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Successes and Failures in the Construction of NESS–States

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Successes and failures in the construction of NESS-states

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Abstract

We construct non-equilibrium states by coupling a large but finite system with quasifree evolution to two temperature baths of free fermions or bosons with different temperature. We are interested in the resulting behaviour of the temperature baths as well as in the consequences for the finite system, especially what happens in the limit when the finite system tends to infinity. As special example we consider the Kronig-Penney model and the tight binding model. Here a heat current remains that changes the baths but does not allow a limiting behaviour when the finite system tends to infinity. For the random versions of these models, especially the Anderson model the heat current disappears exponentially with the size of the system. We show that a limit state is attained that is independent of the time direction and does not show any kind of symmetry breaking.

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1 Introduction

We know how to construct equilibrium states in the thermodynamic limit as limit of Gibbs states. However we are also interested whether there exist other time invariant states, especially states that permit a heat flux. For quasifree states we can construct such states by considering quasifree states determined by a two point function corresponding to a one particle operator that commutes with the one particle Hamiltonian. If this operator breaks reflection symmetry it leads to a heat current. However these states are also invariant under space translation and we cannot obtain any temperature gradient that is related to the heat current.

The general idea to construct time invariant states as it is offered in [1], [2] is to start with an arbitrary state and take its (or some) invariant mean with respect to time evolution. To mimic a temperature gradient we can take as initial state some arbitrary state on a large but finite system coupled to two infinite heat baths of different temperature. In these heat baths the time evolution is free. For an appropriate local Hamiltonian and an appropriate coupling we can assume that scattering theory between the real time evolution and the uncoupled time evolution applies. Then the invariant means correspond to the limits $t \to \pm\infty$ and depending on the time direction we obtain two different time invariant states reflecting the existence of a heat current.

If the heat current corresponds to a temperature gradient we have to expect that it will decrease when the subsystem increases. If there is enough interaction in the subsystem one might expect that locally up to a negligible amount an equilibrium is obtained corresponding to a temperature that varies and leads to a temperature gradient of size $1/N$ if N is the length of the subsystem. Finally also the size of the heat current should be of size $1/N$.

Of course so far I do not know the tools how to handle such an interacting system. Therefore I reduce my research to systems I can control. These are those where also the increasing subsystem is quasifree and the calculations take place on the one particle level. This is still of some interest, because in general for these systems the limit is taken without any coupling to heatbaths, and one is just interested in the spectral properties in the thermodynamic limit. Now these spectral properties will give the main information on the limiting state, however the coupling to the heatbaths is still powerful enough so that we can apply scattering theory, where of course the time the system needs to reach its final state will in general increase with the size of the subsystem. The Hamiltonian corresponding to an infinite system with periodic interaction will determine which states are possible as time invariant states, but which of these permitted states will be reached will be determined by the heat baths.

A natural choice for such subsystems are the regular ones, that is the Kronig Penney model with a differential equation and the tight binding model with a difference equation. In both cases scattering theory works. The Kronig Penney model is more interesting insofar as some energy regions are forbidden and in these regions we have nearly reflection. In the permitted regions in both cases we have a heat current whose direction depends on the time direction. Inside of the system as inside of the heat baths we have space translation invariance on a microscopic scale, if we neglect the boundary region. The relevant scattering essentially only occurs at the boundary of the system to the heat baths, but we cannot get rid of the size of the system for increasing size but keep a heat current whose strength slightly fluctuates with the size of the system and does not tend to 0 when the size of the interacting systems tend to infinity.

The next examples are again the Kronig Penney model and the tight binding model, but now with random parameters and thus mimicking interaction. The tight binding model is known as Anderson model. These models and variations of them are well studied in the literature [3]. The main tool is Fuerstenbergs theorem [4], that guarantees, that for almost all models in this context the thermodynamic limit allows only a point spectrum and therefore a heatcurrent is not possible. But as long as the system is finite and coupled to the heat baths we still have an absolutely continuous spectrum and can apply scattering theory. As for the Kronig Penney model in the forbidden region in the thermodynamic limit the heat current will decrease exponentially with the size of the system, but now for all energies. But this does not imply that inside of the system we have a vacuum. But neither will we get a state that we can interpret as a state with locally varying temperature and therefore reflecting the mixing power of interaction. We obtain a unique state, now independent of the time direction, that is quasifree with a two point function which is a mean corresponding to the two temperatures and does not break the symmetry relation of the Hamiltonian. Neither can we speak of a temperature gradient as the state depends on space only insofar as the time evolution itself does not commute with space translation. Altogether we observe, that it is possible to construct time invariant states that reflect in some way the coupling to different temperature baths in quite different ways, but never in a way that we expect to happen in realistic models.

