ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Boltzmanngasse 9
A-1090 Wien, Austria

A New Proof of the Newlander–Nirenberg Theorem

Joachim Michel

Vienna, Preprint ESI 2239 (2010)

April 5, 2010

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available online at http://www.esi.ac.at

A NEW PROOF OF THE NEWLANDER-NIRENBERG THEOREM

JOACHIM MICHEL

Abstract. The proof of the Newlander-Nirenberg theorem is given by integral formula methods and a functional analytic argument on continuous linear operators with small operator norms. No Nash-Moser iteration is needed. These formulae might be also useful for other applications.

1. Introduction

We give a new proof of the following theorem of S. Webster [8]. [1]

Theorem 1.1. *In a neighborhood of* $0 \in \mathbb{C}^n$, $n \geq 2$, *let be given an almost complex integrable structure of class* $C^{r+\lambda}$, *where* $r \geq 1$ *is an integer and* $0 < \lambda < 1$. *Then there exists a coordinate map of class* $C^{r+1+\lambda}$ *in a neighborhood of* 0 *which is holomorphic with respect to the structure.*

Webster's proof is based on the Nash-Moser Newton method which necessitates an infinite iteration procedure by controlling the convergence with respect to the desired $C^{r+\lambda}$-norms of approximate solutions. Convergence is achieved by shrinking in each step the domains where the approximate solutions are given. In our approach we show that the norm of the error term of the approximate solution operator is smaller than 1 if a certain parameter τ is chosen sufficiently small. Here we work on a fixed domain of definition. Moreover, our approach can be generalized to some extent to non-integrable complex structures. So we hope to give interesting applications of our method in further articles. In the proofs we work with approximate solution operators. These are modifications of the well-known solution operators of Cauchy-Fantappiè type for the $\overline{\partial}$-equation in $\mathbb{C}^n$. They are constructed by taking in each point p the classical integral kernels derived from the frozen complex structure at p. The main problem is to control the norms of the integrals up to the boundary without any loss of regularity. This causes some technical problems since even the so-called Bochner-Martinelli volume integral, which is trivial in $\mathbb{C}^n$, needs an elaborate reasoning.

2000 *Mathematics Subject Classification.* 32000.

Key words and phrases. Integrable almost complex structures, integral formulas, $\overline{\partial}$ problem.

[1] The paper was finished during the author's stay at ESI, Vienna (Austria), from october 27, to november 5, 2009

We shall denote canonical coordinates of $\mathbb{C}^n$ by ζ_i or z_i, $i = 1, 2, ..., n$. Let be given for $j, \alpha = 1, 2, ..., n$ functions $A_j^\alpha \in \mathcal{C}^{r+\lambda}(\overline{B})$ on the closed unit ball $\overline{B}$ of $\mathbb{C}^n$, with $r \in \mathbb{N}$, $0 < \lambda < 1$, such that

$$Y_j = \frac{\partial}{\partial z_j} + \sum_{\alpha=1}^{n} A_j^\alpha(.) \frac{\partial}{\partial \overline{z}_\alpha}, \quad j = 1, 2, ..., n, \tag{1.1}$$

defines an integrable almost complex structure J, i. e. $[Y_j, Y_k] = 0$ for all $j, k = 1, 2, ..., n$. Following Webster we can assume that the A_j^α 's vanish at 0. Denote by J^τ, for $0 \leq \tau \leq 1$, the integrable almost complex structure given by

$$Y_j^\tau(z) = \frac{\partial}{\partial z_j} + \sum_{\alpha=1}^{n} A_j^\alpha(\tau z) \frac{\partial}{\partial \overline{z}_\alpha}, \quad j = 1, 2, ..., n. \tag{1.2}$$

It is obvious that we can find in a neighborhood of 0 a holomorphic coordinate map for $J = J^1$ if and only if this can be done for some J^τ, with $0 < \tau < 1$. Therefore, we want to find such a J$^\tau$ for sufficiently small τ. Let the A_j^α 's be chosen once and for all. Uniform constants may depend on the A_j^α 's (and their derivatives). Most constants in the estimates will be uniform with respect to τ. In order to avoid complicated notations we shall often drop τ. So, for example, we write $A_j^\alpha(z)$ instead of $A_j^\alpha(\tau z)$. The difference is immaterial and only consists in the presence of an additional factor τ in the final estimates of terms derived from the $A_j^\alpha(\tau z)$. Note that $|A_j^\alpha(z)| \lesssim \tau$ with a constant independent of τ. Also we tacitly assume that assertions should be true only for sufficiently small τ. To any τ correspond differential operators $\partial = \partial^\tau$, $\overline{\partial} = \overline{\partial}^\tau$ such that $d = \partial + \overline{\partial}$, $\overline{\partial}\partial = \partial\overline{\partial} = 0$. Therefore it makes sense to consider the $\overline{\partial}$ equation (in the distributional sense)

$$\overline{\partial} u = f, \quad \text{with} \quad \overline{\partial} f = 0,$$

where f is a $(0, 1)$ form. A function g is $(J^\tau-)$ holomorphic if $\overline{\partial}^\tau g = 0$. We want to solve the above $\overline{\partial}$ equation by a solution operator with estimates for sufficiently small $\tau > 0$. Proofs which are elementary for a constant complex structure are quite technical in our case. For the sake of a quick understanding of the main ideas we have shifted them to section 5.

2. A generalized Bochner-Martinelli-Koppelman formula for the ball

Let $0 \leq \tau \leq 1$ be fixed. We remind the reader that we drop the τ in the notations. Let $f \in \mathcal{C}_{0,1}^\lambda(\overline{B})$, with $\overline{\partial} f = 0$. At first, we want to establish a Bochner-Martinelli-Koppelman like formula for the almost complex structure. We adopt the notations of [5] for differential forms. In particular, the $d\zeta_i$'s will commute with the dz_j 's in double differential forms.

Let $\zeta \in \overline{B}$ be fixed. We choose new complex coordinates on $\mathbb{C}^n$ centered at ζ, denoted by

$$z_\zeta = (z_{\zeta,1}, z_{\zeta,2}, ..., z_{\zeta,n}) : \mathbb{C}^n \longrightarrow \mathbb{C}^n,$$

with

$$z_{\zeta,j}(z) = \sum_{k=1}^{n} \Lambda_j^k(\zeta)(z_k - \zeta_k) + \Gamma_j^k(\zeta)(\overline{z}_k - \overline{\zeta}_k), \tag{2.1}$$

where the coefficients are determined by

$$\Lambda_j^k(\zeta) + \sum_{l=1}^{n} A_k^l(\zeta)\Gamma_j^l(\zeta) = \delta_{k,j}, \quad \Gamma_j^k(\zeta) + \sum_{l=1}^{n} \overline{A_k^l(\zeta)}\Lambda_j^l(\zeta) = 0, \tag{2.2}$$

for $j, k = 1, 2, ..., n$. Since $A_k^l(0) = 0$ the above system is uniquely solvable on $\overline{B}$, the coefficients are in $\mathcal{C}^{r+\lambda}(\overline{B})$ and

$$\left|\Lambda_j^k(\zeta) - \delta_{k,j}\right| + \left|\Gamma_j^k(\zeta)\right| \lesssim \tau.$$

Moreover, $\Lambda_j^k(0) = \delta_{k,j}$, $\Gamma_j^k(0) = 0$, and the matrix $\{\Lambda_j^k(\zeta)\}_{j,k}$ is invertible. We shall also need to replace z by ζ. Set (we write $A_j^\alpha(\zeta)$ for $A_j^\alpha(\tau\zeta)$)

$$X_j(\zeta) = \frac{\partial}{\partial \zeta_j} + \sum_{\alpha=1}^{n} A_j^\alpha(\zeta)\frac{\partial}{\partial \overline{\zeta}_\alpha}, \quad j = 1, 2, ..., n. \tag{2.3}$$

Also we consider for fixed $z \in \overline{B}$ coordinates

$$\zeta_{z,j}(\zeta) = \sum_{k=1}^{n} \Lambda_j^k(z)(\zeta_k - z_k) + \Gamma_j^k(z)(\overline{\zeta}_k - \overline{z}_k). \tag{2.4}$$

Obviously

$$\begin{aligned} Y_i(z)z_{\zeta,j}(z) &= \delta_{i,j} + \sum_{\alpha=1}^{n} \left(A_i^\alpha(z) - A_i^\alpha(\zeta)\right)\Gamma_j^\alpha(\zeta), \\ \overline{Y}_i(z)z_{\zeta,j}(z) &= \sum_{\alpha=1}^{n} \left(\overline{A}_i^\alpha(z) - \overline{A}_i^\alpha(\zeta)\right)\Lambda_j^\alpha(\zeta). \end{aligned}$$

Similar formulas are obtained for $X_i(\zeta)\zeta_{z,j}$ and $\overline{X}_i(\zeta)\zeta_{z,j}$. We note for later use

(2.5)

$$z_{\zeta,j}(z) + \zeta_{z,j}(\zeta) = \sum_{k=1}^{n} \left(\Lambda_j^k(\zeta) - \Lambda_j^k(z)\right)(z_k - \zeta_k) + \left(\Gamma_j^k(\zeta) - \Gamma_j^k(z)\right)(\overline{z}_k - \overline{\zeta}_k)$$

and

$$\begin{aligned}
X_i(\zeta)z_{\zeta,j}(z) &= -X_i(\zeta)\zeta_{z,j}(z) + X_i(\zeta)\left(z_{\zeta,j}(z)+\zeta_{z,j}(\zeta)\right) \\
&= -\delta_{i,j} - \sum_{\alpha=1}^{n}\left(A_j^\alpha(z)-A_j^\alpha(\zeta)\right)\Gamma_j^\alpha(z) + X_i(\zeta)\left(z_{\zeta,j}(z)+\zeta_{z,j}(\zeta)\right),
\end{aligned}$$

$$\overline{X}_i(\zeta)z_{\zeta,j}(z) = -\sum_{\alpha=1}^{n}\left(\overline{A}_j^\alpha(z)-\overline{A}_j^\alpha(\zeta)\right)\Gamma_j^\alpha(z) + \overline{X}_i(\zeta)\left(z_{\zeta,j}(z)+\zeta_{z,j}(\zeta)\right).$$

