

Tangent Lifts of Poisson and Related Structures

**Janusz Grabowski
Paweł Urbański**

Vienna, Preprint ESI 224 (1995)

April 25, 1995

Supported by Federal Ministry of Science and Research, Austria
Available via <http://www.esi.ac.at>

TANGENT LIFTS OF POISSON AND RELATED STRUCTURES

J GRABOWSKI¹ AND P URBAŃSKI²

¹ Erwin Schrödinger Institute for Mathematical Physics
Pasteurgasse 6/7, 1090 Wien, Austria
Institute of Mathematics, University of Warsaw
Banacha 2, 02-097 Warsaw, Poland

² Division of Mathematical Methods
in Physics, University of Warsaw
Hoża 74, 00-682 Warsaw, Poland

ABSTRACT. The derivation d_T on the exterior algebra of forms on a manifold M with values in the exterior algebra of forms on the tangent bundle TM is extended to multivector fields. These tangent lifts are studied with applications to the theory of Poisson structures, canonical vector fields and Poisson-Lie groups.

0. Introduction. A derivation d_T on the exterior algebra of forms on a manifold M with values in the exterior algebra of forms of the tangent bundle TM plays essential role in the calculus of variations ([Tu]) and, in particular, in analytical mechanics. The derivation $d_T\omega$ of the symplectic 2-form of a symplectic manifold (M, ω) provides the tangent bundle TM with a symplectic structure. The concept of a generalized hamiltonian system can be introduced as a Lagrangian submanifold of $(TM, d_T\omega)$. The infinitesimal dynamics of a relativistic particle is an example of such a system. The derivation d_T has also an aspect of the total Lie derivative: $\mathcal{L}_X\mu = X^*d_T\mu$ (Theorem 3.2). The aim of this paper is to extend d_T to the exterior algebra of multivector fields and to establish relations related to Poisson structures which correspond to the mentioned above relations in symplectic geometry. In first two sections we concentrate on the definition of d_T in a more general than usual (i. e., as a commutator $[i_T, d]$) framework, which gives a possibility to define d_T on multivector fields. The basic property of d_T is that it commutes with the Schouten bracket, what corresponds to the fact that d_T commutes with d on forms. In Section 3 we show that, as in the case of forms, d_T plays the role of the total Lie derivative. As a consequence, we can describe in Section 6 a canonical vector field on a Poisson manifold as a Lagrangian submanifold with respect to the tangent Poisson structure introduced in Section 5.

The derivation d_T helps to identify vector bundle morphisms $\nu: T^*M \rightarrow TM$, which correspond to Poisson structures (Theorem 4.4). This identification is complementary to ones expressed in terms of the Jacobi identity and the Schouten bracket. What is important, the condition for ν is expressed in terms of objects and morphisms and does not require any additional, general operations like exterior derivative and the Schouten bracket. Hence, it is a subject for functorial treatments.

The presentation of the tangent Poisson structure in Section 5 is close to the one given Courant in [Co1, Co2, Co3]. The remaining part of the paper is devoted to the tangent lift of Poisson-Lie structures and to the analysis of the symplectic foliations of tangent Poisson manifolds.

Classification PACS: 02.40+m, 03.20+i.

¹ Partially supported by the Isaac Newton Institute for Mathematical Sciences, Cambridge, England.

² Supported by KBN, grant No 2 11129 91 01.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

This work is a contribution to a program of geometric formulations of physical theories conducted jointly with W. M. Tulczyjew.

1. Geometric preliminaries.

Let M be a smooth manifold. By $\tau_M: TM \rightarrow M$ we denote the tangent fibration and by $\pi_M: T^*M \rightarrow M$ the cotangent fibration. For $r = 0, 1, 2, \dots$, we define exterior product bundles $\bigwedge^r TM$ and $\bigwedge^r T^*M$ with the canonical projections $\tau_M^r: \bigwedge^r TM \rightarrow M$ and $\pi_M^r: \bigwedge^r T^*M \rightarrow M$ respectively. For $r = 0$ we have $\bigwedge^0 TM \simeq \bigwedge^0 T^*M \simeq M \times \mathbb{R}$.

There is a collection of canonical pairings

$$\langle \cdot, \cdot \rangle_M^r: \bigwedge^r TM \times_M \bigwedge^r T^*M \rightarrow \mathbb{R}.$$

By applying the tangent functor to these pairings, we obtain tangent pairings

$$\langle \cdot, \cdot \rangle_{TM}^r: T\bigwedge^r TM \times_{TM} T\bigwedge^r T^*M \rightarrow T\mathbb{R} \rightarrow T_0\mathbb{R} = \mathbb{R}.$$

We used the canonical identification of bundles

$$T\bigwedge^r TM \times_{TM} T\bigwedge^r T^*M \simeq T(\bigwedge^r TM \times_M \bigwedge^r T^*M).$$

Let (x^i) be a local coordinate system in M . In bundles $\bigwedge^r TM$, $\bigwedge^r T^*M$, $T\bigwedge^r TM$ and $T\bigwedge^r T^*M$ we have adopted coordinate systems

$(x^i, \dot{x}^{j_1 \dots j_r})$, $(x^i, p_{j_1 \dots j_r})$, $(x^i, \dot{x}^{j_1 \dots j_r}, \delta x^k, \delta \dot{x}^{l_1 \dots l_r})$ and $(x^i, p_{j_1 \dots j_r}, \dot{x}^k, \dot{p}_{l_1 \dots l_r})$ respectively, where $j_1 < j_2 < \dots < j_r$, etc. . In these coordinates the introduced pairings read as follows:

$$\langle \cdot, \cdot \rangle_M^r: ((x^i, \dot{x}^{j_1 \dots j_r}), (x^i, p_{l_1 \dots l_r})) \mapsto \sum_{j_1 < j_2 < \dots < j_r} \dot{x}^{j_1 \dots j_r} p_{j_1 \dots j_r}$$

and

$$\langle \cdot, \cdot \rangle_{TM}^r: ((x^i, \dot{x}^{j_1 \dots j_r}, \delta x^k, \delta \dot{x}^{l_1 \dots l_r}), (x^i, p_{j_1 \dots j_r}, \delta x^k, \dot{p}_{l_1 \dots l_r})) \mapsto \sum_{j_1 < \dots < j_r} (\delta \dot{x}^{j_1 \dots j_r} \cdot p_{j_1 \dots j_r} + \dot{x}^{j_1 \dots j_r} \cdot \dot{p}_{j_1 \dots j_r}).$$

For each manifold M there is a canonical diffeomorphism (cf. [Tu1])

$$\kappa_M: TTM \rightarrow TTM$$

which is an isomorphism of vector bundles

$$\tau_{TM}: TTM \rightarrow TM \quad \text{and} \quad T\tau_M: TTM \rightarrow TM.$$