2 The models

We follow essentially the treatment in [3]. We only add that the system is infinite but outside of a finite region free. We arrange the free system in such a way that the total Hamiltonian has only absolutely continuous spectrum. In this way the state on the free part of the system determines in the long run also the state inside of the finite part where the random potential mimics interaction between the particles. More precisely we consider the following models represented by their one-particle Hamiltonian:

Model 1: Kronig-Penney model in a finite region

$$H = -\frac{d^2}{dx^2} + \sum_{n=-N}^{n=N} \lambda\delta(x-nl), \quad (-\infty < x < \infty) \quad \lambda > 0 \tag{1}$$

Model 2: Kronig-Penney model with varying strength and distance

$$H = -\frac{d^2}{dx^2} + \sum_{n=-N}^{n=N} \lambda_n\delta(x-l_n), \quad (-\infty < x < \infty) \quad \lambda_n - \lambda_{n-1} > 0 \tag{2}$$

Model 3: Tight binding electron model: We describe the Hamiltonian as quadratic form, so that it is evident that it is positive definite:

$$< \Psi|H|\Psi >= \sum_{|n|>N} |\Psi_n - \Psi_{n-1}|^2 + \sum_{|n|\leq N} (\alpha|\Psi_n - \Psi_{n-1}|^2 + \beta_n|\Psi_n|^2) \tag{3}$$

where $\alpha < 1$ and $4\alpha + \beta < 4, \quad 0 \leq \beta_n = \beta \quad \forall -N \leq n \leq N-1, \quad \beta_N = 0.$

Model 4: Anderson model

$$< \Psi|H|\Psi >= \sum_{|n|>N} |\Psi_n - \Psi_{n-1}|^2 + \sum_{|n|\leq N} (\alpha|\Psi_n - \Psi_{n-1}|^2 + \beta_n|\Psi_n|^2) \tag{4}$$

where now $0 \leq \beta_n$ can vary but still has to satisfy $4\alpha + \beta_n < 4$.

All these Hamiltonians are selfadjoint with absolutely continuous spectrum and can be characterized by their generalized eigenfunctions. We first concentrate on the models 1 and 2. Following [3] we look for the generalized eigenfunctions. In the interval $l_n < x < l_{n+1}$ we write them as $\Psi(x) = A_n cos(kx) + B_n sin(kx)$ so that the values A_n, B_n are calculated by the transfer matrix T_n

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = T_n \begin{pmatrix} A_n \\ B_n \end{pmatrix}$$

$$T_n = \begin{bmatrix} cos(k((l_{n+1} - l_n)) & \frac{1}{k} sin(k(l_{n+1} - l_n)) \\ -k sin(k(l_{n+1} - l_n)) + \lambda_n cos(k(l_{n+1} - l_n)) & cos(k(l_{n+1} - l_n)) + \frac{\lambda_n}{k} sin(k(l_{n+1} - l_n)) \end{bmatrix} \tag{5}$$

where we can take $l_{n+1} - l_n = 1 \quad \forall |n| > N$. The choice how to describe the eigenfunctions is done in such a way that the transfer matrix T_n is real with determinant 1 so that we can apply Fuerstenbergs theorem [4]. Since the determinant is $\neq 0$ eigenfunctions cannot have finite support. The energy corresponding to the eigenfunctions is $E(k) = k^2$. The Hamiltonian is selfadjoint and by our condition on λ positive definite, therefore k has to be real both in model 1 and 2.

Model 3 and 4 have essentially the same structure. Our choice of parameters follows from the fact that

$$0 \leq < \Psi|H_0|\Psi > = \sum_n |\Psi_n - \Psi_{n-1}|^2 \leq 4 \sum_n |\Psi_n|^2 \tag{6}$$

and therefore the energy spectrum is $[0, 4]$. The energy spectrum of H should coincide with the energy spectrum of H_0. This follows on one hand because for functions localized outside of a finite region the expectation values coincide and therefore the range of the spectrum can only be larger. That this does not happen is guaranteed by

$$0 \leq \alpha < \Psi|H_0|\Psi > \leq < \Psi|H|\Psi > \leq$$

$$\sum_{|n|>N} (|\Psi_n| + |\Psi_{n-1}|)^2 + \sum_{|n|\neq N} \alpha((|\Psi_n| + |\Psi_{n-1}|)^2 + \beta_n |\Psi_n|^2) \leq 4 \sum_n |\Psi_n|^2 \tag{7}$$