Note that $X_i(\zeta)z_{\zeta,j}$ and $\overline{X}_i(\zeta)z_{\zeta,j}$ are of class $\mathcal{C}^\infty$ with respect to z.
We introduce the following symbolic notations for double differential forms on $\overline{B}\times\overline{B}$. We denote by L the class of differential forms on $\overline{B}\times\overline{B}$ having coefficients which are finite sums of products $U_1\cdot U_2\cdots U_k$. Here the factors U_i are of the following two types
i) $U_i \in \mathcal{C}^\infty(\overline{B}\times\overline{B})$,
ii) U_i or $\overline{U}_i \in \{A_j^\alpha, \Lambda_j^\alpha - \delta_{\alpha j}, \Gamma_j^\alpha\}$ with respect to ζ or z, or a derivative of these expressions with respect to the canonical coordinates of $\mathbb{C}^n$.
At least one of the factors must be of type (ii).
We write $U \in L^{(a,b)}$ if U is of class $\mathcal{C}^{a+\lambda}$ with respect to ζ and of class $\mathcal{C}^{b+\lambda}$ with respect to z. This implies that for all $0\le k_1\le a$, $0\le k_2\le b$ $D_\zeta^{k_1}D_z^{k_2}U \in \mathcal{C}^\lambda(\overline{B}\times\overline{B})$ and

$$\left|D_\zeta^{k_1}D_z^{k_2}U\right| \lesssim \tau,$$

with a constant independent of ζ, z and τ.
We write $U \in \widetilde{L}_s^{(a,b)}$ if $U \in L^{(a,b)}$ has coefficients which are finite sums of products $\widetilde{U}_1\cdot\widetilde{U}_2\cdots\widetilde{U}_k$ where the factors are of the following three types
i) $\widetilde{U}_i \in \mathcal{C}^\infty(\overline{B}\times\overline{B})$,
ii) $\widetilde{U}_i$ or $\overline{\widetilde{U}}_i \in \{z_k-\zeta_k \mid 1\le k\le n\}$,
iii) either $\widetilde{U}_i$ is $U_i(\zeta)-U_i(z)$ or a derivative of U_i, where U_i or $\overline{U}_i \in \{A_j^\alpha, \Lambda_j^\alpha, \Gamma_j^\alpha\}$. At least one of the factors must be of third type.
Moreover $\widetilde{U}_1\cdot\widetilde{U}_2\cdots\widetilde{U}_k$ must vanish of order s if $\zeta=z$. That means, that at least s factors are of type (ii) or type $U_i(\zeta)-U_i(z)$.
In particular this implies that for all $0\le k_1\le a$, $0\le k_2\le b$ $D_\zeta^{k_1}D_z^{k_2}U \in \mathcal{C}^\lambda(\overline{B}\times\overline{B})$ and

$$\left|D_\zeta^{k_1}D_z^{k_2}U(\zeta,z)\right| \lesssim \tau|\zeta-z|^{\max(0,s-k_1-k_2)}.$$

Here $D_\zeta^{k_1}D_z^{k_2}$ are mixed derivatives with respect to the underlying real coordinates of ζ and z of the indicated order. Typical examples are $A_j^\alpha \in L^{(r,\infty)}$, $A_j^\alpha(\zeta)-A_j^\alpha(z) \in \widetilde{L}_1^{(r,\infty)}$. We shall shortly write $U = L^{(a,b)}$ if $U \in L^{(a,b)}$ and $U = \widetilde{L}_s^{(a,b)}$ if $U \in \widetilde{L}_s^{(a,b)}$.

Then we have

$$\begin{aligned}
\Gamma_j^\alpha(\zeta) &= L^{(r,\infty)}, \quad \Lambda_j^\alpha(\zeta) - \delta_{\alpha,j} = L^{(r,\infty)}, \\
z_{\zeta,j} &= \widetilde{L}_1^{(r,\infty)}, \ \zeta_{z,j} = \widetilde{L}_1^{(\infty,r)}, \ z_{\zeta,j}(z) + \zeta_{z,j}(\zeta) = \widetilde{L}_2^{(r,r)}, \\
Y_i z_{\zeta,j} &= \delta_{i,j} + \widetilde{L}_1^{(r,r)}, \quad \overline{Y}_i z_{\zeta,j} = \widetilde{L}_1^{(r,r)}, \\
X_i(\zeta) z_{\zeta,j} &= -\delta_{i,j} + \widetilde{L}_1^{(r-1,r)}, \quad \overline{X}_i(\zeta) z_{\zeta,j} = \widetilde{L}_1^{(r-1,r)}.
\end{aligned}$$

We denote the dual $(1,0)$ forms of $\{X_i(\zeta)\}_{1\le i\le n}$ by $\{\omega_i(\zeta)\}_{1\le i\le n}$.

$$\omega_i(\zeta) = \sum_{\alpha=1}^{n} \Lambda_i^\alpha(\zeta) d\zeta_\alpha + \Gamma_i^\alpha(\zeta) d\overline{\zeta}_\alpha, \quad i = 1,2,...,n. \tag{2.6}$$

Note that $\omega_i = d\zeta_i + L^{(r,\infty)}$, $\overline{\partial}\omega_i = L^{(r-1,\infty)}$. We set

$$\begin{aligned}
\eta_0^*(\zeta,z) &= -\sum_{j=1}^{n} \overline{z_{\zeta,j}(z)}\omega_j(\zeta), \\
\Phi_0(\zeta,z) &= \sum_{j=1}^{n} |z_{\zeta,j}(z)|^2, \\
\eta_0(\zeta,z) &= \frac{\eta_0^*(\zeta,z)}{\Phi_0(\zeta,z)} = \sum_{j=1}^{n} \eta_{0,j}(\zeta,z)\omega_j(\zeta).
\end{aligned}$$

We define the Bochner-Martinelli-Koppelman kernel for $(0,q)$ forms by

$$B_{n,q}(\zeta,z) = C_{n,q}\eta_0(\zeta,z) \wedge \left(\overline{\partial}_\zeta \eta_0(\zeta,z)\right)^{n-q-1} \wedge \left(\overline{\partial}_z \eta_0(\zeta,z)\right)^{q}$$

for $q = 0,1,2,....,n-1$, with $\mathrm{C}_{n,q} = (-1)^{q(q-1)/2}\begin{pmatrix} n-1 \\ q \end{pmatrix}$, and by $B_{n,-1} = B_{n,n} = 0$.

Our first goal is to calculate $E_1 = \overline{\partial}_\zeta B_{n,1} + \overline{\partial}_z B_{n,0}$. In the classical setting E_1 vanishes. Here we apply the following trick. We consider double differential forms on $(\mathbb{C}^n \times \mathbb{C}^n) \times \mathbb{C}^n$ with variables $(\zeta,\mu,\kappa) \in \overline{B} \times \overline{B} \times \overline{B}$ and $z \in \mathbb{C}^n$ and set

$$\begin{aligned}
\omega_j(\kappa) &= \sum_{\alpha=1}^{n} \left(\Lambda_j^\alpha(\kappa) d\kappa_\alpha + \Gamma_j^\alpha(\kappa) d\overline{\kappa}_\alpha\right), \\
\eta_0^{\mu,\kappa}(\zeta,z) &= -\frac{\sum_{j=1}^n \overline{z_{\zeta,j}(z)}\omega_j(\kappa)}{\sum_{j=1}^n \overline{z_{\zeta,j}(z)}\,\mu_i} = \sum_{j=1}^{n} \eta_{0,j}^{\mu,\kappa}(\zeta,z)\omega_j(\kappa), \\
B_{n,q}^{\mu,\kappa}(\zeta,z) &= C_{n,q}\eta_0^{\mu,\kappa}(\zeta,z) \wedge \left(\overline{\partial}_\zeta \eta_0^{\mu,\kappa}(\zeta,z)\right)^{n-q-1} \wedge \left(\overline{\partial}_z \eta_0^{\mu,\kappa}(\zeta,z)\right)^{q}.
\end{aligned}$$

When we calculate $E_1^{\mu,\kappa} := \overline{\partial}_\zeta B_{n,1}^{\mu,\kappa}(\zeta,z) + \overline{\partial}_z B_{n,0}^{\mu,\kappa}(\zeta,z)$ we obtain a constant times $\left(\overline{\partial}_\zeta \eta_0^{\mu,\kappa}(\zeta,z)\right)^{n-1} \wedge \overline{\partial}_z \eta_0^{\mu,\kappa}(\zeta,z)$. Other terms cancel by the choice of the coefficients $C_{n,q}$.

Remark. For non-integrable structures terms comprising factors $\overline{\partial}_\zeta^2 \eta_0^{\mu,\kappa}(\zeta,z)$ occur.

Since $\sum_{j=1}^n \eta_{0,j}^{\mu,\kappa}(\zeta,z)\mu_j = -1$ we have

$$\sum_{j=1}^n \overline{\partial}_\zeta \eta_{0,j}^{\mu,\kappa}(\zeta,z)\mu_j = 0, \quad \sum_{j=1}^n \overline{\partial}_z \eta_{0,j}^{\mu,\kappa}(\zeta,z)\mu_j = 0.$$

So by linear algebra reasons we obtain $E_1^{\mu,\kappa} = 0$. (See especially ([6], section 2) and also Norguet [4] or Koppelman [3]) for this part of reasoning.) Thus with $\psi(\zeta,z) = (\zeta, z_\zeta(z), \zeta; z) = (\zeta, \mu, \kappa; z)$

$$\psi^*(E_1^{\mu,\kappa}) = \psi^*(\overline{\partial}_\zeta B_{n,1}^{\mu,\kappa}) + \psi^*(\overline{\partial}_z B_{n,0}^{\mu,\kappa}) = 0.$$

Now $\psi^*(\mu) = z_\zeta(z)$, $\psi^*(\omega_j(\kappa)) = \omega_j(\zeta)$. Therefore, when we expand $E_1(\zeta,z)$ we obtain $\psi^*(E_1^{\mu,\kappa})$ and a sum of junk terms coming from differentiation of $\overline{\partial}_\zeta \omega_j$, $\overline{\partial}_\zeta z_{\zeta,j} = \widetilde{L}_1^{(r,r)}$ or $\overline{\partial}_z z_{\zeta,j} = \widetilde{L}_1^{(r-1,r)}$. An easy but tedious calculation gives

Lemma 2.1. $E_1 = \frac{\widetilde{L}_1^{(r-1,r)}}{|z_\zeta(z)|^{2n}} + \frac{\widetilde{L}_3^{(r-1,r)}}{|z_\zeta(z)|^{2n+2}}$.

We show for $E_1 f(z) = \int_B f(\zeta) \wedge E_1(\zeta,z)$:

Lemma 2.2. *E_1 is a continuous operator from $\mathcal{C}_{0,1}^\lambda(\overline{B})$ to $\mathcal{C}^\lambda(\overline{B})$ and its $\mathcal{C}^\lambda-$operator norm tends to 0 if $\tau \to 0$.*

Lemma 2.3. *Let $f \in \mathcal{C}_{0,1}^0(\overline{B})$ with $\overline{\partial} f \in \mathcal{C}_{0,2}^0(\overline{B})$. Then for all $z \in B$*

$$f(z) = E_1 f(z) + \int_{bB} f(\zeta) \wedge B_{n,1}(\zeta,z) - \int_B \overline{\partial} f(\zeta) \wedge B_{n,1}(\zeta,z) - \overline{\partial}_z \int_B f(\zeta) \wedge B_{n,0}(\zeta,z).$$

Remark. B can be replaced by a convex domain with smooth boundary.