By α_M we denote an isomorphisms

$$\alpha_M: TT^*M \rightarrow T^*TM$$

of vector bundles

$$T\pi_M: TT^*M \rightarrow TM \quad \text{and} \quad \pi_{TM}: T^*TM \rightarrow TM,$$

dual to κ_M with respect to pairings $\langle \cdot, \cdot \rangle_{TM}^r = \langle \cdot, \cdot \rangle_{TM}^1$ and $\langle \cdot, \cdot \rangle_{TM} = \langle \cdot, \cdot \rangle_{TM}^1$:

$$\langle v, \alpha_M(w) \rangle_{TM} = \langle \kappa_M(v), w \rangle_{TM}^r.$$

In the following, by ε_M we denote α_M^{-1} . In the introduced above local coordinates in $\mathbb{T}TM$, $\mathbb{T}\mathbb{T}^*M$ and the adopted from (x^i) coordinates $(x^i, \dot{x}^j, \pi_k, \dot{\pi}_l)$ in $\mathbb{T}^*\mathbb{T}M$, we have

$$(x^i, \dot{x}^j, \delta x^k, \delta \dot{x}^l) \circ \kappa_M = (x^i, \delta x^j, \dot{x}^k, \delta \dot{x}^l)$$

and

$$(x^i, \dot{x}^j, \pi_k, \dot{\pi}_l) \circ \alpha_M = (x^i, \dot{x}^j, \dot{p}_k, p_l).$$

Now, we generalize these morphisms to the multilinear case. Let \bigwedge_M^r be the wedge product mapping

$$\bigwedge_M^r: \times_{\mathbb{T}M}^r \mathbb{T}M \rightarrow \bigwedge^r \mathbb{T}M.$$

We apply the tangent functor to this mapping and we get

$$\mathbb{T}\bigwedge_M^r: \mathbb{T} \times_{\mathbb{T}M}^r \mathbb{T}M \rightarrow \mathbb{T}\bigwedge^r \mathbb{T}M.$$

Since κ_M extends to an isomorphism of vector bundles

$$\times^r \kappa_M: \times_{\mathbb{T}TM}^r \mathbb{T}TM \rightarrow \times_{\mathbb{T}TM}^r \mathbb{T}TM \simeq \mathbb{T} \times_{\mathbb{T}M}^r \mathbb{T}M,$$

we get also

$$\mathbb{T}\bigwedge_M^r \circ (\times^r \kappa_M): \times_{\mathbb{T}TM}^r \mathbb{T}TM \rightarrow \mathbb{T}\bigwedge^r \mathbb{T}M.$$

It is easy to verify that this mapping is multilinear and skew-symmetric and, consequently, defines a morphism κ_M^r of vector bundles over $\mathbb{T}M$:

$$\kappa_M^r: \bigwedge^r \mathbb{T}TM \rightarrow \mathbb{T}\bigwedge^r \mathbb{T}M. \quad (1.1)$$

In other words,

$$\kappa_M^r \circ \bigwedge_{\mathbb{T}M}^r = \mathbb{T}\bigwedge_M^r \circ \times^r \kappa_M,$$

i. e., the following diagram is commutative

$$\begin{array}{ccc} \times_{\mathbb{T}TM}^r \mathbb{T}TM & \xrightarrow{\bigwedge_{\mathbb{T}M}^r} & \bigwedge^r \mathbb{T}TM \\ \times^r \kappa_M \downarrow & & \downarrow \kappa_M^r \\ \mathbb{T} \times_{\mathbb{T}M}^r \mathbb{T}M & \xrightarrow{\mathbb{T}\bigwedge_M^r} & \mathbb{T}\bigwedge^r \mathbb{T}TM \end{array} \quad (1.2)$$

Of course, $\kappa_M^1 = \kappa_M$ and for a simple r -vector $v_1 \wedge \cdots \wedge v_r$ on $\mathbb{T}M$ we have

$$\kappa_M^r(v_1 \wedge \cdots \wedge v_r) = \mathbb{T}\bigwedge_M^r(\kappa_M(v_1), \dots, \kappa_M(v_r)).$$

In local coordinates κ_M^r reads as follows:

$$(x^i, \dot{x}^{j_1 \dots j_r}, \delta x^k, \delta \dot{x}^{l_1 \dots l_r}) \circ \kappa_M^r = (x_i, \delta x^{j_1 \dots j_r}, \dot{x}^k, \sum_m \delta x^{l_1 \dots l_{m-1}} \wedge \delta \dot{x}^{l_m} \wedge \delta x^{l_{m+1} \dots l_r}),$$

where $(x^i, \dot{x}^j, \delta x^{k_1 \dots k_n} \wedge \delta \dot{x}^{l_{n+1} \dots l_r})$ are adopted coordinates in $\bigwedge^r \mathbb{T}TM$, with the obvious identification

$$\delta x^{l_1 \dots l_{m-1}} \wedge \delta \dot{x}^{l_m} \wedge \delta x^{l_{m+1} \dots l_r} = (-1)^{r-m} \delta x^{l_1 \dots l_{m-1} l_{m+1} \dots l_r} \wedge \delta \dot{x}^{l_m}.$$

Let ${}^i \bigwedge_M^{r-i}$ be the wedge product

$${}^i \bigwedge_M^{r-i}: \bigwedge^i \mathbb{T}M \times_M \bigwedge^{r-i} \mathbb{T}M \rightarrow \bigwedge^r \mathbb{T}M.$$

From the diagram (1.2) we easily get that the diagram

$$\begin{array}{ccc}
\bigwedge^i \mathbb{T}TM \times_{\mathbb{T}M} \bigwedge^{r-i} \mathbb{T}TM & \xrightarrow{\kappa_M^i \times \kappa_M^{r-i}} & \mathbb{T} \bigwedge^i \mathbb{T}M \times_{\mathbb{T}M} \mathbb{T} \bigwedge^{r-i} \mathbb{T}M \\
\downarrow \bigwedge^i \mathbb{T}M^{r-i} & & \downarrow \mathbb{T} \bigwedge^i M^{r-i} \\
\bigwedge^r \mathbb{T}TM & \xrightarrow{\kappa_M^r} & \mathbb{T} \bigwedge^r \mathbb{T}M
\end{array} \tag{1.3}$$

is commutative.

In order to define ε_M^r , a counterpart to κ_M^r , we consider first the mapping

$$\mathbb{T} \bigwedge_{\mathbb{T}^*M}^r \circ (\times^r \varepsilon_M): \times^r \pi_{\mathbb{T}M}^* \mathbb{T}^* \mathbb{T}M \rightarrow \mathbb{T} \bigwedge^r \mathbb{T}^* \mathbb{T}M.$$

Since it is multilinear and skew-symmetric, it defines a mapping

$$\varepsilon_M^r: \bigwedge^r \mathbb{T}^* \mathbb{T}M \rightarrow \mathbb{T} \bigwedge^r \mathbb{T}^* M$$

and, as in the case of κ_M^r , we have

$$\varepsilon_M^r \circ \bigwedge_{\pi_{\mathbb{T}M}}^r = \mathbb{T} \bigwedge_{\pi_M}^r \circ \times^r \varepsilon_M,$$

i. e., the diagram

$$\begin{array}{ccc}
\times^r \pi_{\mathbb{T}M}^* \mathbb{T}^* \mathbb{T}M & \xrightarrow{\bigwedge_{\mathbb{T}M}^r} & \bigwedge^r \mathbb{T}^* \mathbb{T}M \\
\downarrow \times^r \varepsilon_M & & \downarrow \varepsilon_M^r \\
\mathbb{T} \times_{\pi_M}^r \mathbb{T}^* M & \xrightarrow{\mathbb{T} \bigwedge_M^r} & \mathbb{T} \bigwedge^r \mathbb{T}^* M
\end{array} \tag{1.4}$$

is commutative.

