \to \epsilon$ (a trivial partition) as $n \to \infty$;
- (ii) P_n permutes the atoms of Ξ_n ;
- (iii) $\sum_{i=1}^{q_n} \mu(P_n C_{n,i} \land TC_{n,i}) < f(q_n).$

Lemma 5.2. Let B be an ordered Bratteli diagram of finite rank with the Vershik map $T: X_B \to X_B$ (not necessarily defined everywhere). Then for any finite ergodic measure μ , there exists a constant $\theta > 0$ such that the system (X_B, μ, T) admits a periodic approximation of type I with the speed $f(n) = \theta/n$.

Proof. (1) Using Theorem 3.7, we can find a subset of vertices $W \subseteq \{1, \ldots, d\}$ such that the support of μ is the subdiagram (denoted by B_{μ}) which is determined by the vertices from W. Let $\{G_n\}_{n\geq 1}$ be the sequence of incidence matrices of the diagram B_{μ} . Denote by μ^0 the restriction of μ to the path-space (denoted by Y_{μ}) of the subdiagram B_{μ} .

Fix any $\varepsilon > 0$. Then, applying Proposition 4.10 to the diagram B_{μ} , we find a probability vector η , defined on the vertices of the subdiagram B_{μ} only, and an integer n_0 such that

$$\mu(B_n(w)) = \mu^0(B_n(w)) < \frac{\eta_w + \varepsilon/k}{||G_{n-1} \cdots G_1||_1} \text{ for all } w \in W \text{ and } n \ge n_0,$$

where k = card(W). Hence,

$$\sum_{w \in W} \mu(B_n(w)) < \frac{1 + \varepsilon}{||G_{n-1} \cdots G_1||_1}.$$
(5.1)

By telescoping the diagram between the top level and the level n_0 we obtain that (5.1) holds for every n.

(2) We note that the support of μ is the saturation $X_{\mu} = \mathcal{E}(Y_{\mu})$. Thus, it is enough to consider the Vershik map T on the invariant set X_{μ} only. Each vertex $w \in V_n \cap W$ defines a T-tower $X_w^{(n)}$ which consists of all paths that go though the vertex w and are cofinal to paths from Y_{μ} . For every n define a partition $\Xi_n = \{C_{n,1}, \ldots, C_{n,q_n}\}$ of the space X_{μ} as the union of all towers $X_w^{(n)}$. Then

$$q_n = \sum_{w \in W} h_w^{(n)}$$

is the number of atoms in the partition Ξ_n , where $h_w^{(n)}$ is the number of paths from the vertex $w \in V_n$ to the top vertex in the diagram B. For each Ξ_n , we construct a periodic transformation P_n which permutes the atoms of $X_w^{(n)}$: P_n coincides with T on all atoms except the top of the tower and maps the top of $X_w^{(n)}$ to its base via $T^{-h_w^{(n)}}$ (see, for example, Corollary 7 in [BDM05]).

It follows from the construction of P_n that

$$\sum_{i=1}^{q_n} \mu(P_n C_{n,i} \vartriangle TC_{n,i}) \le \sum_{w \in W} \mu(B_n(w))) < \frac{1+\varepsilon}{||G_{n-1} \cdots G_1||_1}.$$

Proposition 4.12(1) implies that there exists a constant K > 0 such that $q_n \leq K ||G_{n-1} \cdots G_1||_1$ for all $n \geq 1$. This proves the result. \square *Remark* 5.3. In the case when the Bratteli diagram is simple and uniquely ergodic, it follows from the proof above that the Vershik map admits a periodic approximation of type I with the speed $f(n) = \theta/n$, where $\theta > 1$ is arbitrary. Indeed, for such a diagram we have $G_n = F_n$ and $q_n = ||G_{n-1} \cdots G_1||_1$ (recall that we are using the entrywise matrix 1-norm (2.4)).

The next theorem shows that systems with relatively high speed of approximation always have zero entropy.

Theorem 5.4 (Theorem 4.3 in [KS67]). Let (X, μ, T) be an ergodic probability measure-preserving system. If the system admits approximation of type I with the speed $\frac{\theta}{\ln n}$, then its measure-theoretical entropy is not greater than $\frac{\theta}{2}$.

Now, the following corollary is immediate from Theorem 5.4 and Lemma 5.2. We should note that a similar result is also established for the class of simple Bratteli diagrams in [BR, Proposition 6.7.2], but with a completely different approach.

Corollary 5.5. Let B be a Bratteli diagram of finite rank. Assume that μ is an ergodic probability measure on B. Then any Vershik map on B has zero μ -entropy.