3. An approximate solution operator for almost complex structures

By introducing an almost holomorphic support function for B we can construct a new integral kernel and replace the boundary integral in lemma 2.3. Since error terms will tend to 0 if $\tau \to 0$ this will give an approximate solution operator for $\overline{\partial}$. A functional analytic argument will lead to a solution operator for $\overline{\partial}$ later on.
A defining function for the unit ball is $r(\zeta) = |\zeta|^2 - 1$. We set

$$\Phi(\zeta,z) = \sum_{i=1}^n X_j r(\zeta) z_{\zeta,i}(z). \tag{3.1}$$

In order to analyze Φ we set for fixed ζ $r_\zeta(z_\zeta(z)) = r(z)$. Then for $y = z_\zeta(z)$ Taylor expansion yields

$$r_\zeta(y) = r_\zeta(0) + 2\mathrm{Re} \sum_{i=1}^n \frac{\partial r_\zeta}{\partial y_i}(0) y_i + H_\zeta(\widetilde{y})[y],$$

where $H_\zeta(\widetilde{y})[y]$ denotes the Hessian of r_ζ at a point $\widetilde{y}$ on the line segment between 0 and y.

$$H_\zeta(\widetilde{y})[y] = 2\mathrm{Re}\sum_{i,j=1}^{n}\frac{\partial^2 r_\zeta}{\partial y_i \partial y_j}(\widetilde{y})y_i y_j + 2\sum_{i,j=1}^{n}\frac{\partial^2 r_\zeta}{\partial y_i \partial \overline{y}_j}(\widetilde{y})y_i\overline{y}_j.$$

Since

$$\frac{\partial}{\partial y_i} = \frac{\partial}{\partial z_i} + \sum_{\alpha=1}^{n} A_i^\alpha(\zeta)\frac{\partial}{\partial \overline{z}_\alpha}$$

we obtain with $\widetilde{y} = z_\zeta(\widetilde{z})$

$$\begin{aligned} H_\zeta(\widetilde{y})[y] &= 2\mathrm{Re}\sum_{i,j=1}^{n}\frac{\partial^2 r}{\partial z_i \partial z_j}(\widetilde{z})z_{\zeta,i}(z)z_{\zeta,j}(z) + 2\sum_{i,j=1}^{n}\frac{\partial^2 r}{\partial z_i \partial \overline{z}_j}(\widetilde{z})z_{\zeta,i}(z)\overline{z_{\zeta,j}}(z) + O_\tau(|\zeta-z|^2) \\ &= 2|z_\zeta(z)|^2 + O_\tau(|\zeta-z|^2) \\ &= 2|\zeta-z|^2 + O_\tau(|\zeta-z|^2), \end{aligned}$$

where

$$|O_\tau(|\zeta-z|^2)| \lesssim \tau|\zeta-z|^2,$$

with a constant independent of τ and ζ. Since Re $\sum_{i=1}^n \frac{\partial r_\zeta}{\partial y_i}(0)y_i = \Phi(\zeta,z)$ this yields

$$-2\mathrm{Re}\,\Phi(\zeta,z) \geq r(\zeta) - r(z) + |\zeta-z|^2 \tag{3.2}$$

for $\zeta, z \in \overline{B}$ and τ sufficiently small. Then Φ can serve as an approximate holomorphic support function for the ball.

From Φ one can construct Cauchy-Fantappiè kernels in the same way as for Φ_0. In the complex structure case of $\mathbb{C}^n$ two such kernels differ by a sum of explicitely given $\overline{\partial}$ exact forms. In order to calculate the error terms appearing in our case we will apply a similar trick as for E_1.

We set with variables $(\zeta,\mu,\kappa;z)$ on an open set of $[\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}^n]\times\mathbb{C}^n$ where it makes sense

$$\eta^{\mu,\kappa}(\zeta,z) = -\frac{\sum_{i=1}^n (X_j r)(\zeta)\omega_i(\kappa)}{\sum_{i=1}^n (X_j r)(\zeta)\mu_i}.$$

Moreover, we set

$$D_{n,q}^{\mu,\kappa} = C_{n,q}\eta^{\mu,\kappa}(\zeta,z)\wedge\left(\overline{\partial}_\zeta\eta^{\mu,\kappa}(\zeta,z)\right)^{n-q-1}\wedge\left(\overline{\partial}_z\eta^{\mu,\kappa}(\zeta,z)\right)^q.$$

Note that $D_{n,q}^{\mu,\kappa}$ vanishes if $q>0$. Since the n-fold product of terms of type $\eta^{\mu,\kappa}-\eta_0^{\mu,\kappa}, \overline{\partial}_\zeta\eta^{\mu,\kappa}, \overline{\partial}_\zeta\eta_0^{\mu,\kappa}, \overline{\partial}_z\eta^{\mu,\kappa}, \overline{\partial}_z\eta_0^{\mu,\kappa}$ vanishes by purely algebraic reasons (see, for example ([6], section 2) there exist universal coefficients $l_{a,b}^\nu$ only depending on n such that

$$B_{n,1}^{\mu,\kappa} = B_{n,1}^{\mu,\kappa} - D_{n,1}^{\mu,\kappa} = \overline{\partial}_\zeta U_1^{\mu,\kappa} + \overline{\partial}_z U_0^{\mu,\kappa}, \tag{3.3}$$

with (for $\nu = 0,1$)

$$U_\nu^{\mu,\kappa} = \sum_{a+b=n-\nu-2} l_{a,b}^\nu \eta_0^{\mu,\kappa}\wedge\eta^{\mu,\kappa}\wedge(\overline{\partial}_\zeta\eta_0^{\mu,\kappa})^a\wedge(\overline{\partial}_\zeta\eta^{\mu,\kappa})^b\wedge(\overline{\partial}_z\eta_0^{\mu,\kappa})^\nu.$$

Now we set

$$\eta(\zeta,z)=\frac{\eta^*(\zeta,z)}{\Phi(\zeta,z)}=\frac{-\sum_{i=1}^n (X_j r)(\zeta)\omega_i(\zeta)}{\sum_{i=1}^n (X_j r)(\zeta) z_{\zeta,i}(z)},$$

$$\begin{aligned} U_\nu &= \sum_{a+b=n-\nu-2} l^\nu_{a,b}\eta_0\wedge\eta\wedge(\overline{\partial}_\zeta\eta_0)^a\wedge(\overline{\partial}_\zeta\eta)^b\wedge(\overline{\partial}_z\eta_0)^\nu,\\ F_1 &= B_{n,1}-\overline{\partial}_\zeta U_1-\overline{\partial}_z U_0. \end{aligned}$$

In order to compare the forms depending on μ,κ with the latter forms we introduce the map

$$\Psi:\mathbb{C}^n\times\mathbb{C}^n\to[\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}^n]\times\mathbb{C}^n,\ (\zeta,z)\longmapsto(\zeta,z_\zeta(z),\zeta;z)=(\zeta,\mu,\kappa;z).$$

Then $\eta=\Psi^*(\eta^{\mu,\kappa})$, $\eta_0=\Psi^*(\eta_0^{\mu,\kappa})$. Since $\Psi^*B^{\mu,\kappa}_{n,1}-\Psi^*\overline{\partial}_\zeta U_1^{\mu,\kappa}-\Psi^*\overline{\partial}_z U_0^{\mu,\kappa}=0$ it is clear that $B_{n,1}-\overline{\partial}_\zeta U_1-\overline{\partial}_z U_0$ is a sum of terms containing at least one of the factors $\overline{\partial}_\zeta\omega_i(\zeta)$, $\overline{\partial}_\zeta z_\zeta(z)$, or $\overline{\partial}_z z_\zeta(z)$. Let us look at the different cases. As before, it is easily seen that terms coming from $B_{n,1}$ give rise to terms of type $\frac{\widetilde{L}_2^{(r-1,\infty)}}{\Phi_0^n}$.
U_ν is a sum of terms

$$\frac{\eta_0^*\wedge\eta^*\wedge(\overline{\partial}_\zeta\eta_0^*)^a\wedge(\overline{\partial}_\zeta\eta^*)^b\wedge(\overline{\partial}_z\eta_0^*)^\nu}{\Phi_0^{a+1+\nu}\Phi^{b+1}}, \tag{3.4}$$

with $a+b=n-2-\nu$. $\overline{\partial}_\zeta U_1$ is a linear combination of the following terms

$$\begin{aligned} U_{1,1} &= \frac{\eta^*\wedge(\overline{\partial}_\zeta\eta_0^*)^{a+1}\wedge(\overline{\partial}_\zeta\eta^*)^b\wedge\overline{\partial}_z\eta_0^*}{\Phi_0^{a+2}\Phi^{b+1}},\\ U_{1,2} &= \frac{\eta_0^*\wedge(\overline{\partial}_\zeta\eta_0^*)^{a}\wedge(\overline{\partial}_\zeta\eta^*)^{b+1}\wedge\overline{\partial}_z\eta_0^*}{\Phi_0^{a+2}\Phi^{b+1}},\\ U_{1,3} &= \frac{\eta_0^*\wedge\eta^*\wedge(\overline{\partial}_\zeta\eta_0^*)^{a}\wedge(\overline{\partial}_\zeta\eta^*)^{b}\wedge\overline{\partial}_\zeta\overline{\partial}_z\eta_0^*}{\Phi_0^{a+2}\Phi^{b+1}},\\ U_{1,4} &= \frac{\overline{\partial}_\zeta\Phi_0\wedge\eta_0^*\wedge\eta^*\wedge(\overline{\partial}_\zeta\eta_0^*)^{a}\wedge(\overline{\partial}_\zeta\eta^*)^{b}\wedge\overline{\partial}_z\eta_0^*}{\Phi_0^{a+3}\Phi^{b+1}},\\ U_{1,5} &= \frac{\overline{\partial}_\zeta\Phi\wedge\eta_0^*\wedge\eta^*\wedge(\overline{\partial}_\zeta\eta_0^*)^{a}\wedge(\overline{\partial}_\zeta\eta^*)^{b}\wedge\overline{\partial}_z\eta_0^*}{\Phi_0^{a+2}\Phi^{b+2}}, \end{aligned}$$

with $a+b=n-3$.

$\overline{\partial}_z U_0$ is a linear combination of the following terms

$$\begin{aligned} U_{0,1} &= \frac{\eta^* \wedge (\overline{\partial}_\zeta \eta_0^*)^a \wedge (\overline{\partial}_\zeta \eta^*)^b \wedge \overline{\partial}_z \eta_0^*}{\Phi_0^{a+1}\Phi^{b+1}}, \\ U_{0,2} &= \frac{\eta_0^* \wedge \eta^* \wedge \overline{\partial}_\zeta \overline{\partial}_z \eta_0^* \wedge (\overline{\partial}_\zeta \eta_0^*)^{a-1} \wedge (\overline{\partial}_\zeta \eta^*)^b}{\Phi_0^{a+1}\Phi^{b+1}}, \quad a \geq 1, \\ U_{0,3} &= \frac{\overline{\partial}_z \Phi_0 \wedge \eta_0^* \wedge \eta^* \wedge (\overline{\partial}_\zeta \eta_0^*)^a \wedge (\overline{\partial}_\zeta \eta^*)^b}{\Phi_0^{a+2}\Phi^{b+1}}, \\ U_{0,4} &= \frac{\overline{\partial}_z \Phi \wedge \eta_0^* \wedge \eta^* \wedge (\overline{\partial}_\zeta \eta_0^*)^a \wedge (\overline{\partial}_\zeta \eta^*)^b}{\Phi_0^{a+1}\Phi^{b+2}}, \end{aligned}$$

with $a+b=n-2$.