Let $(x^i, \dot{x}^j, \pi^{k_1 \dots k_n} \wedge \dot{\pi}^{l_{n+1} \dots l_r})$ be the adopted coordinate system in $\bigwedge^r \mathbb{T}^* \mathbb{T}M$. We have

$$(x^i, p_{j_1 \dots j_r}, \dot{x}^k, \dot{p}_{l_1 \dots l_r}) \circ \varepsilon_M^r = (x^i, \dot{\pi}_{j_1 \dots j_r}, \dot{x}^k, \sum_m \dot{\pi}_{l_1 \dots l_{m-1}} \wedge \pi_{l_m} \wedge \dot{\pi}_{l_{m+1} \dots l_r}).$$

Here we identified $\dot{\pi}_{l_1 \dots l_{m-1}} \wedge \pi_{l_m} \wedge \dot{\pi}_{l_{m+1} \dots l_r}$ and $(-1)^{m-1} \pi_{l_m} \wedge \dot{\pi}_{l_1 \dots l_{m-1} l_{m+1} \dots l_r}$.

We have also a commutative diagram

$$\begin{array}{ccc}
\bigwedge^i \mathbb{T}^* \mathbb{T}M \times_{\mathbb{T}M} \bigwedge^{r-i} \mathbb{T}^* \mathbb{T}M & \xrightarrow{\varepsilon_M^i \times \varepsilon_M^{r-i}} & \mathbb{T} \bigwedge^i \mathbb{T}^* M \times_{\mathbb{T}M} \mathbb{T} \bigwedge^{r-i} \mathbb{T}^* M \\
\downarrow \bigwedge^i \mathbb{T}M^{r-i} & & \uparrow \mathbb{T} \bigwedge^i M^{r-i} \\
\bigwedge^r \mathbb{T}^* \mathbb{T}M & \xrightarrow{\varepsilon_M^r} & \mathbb{T} \bigwedge^r \mathbb{T}^* M
\end{array} \tag{1.5}$$

By $\kappa_M^{r'}$ and $\varepsilon_M^{r'}$ we denote vector bundle morphism

$$\kappa_M^{r'}: \mathbb{T} \bigwedge^r \mathbb{T}^* M \rightarrow \bigwedge^r \mathbb{T}^* \mathbb{T}M,$$

and

$$\varepsilon_M^{r'}: \mathbb{T} \bigwedge^r \mathbb{T}M \rightarrow \bigwedge^r \mathbb{T} \mathbb{T}M,$$

dual to κ_M^r and ε_M^r with respect to pairings $\langle \cdot, \cdot \rangle_{\mathbb{T}M}^r$ and $\langle \cdot, \cdot \rangle_{\mathbb{T}M}^r$. We have, in particular, $\kappa_M^{1'} = \kappa_M' = \varepsilon_M^{-1} = \alpha_M$ and $\varepsilon_M^{1'} = \kappa_M$.

Functorial properties

It is known that κ_M and α_M are natural transformations of iterated functors $\mathbb{T}\mathbb{T}$, $\mathbb{T}\mathbb{T}^*$ and $\mathbb{T}^*\mathbb{T}$, i. e., that for every morphism $\varphi: M \rightarrow N$ we have

$$\begin{aligned}\kappa_N \circ \mathbb{T}\mathbb{T}\varphi &= \mathbb{T}\mathbb{T}\varphi \circ \kappa_M, \\ \alpha_M \circ \mathbb{T}\mathbb{T}^*\varphi &= \mathbb{T}^*\mathbb{T}\varphi \circ \alpha_N, \\ \mathbb{T}\mathbb{T}^*\varphi \circ \varepsilon_N &= \varepsilon_M \circ \mathbb{T}^*\mathbb{T}\varphi.\end{aligned}\tag{1.6}$$

Note that $\mathbb{T}^*\varphi$ is, in general, not a mapping but a relation only, with the domain $\mathbb{T}_{\varphi(M)}^*N$ and codomain $(\ker \mathbb{T}\varphi)^\circ$. Morphisms

$$\bigwedge^r \mathbb{T}\varphi: \bigwedge^r \mathbb{T}M \rightarrow \bigwedge^r \mathbb{T}N$$

and

$$\bigwedge^r \mathbb{T}^*\varphi: \bigwedge^r \mathbb{T}^*N \rightarrow \bigwedge^r \mathbb{T}^*M$$

are defined by relations

$$\bigwedge^r \mathbb{T}\varphi \circ \bigwedge_M^r = \bigwedge_N^r \circ \times^r \mathbb{T}\varphi$$

and

$$\bigwedge^r \mathbb{T}^*\varphi \circ \bigwedge_N^r = \bigwedge_M^r \circ \times^r \mathbb{T}^*\varphi.$$

Theorem 1.1. *For $\varphi: M \rightarrow N$ we have*

$$\kappa_N^r \circ \bigwedge^r \mathbb{T}\mathbb{T}\varphi = \mathbb{T}\bigwedge^r \mathbb{T}\varphi \circ \kappa_M^r$$

and

$$\mathbb{T}\bigwedge^r \mathbb{T}^*\varphi \circ \varepsilon_N^r = \varepsilon_M^r \circ \bigwedge^r \mathbb{T}^*\mathbb{T}\varphi.$$

Proof. From the definition of κ_N^r it follows that

$$\begin{aligned}\kappa_N^r \circ \bigwedge^r \mathbb{T}\mathbb{T}\varphi \circ \bigwedge_{\mathbb{T}M}^r &= \kappa_N^r \circ \bigwedge_{\mathbb{T}N}^r \circ \times^r \mathbb{T}\mathbb{T}\varphi = \mathbb{T}\bigwedge_N^r \circ \times^r \kappa_N \circ \times^r \mathbb{T}\mathbb{T}\varphi \\ &= \mathbb{T}\bigwedge_N^r \circ \times^r (\kappa_N \circ \mathbb{T}\mathbb{T}\varphi) = \mathbb{T}\bigwedge_N^r \circ \times^r (\mathbb{T}\mathbb{T}\varphi \circ \kappa_M) = \mathbb{T}\bigwedge_N^r \circ \times^r \mathbb{T}\mathbb{T}\varphi \circ \times^r \kappa_M.\end{aligned}$$