Next we show that Bratteli-Vershik systems on finite rank diagrams are not strongly mixing. This result is achieved, roughly speaking, by establishing the low complexity of such systems.

Theorem 5.6. Let $B = (V, E, \geq)$ be an ordered Bratteli diagram of finite rank. Let $T : X_B \to X_B$ be a Vershik map defined by an order \geq on B (T is not necessarily continuous everywhere). Then for any invariant probability ergodic measure μ , the dynamical system (X_B, μ, T) is not strongly mixing. *Proof.* (I) Denote by $X_v^{(n)}$ the set of paths going through the vertex v at the level n. In the proof we will interpret the family $\{X_v^{(n)} : v \in V_n\}$ as a Kakutani-Rokhlin partion of X_B . Then $X_v^{(n)} = \{B_n(v), \ldots, T^{h_v^{(n)}-1}B_n(v)\}$ is a T-tower, where $h_v^{(n)}$ is the number of finite paths from the top vertex v_0 to a vertex v of level n, and $B_n(v)$ is the clopen set generated by the finite minimal path connecting the vertices v_0 and v.

It follows from Theorem 3.7 that for a given finite ergodic measure μ there is a subset of vertices $W \subseteq \{1, \ldots, d\}$ such that the support of μ is a simple uniquely ergodic subdiagram (denoted by B_{μ}) spanned by the vertices from W. Set

$$X_{\epsilon}^{(n)} = \bigcup_{v \notin W} X_v^{(n)}.$$

Then, as in the proof of Theorem 3.7, we can show that

$$\lim_{n \to \infty} \mu(X_{\epsilon}^{(n)}) = 0.$$
(5.2)

For every *n* we consider the Kakutani-Rokhlin partition $\{X_v^{(n)}, v \in W; X_{\epsilon}^{(n)}\}$ where the tower $X_{\epsilon}^{(n)}$ has a height $h_{\epsilon}^{(n)} = 1$ by definition.

Denote by W_{r+1}^* the set of words of length r+1 over the alphabet $W \cup \{\epsilon\}$. For a word $\omega = \omega_0 \dots \omega_r \in W_{r+1}^*$, define a clopen set

$$\widehat{B}_n(\omega) = B_n(\omega_0) \cap T^{-h_{\omega_0}^{(n)}} B_n(\omega_1) \cap \ldots \cap T^{-h_{\omega_0}^{(n)}-\ldots-h_{\omega_{r-1}}^{(n)}} B_n(\omega_r).$$

In other words, the set $\widehat{B}_n(\omega)$ consists of the points from $B_n(\omega_0)$ that visit the towers $X_{\omega_1}^{(n)}, \ldots, X_{\omega_r}^{(n)}$ consecutively under the action of T. Clearly, some of the sets $\widehat{B}_n(\omega)$ may be empty. Set also

$$q_n(\omega) = h_{\omega_0}^{(n)} + \ldots + h_{\omega_r}^{(n)}$$
(5.3)

and

$$C_n(\omega) = \bigcup_{\ell=0}^{q_n(\omega)-1} T^\ell \widehat{B}_n(\omega).$$
(5.4)

Considering the *T*-orbit of any point $x \in X_B$, we see that

$$X_B = \bigcup_{\omega \in W_{r+1}^*} C_n(\omega)$$

We have

$$\sum_{\omega \in W_{r+1}^*} \limsup_{n \to \infty} \mu(C_n(\omega)) \ge \limsup_{n \to \infty} \sum_{\omega \in W_{r+1}^*} \mu(C_n(\omega)) \ge 1.$$

Thus, there exists a word $\overline{\omega} \in W_{r+1}^*$ with $\limsup_{n\to\infty} \mu(C_n(\overline{\omega})) = 2\delta$ for some $\delta > 0$. In view of (5.2), the word $\overline{\omega}$ consists only of the letters from W. By telescoping, we may assume that $\mu(C_n(\overline{\omega})) \geq \delta$ for all levels n.

(II) Now let r = |W|, and observe that the word $\overline{\omega}$ has at least two identical letters. Suppose that one such pair appears at positions i and j, i < j. Set $u = \overline{\omega}[i, j - 1] = u_0 \dots u_p$, where $u_0, \dots, u_p \in W$ and p = j - i - 1. Define

$$C_n = \bigcup_{\ell=q_n(\overline{\omega}[0,i-1])}^{q_n(\overline{\omega}[0,i-1])+h_{u_0}^{(n)}-1} T^{\ell}\widehat{B}_n(\overline{\omega}).$$

In other words, the set C_n is constructed from the orbit of the set $\widehat{B}_n(\overline{\omega})$ when it intersects the tower corresponding to the letter u_0 at position *i*. Note that this is a disjoint union, C_n being a subtower of the tower over $X_{u_0}^{(n)}$.