Because of (3.3) most terms of F_1 cancel out except those of $U_{\nu,i}$ when at least one of the operators $\overline{\partial}_\zeta$ or $\overline{\partial}_z$ in the kernels falls on $\omega_i(\zeta)$ or $z_\zeta(z)$. Therefore the F_1-terms generating factors in $U_{\nu,i}$ are

$$\sum_{j=1}^n \overline{z_{\zeta,j}}\,\overline{\partial}_\zeta \omega_j = \widetilde{L}_1^{(r-1,\infty)} \quad \text{in} \quad \overline{\partial}_\zeta \eta_0^*, \quad \sum_{j=1}^n (X_i r)\overline{\partial}_\zeta \omega_j = L^{(r-1,r)} \quad \text{in} \quad \overline{\partial}_\zeta \eta^*,$$

$$\sum_{j=1}^n \overline{z_{\zeta,j}}\,\overline{\partial}_\zeta z_{\zeta,j} = \widetilde{L}_2^{(r,r)} \quad \text{in} \quad \overline{\partial}_\zeta \Phi_0, \quad \sum_{j=1}^n (X_i r)\overline{\partial}_z z_{\zeta,j} = \widetilde{L}_1^{(r-1,r)} \quad \text{in} \quad \overline{\partial}_z \Phi.$$

This leads to F_1-terms of the following types

$$\begin{aligned} I &= \frac{L^{(r-1,r)}}{\Phi_0^{n-b-1}\Phi^{b+1}}, \qquad II = \frac{\widetilde{L}_2^{(r-1,r)}}{\Phi_0^{n-b}\Phi^{b+1}}, \quad \text{with } b \geq 1, \\ III &= \frac{\widetilde{L}_1^{(r-1,r)}}{\Phi_0^{n-1}\Phi}, \qquad IV = \frac{\widetilde{L}_3^{(r-1,r)}}{\Phi_0^{n}\Phi}. \end{aligned}$$

Note that the singularities of terms of type I and II are higher than those of the respective terms III or IV. A simple application of lemma 2.3 and Stokes theorem yield

Lemma 3.1. *Let $f \in \mathcal{C}^0_{0,1}(\overline{B})$ such that $\overline{\partial} f \in \mathcal{C}^0_{0,2}(\overline{B})$. Then for all $z \in B$*

$$f(z) = E_1 f(z) + F_1 f(z) + \int_{bB} \overline{\partial}_\zeta f \wedge U_1 + \overline{\partial}_z \int_{bB} f \wedge U_0 - \int_B \overline{\partial} f \wedge B_{n,1} - \overline{\partial}_z \int_B f \wedge B_{n,0}.$$

In section 5 we show

Lemma 3.2. *F_1 is a continuous operator from $\mathcal{C}^\lambda_{0,1}(\overline{B})$ to $\mathcal{C}^\lambda(\overline{B}) \cap \mathcal{C}^{r+\lambda}(B)$ whose $\mathcal{C}^\lambda-$operator norm tends to 0 if $\tau \to 0$.*

Lemma 3.3. *Let for $g \in \mathcal{C}^\lambda_{0,1}(\overline{B})$*

$$S_1 g(z) = \int_B g(\zeta) \wedge B_{n,0}(\zeta, z).$$

Then for any compact set K with $0 \in K \subset\subset B$

$$|S_1 g|_{\mathcal{C}^1(K)} \lesssim |g|_{\mathcal{C}^\lambda(\overline{B})}.$$

4. Construction of holomorphic coordinates

We set $K_1 = E_1 + F_1$ and

$$T_1 f = \int_{bB} f \wedge U_0 - \int_B f \wedge B_{n,0}.$$

We consider the Banach space $X_\lambda = \{g \in \mathcal{C}^\lambda_{0,1}(\overline{B}) \mid \overline{\partial} g = 0\}$. Then $K_1 g \in \mathcal{C}^\lambda_{0,1}(\overline{B})$ for $g \in X_\lambda$. Since by (3.1) $K_1 g \in \mathcal{C}^\lambda_{0,1}(\overline{B})$ and

$$g = K_1 g + \overline{\partial} T_1 g,$$

it follows $\overline{\partial} T_1 g \in X_\lambda$ and $K_1 g \in X_\lambda$. Since for small τ the operator norm of K_1 tends to 0 $1 - K_1$ is an automorphism of the Banach space X_λ. Let $f \in X_\lambda$. The above decomposition for $g = (1-K_1)^{-1} f$ gives

$$f = \overline{\partial} T_1 (1-K_1)^{-1} f.$$

Now lemma 3.3 implies

$$|S_1 g|_{\mathcal{C}^1(K)} \lesssim |g|_{\mathcal{C}^\lambda(\overline{B})} \lesssim |f|_{\mathcal{C}^\lambda(\overline{B})}.$$

Since the corresponding estimate for

$$\int_{bB} (1-K_1)^{-1} f \wedge U_0$$

is trivial because of $bB \cap \overline{K} = \emptyset$ we obtain

$$\left|T_1 (1-K_1)^{-1} f\right|_{\mathcal{C}^1(K)} \lesssim |f|_{\mathcal{C}^\lambda(\overline{B})}.$$

Now we set for fixed $1 \leq l \leq n$ $f = \overline{\partial} z_l$. Then $Z_l = z_l - T_1\left((1-K_1)^{-1}\overline{\partial} z_l\right)$ is $J^\tau-$holomorphic. Moreover, since $\mid \overline{\partial} z_l \mid_{\mathcal{C}^\lambda(\overline{B})} \to 0$ for $\tau \to 0$ the gradients of the $Z_1, ..., Z_n$ are linearly independent for small τ. This implies that $Z = (Z_1, ..., Z_n)$ defines a holomorphic coordinate map for J^τ on a neighborhood of 0.

Now the Z_l are solutions of the linear first order elliptic system

$$\overline{Y}_i Z_l = 0, \quad i = 1, 2, ..., n.$$

Since the coefficients of the system are of class $\mathcal{C}^{r+\lambda}$ the Z_l are of class $\mathcal{C}^{r+1+\lambda}$ on K. For this argument see [2].

5. Main proofs

Proof of lemma 2.2
Here we apply the Hardy-Littlewood lemma (see [5], ch. V, §3, Lemma 3.1). Since $|E_1(\zeta,0)| \lesssim \tau |\zeta|^{1-2n}$ it follows $|E_1 f(0)| \lesssim \tau |f|_\infty$ with a constant independent of f and τ. So it will be sufficient to show

$$|\mathrm{grad}(E_1 f)(z)| \lesssim \tau |f|_{\mathcal{C}^\lambda(B)} (1- \mid z \mid)^{\lambda-1},$$

with a constant independent of f and τ.
Let $G : \mathcal{C}^\lambda(\overline{B}) \to \mathcal{C}^\lambda(\mathbb{C}^n)$ be a bounded linear extension operator with $\operatorname{supp} G(f) \subset\subset B(0,1+\kappa)$, for $\kappa > 0$, and

$$|Gf|_{\mathcal{C}^\lambda(\mathbb{C}^n)} \lesssim |f|_{\mathcal{C}^\lambda(B)}$$

with a constant independent of f. For κ sufficiently small we can assume that all the above kernels make sense for $z \in \overline{B}$ and $\zeta \in \overline{B(0,1+\kappa)}$. We apply G to forms coefficientwise. Now

$$\int_B f(\zeta) \wedge E_1(\zeta,z) = I_1(z) - I_2(z),$$

with

$$I_1(z) = \int_{\mathbb{C}^n} Gf(\zeta) \wedge E_1(\zeta,z)$$
$$I_2(z) = \int_{\mathbb{C}^n \setminus B} Gf(\zeta) \wedge E_1(\zeta,z)$$

$I_2(z)$ we can differentiate under the integral sign. By the Hardy-Littlewood lemma we obtain easily $I_2 \in \mathcal{C}^\mu(\overline{B})$ for all $\mu < 1$. Now $I_1(z) \in \mathcal{C}^1(\mathbb{C}^n)$ as can be seen exactly as for I_ε in the proof of lemma 3.3 with the slight modification that $P_i(\zeta - z)$ may have factors $U(\zeta) - U(z)$, with $U \in \{A_j^\alpha, \Lambda_j^\alpha, \Gamma_j^\alpha\}$. The proof works nevertheless since

$$\left| \left(\frac{\partial}{\partial y_l} + \frac{\partial}{\partial x_l} \right) (U(x) - U(y)) \right| \lesssim \tau \mid x - y \mid^\lambda .$$

That the $\mathcal{C}^\lambda$−operator norm of E_1 tends to 0 for $\tau \to 0$ is an easy consequence of the presence of $U(\zeta) - U(z)$ in the numerator of the kernel.
Proof of lemma 2.3. Standard regularisation allows to assume that $f \in \mathcal{C}^1_{0,1}(\overline{B})$. We set for a domain $D \subset B$ with smooth boundary and $z \in D$

$$((*)_D f)(z) = \int_{bD} f(\zeta) \wedge B_{n,1}(\zeta,z) - \int_D \overline{\partial} f(\zeta) \wedge B_{n,1}(\zeta,z) - \overline{\partial}_z \int_D f(\zeta) \wedge B_{n,0}(\zeta,z).$$

Let for $z_0 \in B$ and $0 < \varepsilon < 1 - |z_0|$ $D_\varepsilon = \{z \in B |\, |z - z_0| < \varepsilon\}$. Then Stokes theorem yields for $z \in D_\varepsilon$, since integration is over a $(n-1,n)$ form with respect ton ζ,

$$\begin{aligned}((*)_B f)(z) &= ((*)_{D_\varepsilon} f)(z) + \int_{b(B-D_\varepsilon)} f \wedge B_{n,1} - \int_{B-D_\varepsilon} \overline{\partial} f \wedge B_{n,1} - \overline{\partial}_z \int_{B-D_\varepsilon} f \wedge B_{n,0} \\ &= ((*)_{D_\varepsilon} f)(z) - \int_{B-D_\varepsilon} f \wedge \overline{\partial}_\zeta B_{n,1} - \int_{B-D_\varepsilon} f \wedge \overline{\partial}_z B_{n,0} \\ &= ((*)_{D_\varepsilon} f)(z) - \int_{B-D_\varepsilon} f(\zeta) \wedge E_1(\zeta, z).\end{aligned}$$

Since E_1 is integrable it remains to show

$$\lim_{\varepsilon \to 0} ((*)_{D_\varepsilon} f)(z_0) = f(z_0).$$

Now the idea is to freeze the almost complex structure at z_0 in order to reduce the expression to the classical Bochner-Martinelli-Koppelman formula for $\mathbb{C}^n$.