Since

$$\bigwedge^r \mathbb{T}\varphi \circ \bigwedge_M^r = \bigwedge_N^r \circ \times^r \mathbb{T}\varphi,$$

we get

$$\mathbb{T}\bigwedge^r \mathbb{T}\varphi \circ \mathbb{T}\bigwedge_M^r = \mathbb{T}\bigwedge_N^r \circ \times^r \mathbb{T}\mathbb{T}\varphi$$

and

$$\mathbb{T}\bigwedge_N^r \circ \times^r \mathbb{T}\mathbb{T}\varphi \circ \times^r \kappa_M = \mathbb{T}\bigwedge^r \mathbb{T}\varphi \circ \mathbb{T}\bigwedge_M^r \circ \times^r \kappa_M = \mathbb{T}\bigwedge^r \mathbb{T}\varphi \circ \kappa_M^r \circ \bigwedge_{\mathbb{T}M}^r.$$

This completes the proof of the first identity. The proof of the second one is analogous. ■

We have also the dual identities:

Theorem 1.2. *For $\varphi: M \rightarrow N$ we have*

$$\kappa_M^{r'} \circ \mathbb{T}\bigwedge^r \mathbb{T}^*\varphi = \bigwedge^r \mathbb{T}^*\mathbb{T}\varphi \circ \kappa_N^{r'}$$

and

$$\bigwedge^r \mathbb{T}\mathbb{T}\varphi \circ \varepsilon_M^{r'} = \varepsilon_N^{r'} \circ \mathbb{T}\bigwedge^r \mathbb{T}\varphi.$$

2. Derivation $d_{\mathbb{T}}$ of differential forms and multivector fields.

In this section we refer to the theory of derivations of differential forms as presented in [PiTu].

Definition. Let $\Phi = \bigoplus_{q=0}^{\infty} \Phi^q$ and $\Psi = \bigoplus_{q=0}^{\infty} \Psi^q$ be commutative graded algebras and let $\rho: \Phi \rightarrow \Psi$ be a graded algebra homomorphism. A linear mapping $a: \Phi \rightarrow \Psi$ is called a ρ -derivation of degree r if:

$$a(\Phi^q) \subset \Psi^{q+r}$$

and

$$a(\mu \wedge \nu) = a(\mu) \wedge \rho(\nu) + (-1)^{qr} \rho(\mu) \wedge a(\nu),$$

where $q = \text{degree } \mu$.

Let M be a manifold and let $\tau: E \rightarrow M$ be a vector fibration. By $\Phi(\tau)$ we denote the graded exterior algebra generated by sections of τ . For $\tau = \pi_M$ we get the graded algebra of forms on the manifold M and for $\tau = \tau_M$ the graded algebra of multivector fields on M .

Let be $\mu \in \Phi^r(\pi_M)$, i. e., μ is an r -form on M . The *vertical lift* of μ is an r -form $v_{\mathbb{T}}(\mu) \in \Phi^r(\pi_{\mathbb{T}M})$, $v_{\mathbb{T}}(\mu) = \tau_M^* \mu$, i. e., $v_{\mathbb{T}}(\mu)$ is the pull-back of μ with respect to the projection τ_M . Since the pull-back commutes with the wedge product, the mapping

$$v_{\mathbb{T}}: \Phi(\pi_M) \rightarrow \Phi(\pi_{\mathbb{T}M})$$

is a homomorphism of graded commutative algebras.

A *second order vector field* Γ on M is a vector field on $\mathbb{T}M$ such that

$$\tau_{\mathbb{T}M} \circ \Gamma = \text{id}_{\mathbb{T}M} = (\mathbb{T}\tau_M) \circ \Gamma$$

or, equivalently, $\kappa_M \circ \Gamma = \Gamma$. In adopted local coordinates,

$$\Gamma(x, \dot{x}) = \dot{x}^k \frac{\partial}{\partial x^k} + f^k(x, \dot{x}) \frac{\partial}{\partial \dot{x}^k}.$$

The contraction of the vertical lift of a form $\mu \in \Phi^r(\pi_M)$, $r > 0$, with Γ does not depend on the choice of the second order field Γ . We define

$$i_{\mathbb{T}}\mu = \begin{cases} i_{\Gamma}(v_{\mathbb{T}}\mu) \in \Phi^{r-1}(\pi_{\mathbb{T}M}), & \text{for } r > 0 \\ 0 & \text{for } r = 0. \end{cases}$$

The *tangent lift* $d_{\mathbb{T}}\mu$ of $\mu \in \Phi(\pi_M)$ is defined by

$$d_{\mathbb{T}}\mu = di_{\mathbb{T}}\mu + i_{\mathbb{T}}d\mu = \mathcal{L}_{\Gamma}v_{\mathbb{T}}(\mu).$$

Since \mathcal{L}_{Γ} is a derivation in $\Phi(\pi_{\mathbb{T}M})$ and $v_{\mathbb{T}}$ is a homomorphism of graded algebras, it follows that $d_{\mathbb{T}}$ is a $v_{\mathbb{T}}$ -derivation of degree 0.

If, in local coordinates, $\mu = \mu_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$, then

$$d_{\mathbb{T}}\mu(x, \dot{x}) = \frac{\partial \mu_{i_1 \dots i_r}}{\partial x^k}(x) \dot{x}^k dx^{i_1} \wedge \dots \wedge dx^{i_r} + \sum_m \mu_{i_1 \dots i_r}(x) dx^{i_1} \wedge \dots \wedge dx^{i_m} \wedge \dots \wedge dx^{i_r} \quad (2.1)$$

for $r > 0$ and

$$d_{\mathbb{T}}\mu(x, \dot{x}) = \frac{\partial \mu}{\partial x^i}(x) \dot{x}^i$$

for $r = 0$.

The operation $i_{\mathbb{T}}$, which is, in fact, a $v_{\mathbb{T}}$ -derivation of degree -1 , can be defined in a more intrinsic way. An r -form μ , $r > 0$, defines a vector bundle morphism

$$\tilde{\mu}^1: \mathbb{T}M \rightarrow \bigwedge^{r-1} \mathbb{T}^*M: v \mapsto i_v \mu$$

and the following formula holds:

$$i_{\mathbb{T}}\mu = (\tilde{\mu}^1)^*\theta_M^{r-1}, \quad (2.2)$$

where θ_M^{r-1} is the canonical (Liouville) $(r-1)$ -form on $\bigwedge^{r-1}\mathbb{T}^*M$. The Liouville form is defined by

$$\theta_M^{r-1}(a)(v_1, \dots, v_{r-1}) = a(\mathbb{T}\pi_M(v_1), \dots, \mathbb{T}\pi_M(v_{r-1})).$$

Let us notice also that for $r=0$ we have $d_{\mathbb{T}}\mu(v) = \langle v, d\mu \rangle$ (μ is a function).