Since $\overline{\omega}_i = \overline{\omega}_j = u_0$, we obtain that for all $n \ge 1$ and $\ell = 0, \dots, h_{u_0}^{(n)} - 1$,

$$T^{q_n(u)+q_n(\overline{\omega}[0,i-1])+\ell}\widehat{B}_n(\overline{\omega}) \subset T^\ell B_n(u_0).$$
(5.5)

We claim that

$$\limsup_{n \to \infty} \mu(C_n) > 0.$$

Indeed, by (5.4) and (5.3) we have

$$\mu(C_n) = h_{u_0}^{(n)} \mu(\widehat{B}_n(\overline{\omega})) \ge (h_{u_0}^{(n)}/q_n(\overline{\omega})) \mu(C_n(\overline{\omega})) \ge (h_{u_0}^{(n)}/q_n(\overline{\omega}))\delta,$$

and it remains to note that $h_{u_0}^{(n)}/q_n(\overline{\omega})$ is uniformly bounded from zero by Proposition 4.12.

By telescoping, we can assume that

$$\mu(C_n) \ge \gamma > 0$$

for some $\gamma = \gamma(\delta)$ and all $n \ge 1$.

(III) Choose a level n_0 such that $\mu(B_{n_0}(v)) < \gamma/2$ for all $v = 1, \ldots, d$. For each level $n \ge n_0$, there is a vertex v_n such that $B_n(u_0) \subset B_{n_0}(v_n)$. By telescoping we may assume that $v_n = v$ for all n. Set $D_n = C_n \cap B_{n_0}(v)$. We note that $\mu(B_{n_0}(v)) \ge \mu(B_n(u_0)) > 0$. It follows from (5.5) that

$$T^{q_n(u)}D_n \subset B_{n_0}(v)$$
 for all $n \ge 1$.

Thus, $D_n \subset B_{n_0}(v) \cap T^{-q_n(u)}B_{n_0}(v)$. As the Vershik map is aperiodic, we conclude that $q_n(u) \to \infty$ as $n \to \infty$. Thus, the theorem would be proved if we show that

$$\limsup_{n \to \infty} \mu(D_n) / \mu(B_{n_0}(v)) \ge \gamma, \tag{5.6}$$

because then for some $n = n_k \to \infty$ we will have

$$\mu(B_{n_0}(v) \cap T^{-q_n(u)}B_{n_0}(v)) \ge \mu(D_n) \ge (\gamma/2)\mu(B_{n_0}(v)) > \mu(B_{n_0}(v))^2.$$

(IV) By ergodicity of μ , the set Z_{u_0} of all paths which visit the vertex u_0 at infinitely many levels has measure 1, see the proof of Theorem 3.3. Hence by the pointwise ergodic theorem we may find a path $x \in Z_{u_0}$ such that

$$\frac{|\{-i_n(x) \le \ell \le j_n(x) : T^i(x) \in B_{n_0}(v)\}|}{h_{v_n(x)}^{(n)}} \to \mu(B_{n_0}(v)) \text{ as } n \to \infty,$$

where $v_n(x)$ is the vertex of the path x at the level n and the numbers $i_n(x)$, $j_n(x)$ are defined as in the proof of Proposition 4.10 (in particular, $i_n(x) + j_n(x) = h_{v_n(x)}^{(n)}$). Let $\mathcal{N} := \{n : v_n(x) = u_0\}$ which is infinite by assumption. Set $S_n = T^{q_n(\overline{\omega}[0,i-1])} \widehat{B}_n(\overline{\omega})$, so that $C_n = \bigcup_{\ell=0}^{h_{u_0}^{(n)}-1} T^{\ell} S_n$ and $\mu(C_n) = \mu(S_n) h_{u_0}^{(n)}$. Then we have for all $n \in \mathcal{N}$:

$$\begin{split} \mu(D_n) &= \frac{|\{\ell = 0, \dots, h_{u_0}^{(n)} - 1 : T^{\ell} B_n(u_0) \subset B_{n_0}(v)\}|}{h_{u_0}^{(n)}} \mu(S_n) h_{u_0}^{(n)} \\ &= \frac{|\{-i_n(x) \le \ell \le j_n(x) : T^{\ell}(x) \in B_{n_0}(v)\}|}{h_{u_0}^{(n)}} \mu(S_n) h_{u_0}^{(n)} \\ &\sim \quad \mu(B_{n_0}(v)) \mu(C_n) \ge \gamma \mu(B_{n_0}(v)) \text{ as } n \to \infty, \ n \in \mathcal{N}. \end{split}$$

This proves (5.6) and hence the theorem.