Let $\phi \in \mathcal{C}^\infty(\mathbb{C}^n)$ with $\phi(z) = 1$ if $|z| < 1/2$ and $\phi(z) = 0$ if $|z| > 3/4$. Set $\phi_\varepsilon(z) = \phi((z - z_0)/\varepsilon)$. Now

$$((*)_{D_\varepsilon} f)(z) = ((*)_{D_\varepsilon} \phi_\varepsilon f)(z) + ((*)_{D_\varepsilon}(1 - \phi_\varepsilon) f)(z).$$

Now, by Lemma 2.1

$$\lim_{\varepsilon \to 0} ((*)_{D_\varepsilon}(1 - \phi_\varepsilon) f)(z_0) = -\lim_{\varepsilon \to 0} \int_{D_\varepsilon} (1 - \phi_\varepsilon(\zeta)) f(\zeta) \wedge E_1(\zeta, z_0) = 0.$$

Since, for $z \in D_\varepsilon$,

$$((*)_{D_\varepsilon} \phi_\varepsilon f)(z) = -\int_{D_\varepsilon} \overline{\partial}(\phi_\varepsilon(\zeta) f(\zeta)) \wedge B_{n,1}(\zeta, z) - \overline{\partial}_z \int_{D_\varepsilon} \phi_\varepsilon(\zeta) f(\zeta) \wedge B_{n,0}(\zeta, z)$$

the main problem is now to get $\overline{\partial}_z$ under the integral sign.

$$\begin{aligned}\overline{\partial}_z \int_{D_\varepsilon} \phi_\varepsilon(\zeta) f(\zeta) \wedge B_{n,0}(\zeta, z) &= \overline{\partial}_z \int_{\mathbb{C}^n} (\phi_\varepsilon f)(\zeta) \wedge B_{n,0}(\zeta, z) \\ &= \overline{\partial}_z \int_{\mathbb{C}^n} (\phi_\varepsilon f)(\zeta + z) \wedge B_{n,0}(\zeta + z, z).\end{aligned}$$

Now

$$B_{n,0}(\zeta + z, z) = B'_{n,0}(\zeta + z, z) + R_{n,0}(\zeta + z, z),$$

with $B'_{n,0}(\zeta + z, z)/C_{n,0}$ equals to

$$\frac{\left(\sum_{i,\alpha=1}^n \left(\overline{\Lambda_i^\alpha}(\zeta)\overline{\zeta}_\alpha + \overline{\Gamma_i^\alpha}(\zeta)\zeta_\alpha\right)\omega_i(\zeta)\right) \wedge \left(\sum_{i,\alpha=1}^n \left(\overline{\Lambda_i^\alpha}(\zeta)d\overline{\zeta}_\alpha + \overline{\Gamma_i^\alpha}(\zeta)d\zeta_\alpha\right)\omega_i(\zeta)\right)^{n-1}}{\left(\sum_{i=1}^n \left|\sum_{\alpha=1}^n \overline{\Lambda_i^\alpha}(\zeta)\overline{\zeta}_\alpha + \overline{\Gamma_i^\alpha}(\zeta)\zeta_\alpha\right|^2\right)^n}$$

and

$$|R_{n,0}(\zeta, z)| \lesssim |\zeta - z|^{2-2n}, \quad \left|\overline{\partial}_z R_{n,0}(\zeta, z)\right| \lesssim |\zeta - z|^{1-2n}.$$

Since $|\overline{\partial}_z(\phi_\varepsilon f)(\zeta+z)| \lesssim 1/\varepsilon$ we obtain

$$\lim_{\varepsilon\to 0} \overline{\partial}_z \int_{\mathbb{C}^n} (\phi_\varepsilon f)(\zeta+z) \wedge R_{n,0}(\zeta+z,z)\Big|_{z=z_0} = 0.$$

It remains to simplify $B'_{n,0}(\zeta+z,z)$. Now $\sum_{\alpha=1}^n \left(\overline{\Lambda_i^\alpha}(\zeta)d\overline{\zeta}_\alpha + \overline{\Gamma_i^\alpha}(\zeta)d\zeta_\alpha\right) = \overline{\omega}_i(\zeta)$. So if we replace any $\Lambda_i^\alpha(\zeta)$ by $\Lambda_i^\alpha(z)$ and any $\Gamma_i^\alpha(\zeta)$ by $\Gamma_i^\alpha(z)$ and $\omega_i(\zeta)$ by $\widetilde{\omega}_i(z) = \sum_{\alpha=1}^n \left(\Lambda_i^\alpha(z)d\zeta_\alpha + \Gamma_i^\alpha(z)d\overline{\zeta}_\alpha\right)$ we obtain

$$B'_{n,0}(\zeta+z,z) = \widetilde{B}_{n,0}(\zeta+z,z) + \widetilde{R}_{n,0}(\zeta+z,z),$$

with

$$\left|\widetilde{R}_{n,0}(\zeta,z)\right| \lesssim |\zeta-z|^{2-2n}, \quad \left|\overline{\partial}_z\widetilde{R}_{n,0}(\zeta,z)\right| \lesssim |\zeta-z|^{1-2n},$$

and

$$\widetilde{B}_{n,0}(\zeta+z,z) = C_{n,0}\frac{\left(\sum_{i,\alpha=1}^n \left(\overline{\Lambda_i^\alpha}(z)\overline{\zeta}_\alpha + \overline{\Gamma_i^\alpha}(z)\zeta_\alpha\right)\widetilde{\omega}_i(z)\right) \wedge \left(\sum_{i=1}^n \overline{\widetilde{\omega}_i}(z)\wedge\widetilde{\omega}_i(z)\right)^{n-1}}{\left(\sum_{i=1}^n \left|\sum_{\alpha=1}^n \overline{\Lambda_i^\alpha}(z)\overline{\zeta}_\alpha + \overline{\Gamma_i^\alpha}(z)\zeta_\alpha\right|^2\right)^n}.$$

Also as before we have for $z = z_0$

$$\lim_{\varepsilon\to 0} \overline{\partial}_z \int_{\mathbb{C}^n} (\phi_\varepsilon f)(\zeta+z) \wedge \widetilde{R}_{n,0}(\zeta+z,z)\Big|_{z=z_0} = 0.$$

Finally we are left with

$$I = \overline{\partial}_z \int_{\mathbb{C}^n} (\phi_\varepsilon f)(\zeta+z) \wedge \widetilde{B}_{n,0}(\zeta+z,z)\Big|_{z=z_0} = I_1(z_0) - I_2(z_0),$$

where

$$\begin{aligned} I_1(z) &= \int_{\mathbb{C}^n} \overline{\partial}_z((\phi_\varepsilon f)(\zeta+z)) \wedge \widetilde{B}_{n,0}(\zeta+z,z), \\ I_2(z) &= \int_{\mathbb{C}^n} (\phi_\varepsilon f)(\zeta+z) \wedge \overline{\partial}_z\widetilde{B}_{n,0}(\zeta+z,z), \end{aligned}$$

since here differentiation under the integral sign does not increase the singularity of the kernel. By the same reason

$$\lim_{\varepsilon\to 0} I_2(z_0) = 0.$$

Now

$$\begin{aligned} I_1(z_0) &= \int_{\mathbb{C}^n} \overline{\partial}_z((\phi_\varepsilon f)(\zeta+z)) \wedge \widetilde{B}_{n,0}(\zeta+z,z)\Big|_{z=z_0} \\ &= \int_{\mathbb{C}^n} \overline{\partial}_z((\phi_\varepsilon f)(\zeta+z))\big|_{z=z_0} \wedge \widetilde{B}_{n,0}(\zeta+z_0,z_0). \end{aligned}$$

Let $(\phi_\varepsilon f)(\zeta) = \sum_{i=1}^n h_i(\zeta)\overline{\omega}_i(\zeta)$. We obtain with $\theta_j(z) = \sum_{\alpha=1}^n \Lambda_j^\alpha(z)dz_\alpha + \Gamma_j^\alpha(z)d\overline{z}_\alpha$

$$\begin{aligned}\overline{Y}_j(z)(h_i(\zeta+z)) &= \left(\frac{\partial}{\partial \overline{z}_j} + \sum_{\alpha=1}^n \overline{A}_j^\alpha(z)\frac{\partial}{\partial z_\alpha}\right) h_i(\zeta+z)\\ &= \left[\left(\frac{\partial}{\partial \overline{\zeta}_j} + \sum_{\alpha=1}^n \overline{A}_j^\alpha(z)\frac{\partial}{\partial \zeta_\alpha}\right) h_i\right](\zeta+z).\end{aligned}$$

Therefore

$$\overline{\partial}_z h_i(\zeta+z) = \sum_{j=1}^n \left[\left(\frac{\partial}{\partial \overline{\zeta}_j} + \sum_{\alpha=1}^n \overline{A}_j^\alpha(z)\frac{\partial}{\partial \zeta_\alpha}\right) h_i\right](\zeta+z)\overline{\theta}_j(z).$$

This gives for $I_1(z_0)$

$$\begin{aligned}&\int_{\mathbb{C}^n}\sum_{i=1}^n\sum_{j=1}^n \left[\left(\frac{\partial}{\partial \overline{\zeta}_j} + \sum_{\alpha=1}^n \overline{A}_j^\alpha(z_0)\frac{\partial}{\partial \zeta_\alpha}\right) h_i\right](\zeta+z_0)\overline{\theta}_j(z_0)\wedge\overline{\omega}_i(\zeta+z_0)\wedge\widetilde{B}_{n,0}(\zeta+z_0,z_0)\\ &= \int_{\mathbb{C}^n}\sum_{i=1}^n\sum_{j=1}^n \left[\left(\frac{\partial}{\partial \overline{\zeta}_j} + \sum_{\alpha=1}^n \overline{A}_j^\alpha(z_0)\frac{\partial}{\partial \zeta_\alpha}\right) h_i\right](\zeta)\overline{\theta}_j(z_0)\wedge\overline{\omega}_i(\zeta)\wedge\widetilde{B}_{n,0}(\zeta,z_0)\\ &= I_3(z_0) + \int_{\mathbb{C}^n}\sum_{i=1}^n\sum_{j=1}^n \left[\left(\frac{\partial}{\partial \overline{\zeta}_j} + \sum_{\alpha=1}^n \overline{A}_j^\alpha(z_0)\frac{\partial}{\partial \zeta_\alpha}\right) h_i\right](\zeta)\overline{\theta}_j(z_0)\wedge\overline{\widetilde{\omega}}_i(z_0)\wedge\widetilde{B}_{n,0}(\zeta,z_0),\end{aligned}$$

with $\lim_{\varepsilon\to 0} I_3(z_0) = 0$.
Now we extend the frozen almost complex structure at z_0 to a neighborhood by setting

$$\widetilde{\zeta}_i = \sum_{\alpha=1}^n \Lambda_i^\alpha(z_0)\zeta_\alpha + \Gamma_i^\alpha(z_0)\overline{\zeta}_\alpha, \quad \widetilde{z}_i = \sum_{\alpha=1}^n \Lambda_i^\alpha(z_0)z_\alpha + \Gamma_i^\alpha(z_0)\overline{z}_\alpha.$$

Then $d\widetilde{\zeta}_i = \widetilde{\omega}_i(z_0)$, $d\widetilde{z}_i = \theta_i(z_0)$,

$$\frac{\partial}{\partial \overline{\widetilde{\zeta}}_i} = \frac{\partial}{\partial \overline{\zeta}_i} + \sum_{\alpha=1}^n \overline{A}_i^\beta(z_0)\frac{\partial}{\partial \zeta_\beta}, \quad \frac{\partial}{\partial \overline{\widetilde{z}}_i} = \frac{\partial}{\partial \overline{z}_i} + \sum_{\alpha=1}^n \overline{A}_i^\beta(z_0)\frac{\partial}{\partial z_\beta}.$$