The quadratic form of the Hamiltonian corresponds to the operator

$$(H\Psi)_n = -\alpha_{n+1}\Psi_{n+1} - \alpha_n \Psi_n + (\alpha_n + \alpha_{n-1} + \beta_n)\Psi_n \tag{8}$$

Now the sequence $\{\Psi_n\}$ corresponding to a generalized eigenfunction of the Hamiltonian is determined by the transfer matrix

$$\begin{pmatrix} \Psi_{n+1} \\ \Psi_n \end{pmatrix} = \begin{bmatrix} \frac{\alpha_n + \alpha_{n-1} + \beta_n - E}{\alpha_{n+1}} & -\frac{\alpha_n}{\alpha_{n+1}} \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \Psi_n \\ \Psi_{n-1} \end{pmatrix} \tag{9}$$

As in model 1 and 2 the transfer matrix is real valued and has determinant $\neq 0$, so that eigenfunctions do not have compact support and therefore do not vanish in a region where the Hamiltonian acts as the free Hamiltonian. For the permitted energy values the absolute value of the eigenfunctions cannot decrease in the region without interaction. Therefore proper eigenfunctions are excluded and the spectrum is determined by the behaviour outside of a finite region and is absolutely continuous. For the model 4 we will apply Fuerstenbergs theorem for random sequences. This demands that the transfer matrix has to be real what is satisfied but also that it has to have determinant 1. This makes it necessary that $\alpha_n = \alpha_{n+1}$ except at the transition points $|n| = N$. At these transition points we have to accept that $\alpha_n \neq \alpha_{n+1}$, otherwise we cannot satisfy $4\alpha_n + \beta_n < 4$ and cannot exclude bound states. Thus the transition matrix at these single points has determinant $\neq 1$, but also $\neq 0$ and can chance details of the scattering, but not create transfer if it is otherwise forbidden.

3 Scattering theory

The idea how to obtain NESS-states is based on the application of scattering theory. As suggested in [1], [2] we consider a finite system with some local Hamiltonian coupled to two heat baths with different temperature. We take all systems to be onedimensional. Therefore we have as algebra the algebra built by bosonic or fermionic creation and annihilation operators over the Hilbertspace

$$L^2(-\infty, -N) \oplus L^2[-N, N] \oplus L^2(N, \infty) \tag{10}$$

where the Hilbertspaces L^2 are either over the continuum (model 1 and 2) or over the lattice (model 3 and 4). The initial state is assumed to decompose into

$$\omega = \omega_{(-\infty,-N)}(\beta_-) \otimes \omega_{[-N,N]} \otimes \omega_{(-N,\infty)}(\beta_+) \tag{11}$$

There is no need to specify how the state looks like in the region $[-N, +N]$ apart that we assume that it is quasifree. Therefore the state is invariant under some Hamiltonian which corresponds to a Hamiltonian on the one particle level of the form

$$\hat{H} = H^0_{(-\infty,-N)} \oplus H_{[-N,N]} \oplus H^0_{(N,\infty)} \tag{12}$$

where H^0 is the free Hamiltonian in the restricted regions corresponding to the four models with appropriately chosen boundary conditions. The spectrum of $\hat{H}$ contains an absolutely continuous part from the contribution of the heat baths and a pure point spectrum corresponding to the finite system. The time evolution on the algebra determined by this one particle Hamiltonian we denote with $\hat{\tau}_t$, whereas the real time evolution τ_t is determined by the one particle Hamiltonians corresponding to the models 1 to 4. The state (11) will evolve in time and tend to the final NESS-state

$$\omega_\infty(A) = \lim_{t\to\infty} \omega(\hat{\tau}_{-t} \circ \tau_t A) \tag{13}$$

if we can show that scattering theory between the automorphisms applies. Since they are quasifree it suffices if it works between the two one particle Hamiltonians.