6 Extension of Measures from Simple Subdiagrams

In this section, we apply the results of preceding sections to find a simple algebraic criterium for the finiteness of the measure extension. Roughly speaking, the established criterium says that the extension of a measure from a subdiagram is finite if and only if the subdiagram contributes the same number of finite paths (up to a factor) as the entire diagram.

To illustrate this idea, we focus on the simplest case when the diagram B has only one minimal and one non-minimal component. In other words, B is assumed to be defined by a sequence of incidence matrices

$$F_n = \begin{pmatrix} D_n & 0\\ A_n & C_n \end{pmatrix}, \quad n \ge 1,$$

where the subdiagrams B(D) and B(C), formed by the incidence matrices D_n and C_n , are simple and A_n is non-zero. By construction, the minimal component of B corresponds to B(D) and the non-minimal one is defined by B(C). We also assume that these components are uniquely ergodic. This assumption allows us to use the asymptotics of the tower heights found in preceding sections. We note that the case when a finite rank diagram has the general form (3.2) (see Section 3) can also be studied in a similar way for every particular diagram by the methods we discuss below. We decided to restrict ourselves to the twocomponent case, since the complexity of the formulas is significantly increased with the number of components. The case under consideration clearly shows the essence of the construction.

Throughout this section, we denote by $V_n(D)$ and $V_n(C)$ the vertex sets of subdiagrams B(D) and B(C), respectively, $n \ge 1$. As above, we assume that B has single edges between the initial vertex v_0 and the vertices of the first level.

Suppose μ is a probability invariant measure on B(C). Denote by $\hat{\mu}$ the extension of measure μ to X_B . We will find purely algebraic conditions in terms of incidence matrices under which the extension $\hat{\mu}$ is a finite measure.

Let $A_i = (a_{v,u}^{(i)})$ and set

$$\alpha_{i} = \max\{a_{v,u}^{(i)} : v \in V_{i+1}(C), \ u \in V_{i}(D)\},\$$
$$\beta_{i} = \min\{a_{v,u}^{(i)} : v \in V_{i+1}(C), \ u \in V_{i}(D)\}.$$

Theorem 6.1. Let the Bratteli diagram B be as above. Suppose that the Bratteli subdiagrams B(C) and B(D) are uniquely ergodic.

(i) If

$$\sum_{i=1}^{\infty} \alpha_i \frac{||D_{i-1} \cdots D_1||_1}{||C_i \cdots C_1||_1} < \infty,$$

then the measure $\widehat{\mu}(X_B)$ is finite. (ii) If $\widehat{\mu}(X_B)$ is finite, then

$$\sum_{i=1}^{\infty} \beta_i \frac{||D_{i-1} \cdots D_1||_1}{||C_i \cdots C_1||_1} < \infty.$$

Proof. To prove the theorem it is enough, in view of Corollary 3.9, to check the convergence of the series

$$\sum_{i=1}^{\infty} \sum_{v \in W_{i+1}} \sum_{w \in W'_i} q_{v,w}^{(i)}, \tag{6.1}$$

where $W_i = V(C) \cap V_i$ and $W'_i = V(D) \cap V_i$.

We observe that it follows from the form of the diagram B that the heights $h_w^{(i)}$, for $w \in V(D)$, are completely determined by the products of the matrices $D_{i-1} \cdots D_1$. Hence, Proposition 4.8 implies that there are positive constants k_1 and k_2 such that

$$k_1 \le \frac{h_w^{(i)}}{||D_{i-1} \cdots D_1||_1} \le k_2$$

for all levels $i \ge 1$ and all $w \in V(D)$. On the other hand, by Proposition 4.12 the finiteness of the extension $\hat{\mu}$ is equivalent to the fact that there exist positive constants r_1 and r_2 such that for all $i \ge 1$ and $v \in V(C)$

$$r_1 \le \frac{h_v^{(i)}}{||C_{i-1}\cdots C_1||_1} \le r_2.$$

Then, for all $i \ge 1$ we have that

$$\sum_{v \in W_{i+1}} \sum_{w \in W'_i} q_{v,w}^{(i)} = \sum_{v \in W_{i+1}} \sum_{w \in W'_i} f_{v,w}^{(i)} \frac{h_w^{(i)}}{h_v^{(i+1)}}$$
$$\leq \alpha_i \frac{k_2 |W'_i| \cdot |W_{i+1}|}{r_1} \frac{||D_{i-1} \cdots D_1||_1}{||C_i \cdots C_1||_1}$$

Thus, statement (1) implies the convergence of (6.1) and, therefore, establishes the finiteness of the extension.