With this notation the remaining term becomes

$$\int_{\mathbb{C}^n}\sum_{i=1}^n\sum_{j=1}^n \frac{\partial}{\partial\overline{\widetilde{\zeta}}_j}\widetilde{h}_i(\widetilde{\zeta})d\overline{\widetilde{z}}_j\wedge d\overline{\widetilde{\zeta}}_i\wedge\widetilde{\widetilde{B}}_{n,0}(\widetilde{\zeta},\widetilde{z}_0),$$

with

$$\widetilde{\widetilde{B}}_{n,0}(\widetilde{\zeta},\widetilde{z}) = C_{n,0}\frac{\left(\sum_i^n\left(\overline{\widetilde{\zeta}}_i - \overline{\widetilde{z}}_i\right)d\widetilde{\zeta}_i\right)\wedge\left(\sum_{i=1}^n d\overline{\widetilde{\zeta}}_i\wedge d\widetilde{\zeta}_i\right)^{n-1}}{\left|\widetilde{\zeta}-\widetilde{z}\right|^{2n}}.$$

Inverting the above reasoning by keeping the constant complex structure we can get $\overline{\partial}$ off the integral sign

$$\lim_{\varepsilon\to 0}\overline{\partial}_z\int_{D_\varepsilon}\phi_\varepsilon(\zeta)f(\zeta)\wedge B_{n,0}(\zeta,z)=\lim_{\varepsilon\to 0}\overline{\partial}_{\widetilde{z}}\int_{\mathbb{C}^n}(\widetilde{\phi_\varepsilon f})(\widetilde{\zeta})\wedge\widetilde{\widetilde{B}}_{n,0}(\widetilde{\zeta},\widetilde{z}),$$

where $\overline{\partial}_{\widetilde{z}}$ denotes the $\overline{\partial}$ operator for the constant structure at z_0. Note that $\widetilde{\widetilde{B}}_{n,0}$ is the classical Bochner-Martinelli kernel $B_{n,0}$ for the constant complex structure.

Now the other term can be treated in the same way

$$\begin{aligned}&\lim_{\varepsilon\to 0}-\int_{D_\varepsilon}\overline{\partial}(\phi_\varepsilon(\zeta)f(\zeta))\wedge B_{n,1}(\zeta,z)\\=&\lim_{\varepsilon\to 0}-\int_{\mathbb{C}^n}\overline{\partial}(\phi_\varepsilon(\zeta)f(\zeta))\wedge B_{n,1}(\zeta,z)\\=&\lim_{\varepsilon\to 0}-\int_{\mathbb{C}^n}\overline{\partial}_{\widetilde{\zeta}}(\widetilde{\phi_\varepsilon f})(\widetilde{\zeta})\wedge\widetilde{\widetilde{B}}_{n,1}(\widetilde{\zeta},\widetilde{z}),\end{aligned}$$

where $\widetilde{\widetilde{B}}_{n,1}$ denotes the Bochner-Martinelli kernel $B_{n,1}$ for the constant complex structure.

Summing up all the remaining terms gives

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left\{-\int_{\mathbb{C}^n}\overline{\partial}_{\widetilde{\zeta}}(\widetilde{\phi_\varepsilon f})(\widetilde{\zeta})\wedge\widetilde{\widetilde{B}}_{n,1}(\widetilde{\zeta},\widetilde{z})+\overline{\partial}_{\widetilde{z}}\int_{\mathbb{C}^n}(\widetilde{\phi_\varepsilon f})(\widetilde{\zeta})\wedge\widetilde{\widetilde{B}}_{n,0}(\widetilde{\zeta},\widetilde{z})\right\}\\=&\lim_{\varepsilon\to 0}(\widetilde{\phi_\varepsilon f})(\widetilde{z}_0)\\=&\,f(z_0)\end{aligned}$$

by the classical Bochner-Martinelli-Koppelman formula.

Proof of lemma 3.2. If t denotes a tangential coordinate which is transversal to the complex tangent space of bB, then $\mid\Phi\mid\gtrsim\mid t\mid+\mid\zeta-z\mid^2$ (see (5.2), (5.3)). Thus Φ has one good direction concerning integration along the boundary of B. After having integrated with respect to t Φ and Φ_0 have comparable vanishing properties. Thus the kernels of type I and II show similar properties as

$$\frac{L^{(r-1,r)}}{\Phi^n}.$$

The latter kernels have been treated in Ahern/Schneider [1]. The proofs can be generalized to our case with slight modifications. The main trick consists in removing too high powers of Φ in the denominators by an integration by parts along a tangential vector field $\mathcal{T}_\zeta=\frac{\partial}{\partial t}$ with $\mathcal{T}_\zeta\Phi\approx 1$. In the following we give the construction of the basic objects of the Ahern/Schneider approach, but in a uniform way with respect to τ.

Let $0<r_0<1$ be fixed and to be chosen later. It is clear from the definition of F_1 that

$$\sup_{|z|\le r_0}\mid F_1f(z)\mid\le C(\tau)\mid f\mid_{\lambda,B},$$

with $C(\tau) \longrightarrow 0$ if $\tau \longrightarrow 0$. Now the vector field

$$\mathcal{T}_\zeta = i(z_1, ..., z_n)$$

is tangential to the spheres centered at the origin and transversal to the complex tangent space with respect to J_τ. As a differential operator $\mathcal{T}_\zeta$ is represented by

$$\mathcal{T}_\zeta = i \sum_{\alpha=1}^{n} \left(\zeta_\alpha \frac{\partial}{\partial \zeta_\alpha} - \overline{\zeta}_\alpha \frac{\partial}{\partial \overline{\zeta}_\alpha} \right). \tag{5.1}$$

Since for $\zeta \in S = bB$

$$\begin{aligned}
\Phi(\zeta, z) &= \sum_{j=1}^{n} \left(X_j \mid \zeta \mid^2 \right) z_{\zeta,j}(z) \\
&= \sum_{j=1}^{n} \left(X_j \mid \zeta \mid^2 \right) \sum_{\alpha=1}^{n} \left(\Lambda_j^\alpha(\zeta)(\zeta_a - z_\alpha) + \Gamma_j^\alpha(\zeta)(\overline{\zeta}_a - \overline{z}_\alpha) \right)
\end{aligned}$$

it follows

$$\begin{aligned}
\mathcal{T}_\zeta \Phi(\zeta, z) &= O(\mid \zeta - z \mid) + \sum_{j=1}^{n} \left(X_j \mid \zeta \mid^2 \right) \sum_{\alpha=1}^{n} \left(i\zeta_\alpha \Lambda_j^\alpha(\zeta) - i\overline{\zeta}_a \Gamma_j^\alpha(\zeta) \right) \\
&= O(\mid \zeta - z \mid) + O(\tau) + \sum_{j=1}^{n} \left(X_j \mid \zeta \mid^2 \right) i\zeta_j \\
&= O(\mid \zeta - z \mid) + O(\tau) + i \mid \zeta \mid^2 \\
&= O(\mid \zeta - z \mid) + O(\tau) + i.
\end{aligned}$$

Thus there exists a $0 < r_0 < 1/2$ such that for all $\zeta \in S$, $z \in \overline{B}$,with $\mid \zeta - z \mid < r_0$ and sufficiently small τ

$$\mid \mathcal{T}_\zeta \Phi(\zeta, z) \mid \geq \frac{1}{2}.$$

Moreover $\mathcal{T}_\zeta \Phi$ is of class $\mathcal{C}^{r-1+\lambda}$ with respect to ζ and $\mathcal{C}^\infty$ with respect to z. For any $\zeta_0 \neq 0$ $\varphi_{\zeta_0}(t) = e^{it}\zeta_0$ is the integral curve of $\mathcal{T}_\zeta$ passing through ζ_0. Next we construct local coordinate maps with t as the first coordinate.
Let $3/4 < 1 - r_0/2 \leq\mid z \mid\leq 1$. We set $\zeta^0 = z/ \mid z \mid\in S$. Then $\mid z - \zeta^0 \mid\leq \frac{r_0}{2}$ and

$$\mid z - \varphi_{\zeta^0} \mid\leq\mid z - \zeta^0 \mid + \mid \zeta^0 - e^{it}\zeta^0 \mid\leq \frac{r_0}{2} + \mid 1 - e^{it} \mid.$$

Let $t_0 > 0$ be so small that for all $\mid t \mid\leq t_0$ $\mid 1 - e^{it} \mid\leq \frac{r_0}{2}$. Then

$$\mid z - \varphi_{\zeta^0} \mid\leq \frac{r_0}{2}$$

for all $\mid t \mid\leq t_0$, and t_0 can be chosen to be independent of z.
Now we define the following real vector space which approximates the complex tangent space to $\mid z \mid \cdot S$ with respect to the canonical complex structure (for small τ).

Let $1 - r_0/2 \leq \mid z \mid \leq 1$. We set

$$\widetilde{T}_z^{\mathbb{C}} = \{V \in \mathbb{C}^n \mid \sum_{\beta=1}^{n} \sum_{j=1}^{n} \left(\overline{z}_j + \sum_{\alpha=1}^{n} A_j^{\alpha}(z) z_\alpha \right) \left(\Lambda_j^{\beta}(z) V_\beta + \Gamma_j^{\beta}(z) \overline{V}_\beta \right).$$

Let $U_1(z), ..., U_{2n-2}(z)$ be $(2n-2)$ orthonormal real vector fields in $\widetilde{T}_z^{\mathbb{C}}$ of class $\mathcal{C}^{r+\lambda}$ with respect to z. Then $\{\zeta^0, \mathcal{T}_{\zeta^0}, U_1(z), ..., U_{2n-2}(z)\}$ is an almost orthogonal real basis of $\mathbb{C}^n$ and ζ^0 and $\mathcal{T}_{\zeta^0}$ are almost orthogonal to $\widetilde{T}_z^{C}$ in the complex sense (for small τ).
Let z be fixed and $\zeta \in S$ be near ζ^0. Since the flow $e^{it}\zeta^0$ is transverse to $\mathrm{span}_{\mathbb{R}}\{\zeta^0, \widetilde{T}_z^{\mathbb{C}}\}$ the map

$$\psi(t + yi, \omega) = \psi(t + yi, \omega_1, ..., \omega_{2n-2}) = \zeta = e^{ti} \left(z + y\zeta^0 + \sum_{\gamma=1}^{2n-2} \omega_\gamma U_\gamma(z) \right)$$

defines a $\mathcal{C}^\infty-$ diffeomorphism of a neighborhood in $\mathbb{R}^{2n}$ of $\{(0, y, 0, ..., 0) \mid - \mid z \mid /2 \leq y \leq 1- \mid z \mid\}$ which is of class $\mathcal{C}^{r+\lambda}$ with respect to z (for sufficiently small τ, r_0, t_0). Moreover, there exists a map

$$h_z(\omega) : \{(z, \omega) \mid 1 - \frac{r_0}{2} \leq \mid z \mid \leq 1, \mid \omega \mid \leq \eta\} \longrightarrow \mathbb{R}$$

which is of class $\mathcal{C}^\infty$ in ω and of $\mathcal{C}^{r+\lambda}$ in z, such that $h_z(0) = 1- \mid z \mid$ and $\psi(t + h_z(\omega)i, \omega) \in S$ (by shrinking r_0, t_0, τ) for a sufficiently small $\eta > 0$. None of the above bounds depends on z or ζ.
Remark. For the standard structure or for $\tau = 0$ one has

$$h_z(\omega) = \sqrt{1- \mid \omega \mid^2} - \mid z \mid .$$

Next we expand $\psi(t + yi, \omega)$ with respect to t and obtain

$$\begin{aligned} \psi(t + yi, \omega) &= z + y\zeta^0 + \sum_{\gamma=1}^{2n-2} \omega_\gamma U_\gamma(z) + itz + O(t^2 + y^2 + \mid \omega \mid^2) \\ &= z + y\zeta^0 + \sum_{\gamma=1}^{2n-2} \omega_\gamma U_\gamma(z) + itz + O(\mid \zeta - z \mid^2). \end{aligned}$$