The tangent lift $d_{\mathbb{T}}\mu$ can be defined more directly by means of the tangent functor. Let us fix $0 \leq i \leq r$. An r -form μ on M defines, in an obvious way, a vector bundle morphism

$$\tilde{\mu}^i: \bigwedge^i \mathbb{T}M \rightarrow \bigwedge^{r-i} \mathbb{T}^*M.$$

Now, we define κ_M^r and ε_M^r for $r=0$. We have $\bigwedge^0 \mathbb{T}M = \bigwedge^0 \mathbb{T}^*M = M \times \mathbb{R}$. We define

$$\kappa_M^0 = \varepsilon_M^0: \bigwedge^0 \mathbb{T}\mathbb{T}M = \mathbb{T}M \times \mathbb{R} \rightarrow \mathbb{T}\bigwedge^0 \mathbb{T}M = \mathbb{T}(M \times \mathbb{R}) = \mathbb{T}M \times \mathbb{R} \times \mathbb{T}_0\mathbb{R}$$

by

$$(x^i, \dot{x}^j, t, \dot{t}) \circ \kappa_M^0 = (x^i, \dot{x}^j, 0, t).$$

The dual mapping $\kappa_M^0{}': \mathbb{T}\bigwedge^0 \mathbb{T}M \rightarrow \bigwedge^0 \mathbb{T}\mathbb{T}M$ is given by

$$(x^i, \dot{x}^j, t) \circ \kappa_M^0{}' = (x^i, \dot{x}^j, t).$$

Theorem 2.1. *Let be $\mu \in \Phi^r(\pi_M)$ and $0 \leq i \leq r$. The following diagram is commutative*

$$\begin{array}{ccc} \mathbb{T}\bigwedge^i \mathbb{T}M & \xrightarrow{\mathbb{T}\tilde{\mu}^i} & \mathbb{T}\bigwedge^{r-i} \mathbb{T}^*M \\ \kappa_M^i \uparrow & & (\kappa_M^{r-i})' \downarrow \\ \bigwedge^i \mathbb{T}\mathbb{T}M & \xrightarrow{\widetilde{d_{\mathbb{T}}\mu}^i} & \bigwedge^{r-i} \mathbb{T}^*\mathbb{T}M \end{array} \quad (2.3)$$

Proof. We show first that there exists $D(\mu) \in \Phi^r(\pi_{\mathbb{T}M})$ such that

$$\widetilde{D(\mu)}^i = (\kappa_M^{r-i})' \circ \mathbb{T}\tilde{\mu}^i \circ \kappa_M^i.$$

In order to do this, we apply the tangent functor to the commutative diagram

$$\begin{array}{ccc} \bigwedge^r \mathbb{T}M & \xrightarrow{\tilde{\mu}^r} & \mathbb{R} \\ \uparrow \kappa_M^{r-i} & & \uparrow \langle \cdot \rangle_M^{r-i} \\ \bigwedge^i \mathbb{T}M \times_M \bigwedge^{r-i} \mathbb{T}M & \xrightarrow{\tilde{\mu}^i \times \text{id}} & \bigwedge^{r-i} \mathbb{T}^*M \times_M \bigwedge^{r-i} \mathbb{T}M \end{array} \quad (2.4)$$

and, since the diagram (1.5) is commutative, we get the following commutative diagram

$$\begin{array}{ccccccc} \bigwedge^r \mathbb{T}\mathbb{T}M & \xrightarrow{\kappa_M^r} & \mathbb{T}\bigwedge^r \mathbb{T}M & \xrightarrow{d_{\mathbb{T}}\tilde{\mu}^r} & \mathbb{R} & & \\ \uparrow \kappa_M^{r-i} & & \uparrow \mathbb{T}\kappa_M^{r-i} & & \uparrow \langle \cdot \rangle_{\mathbb{T}M}^{r-i} & & (2.5). \\ \bigwedge^i \mathbb{T}\mathbb{T}M \times_{\mathbb{T}M} \bigwedge^{r-i} \mathbb{T}\mathbb{T}M & \xrightarrow{\kappa_M^i \times \kappa_M^{r-i}} & \mathbb{T}\bigwedge^i \mathbb{T}M \times_{\mathbb{T}M} \mathbb{T}\bigwedge^{r-i} \mathbb{T}M & \xrightarrow{\mathbb{T}\tilde{\mu}^i \times \text{id}} & \mathbb{T}\bigwedge^{r-i} \mathbb{T}^*M \times_{\mathbb{T}M} \mathbb{T}\bigwedge^{r-i} \mathbb{T}M & & \end{array}$$

Here we regard 0-forms as functions rather than sections of $\bigwedge^0 \mathbb{T}$ -bundles. It shows that for $u \in \bigwedge^i \mathbb{T}TM$ and $v \in \bigwedge^{r-i} \mathbb{T}TM$ we have

$$\langle \mathbb{T}\tilde{\mu}^i \circ \kappa_M^i(u), \kappa_M^{r-i}(v) \rangle'_{\mathbb{T}M} = d_{\mathbb{T}}\tilde{\mu}^r \circ \kappa_M^r(u \wedge v)$$

and, consequently,

$$d_{\mathbb{T}}\tilde{\mu}^r \circ \kappa_M^r(u \wedge v) = \langle (\kappa_M^{r-i})' \circ \mathbb{T}\tilde{\mu}^i \circ \kappa_M^i(u), v \rangle'_{\mathbb{T}M}.$$

It follows that $D(\mu)$ exists and $D(\mu)(\cdot) = d_{\mathbb{T}}\tilde{\mu}^r \circ \kappa_M^r$. Since in local coordinates

$$\tilde{\mu}^r(x, \dot{x}) = \mu_{i_1 \dots i_r} \dot{x}^{i_1 \dots i_r},$$

we get

$$d_{\mathbb{T}}\tilde{\mu}^r(x, \dot{x}, \delta x, \delta \dot{x}) = \frac{\partial \mu_{i_1 \dots i_r}}{\partial x^k}(x) \delta x^k \dot{x}^{i_1 \dots i_r} + \mu_{i_1 \dots i_r}(x) \delta \dot{x}^{i_1 \dots i_r}$$

and

$$d_{\mathbb{T}}\tilde{\mu}^r \circ \kappa_M^r = \frac{\partial \mu_{i_1 \dots i_r}}{\partial x^k}(x) \dot{x}^k \delta x^{i_1 \dots i_r} + \sum_m \mu_{i_1 \dots i_r}(x) \delta x^{i_1 \dots i_{m-1}} \wedge \delta \dot{x}^{i_m} \wedge \delta x^{i_{m+1} \dots i_r}.$$

The right hand side in this formula corresponds precisely the right hand side in (2.1). It follows that $d_{\mathbb{T}}\tilde{\mu}^r \circ \kappa_M^r = \widetilde{d_{\mathbb{T}}\mu}^r$ and this completes the proof. \blacksquare

Derivations of multivector fields and the Schouten bracket.