The statement (2) is proved analogously from the lower bound for the sum $\sum_{v \in W_{i+1}} \sum_{w \in W'_i} q_{v,w}^{(i)}$.

Corollary 6.2. If there are positive integers N_1 and N_2 such that $N_1 \leq \beta_i \leq \alpha_i \leq N_2$ for all $i \geq 1$, then

$$\mu(X_B) < \infty \iff \sum_{i=1}^{\infty} \frac{||D_{i-1} \cdots D_1||_1}{||C_i \cdots C_1||_1} < \infty.$$
(6.2)

Remark 6.3. (1) The condition $N_1 \leq \beta_i \leq \alpha_i \leq N_2$ $(i \geq 1)$ is equivalent to the property of finiteness of the set $\{A_i : i \geq 1\}$. In particular, this is the case when the matrices F_i are taken from a finite set of matrices (linearly recurrent case, which is discussed below).

(2) For any fixed sequences $\{D_i\}$ and $\{C_i\}$, the condition $\mu(X_B) = \infty$ can be obtained by an appropriate choice of matrices A_i .

(3) In the case of stationary diagrams, Corollary 6.2 gives another proof of the fact that the measure extension is finite if and only if the spectral radius of $C = C_n$ is strictly greater than that of $D = D_n$, see Theorem 4.3 in [BKM09]. *Example* 6.4. Let the diagram *B* be defined by the incidence matrices

$$F_n = \begin{pmatrix} \tau_n & 0\\ a_n & \omega_n \end{pmatrix}, \quad n \ge 1,$$

where the entries of F_n are positive integers (greater than one). For this diagram we can give the exact formulas for $\hat{\mu}(X_B)$ and values of measure on clopen sets.

Let μ be the probability measure defined by the odometer $\{\omega_i\}$. It can be easily shown that

$$\widehat{\mu}(X_B) = 1 + \sum_{i=1}^{\infty} a_i \frac{\tau_{i-1} \cdots \tau_1}{\omega_i \cdots \omega_1}.$$
(6.3)

(we skipped a routine computation). Suppose $\hat{\mu}(X_B)$ is finite and set $q = \sum_{i=1}^{\infty} a_i(\tau_{i-1}\cdots\tau_1)(\omega_i\cdots\omega_1)^{-1}$. To determine the values of measure $\hat{\mu}$ on clopen sets, we have to find the vectors $p^{(n)} = (p_{\tau}^{(n)}, p_{\omega}^{(n)})^T$ where the entries of $p^{(n)}$ are the measures of the corresponding the bases $B_n(\tau)$ and $B_n(\omega)$ of towers $X_{\tau}^{(n)}$ and $X_{\omega}^{(n)}$. Clearly, $p^{(1)} = (q, 1)^T$. To find $p^{(n)}$, we use the relation

$$A_n^{-1} \cdots A_1^{-1} p^{(1)} = p^{(n+1)}$$

2

where $A_i = F_i^T$. We have that

$$A_n^{-1}\cdots A_1^{-1} = \frac{1}{\omega_n\cdots\omega_1\tau_n\cdots\tau_1} \left(\begin{array}{cc} \omega_n\cdots\omega_1 & \sigma_n\\ 0 & \tau_n\cdots\tau_1 \end{array}\right),$$

where

 $\sigma_n = -\omega_n \cdots \omega_2 a_1 - \omega_n \cdots \omega_3 a_2 \tau_1 - \ldots - \omega_n a_{n-1} \tau_{n-2} \cdots \tau_1 - a_n \tau_{n-1} \cdots \tau_1.$

Thus,

$$A_n^{-1} \cdots A_1^{-1} \begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{q}{\tau_n \cdots \tau_1} + \frac{\sigma_n}{\omega_n \cdots \omega_1 \tau_n \cdots \tau_1} \\ \frac{1}{\omega_n \cdots \omega_1} \end{pmatrix}$$

Using the formula for σ_n we finally obtain that

$$p_{\tau}^{(n+1)} = \frac{1}{\tau_n \cdots \tau_1} \sum_{i=n+1}^{\infty} \frac{a_i \tau_{i-1} \cdots \tau_1}{\omega_i \cdots \omega_1}$$
$$p_{\omega}^{(n+1)} = \frac{1}{\omega_n \cdots \omega_1}$$

,

These formulas allow us to find the set of measure values on clopen sets.