Thus with $(\diamondsuit) = \Phi(\psi(t+yi,\omega),z)$, $O_2 = O(|\zeta - z|^2)$, $\zeta = \psi(t+yi,\omega)$

$$\begin{aligned}
(\diamondsuit) &= \sum_{j=1}^{n} \big(X_j \mid \zeta \mid^2\big) \sum_{\alpha=1}^{n} \big(\Lambda_j^\alpha(\zeta)(\psi_a - z_\alpha) + \Gamma_j^\alpha(\zeta)(\overline{\psi}_a - \overline{z}_\alpha)\big) \\
&= \sum_{j=1}^{n} \big(X_j \mid \zeta \mid^2\big) \sum_{\alpha=1}^{n} \big(\Lambda_j^\alpha(z)(\psi_a - z_\alpha) + \Gamma_j^\alpha(z)(\overline{\psi}_a - \overline{z}_\alpha)\big) + O_2 \\
&= \sum_{j=1}^{n} \left(\overline{z}_j + \sum_{\beta=1}^{n} A_j^\beta(z) z_\beta\right) \sum_{\alpha=1}^{n} \big(\Lambda_j^\alpha(z)(\psi_a - z_\alpha) + \Gamma_j^\alpha(z)(\overline{\psi}_a - \overline{z}_\alpha)\big) + O_2 \\
&= \sum_{j=1}^{n} \left(\overline{z}_j + \sum_{\beta=1}^{n} A_j^\beta(z) z_\beta\right) \sum_{\alpha=1}^{n} \Big(\Lambda_j^\alpha(z)(y\zeta_a^0 + itz_\alpha) + \Gamma_j^\alpha(z)(y\overline{\zeta}_a^0 - it\overline{z}_\alpha)\Big) + O_2.
\end{aligned}$$

Now $\Lambda_j^\alpha(z) = \delta_{\alpha j} + O(\tau)$, $\quad \Gamma_j^\alpha(z) = O(\tau)$, $\quad A_j^\alpha(z) = O(\tau)$, implies

$$\begin{aligned}
(\diamondsuit) &= O(|\zeta - z|^2) + O(\tau)O(|y| + |t|) + \sum_{\alpha,j=1}^{n} \overline{z}_j \delta_{\alpha j}(y\zeta_a^0 + itz_\alpha) \\
&= O(|\zeta - z|^2) + O(\tau)O(|y| + |t|) + y \mid z \mid + it \mid z \mid^2 .
\end{aligned}$$

Consequently, we obtain for small τ, since $|\Phi(\zeta,z)| \gtrsim |\zeta - z|^2$,

$$|\Phi(\zeta,z)| \gtrsim |y| + |t| \tag{5.2}$$

for all $\zeta = \psi(t+yi,\omega) \in S$ and $|\zeta - z|$ sufficiently small with bounds and constants independent of ζ, z and τ. Therefore

$$|\Phi(\zeta,z)| \gtrsim |\zeta - z|^2 + |y| + |t| + |r(z)| \gtrsim 1 - |z| + |t| + |\omega|^2 \tag{5.3}$$

for $|t| \le t_0/2$, $|\omega| \le \eta_0$, $1 - r_0/4 \le |z| \le 1$, $\zeta = \psi(t+yi,\omega)$, $y = h_z(\omega)$. From the construction of $\psi(t+yi,\omega)$ it is equally obvious that for small τ

$$|\psi(t+yi,\omega) - z|^2 \gtrsim |t|^2 + y^2 + |\omega|^2 \gtrsim |\zeta - z|^2 .$$

Since $|\zeta - z| \gtrsim |\zeta| - |z| = 1 - |z|$ it follows

$$|\psi(t+yi,\omega) - z|^2 \gtrsim (1 - |z|)^2 + t^2 + |\omega|^2 .$$

Therefore, we obtain with uniform constants $t_1 > 0$, $\eta_1 > 0$, $0 < r_2 < 1$ that for $|t| \le t_1$, $|\omega| \le \eta_1$, $1 - r_2 \le |z| \le 1$, $\zeta = \psi(t + h_z(\omega)i,\omega)$

$$\begin{aligned}
&|\psi(t + h_z(\omega)i,\omega) - z| < r_1, \\
&|\Phi(\zeta,z)| \gtrsim 1 - |z| + |t| + |\omega|^2, \\
&|\psi(t + h_z(\omega)i,\omega) - z|^2 \gtrsim (1 - |z|)^2 + t^2 + |\omega|^2 .
\end{aligned}$$

Moreover, $(t,\omega) \longmapsto \zeta = \psi(t + h_z(\omega)i,\omega)$ defines a $\mathcal{C}^{r+\lambda}$ diffeomorphism of a neighborhood of $0 \in \mathbb{R}^{2n-1}$ onto a neighborhood of $\zeta^0 = z/|z|$ in S. Its jacobian is bounded and bounded away from 0 uniformly for all sufficiently small τ.

To achieve the proof of lemma 3.2 we again apply the Hardy-Littlewood lemma. So let $f \in \mathcal{C}^{\lambda}_{0,1}(\overline{B})$. We have to show that

$$|\mathrm{grad}(F_1 f)(z)| \leq C(\tau) \mid f \mid_{\mathcal{C}^{\lambda}(\overline{B})} (1- \mid z \mid)^{\lambda-1}, \tag{5.4}$$

with $C(\tau) \to 0$ if $\tau \to 0$.
Now

$$F_1 f(z) = \int_S f(\zeta) \wedge \mathcal{F}_1(\zeta, z),$$

where $\mathcal{F}_1$ is a sum of terms of type I, II, III or IV.
Calculating $\mathrm{grad}\mathcal{F}_1$ we obtain a sum of terms

$$\begin{aligned}
I_1 &= \frac{L^{(r-1,r-1)}}{\Phi_0^{n-b}\Phi^{b+1}}, \quad I_2 = \frac{L^{(r-1,r-1)}}{\Phi_0^{n-b-1}\Phi^{b+2}}, \quad \textit{with } b \geq 1,\\
II_1 &= \frac{\widetilde{L}_1^{(r-1,r-1)}}{\Phi_0^{n-b}\Phi^{b+1}}, \quad II_2 = \frac{\widetilde{L}_2^{(r-1,r-1)}}{\Phi_0^{n-b+1}\Phi^{b+1}}, \quad II_3 = \frac{\widetilde{L}_2^{(r-1,r-1)}}{\Phi_0^{n-b}\Phi^{b+2}}, \quad \textit{with } b \geq 1,\\
III_1 &= \frac{L^{(r-1,r-1)}}{\Phi_0^{n-1}\Phi}, \quad III_2 = \frac{\widetilde{L}_1^{(r-1,r-1)}}{\Phi_0^{n}\Phi}, \quad III_3 = \frac{\widetilde{L}_1^{(r-1,r-1)}}{\Phi_0^{n-1}\Phi^2},\\
IV_1 &= \frac{\widetilde{L}_2^{(r-1,r-1)}}{\Phi_0^{n}\Phi}, \quad IV_2 = \frac{\widetilde{L}_3^{(r-1,r-1)}}{\Phi_0^{n+1}\Phi}, \quad IV_3 = \frac{\widetilde{L}_3^{(r-1,r-1)}}{\Phi_0^{n}\Phi^2}.
\end{aligned}$$

The worst cases are when the differentiation falls on Φ that is I_2, II_3, III_3, IV_3. The cases III_3, IV_3 are easier than I_2 and II_3 and these are comparable to each other. Therefore, as a typical example, it suffices to handle

$$II_3 = \int_S f(\zeta) \wedge \frac{\widetilde{L}_2^{(r-1,r-1)}(\zeta, z)}{\Phi_0^{n-b}(\zeta, z)\Phi^{b+2}(\zeta, z)}, \quad b \geq 1.$$

Since the kernel of F_1 has point singularities we can assume that f is supported in a neighborhood of ζ^0, where the $(t, \omega)-$ coordinates are defined. Then $\zeta = \psi(t + h_z(\omega)i, \omega)$, $\zeta^0 = \psi(0, 0)$. It follows

$$\begin{aligned}
II_3 &= \int_{t,\omega} (f(t,\omega) - f(0,\omega) \wedge \frac{\widetilde{L}_2^{(r-1,r-1)}(\zeta, z)}{\Phi_0^{n-b}(\zeta, z)\Phi^{b+2}(\zeta, z)} + \int_{t,\omega} f(0,\omega) \wedge \frac{\widetilde{L}_2^{(r-1,r-1)}(\zeta, z)}{\Phi_0^{n-b}(\zeta, z)\Phi^{b+2}(\zeta, z)}\\
&= II_{3,1} + II_{3,2}.
\end{aligned}$$

Since

$$\left| (f(t,\omega) - f(0,\omega)) \wedge \frac{\widetilde{L}_2^{(r-1,r-1)}}{\Phi_0^{n-b}\Phi^{b+2}} \right| \lesssim \frac{\mid t \mid^{\lambda} C(\tau) \mid f \mid_{\mathcal{C}^{\lambda}}}{\mid \Phi_0 \mid^{n-b} \mid \Phi \mid^{b+2}}$$

by standard estimates (see [1] or [5]) we obtain

$$|II_{3,1}| \lesssim \frac{C(\tau) \mid f \mid_{\mathcal{C}^{\lambda}}}{(1- \mid z \mid)^{1-\lambda}},$$

with $C(\tau) \to 0$. Now the coefficients of $\widetilde{L}_2^{(r-1,r-1)}$ are sums of products $\widetilde{U}_1 \cdot \widetilde{U}_2 \cdots \widetilde{U}_k$ which we have described above. We shall decompose the factors in the following way. When $\widetilde{U}_i$ is of class $\mathcal{C}^1$ or better with respect to ζ we let it unchanged. When it is only $\mathcal{C}^\lambda$ $(r=1)$ we write

$$\widetilde{U}_i(t,\omega) = (\widetilde{U}_i(t,\omega) - \widetilde{U}_i(0,\omega)) + \widetilde{U}_i(0,\omega).$$

By decomposing all these terms in the product we obtain $II_{3,2} = II_{3,2,1} + II_{3,2,2}$, with

$$|II_{3,2,1}| \lesssim \frac{C(\tau)\sup \mid f \mid}{(1- \mid z \mid)^{1-\lambda}},$$

by estimating exactly as for $II_{3,1}$, and

$$II_{3,2,2} = \int_{t,\omega} f(0,\omega) \wedge \frac{\widetilde{L}_2^{(r-1,r-1)}((0,\omega),z)}{\Phi_0^{n-b}(\zeta,z)\Phi^{b+2}(\zeta,z)}.$$

We remind the reader that we did not decompose factors like $z_k - \zeta_k$ or $A_j^\alpha(\zeta) - A_j^\alpha(z)$, for example, in order to not destroy vanishing orders for $\zeta = z$.