A similar construction can be done in the case of multivector fields. Let be $X \in \Phi^r(\tau_M)$. As in the case of forms, we have a family of contraction mappings

$$\tilde{X}^i: \bigwedge^i \mathbb{T}^*M \rightarrow \bigwedge^{r-i} \mathbb{T}M.$$

Theorem 2.2. *There is a uniquely defined multivector field $d_{\mathbb{T}}X \in \Phi^r(\tau_{\mathbb{T}M})$ such that*

$$\widetilde{d_{\mathbb{T}}X}^i = (\varepsilon_M^{r-i})' \circ \mathbb{T}\tilde{X}^i \circ \varepsilon_M^i$$

for $i = 0, 1, \dots, r$, i. e., the following diagram is commutative

$$\begin{array}{ccc} \mathbb{T}\bigwedge^i \mathbb{T}^*M & \xrightarrow{\mathbb{T}\tilde{X}^i} & \mathbb{T}\bigwedge^{r-i} \mathbb{T}M \\ \varepsilon_M^i \uparrow & & (\varepsilon_M^{r-i})' \downarrow \\ \bigwedge^i \mathbb{T}^* \mathbb{T}M & \xrightarrow{\widetilde{d_{\mathbb{T}}X}^i} & \bigwedge^{r-i} \mathbb{T} \mathbb{T}M \end{array} \quad (2.6)$$

The proof goes on like the proof of Theorem 2.1. In particular, we get

$$\widetilde{d_{\mathbb{T}}X}^r = d_{\mathbb{T}}\tilde{X}^r \circ \varepsilon_M^r,$$

i. e.,

$$d_{\mathbb{T}}X(a_1, \dots, a_r) = d_{\mathbb{T}}\tilde{X}^r(\varepsilon_M a_1 \wedge \dots \wedge \varepsilon_M a_r).$$

Now, writing in local coordinates

$$X = X^{i_1 \dots i_r}(x) \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_r}}$$

and $p = p_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$, we have

$$\tilde{X}^r(p) = X^{i_1 \dots i_r} p_{i_1 \dots i_r}$$

and

$$\begin{aligned} d_{\mathbb{T}} \tilde{X}^r(x^i, \dot{x}^k, \pi_{i_1 \dots i_m} \wedge \dot{\pi}_{i_{m+1} \dots i_r}) &= \frac{\partial X^{i_1 \dots i_r}}{\partial x^{i_k}}(x) \dot{x}^k \dot{\pi}_{i_1 \dots i_r} + \\ &+ \sum_m X^{i_1 \dots i_r}(x) \dot{\pi}_{i_1 \dots i_{m-1}} \wedge \pi_{i_m} \wedge \dot{\pi}_{i_{m+1} \dots i_r}, \end{aligned}$$

i. e.,

$$d_{\mathbb{T}} X = \frac{\partial X^{i_1 \dots i_r}}{\partial x^{i_k}}(x) \dot{x}^k \frac{\partial}{\partial \dot{x}^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial \dot{x}^{i_r}} + \sum_m X^{i_1 \dots i_r}(x) \frac{\partial}{\partial \dot{x}^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial \dot{x}^{i_m}} \wedge \dots \wedge \frac{\partial}{\partial \dot{x}^{i_r}}.$$

Now, let X be a simple r -vector field, i. e., $X = X_1 \wedge \dots \wedge X_r$ for some vector fields $X_i \in \Phi(\tau_M)$. We can consider \tilde{X}^r as a multilinear, skewsymmetric function on $\times_M^r \mathbb{T}^*M$

$$\tilde{X}^r(a_1, \dots, a_r) = \sum_{\sigma \in S_n} (-1)^\sigma \widetilde{X_{\sigma(1)}}^1(a_1) \cdots \widetilde{X_{\sigma(r)}}^1(a_r).$$

Since $d_{\mathbb{T}}$ is a $v_{\mathbb{T}} = \tau_{\times_M^r \mathbb{T}^*M}^*$ -derivation on forms, we have

$$d_{\mathbb{T}} \tilde{X}^r = \sum_{\sigma \in S_n} (-1)^\sigma \sum_m v_{\mathbb{T}}(\widetilde{X_{\sigma(1)}}^1) \cdots d_{\mathbb{T}}(\widetilde{X_{\sigma(m)}}^1) \cdots v_{\mathbb{T}}(\widetilde{X_{\sigma(r)}}^1)$$

and

$$\widetilde{d_{\mathbb{T}} \tilde{X}^r} = \sum_{\sigma \in S_n} (-1)^\sigma \sum_m (v_{\mathbb{T}}(\widetilde{X_{\sigma(1)}}^1)) \circ \varepsilon_M \cdots (d_{\mathbb{T}}(\widetilde{X_{\sigma(m)}}^1)) \circ \varepsilon_M \cdots (v_{\mathbb{T}}(\widetilde{X_{\sigma(r)}}^1)) \circ \varepsilon_M.$$

Let $Y \in \Phi^1(\tau_M)$ be a vector field on M . Functions $v_{\mathbb{T}}(\tilde{Y}^1) \circ \varepsilon_M$ and $d_{\mathbb{T}}(\tilde{Y}^1) \circ \varepsilon_M$ are linear functions on \mathbb{T}^*TM and define vector fields on $\mathbb{T}M$. In local coordinates we have for $Y = Y^i \frac{\partial}{\partial x^i}$:

$$\tilde{Y}^1(x, p) = Y^i(x) p_i \quad \text{and} \quad v_{\mathbb{T}}(\tilde{Y}^1)(x, p, \dot{x}, \dot{p}) = Y^i(x) p_i.$$

Hence, from the definition of ε_M ,

$$v_{\mathbb{T}}(\tilde{Y}^1) \circ \varepsilon_M(x, \dot{x}, \pi, \dot{\pi}) = Y^i(x) \dot{\pi}_i.$$

It follows that $v_{\mathbb{T}}(\tilde{Y}^1) \circ \varepsilon_M = \tilde{Y}^v$, where $Y^v = Y^i(x) \frac{\partial}{\partial x^i}$ is the *vertical lift* of Y . The vertical lift $\Phi^1(\tau_M) \ni Y \mapsto Y^v \in \Phi^1(\tau_{\mathbb{T}M})$ is linear and can be extended in a unique way to a homomorphism $v_{\mathbb{T}}$ of graded algebras

$$v_{\mathbb{T}}: \Phi(\tau_M) \rightarrow \Phi(\tau_{\mathbb{T}M}).$$

The vertical lift Y^v is the generator of a one-parameter group (ψ^t) of diffeomorphisms (a flow) of $\mathbb{T}M$ defined by

$$\psi^t(v) = v + tY(\tau_M(v)).$$

In a similar way we get

$$d_{\mathbb{T}} \tilde{Y}^1(x, p, \dot{x}, \dot{p}) = Y^i(x) \dot{p}_i + \frac{\partial Y^i}{\partial x^k}(x) p_i \dot{x}^k$$

and

$$d_{\mathbb{T}}\tilde{Y}^1 \circ \varepsilon_M(x, \dot{x}, \pi, \dot{\pi}) = Y^i(x)\pi_i + \frac{\partial Y^i}{\partial x^k}(x)\dot{x}^k \dot{\pi}_i.$$

It follows that $d_{\mathbb{T}}\tilde{Y}^1 \circ \varepsilon_M = \tilde{Y}^c$, where

$$Y^c(x, \dot{x}) = Y^i(x)\frac{\partial}{\partial x^i} + \frac{\partial Y^i}{\partial x^k}(x)\dot{x}^k \frac{\partial}{\partial \dot{x}^i}$$

is the *complete lift* of Y . The vector field Y^c is the generator of the (local) one-parameter group of diffeomorphisms $\mathbb{T}\varphi^t: \mathbb{T}M \rightarrow \mathbb{T}M$, where (φ^t) is the flow generated by the vector field Y .