We finish the example by considering a particular case of the studied class. Let now

$$F_n = \left(\begin{array}{cc} 2 & 0\\ 1 & \omega_n \end{array}\right), \quad n \ge 1$$

where $\omega_n = 2$ or 3. Let $k(n) = |\{1 \le i \le n : \omega_i = 3\}|$. By (6.3), we have that

$$\widehat{\mu}(X_B) = 1 + \sum_{i=1}^{\infty} \frac{2^{i-1}}{\omega_i \cdots \omega_1} = 1 + \frac{1}{2} \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{k(i)}.$$

Let $i_0 = 1, i_1, \ldots, i_n, \ldots$, be such numbers that

$$k(i) = n, \quad i_n \le i < i_{n+1}, \quad n = 0, 1, \dots$$

Then

$$\widehat{\mu}(X_B) = 1 + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n (i_{n+1} - i_n).$$
(6.4)

Relation (6.4) serves yields a number of sufficient conditions for finiteness of $\hat{\mu}(X_B)$. In particular, suppose that

$$i_{n+1} - i_n \leq K n^c, \quad K, c \in \mathbb{R}_+$$

for sufficiently large n. Then $\widehat{\mu}(X_B) < \infty$. Another example is obtained if $k(n)n^{-1} \to \alpha$ as $n \to \infty$ where $\alpha \in (0, 1)$. Then again $\widehat{\mu}(X_B)$ is finite.

We can extend the last part of the previous example to the case of diagrams which have finitely many different incidence matrices.

Definition 6.5. A Bratteli diagram is called *linearly recurrent* if it has a finitely many different incidence matrices.

Remark 6.6. Simple linearly recurrent diagrams were studied in the papers [CDHM03] and [DHS99]. These diagrams appeared there as Bratteli-Vershik models for minimal dynamical system whose time of recurrence behaves as a linear function. We should emphasize that for the needs of our paper the term "linearly recurrent" just means that the set of matrices is finite and we are not interested here in the time of recurrence.

Let B = (V, E) be a linearly recurrent Bratteli diagram with incidence matrices $\{F_n\}_{n\geq 1}$. Denote by \mathcal{A} the set of all different incidence matrices. Then the diagram B naturally defines a sequence $\omega \in \mathcal{A}^{\mathbb{N}}$ with $\omega_i = F_i$. It turns out that the growth rate of the product $||F_n \cdots F_1||_1$ heavily depends on the combinatorial properties of the sequence ω . The next proposition, which was essentially proved in [JB90], is a crucial step for getting estimates for the growth of matrix products.

Let R be a diagonal matrix with positive diagonal entries. Set

$$M(R) = \max_{i,j} R_{i,i} R_{j,j}^{-1}, \quad m(R) = \min_{i,j} R_{i,i} R_{j,j}^{-1}.$$

Then for any non-negative matrix A, we have the inequalities

 $m(R)||A||_1 \le ||R^{-1}AR||_1 \le M(R)||A||_1.$

For a positive vector x, denote by D_x the diagonal $d \times d$ matrix whose diagonal entries are the entries of x written in the same order. For two positive vectors x and y, denote by x/y their componentwise ratio, i.e., $x/y = (x_1/y_1, ..., x_d/y_d)$. For a vector x > 0, let x_{max} be the maximal entry of x and x_{min} the minimal one.

Proposition 6.7. Let A_1, \ldots, A_n be primitive matrices. Let x_i denote a Perron-Frobenius eigenvector for the matrix A_i and $\rho(A_i)$ its spectral radius. Then

$$\frac{||A_1A_2\cdots A_n||_1}{\rho(A_1)\rho(A_2)\cdots\rho(A_n)} \le \frac{1}{m(D_{x_n})} \left(\frac{x_n}{x_{n-1}}\right)_{\max} \cdots \left(\frac{x_2}{x_1}\right)_{\max} \left(\frac{x_1}{x_n}\right)_{\max}$$

and

$$\frac{||A_1A_2\cdots A_n||_1}{\rho(A_1)\rho(A_2)\cdots\rho(A_n)} \ge \frac{1}{M(D_{x_n})} \left(\frac{x_n}{x_{n-1}}\right)_{\min} \cdots \left(\frac{x_2}{x_1}\right)_{\min} \left(\frac{x_1}{x_n}\right)_{\min}$$

Proof. It was shown in the proof of Theorem 1 from [JB90] that

$$D_{x_n}^{-1} A_1 A_2 \dots A_n D_{x_n} 1$$

$$\leq \rho(A_1) \rho(A_2) \dots \rho(A_n) \left(\frac{x_n}{x_{n-1}}\right)_{\max} \dots \left(\frac{x_2}{x_1}\right)_{\max} \left(\frac{x_1}{x_n}\right)_{\max} \overline{1}$$

where $\overline{1} = (1, \ldots, 1)^T$. We note that

$$||A_1A_2\cdots A_n||_1 \le \frac{1}{m(D_{x_n})}||D_{x_n}^{-1}A_1A_2\cdots A_nD_{x_n}||_1$$

and then we use (2.5) to prove the first inequality. The second one follows from the proof of [JB90, Theorem 1] in a similar way by reversing the inequalities. \Box