The state is fixed by the two point function

$$\omega(a^*(f)a(g)) =< g|\hat{\rho}|f >, \quad [\hat{H}, \hat{\rho}] = 0 \tag{14}$$

Therefore

$$\omega_\infty(a^*(f)a(g)) = \lim_{t\to\infty} < g|e^{iHt}e^{-i\hat{H}t}\hat{\rho}e^{i\hat{H}t}e^{-iHt}|f > \tag{15}$$

Thus our only task is to evaluate the wave operator $\Omega = \lim_{t\to\infty} e^{i\hat{H}t}e^{-iHt}$. H has absolutely continuous spectrum and so has $\hat{H}$ apart from finitely many bound states. Therefore the limit exists in the strong sense and has range on the absolutely continuous part of $\hat{H}$. The adjoint therefore converges weakly which is sufficient for the existence of the limit in (15). In complete generality the wave operator can be expressed as integral kernel in terms of the generalized eigenfunctions Ψ corresponding to H and Φ corresponding to $\hat{H}$.

$$\Omega(x,y) = \int_0 dk \Phi_+(x,k)\bar{\Psi}_+(y,k) + \int_0 dk \Phi_-(x,k)\bar{\Psi}_-(y,k) \tag{16}$$

The upper limit in the integral is $+\infty$ in model 1 and 2 and 2 in model 3 and 4 according to the energy range of the absolutely continuous spectrum. According to (12) $\Phi_+(x,k) = \chi_{(N,\infty)}(x) sin(k(x-N)), \Phi_-(x,k) = \chi_{(-\infty,-N)}(x) sin(k(x+N))$ are the generalized eigenfunctions corresponding to $\hat{H}$, the other eigenfunctions are proper eigenfunctions and do not contribute to the wave operator. $\Psi_\pm(y,k)$ are the generalized eigenfunctions corresponding to H where $\pm$ indicates whether the waves are going to the right or to the left. More precisely they have the form

$$\Psi_+(x,k) = e^{ikx} + a(k)e^{-ikx}, \quad -\infty < x < -N, \quad \Psi_+(x,k) = b(k)e^{ikx}, \quad N < x < \infty, \quad k > 0 \tag{17}$$

$$\Psi_-(x,k) = c(k)e^{-ikx}, \quad -\infty < x < -N, \quad \Psi_-(x,k) = e^{-ikx} + d(k)e^{ikx}, \quad N < x < \infty, \quad k > 0 \tag{18}$$

where we do not specify how they look like in the interval $-N < x < N$, though from there the values $a(k), b(k), c(k), d(k)$ have to be calculated.

They are complete in the sense that

$$\int_0 dk[\bar{\Psi}_+(x,k)\Psi_+(y,k) + \bar{\Psi}_-(x,k)\Psi_-(y,k)] = \delta(x-y) \tag{19}$$

and they are orthogonal in the sense that

$$\int dx \bar{\Psi}_+(k,x)\Psi_+(q,x) = \delta(k-q); \quad \int dx \bar{\Psi}_+(k,x)\Psi_-(q,x) = 0 \tag{20}$$

respectively replacing appropriately the integral by a sum in models 3 and 4. That the wave operator has this form can be seen by the ansatz

$$\begin{aligned}\Omega(x,y) = \lim_{t\to\infty}[&\int_0 dk \int_0 dq e^{it(E(q)-E(k))}\bar{\Phi}_+(x,k) \int dz \Phi_+(z,k)\bar{\Psi}_+(z,q)\Psi_+(y,q) + \\ &+ \int_0 dk \int_0 dq e^{it(E(q)-E(k))}\bar{\Phi}_+(x,k) \int dz \Phi_+(z,k)\bar{\Psi}_-(z,q)\Psi_- y,q) + \\ &+ \int_0 dk \int_0 dq e^{it(E(q)-E(k))}\bar{\Phi}_-(x,k) \int dz \Phi_-(z,k)\bar{\Psi}_+(z,q)\Psi_+(y,q) + \\ &+ \int_0 dk \int_0 dq e^{it(E(q)-E(k))}\bar{\Phi}_-(x,k) \int dz \Phi_-(z,k)\bar{\Psi}_-(z,q)\Psi_-(y,q)]\end{aligned}$$

and evaluating

$$\lim_{t\to\infty} e^{it(E(q)-E(k))} \int dz \Phi_\pm(z,k)\bar{\Psi}_\pm(z,q) = \delta(k-q)\delta(\pm)$$

where $E(k), E(q)$ are the energies of the generalized eigenfunctions characterized by k, q. For the models 1 and 2 this reduces to concentrating on Gauss functions (compare [5]), for the models 3 and 4 it follows from the convexity of $E(k)$ together with the distribution properties of $\lim_{y\to\pm\infty}\sum_{x=0}^{\infty}e^{ik(y+x)}$. Only the asymptotic behaviour of the generalized eigenfunctions contributes to the integral. Local parts do not contribute in the limit by Riemann-Lebesgue. The essential contribution has the form