Thus we have proved the following theorem.

Theorem 2.3. *The mapping*

$$d_{\mathbb{T}}: \Phi(\tau_M) \rightarrow \Phi(\tau_{\mathbb{T}M})$$

is a $\nu_{\mathbb{T}}$ -derivation of degree 0 with $d_{\mathbb{T}}(Y) = Y^c$ for $Y \in \Phi^1(\tau_M)$.

The following is a well-known theorem.

Theorem 2.4. *The Lie bracket of vertical and complete lifts satisfy the following commutation relations*

$$\begin{aligned} [X^v, Y^v] &= 0, \\ [X^c, Y^c] &= [X, Y]^c, \\ [X^v, Y^c] &= [X, Y]^v. \end{aligned} \tag{2.7}$$

The Lie bracket of vector fields can be extended to a graded Lie bracket on the graded space $\Phi(\tau_M)$ of multivector fields – the Schouten bracket $[\cdot, \cdot]$. In this graded Lie algebra the space $\Phi^r(\tau_M)$ is of degree $(r-1)$. Let be $X \in \Phi^r(\tau_M)$. Then

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{s(r-1)} Y \wedge [X, Z]$$

for $Y \in \Phi^s(\tau_M)$, i. e., ad_X is a graded derivation of degree $(r-1)$ of the graded commutative algebra $\Phi(\tau_M)$. The mapping

$$\text{ad}: (\Phi(\tau_M), [\cdot, \cdot]) \rightarrow \text{Der}(\Phi(\tau_M), \wedge)$$

is a homomorphism of graded algebras. For simple multivectors we have the following formula (cf. [Mi]):

$$[X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge X_{i-1} \wedge X_{i+1} \wedge \cdots \wedge Y_{j-1} \wedge Y_{j+1} \wedge \cdots \wedge Y_q. \tag{2.8}$$

Theorem 2.5. *The derivation $d_{\mathbb{T}}$ commutes with the Schouten bracket, i. e.,*

$$[d_{\mathbb{T}}X, d_{\mathbb{T}}Y] = d_{\mathbb{T}}[X, Y].$$

Proof. Let be $X = X_1 \wedge \cdots \wedge X_p$. By $(X_1, \dots, X_p)_n$ we denote a p -vector field $X_1^v \wedge \cdots \wedge X_n^c \wedge \cdots \wedge X_p^v$ and by $(X_1, \dots, X_p)_n^i$ a $(p-1)$ -vector field as above with the i -th factor

omitted. We have then $d_{\tau}X = \sum_{i=1}^p (X_1, \dots, X_p)_i$. From the formula (2.8) and Theorem 2.3 we get, for $Y = Y_1 \wedge \dots \wedge Y_q$,

$$\begin{aligned}
[d_{\tau}X, d_{\tau}Y] &= \left[\sum_{n=1}^p (X_1, \dots, X_p)_n, \sum_{m=1}^q (Y_1, \dots, Y_q)_m \right] \\
&= \sum_{n=1}^p \sum_{m=1}^q [(X_1, \dots, X_p)_n, (Y_1, \dots, Y_q)_m] = \\
&= \sum_{n=1}^p \sum_{m=1}^q \left(\sum_{j \neq m} (-1)^{n+j} [X_n^c, Y_j^v] \wedge (X_1, \dots, X_p)_n^j \wedge (Y_1, \dots, Y_q)_m^j + \right. \\
&\quad \left. + (-1)^{n+m} [X_n^c, Y_m^c] \wedge (X_1, \dots, X_p)_n^m \wedge (Y_1, \dots, Y_q)_m^m + \right. \\
&\quad \left. + \sum_{i \neq m} (-1)^{m+i} [X_i^v, Y_m^c] \wedge (X_1, \dots, X_p)_n^i \wedge (Y_1, \dots, Y_q)_m^m \right) = \\
&= \sum_{n,j} (-1)^{n+j} [X_n, Y_j]^v \wedge \sum_{m \neq j} (X_1, \dots, X_p)_n^m \wedge (Y_1, \dots, Y_q)_m^j + \\
&\quad + (-1)^{n+m} [X_n, Y_m]^c \wedge (X_1, \dots, X_p)_n^m \wedge (Y_1, \dots, Y_q)_m^m + \\
&\quad + \sum_{m,i} (-1)^{i+m} [X_i, Y_m]^v \wedge \sum_{n \neq i} (X_1, \dots, X_p)_n^i \wedge (Y_1, \dots, Y_q)_m^m.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d_{\tau}[X, Y] &= \sum_{i,j} (-1)^{i+j} [X_i, Y_j]^c \wedge (X_1, \dots, X_p)_i^j \wedge (Y_1, \dots, Y_q)_j^j + \\
&\quad + \sum_{i,j} (-1)^{i+j} [X_i, Y_j]^v \wedge \sum_{m \neq i} (X_1, \dots, X_p)_m^i \wedge (Y_1, \dots, Y_q)_j^j + \\
&\quad + \sum_{i,j} (-1)^{i+j} [X_i, Y_j]^v \wedge \sum_{n \neq j} (X_1, \dots, X_p)_i^i \wedge (Y_1, \dots, Y_q)_n^j
\end{aligned}$$

and the required equality follows. \blacksquare

The rules of contractions for lifts of forms and vector fields we list in the following Proposition.