Next, consider the sequence $\omega \in \mathcal{A}^{\mathbb{N}}$ defined by a linearly recurrent Bratteli diagram *B* as above. Let $I_A(n)$ be the number of occurrences of the letter *A* in the word $\omega_1 \omega_1 \dots \omega_n$. Let $\mathcal{A}^{(2)}$ be the set of all words of length two from the sequence ω . Denote by $I_{AB}(n)$ the number of occurrences of the pair *AB* in the word $(\omega_1 \omega_2)(\omega_2 \omega_3) \dots (\omega_n \omega_{n+1})$.

Definition 6.8. We will say that the linearly recurrent diagram *B* is *regular* if for every matrix $A \in \mathcal{A}$ and every pair $AB \in \mathcal{A}^{(2)}$ the limits

$$d(A) = \lim_{n \to \infty} \frac{I_A(n)}{n}, \quad d(AB) = \lim_{n \to \infty} \frac{I_{AB}(n)}{n}$$

exist. We call d(A) the density of A in ω and d(AB) the density of AB in the sequence $(\omega_1\omega_2)(\omega_2\omega_3)(\omega_3\omega_4)\ldots$

Let x_A be a Perron-Frobenius eigenvector of $A \in \mathcal{A}$. For any pair of matrices A and B with $AB \in \mathcal{A}^{(2)}$, denote by $\overline{r}(A, B)$ the ratio $(x_B/x_A)_{\text{max}}$. Similarly, we set $\underline{r}(A, B)$ to be the ratio $(x_B/x_A)_{\text{min}}$. Finally, we set

$$\overline{\rho}(\omega) = \prod_{A \in \mathcal{A}} \rho(A)^{d(A)} \times \prod_{AB \in \mathcal{A}^{(2)}} \overline{r}(A, B)^{d(AB)}$$

We refer to the number $\overline{\rho}(\omega)$ as the upper spectral radius along the sequence ω . The number $\rho(\omega)$ is defined similarly by using the values $\underline{r}(A, B)$.

The next lemma shows that $\overline{\rho}(\omega)$ and $\underline{\rho}(\omega)$ are well-defined and may serve as the upper and lower bounds for the products of incidence matrices, respectively.

Lemma 6.9. Let B be a regular linearly recurrent diagram with the sequence of primitive incidence matrices $\omega \in \mathcal{A}^{\mathbb{N}}$. Then

(1) $\overline{\rho}(\omega)$ and $\underline{\rho}(\omega)$ do not depend on the choice of eigenvectors x_A , $A \in \mathcal{A}$; (2) the following inequalities hold

$$\liminf_{n \to \infty} \left(||\omega_1 \omega_2 \dots \omega_n||_1 \right)^{\frac{1}{n}} \ge \underline{\rho}(\omega)$$

and

$$\limsup_{n \to \infty} (||\omega_1 \omega_2 \dots \omega_n||_1)^{\frac{1}{n}} \le \overline{\rho}(\omega).$$

Proof. (1) Let x_A be a Perron-Frobenius eigenvector of A and $x'_A = c_A x_A$, $c_A > 0$. For each n, define

$$\rho_n = \prod_{i=1}^n \rho(\omega_i) \cdot \overline{r}(\omega_i, \omega_{i+1}).$$

Let the number ρ'_n be defined similarly to ρ_n , but with the eigenvectors x_A and x_B replaced by x'_A and x'_B . Then, it is not hard to check that

$$\rho'_n = \frac{c_{w_{n+1}}}{c_{w_1}} \rho_n \text{ for all } n.$$

Since the set $\{c_A : A \in \mathcal{A}\}$ is finite, we get that

$$\lim_{n \to \infty} \left(\frac{\rho_n}{\rho'_n} \right)^{\frac{1}{n}} = 1.$$

On the other hand, we see that

$$(\rho_n)^{\frac{1}{n}} = \prod_{A \in \mathcal{A}} \rho(A)^{\frac{I_A(n)}{n}} \times \prod_{AB \in \mathcal{A}^{(2)}} \overline{r}(A, B)^{\frac{I_{AB}(n)}{n}} \to \overline{\rho}(\omega)$$

as $n \to \infty$. This shows that the definition of $\overline{\rho}(\omega)$ does not depend on the choice of Perron-Frobenius eigenvectors. The proof for $\rho(\omega)$ is similar and left to the reader.