$$\lim_{t\to+\infty}\lim_{N\to\infty}\sum_{n=M}^{N}e^{i(k-q)n-it(cosk-cosq)}=$$

$$=\lim_{t\to+\infty}\lim_{N\to\infty}\frac{e^{i(k-q)(N-t\frac{sin(k/2-q/2)}{k-q}sin(k/2+q/2))}-e^{-it(k-q)\frac{sin(k/2-q/2)}{k-q}sin(k/2+q/2)}}{e^{i(k-q)}-1}e^{i(k-q)M}=$$

$$=\delta(k-q)$$

$$\lim_{t\to-\infty}\lim_{N\to\infty}\sum_{n=M}^{N}e^{i(k-q)n-it(cosk-cosq)}=$$

$$=\lim_{t\to-\infty}\lim_{N\to\infty}\frac{e^{i(k-q)(N-t\frac{sin(k/2-q/2)}{k-q}sin(k/2+q/2))}-e^{-it(k-q)\frac{sin(k/2-q/2)}{k-q}sin(k/2+q/2)}}{e^{i(k-q)}-1}e^{i(k-q)M}=0$$

where we notice that in the permitted region of k,q $\frac{sin(k/2-q)/2}{k-q}sin(k/2+q/2)$ is positive and bounded. The arguments for the Kronig Penney model are essentially the same. Completeness and orthogonality is then a consequence of applying scattering theory with respect to the free evolution where both properties are well established. Notice that completeness when examined for x,y both in the left or the right region implements

$$|a(k)|^2+|c(k)|^2=1,\quad |b(k)|^2+|d(k)|^2=1 \tag{21}$$

With our ansatz (11) the limit state will therefore read

$$\omega_\infty(a^*(f)a(g))=<g|P_+\frac{1}{e^{\beta_+H}\pm 1}+P_-\frac{1}{e^{\beta_-H}\pm 1}|f> \tag{22}$$

depending whether we consider fermions or bosons. Here $P_\pm$ are the projections on the generalized eigenfunctions $\Psi_\pm$. Therefore we can evaluate the heat current, that is determined by the transition values $|b(k)|^2$ and $|c(k)|^2$. We are interested how this heat current depends on N. But in addition we are interested how the effect of $P_\pm$ is reflected locally and in which sense we can talk of a thermodynamic limit $N\to\infty$.

4 The thermodynamic limit

We can calculate the heat current in the left and right heat baths by

$$j_l=\int_0 dkk(\rho_{\beta_+}(k)(1-|a(k)|^2)-\rho_{\beta_-}(k)|c(k)|^2)$$

$$j_r=\int_0 dkk(\rho_{\beta_+}(k)|b(k)|^2-\rho_{\beta_-}(k)(1-|d(k)|^2). \tag{23}$$

Evidently from (21) it vanishes for equal temperature. From symmetry relations of the Hamiltonian it follows that it is the same in the right and left region. However the state in the heat baths will depend on N in the sense that $a(k)$ and $d(k)$ in (17) can still depend on N and therefore also the generalized eigenfunctions will depend on N. How they depend in detail on N in the various models we have to clarify.

In the interacting region we are especially interested how the state looks like far away from the boundary. First we have to argue that the dynamics will converge to a limit. This happens if H_N converges to H_∞ in the strong resolvent sense. This can be seen by the following arguments:

Take a sequence of projections P_L such that $< \Psi|P_L H_N P_L|\Psi >=< \Psi|P_L H_{\bar{N}} P_L|\Psi >$ for $N < \bar{N}$. Therefore also for $z > 0$

$$< \Psi|P_L \frac{1}{P_L H_N P_L + z} P_L|\Psi >=< \Psi|P_L \frac{1}{P_L H_{\bar{N}} P_L + z} P_L|\Psi > \tag{24}$$

Now we can apply

$$P\frac{1}{A}P = P\frac{1}{PAP}P + P\frac{1}{A}(1-P)\frac{1}{(1-P)\frac{1}{A}(1-P)}(1-P)\frac{1}{A}P.$$