Next we diminish the exponent of Φ in the denominator. Since

$$\frac{1}{\Phi^{b+2}} = \frac{-1}{(b+1)\mathcal{T}_\zeta\Phi}\mathcal{T}_\zeta\left(\frac{1}{\Phi^{b+1}}\right) = \frac{-1}{(b+1)\mathcal{T}_\zeta\Phi}\frac{\partial}{\partial t}\left(\frac{1}{\Phi^{b+1}}\right),$$

it follows

$$\frac{1}{\Phi(t,\omega)^{b+2}} = \frac{(\mathcal{T}_\zeta\Phi(0,\omega,z) - \mathcal{T}_\zeta\Phi(t,\omega,z))}{\mathcal{T}_\zeta\Phi(0,\omega,z)}\frac{1}{\Phi^{b+2}} - \frac{1}{(b+1)\mathcal{T}_\zeta\Phi(0,\omega,z)}\frac{\partial}{\partial t}\left(\frac{1}{\Phi^{b+1}}\right).$$

This gives rise to the decomposition $II_{3,2,2} = II_{3,2,2,1} + II_{3,2,2,2}$, with

$$|II_{3,2,2,1}| \lesssim \frac{C(\tau)\sup \mid f \mid}{(1- \mid z \mid)^{1-\lambda}}$$

and

$$II_{3,2,2,2} = \int_{t,\omega} f(0,\omega) \wedge \frac{\widetilde{L}_2^{(r-1,r-1)}((0,\omega),z)}{\mathcal{T}_\zeta\Phi(0,\omega,z)\Phi_0^{n-b}(\zeta,z)}\frac{\partial}{\partial t}\left(\frac{1}{\Phi^{b+1}}\right).$$

By an integration by parts we get

$$II_{3,2,2,2} = \int_{t,\omega} f(0,\omega) \wedge \frac{\widetilde{L}_2^{(r-1,r-1)}((0,\omega),z)\mathcal{T}_\zeta\Phi_0(t,\omega,z)}{\mathcal{T}_\zeta\Phi(0,\omega,z)\Phi_0^{n-b+1}(\zeta,z)\Phi^{b+1}(\zeta,z)}.$$

Note that $\mathcal{T}_\zeta\Phi_0$ is a sum of 2 terms. In the first, denoted by $V(t,\omega,z)$, $\mathcal{T}_\zeta$ has fallen on $\zeta - z$ term. Therefore V has vanshing order 1 for $\zeta = z$ but there is no loss of regularity. In the second term $\mathcal{T}_\zeta$ has fallen on a a structure coefficient. This term is only $\mathcal{C}^\lambda$ if $r=1$, but vanishes of order 2. This latter case gives rise to an integral which can be estimated by $C(\tau)\sup \mid f \mid\mid \log(1- \mid z \mid) \mid \leq C(\tau)\sup \mid f \mid /(1- \mid z \mid)^{1-\lambda}$. V gives rise, after having been splitted into $(V(t,\omega,z) - V(0,\omega,z)) + V(0,\omega,z)$, to an integral

$$\int_{t,\omega} f(0,\omega) \wedge \frac{\widetilde{L}_3^{(r-1,r-1)}((0,\omega),z)}{\mathcal{T}_\zeta\Phi(0,\omega,z)\Phi_0^{n-b+1}(\zeta,z)\Phi^{b+1}(\zeta,z)}.$$

Now by repeating this method at most b times we end up with a critical integral of type

$$I(z)=\int_{t,\omega} f(0,\omega)\wedge\frac{\widetilde{L}^{(r-1,r-1)}_{3+b}((0,\omega),z)}{\mathcal{T}_\zeta\Phi(0,\omega,z)^{b+1}\Phi_0^{n+1}(\zeta,z)\Phi(\zeta,z)}.$$

Expressing everything with respect to the (t,ω) coordinates we get in the worst case the following estimate (with respect to polar coordinates in the ω space, $a=\mid\omega\mid$, $d=1-\mid z\mid$)

$$\begin{aligned}&\mid\ I(z)\mid\leq C(\tau)\sup\mid f\mid\int_{t,a}\frac{a^{2n-3+b+3}dtda}{(d+a+\mid t\mid)^{2n+2}(d+\mid t\mid+a^2)}\\&\leq C(\tau)(\sup\mid f\mid)\mid\log d\mid^2\\&\leq C(\tau)\sup\mid f\mid/(1-\mid z\mid)^{1-\lambda}\end{aligned}$$

since $b\geq 1$. This achieves the proof of lemma 3.2.

Proof of lemma 3.3. By its definition $B_{n,0}$ can be decomposed into a linear combination of terms

$$\frac{h(\zeta)P_i(\zeta-z)}{\Phi_0(\zeta,z)^n},$$

where $i\geq 1$, $h\in\mathcal{C}^{r+\lambda-1}_{n,n-1}(\overline{B})$ is a polynomial of the A^i_j and some of their derivatives and $P_i(\zeta-z)$ is a homogenous polynomial of $\zeta-z$ and $\overline{\zeta}-\overline{z}$ of degree i with constant coefficients. If $i\geq 2$ the corresponding integral can be differentiated under the integral sign and the assertion follows by standard estimates. So we are left with

$$\int_B g(\zeta)\wedge\frac{h(\zeta)P_1(\zeta-z)}{\Phi_0(\zeta,z)^n}.$$

Let $\varphi\in\mathcal{C}^\infty(\mathbb{C}^n)$ have compact support in B with $\varphi\equiv 1$ on a neighborhood of K. Then the assertion for

$$\int_B(1-\varphi(\zeta))g(\zeta)\wedge\frac{h(\zeta)P_1(\zeta-z)}{\Phi_0(\zeta,z)^n}$$

is trivial. So we consider

$$\int_B\varphi(\zeta)g(\zeta)\wedge\frac{h(\zeta)P_1(\zeta-z)}{\Phi_0(\zeta,z)^n}.$$

We denote the real and imaginary parts of the ζ_i´s by $x_1,...,x_{2n}$, and correspondingly for the z_i´s by $y_1,...,y_{2n}$. Then the above integral equals

$$\int_B\varphi(x)g(x)\wedge\frac{h(x)P_1(x_k-y_k)}{\left(\sum_{i,j=1}^n a_{ij}(x)(x_i-y_i)(x_j-y_j)\right)^n},$$

with $a_{ij}\in\mathcal{C}^{r+\lambda}(\overline{B})$.

Let $\varepsilon>0$ and set

$$I_\varepsilon=\int_B\varphi(x)g(x)\wedge\frac{h(x)P_1(x_k-y_k)}{\left(\sum_{i,j=1}^n a_{ij}(x)(x_i-y_i)(x_j-y_j)+\varepsilon\right)^n}.$$

Since for fixed $z \quad lim_{\varepsilon\to 0} I_\varepsilon(z) = I(z)$ it suffices to show $\mathcal{C}^\lambda - \mathcal{C}^1-$estimates for I_ε which are uniform with respect to ε.
Now $\frac{\partial}{\partial y_l} I_\varepsilon = I_{\varepsilon,1} - I_{\varepsilon,2}$, with

$$I_{\varepsilon,1} = \int_B \varphi(x) g(x) h(x) \wedge \left(\frac{\partial}{\partial y_l} + \frac{\partial}{\partial x_l}\right)\left(\frac{P_1(x_k - y_k)}{\left(\sum_{i,j=1}^n a_{ij}(x)(x_i - y_i)(x_j - y_j) + \varepsilon\right)^n}\right)$$

$$I_{\varepsilon,2} = \int_B \varphi(x) g(x) h(x) \wedge \frac{\partial}{\partial x_l}\left(\frac{P_1(x_k - y_k)}{\left(\sum_{i,j=1}^n a_{ij}(x)(x_i - y_i)(x_j - y_j) + \varepsilon\right)^n}\right).$$

The estimates for $I_{\varepsilon,1}$ are trivial since $\left(\frac{\partial}{\partial y_l} + \frac{\partial}{\partial x_l}\right)(x_i - y_i) = 0$. Now $I_{\varepsilon,2} = I_{\varepsilon,2,1} + I_{\varepsilon,2,2}$, with

$$I_{\varepsilon,2,1} = \int_B \varphi(x)((gh)(x) - (gh)(y)) \wedge \frac{\partial}{\partial x_l}\left(\frac{P_1(x_k - y_k)}{\left(\sum_{i,j=1}^n a_{ij}(x)(x_i - y_i)(x_j - y_j) + \varepsilon\right)^n}\right),$$

$$\begin{aligned} I_{\varepsilon,2,2} &= \int_B \varphi(x)(gh)(y) \wedge \frac{\partial}{\partial x_l}\left(\frac{P_1(x_k - y_k)}{\left(\sum_{i,j=1}^n a_{ij}(x)(x_i - y_i)(x_j - y_j) + \varepsilon\right)^n}\right) \\ &= -\int_B \frac{\partial \varphi(x)}{\partial x_l}(gh)(y) \wedge \left(\frac{P_1(x_k - y_k)}{\left(\sum_{i,j=1}^n a_{ij}(x)(x_i - y_i)(x_j - y_j) + \varepsilon\right)^n}\right). \end{aligned}$$

The estimates for $I_{\varepsilon,2,2}$ are also trivial and $|I_{\varepsilon,2,1}|$ can be majorized by

$$| \, g \, |_{\mathcal{C}^\lambda(\overline{B})} \int_B \frac{| \, x - y \, |^\lambda \, dV(x)}{| \, x - y \, |^{2n}} \lesssim | \, g \, |_{\mathcal{C}^\lambda(\overline{B})} \, .$$

References

[1] P. Ahern/R. Schneider. A Smoothing property of the Henkin and Szegö projections. Duke Math. J. vol. 47, No. 1 (1980), p. 135 - 143.

[2] A. Douglis/L. Nirenberg. Interior Estimates for Elliptic Systems of Partial Differential Equations. Comm. Pure Appl. Math., Vol. 8 (1955), p. 503 - 538.

[3] J. Koppelman. The Cauchy Integral for Differential Forms. Bull. Amer. Math. Soc. 73 (1967), p. 554 - 556.

[4] F. Norguet. Problèmes sur les formes différentielles et les courants. Ann. Inst. Fourier 11 (1960), p. 1 - 88.

[5] R. M. Range. Holomorphic Functions and Integral Representations in Several Complex Variables. Springer-Verlag.

[6] R. M. Range/Y. T. Siu. Uniform Estimates for the $\overline{\partial}$-Equation on Domains with Piecewise Smooth Strictly Pseudoconvex Domains. Math. Ann. 206 (1973), p. 325 - 354.

[7] E. M. Stein. Singular Integrals and Differentiability Properties of Functions. Princeton University Press (1970)

[8] S. M. Webster. A New Proof of the Newlander-Nirenberg Theorem. Math. Zeit. 201 (1989), p. 303 - 316.

Joachim Michel, LMPA EA n° 2597, Centre universitaire de la Mi-Voix, 50, rue F. Buisson, F-62100 Calais, France
E-mail address: michel@lmpa.univ-littoral.fr