(2) Using Proposition 6.7 and the fact that the set of matrices is finite, we can find a constant K > 0, which does not depend on n, such that

$$(||\omega_{1}\omega_{2}\dots\omega_{n}||_{1})^{\frac{1}{n}} \leq \left(K\prod_{i=1}^{n}\rho(\omega_{i})\cdot\overline{r}(\omega_{i},\omega_{i+1})\right)^{\frac{1}{n}}$$
$$= K^{\frac{1}{n}}\prod_{A\in\mathcal{A}}\rho(A)^{\frac{I_{A}(n)}{n}}\times\prod_{AB\in\mathcal{A}^{(2)}}r(A,B)^{\frac{I_{AB}(n)}{n}}$$
$$\to \overline{\rho}(\omega)$$

as $n \to \infty$. Thus, $\overline{\rho}(\omega) \ge \limsup_{n \to \infty} (||\omega_1 \omega_2 \dots \omega_n||_1)^{\frac{1}{n}}$. The other inequality is established in a similar way.

Let B be a regular linearly recurrent Bratteli diagram whose incidence matrices have the form

$$F_n = \left(\begin{array}{cc} D_n & 0\\ A_n & C_n \end{array}\right)$$

with D_n and C_n being primitive matrices. By definition of B, the sequences $\{D_n\}_{n\geq 1}$ and $\{C_n\}_{n\geq 1}$ have only finitely many different matrices.

The following theorem shows that the spectral radii along the sequences $\{D_n\}_{n\geq 1}$ and $\{C_n\}_{n\geq 1}$ can distinguish the growth rates of the minimal and non-minimal components of B. This, in particular, answers the question of finiteness of the measure extension from the subdiagram B(C).

Theorem 6.10. Let B be a regular linearly recurrent diagram as above.

(i) If $\underline{\rho}(\{D_n\}_{n\geq 1}) > \overline{\rho}(\{C_n\}_{n\geq 1})$, then the extension of the measure from B(C) is infinite.

(ii) $\overline{\rho}(\{D_n\}_{n\geq 1}) < \underline{\rho}(\{C_n\}_{n\geq 1})$, then the extension of the measure from B(C)is finite.

Proof. In view of Corollary 6.2 and Remark 6.3, it is sufficient to verify whether the series

$$\sum_{n=1}^{\infty} \frac{||D_{n-1}\cdots D_1||_1}{||C_n\cdots C_1||_1}$$

is convergent or not.

(i) Fix
$$\varepsilon > 0$$
 so that $\underline{\rho}(\{D_n\}_{n \ge 1}) - \varepsilon > \overline{\rho}(\{C_n\}_{n \ge 1}) + \varepsilon$. Set

$$r = \limsup_{n \to \infty} \left(\frac{||D_{n-1} \cdots D_1||_1}{||C_n \cdots C_1||_1} \right)^{\frac{1}{n}}.$$

Then, by Lemma 6.9, we get that

$$\sup_{n \ge k} \left(\frac{||D_{n-1} \cdots D_1||_1}{||C_n \cdots C_1||_1} \right)^{\frac{1}{n}} \ge \frac{\inf_{n \ge k} (||D_{n-1} \cdots D_1||_1)^{1/n}}{\sup_{n \ge k} (||C_n \cdots C_1||_1)^{1/n}} \\ \ge \frac{\underline{\rho}(\{D_n\}_{n \ge 1}) - \varepsilon}{\overline{\rho}(\{C_n\}_{n \ge 1}) + \varepsilon} \\ > 1$$

for all k large enough. This implies that r > 1 and, hence, the series diverges due to the root test.

The fact that the condition (ii) leads to the convergent series (where r < 1) is proved similarly.

Remark 6.11. (1) We observe that the statement (i) in Theorem 6.10 implies that the diagram B has a unique invariant measure supported by the minimal component only. On the other hand, the statement (ii) guarantees the existence of a fully supported invariant measure (along with the measure on the minimal component).

(2) We also note that it is possible to treat the numbers

$$\overline{\lambda}(\omega) = \limsup_{n \to \infty} (||\omega_1 \dots \omega_n||)_1^{\frac{1}{n}} \text{ and } \underline{\lambda}(\omega) = \liminf_{n \to \infty} (||\omega_1 \dots \omega_n||)_1^{\frac{1}{n}}$$

as the growth rate for matrix products. Then Theorem 6.10 still holds if we replace $\overline{\rho}(\omega)$ with $\overline{\lambda}(\omega)$ and $\rho(\omega)$ with $\underline{\lambda}(\omega)$.

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