Exponential decay of the kernels corresponding to unitaries and resolvents of the relevant operators are studied in detail in the literature (e.g. [6]). Therefore we know that for every ϵ we can find L_0 such that

$$||P_L \frac{1}{H_N}(1-P_M)|| < \epsilon \forall (M-L) > L_0, N > M \tag{25}$$

This fact allows us to conclude that

$$\lim_{N\to\infty} < \Psi|P_L \frac{1}{H_N + z} P_L|\Psi >=< \Psi|P_L \frac{1}{H_\infty + z} P_L|\Psi > \tag{26}$$

With increasing N also P_L can be chosen to increase to 1. From (26) we have weak resolvent convergence. Estimating for $L - M > L_0, L < N$ and $||\Psi|| < \epsilon + ||P_M\Psi||$

$$||(P_L + (1-P_L))(\frac{1}{H_\infty + z} - \frac{1}{H_N + z})P_L\Psi|| \leq ||(1-P_L))(\frac{1}{H_\infty + z} - \frac{1}{H_N + z})P_M\Psi|| + \epsilon$$

with using once more (24) and (25) we can strengthen the result to strong resolvent convergence. That permits that we can take functions of H_N such that the limit of their local expectation values exist and can be written as

$$\lim_{N\to\infty} < f|\rho(H_N)|g >=< f|\rho(H_\infty)|g > \tag{27}$$

where H_∞ corresponds to our models 1,2 3,4 with N replaced by ∞.

The systems corresponding to H_∞ in the various models are well analyzed in the literature, also in broader context, e.g. in [6] or more recently [7]. Especially the clustering behaviour (25) is well under control. If H_∞ has a continuous spectrum as in model 1 and 3 (25) is a consequence of Riemann Lebesgue in essentially the same way as for free systems. If the spectrum is pure point, then it is a consequence of the exponential decay of the eigenfunctions. Detailed analysis even for more general models can be found in [6]. We examine the consequences for the different models with the main interest how far the models allow to take the limit $N \to \infty$ in

$$\lim_{N\to\infty}\lim_{t\to\infty} < g|e^{iH_N t}e^{-i\hat{H}_N t}\hat{\rho_N}e^{i\hat{H}_N t}e^{-iH_N t}|f > .$$

Model 1 is the Kronig Penney model. The transfer matrix at every point is the same. For given energy parameter $E(k) = k^2$ its eigenvalues are either $e^{\pm i\kappa}$ satisfying

$$cos(\kappa) = cos(kl) + \frac{\lambda}{2k} sin(kl) \tag{28}$$

or they are $e^{\pm\kappa}$ satisfying

$$Cosh(\kappa) = cos(kl) + \frac{\lambda}{2k} sin(kl)$$

For the infinite systems Bloch's theorem tells us that only the first case is permitted and we are reduced to energy bands. For finite N we conclude that the generalized eigenvalues corresponding to the forbidden regions vanish exponentially fast at the boundaries $x = \pm N$. This means especially that $b(k), c(k)$ are of order $e^{-\kappa N}$ and the corresponding current vanishes exponentially fast. For the remaining part however we cannot get rid of the N-dependence. The easiest way to see this is to write the transfer matrix in bra-ket notation

$$T^{(n)} = \prod_{N}^{n+N} T_k = T^{(n+N)} = e^{i\kappa(n+N)}|\Psi_+ >< \Phi_+| + e^{-i\kappa(n+N)}|\Psi_- >< \Phi_-| \tag{29}$$

where $|\Psi_\pm >$ are the eigenfunctions of T and $|\Phi_\pm >$ are the eigenfunctions of T^* corresponding to the eigenvalues $e^{\pm i\kappa}$ and normalized such that $< \Phi_\pm|\Psi_\pm >= 1, < \Phi_\mp|\Psi_\pm >= 0$. $b(k)$ is then calculated from (17) via $T^{(N)}$ and therefore

$$\left\langle \frac{1}{i} |(e^{i\kappa 2N}|\Psi_+ >< \Phi_+| + | + e^{-i\kappa 2N}|\Psi_- >< \Phi_-|)| \frac{1}{i} \right\rangle b(k) = 1. \tag{30}$$

This shows that $|b(k)|$ and therefore the contribution to the heat current (23) will depend on N without tending to any limit. Transfer in the forbidden region vanishes exponentially fast with N, here we can take the thermodynamic limit. In the region in which energy is transferred the amount will however oscillate in N. This has a consequence both for the heat baths as for the interacting system. We can get rid of the dependence on N only by averaging over a region $N \pm L$ where $L \ll N$ but tends to infinity. More precisely we can consider the limit

$$\bar{\omega}(a^*(f)a(g)) = \lim_{N\to\infty} \frac{1}{2N^{1/2}} \sum_{c=-N^{1/2}}^{c=N^{1/2}} \omega_{N+c}(a^*(f)a(g)). \tag{31}$$

In this way a finite heat current will remain. According to our remarks on the thermodynamic limit (25),(26), as long as we stay away from the boundary, both inside of the baths as inside of the interior of the interacting region the state will be invariant under the time evolution (considered as time evolution in the sense of the thermodynamic limit) permitting a heat current and being not reflection invariant. We average among states that are time invariant with respect to different time evolutions, in this sense we cannot talk about time invariance. If however we take into account that (26) holds far away from the boundary the dependence on N disappears and the time evolutions coincide so that in this limit away from the boundary the state has to be considered time and space translation invariant but neither extremely space translation invariant nor extremely time translation invariant.

Model 3 is called "Tight binding electron model" in [3]. Again we have to calculate the eigenvalues of the transfer matrix (9)

$$cos(\kappa) = cos(\frac{2\alpha_n + \beta_n - E}{2\alpha_n}) \tag{32}$$

$$Cosh(\kappa) = Cosh(\frac{2\alpha_n + \beta_n - E}{2\alpha_n})$$

depending whether $\frac{2\alpha_n+\beta_n-E}{2\alpha_n}$ is smaller or larger than 1. The final reflection coefficient is slightly more complicated than (30) because there is the additional term when α_n, β_n change. However as before for $\frac{2\alpha_n+\beta_n-E}{2\alpha_n} > 1$ the transfer coefficient decays exponentially fast with N whereas for $\frac{2\alpha_n+\beta_n-E}{2\alpha_n} < 1$ the transfer coefficient is positive but fluctuates with N because again (30) gives the correct N-dependence of the matrix, only the vectors have to be adjusted. Therefore the conclusions for the heat current and the limiting state remain unchanged.

Our main interest lies on models 2 and 4. If the parameters are sufficiently random then we will argue that the transfer coefficient will vanish exponentially fast. Therefore the state for the heat baths sufficiently away from the boundary will not change and we do not obtain a NESS-state there. At the boundary the coupling to the Kronig Penney model respectively to the Anderson Model will affect the boundary conditions, but due to the exponential decay these boundary conditions do not affect the state far away from the boundary.

Inside of the interacting region again there will be no heat current. Nevertheless according to (22) the projection operators for finite N $P_\pm$ remain with different weights. However they depend on N, and again we have to clarify what are the consequences of this $N-$dependence in the thermodynamic limit.

The main tool in the analysis is Fuerstenbergs theorem:

Theorem 1. *Fuerstenberg theorem*

Let μ be a measure on $SL(2,R)$ which is the group of 2-dimension unimodular matrices transforming the real vector space R^2 into itself. Let G be the smallest subgroup of $SL(2,R)$ containing the support of μ. Assume that G is noncompact and no subgroup of G of finite index is reducible. Let $\{T_n; n = 1, 2, ...\}$ denote the sequence of mutually independent G-valued random variables with the common distribution μ. For every vector $|\Psi>$ with $||\Psi|| = 1$

$$\lim_{n\to\infty} \frac{1}{n} log||T_n T_{n-1}..T_1\Psi|| = \gamma > 0 \tag{33}$$

with probability 1 where γ depends only on μ.

Notice that the conditions on G are satisfied if it contains at least two elements of $SL(2,R)$ with no common eigenvector. Therefore varying λ_n, l_n in model 2 or α, β_n in model 4 we can easily meet the conditions. It follows that (29), the transfer matrix $T^{(N)} = \Pi T_N ... T_1$ between the incoming and outgoing wave becomes for almost all sequences $\{\lambda_n, l_n\}, \{\alpha, \beta_n\}$

$$T^{(N)} = e^{\gamma N}|\Psi_{+,N}><\Phi_{+,N}| + e^{-\gamma N}|\Psi_{-,N}><\Phi_{-,N}| \tag{34}$$

where $|\Psi_{\pm,N}>, |\Phi_{\pm,N}>$ are also random vectors and γ is determined by the sequence. If the vector corresponding to the incoming wave is not orthogonal to $|\Phi_{+,N}>$ then its weight $b(k)$ has to be exponentially small. That it is orthogonal happens with vanishing probability on N, according to (33). Altogether the heat current will decrease exponentially fast with N. At the boundary the phase relations between the incoming and the reflected wave may change, however this does not affect the state of the heat bath far away from the boundary.

Considering the state inside of the interacting region we have to be more careful than for the regular Kronig Penney model or the regular tight binding model. Remember that also inside the interaction region (19) has to hold. Whereas in the previous case we could concentrate on the scattered part, i.e. the part of the continuous spectrum and the eigenfunctions corresponding to the forbidden region contributed only to an exponentially negligible amount, now the continuous part does not exist. Therefore we remain after having taken the limit $t \to \infty$ with

$$\omega_{\infty,N}(a^*(f)a(g)) =< g|P_{+,N}\frac{1}{e^{\beta_+ H_N} \pm 1} + P_{-,N}\frac{1}{e^{\beta_- H_N} \pm 1}|f > \tag{35}$$

We can concentrate on local f, g so that in the thermodynamic limit H_N can be replaced by H_∞, but we still have to worry about the dependence on $P_{\pm,N}$. The generalized eigenfunctions of the Hamiltonian H_N are twofold degenerate: with $\Psi(x)$ resp. Ψ_n also $\bar{\Psi}(x)$ or $\bar{\Psi}_n$ is an eigenfunction. These eigenfunctions cannot be real, as can be seen in the free regime. Therefore we have degeneracy, however the decomposition $P_{\pm,N}$ highly depends on N. As stated we have strong resolvent convergence of H_N. To control (35) it is sufficient to argue that for localized $|f>$ we can take the weak limit

$$\lim P_{\pm,N}|f >= \frac{1}{2}|f > \tag{36}$$

Proof: We use the fact that H_N is two fold degenerate. We can write

$$< f|P_{\pm,N}g(H)|f >= \int dE|f(E,\omega_0)\Psi_{\pm,N}(E)|^2 g(E) \tag{37}$$

where $\Psi_{\pm,N}(E)$ is the generalized eigenfunction corresponding to the energy E and $f(E,\omega_0)$ corresponds to the spectral decomposition of f where we indicate by ω_0 its dependence on the fixed sequence. However we know from resolvent convergence that its dependence on the part of the sequence sufficiently far apart can be ignored. Therefore we can replace it by

$$f(E,\omega_0) = \int d\mu_r(\omega) f(E,\omega) \tag{38}$$

where we integrate over all sequences restricted to those that coincide with the fixed sequence in a neighborhood of the region in which f is located. Now we can use the fact that the sequences are random and the measure $d\mu_r(\omega)$ is ergodic. Therefore we can interpret (37) as the expectation value of projections in a two dimensional Hilbert space averaged by the ergodic measure to a c-number, so that we can argue that in (37) the dependence on $\pm$ disappears and therefore (36) holds.

From (35) and (36) it follows that

$$\omega_\infty(a^*(f)a(g)) = \lim_{\Delta\to\infty} < g|P_{+,N}\frac{1}{e^{\beta_+ H_N} \pm 1} + P_{-,N}\frac{1}{e^{\beta_- H_N} \pm 1}|f >=$$

$$\frac{1}{2} < g|\frac{1}{e^{\beta_+ H_\infty} \pm 1} + \frac{1}{e^{\beta_- H_\infty} \pm 1}|f > \quad \forall f, g, \Delta = Distance[supp(f,g), N] \tag{39}$$

since $\Delta \to \infty$ implies $N \to \infty$. Therefore after taking the limit $t \to \infty$ we can also take the limit $N \to \infty$ and obtain

$$\omega_\infty = \frac{1}{2} < g|\frac{1}{e^{\beta_+ H_\infty} \pm 1} + \frac{1}{e^{\beta_- H_\infty} \pm 1}|f > \tag{40}$$

Evidently the state that finally is reached when we couple the system to two heat baths with different temperature is not a temperature state nor does it carry a heat current. But there is also no symmetry breaking that reflects the orientation and location in space. We have no possibility to assign something like a temperature gradient to the system. The random interaction with the background cannot replace the random interaction between the particles themselves that should produce a temperature gradient.

5 Conclusion

We studied the effect of coupling a system to two heat baths of different temperature in the thermodynamic limit. The coupled systems were quasifree, either translation invariant or with random one particle Hamiltonians. As a consequence in the thermodynamic limit the spectra were either absolutely continuous or pure point. In the former case they permit a heat current that depends on the temperature of the heat baths, but also on the size of the system, so that it does not allow a thermodynamic limit. In the latter case no heat current remains, though the final state of the system is still determined by the heat baths but without any kind of symmetry breaking.

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