Proposition 2.6. *For $\mu \in \Phi^1(\pi_M)$ and $X \in \Phi^1(\tau_M)$ we have*

$$\begin{aligned}
\langle v_{\tau}(\mu), v_{\tau}(X) \rangle &= \langle \tau_M^* \mu, X^v \rangle = 0, \\
\langle d_{\tau} \mu, v_{\tau}(X) \rangle &= \langle v_{\tau} \mu, d_{\tau}(X) \rangle = v_{\tau}(\langle \mu, X \rangle) = \tau_M^* \langle \mu, X \rangle, \\
\langle d_{\tau} \mu, d_{\tau} X \rangle &= d_{\tau}(\langle \mu, X \rangle).
\end{aligned}$$

These formulas can be easily generalized for $X \in \Phi^r(\tau_M)$ (or $\mu \in \Phi^r(\pi_M)$). We obtain

$$\begin{aligned}
\langle d_{\tau} X, v_{\tau} \mu \rangle &\stackrel{(def)}{=} i_{v_{\tau}(\mu)} d_{\tau} X = v_{\tau}(i_{\mu} X), \\
\langle d_{\tau} X, d_{\tau} \mu \rangle &\stackrel{(def)}{=} i_{d_{\tau}(\mu)} d_{\tau} X = d_{\tau}(i_{\mu} X), \\
\langle v_{\tau}(X), v_{\tau}(\mu) \rangle &\stackrel{(def)}{=} i_{v_{\tau}(\mu)} v_{\tau} X = 0
\end{aligned} \tag{2.9}$$

for $X \in \Phi^r(\tau_M)$, $\mu \in \Phi^1(\pi_M)$ and also

$$\langle d_{\tau} \mu, d_{\tau} X \rangle = 0 \text{ for } \mu \in \Phi^r(\pi_M), X \in \Phi^r(\tau_M), r > 2.$$

3. Lie derivations of forms and multivector fields. The derivation $d_{\mathbb{T}}$ is strictly related to the Lie derivation in the algebra of differential forms.

Proposition 3.1. [PiTu] *Let $X: M \rightarrow \mathbb{T}M$ be a vector field on M . Then we have*

$$\mathcal{L}_X \mu = X^* d_{\mathbb{T}} \mu. \quad (3.1)$$

Proof. Since $X^*: \Phi(\pi_{\mathbb{T}M}) \rightarrow \Phi(\pi_M)$ is a homomorphism and $d_{\mathbb{T}}: \Phi(\pi_{\mathbb{T}M}) \rightarrow \Phi(\pi_{\mathbb{T}M})$ is a $v_{\mathbb{T}}$ -derivation, the mapping

$$X^* d_{\mathbb{T}}: \Phi(\pi_{\mathbb{T}M}) \rightarrow \Phi(\pi_M) \quad (3.2)$$

is also a derivation. It follows that it is enough to verify the formula for functions and their differentials. We have for a function f on M

$$X^* d_{\mathbb{T}} f = d_{\mathbb{T}} f \circ X = \langle X, df \rangle = X(f)$$

and

$$X^* d_{\mathbb{T}} df = X^* dd_{\mathbb{T}} f = d(X^* d_{\mathbb{T}} f) = d(X(f)) = \mathcal{L}_X df.$$

■

To get an analogue to this proposition in the case of vector fields, we need an analogue of the pull-back of forms with respect to a vector field. Let X be a vector field on M . We have the decomposition of $\mathbb{T}_{X(M)} \mathbb{T}M$ into horizontal (tangent to $X(M)$) and vertical parts

$$\mathbb{T}_{X(M)} \mathbb{T}M = H_X \mathbb{T}M \oplus V_X \mathbb{T}M. \quad (3.3)$$

The canonical projection $\mathbb{T}_X \mathbb{T}M \rightarrow V_X \mathbb{T}M$ we denote by P_X . Let $Y \in \Phi^1(\tau_{\mathbb{T}M})$ be a vector field on $\mathbb{T}M$. By $X^* Y$ we denote the unique vector field on M such that

$$v_{\mathbb{T}}(X^* Y)|_{X(M)} = P_X Y. \quad (3.4)$$

The mapping $Y \mapsto X^* Y$ has the unique extension to a morphism of graded algebras

$$X^*: \Phi(\tau_{\mathbb{T}M}) \rightarrow \Phi(\tau_M). \quad (3.5)$$

Theorem 3.2. *Let $X: M \rightarrow \mathbb{T}M$ be a vector field on M . For each $\lambda \in \Phi(\tau_M)$ we have the following formulae:*

$$\mathcal{L}_X \lambda = X^* d_{\mathbb{T}} \lambda$$

and

$$\lambda = X^* v_{\mathbb{T}} \lambda.$$

Proof. Since $X^*: \Phi(\tau_{\mathbb{T}M}) \rightarrow \Phi(\tau_M)$ is a homomorphism and $d_{\mathbb{T}}: \Phi(\tau_{\mathbb{T}M}) \rightarrow \Phi(\tau_{\mathbb{T}M})$ is a $v_{\mathbb{T}}$ -derivation the mapping

$$X^* d_{\mathbb{T}}: \Phi(\tau_{\mathbb{T}M}) \rightarrow \Phi(\tau_M) \quad (3.6)$$

is also a derivation. It follows that it is enough to verify the formula for functions and vector fields. We have for a function f on M and a vector field Y on M

$$X^* d_{\mathbb{T}} f = d_{\mathbb{T}} f \circ X = \langle X, df \rangle = X(f) = \mathcal{L}_X f$$

and (Proposition 2.6)

$$\tau_M^*(X^* d_{\mathbb{T}} Y(f)) = v_{\mathbb{T}}((X^* d_{\mathbb{T}} Y)(d_{\mathbb{T}} f)) = v_{\mathbb{T}}(X^* d_{\mathbb{T}} Y)(d_{\mathbb{T}} f).$$

On the other hand,

$$(P_X d_{\mathbb{T}} Y) \circ X = d_{\mathbb{T}} Y \circ X - \mathbb{T}X \circ Y$$

and, consequently,

$$(X^*d_{\mathbb{T}}Y)(f) = \langle d_{\mathbb{T}}Y, d_{\mathbb{T}}df \rangle \circ X - \langle \mathbb{T}X \circ Y, d_{\mathbb{T}}df \rangle.$$

Since

$$\langle d_{\mathbb{T}}Y, d_{\mathbb{T}}df \rangle \circ X = (d_{\mathbb{T}}(\langle Y, df \rangle)) \circ X = X(Y(f))$$

and

$$\langle \mathbb{T}X \circ Y, d_{\mathbb{T}}df \rangle = \langle Y, X^*dd_{\mathbb{T}}f \rangle = \langle Y, dX(f) \rangle = Y(X(f)),$$

we get

$$(X^*d_{\mathbb{T}}Y)(f) = X(Y(f)) - Y(X(f)) = [X, Y](f) = (\mathcal{L}_X Y)(f).$$

The second identity follows directly from the definition. \blacksquare

The projection $\mathbb{T}\tau_M$ induces an isomorphism $\mathbb{T}\tau_M: V_X\mathbb{T}M \rightarrow \mathbb{T}M$ and the dual isomorphism $(\mathbb{T}\tau_M)': \mathbb{T}^*M \rightarrow V_X^*\mathbb{T}M$.

When composed with $(I - P_X)'$, this isomorphism gives an isomorphism $\mathbb{T}^*M \rightarrow H_X^o\mathbb{T}M$, where $H_X^o\mathbb{T}M$ is the annihilator of $H_X\mathbb{T}M$ in $\mathbb{T}_{X(M)}^*\mathbb{T}M$. It follows that for $\nu \in \Phi^1(\pi_{\mathbb{T}M})$ the decomposition

$$\nu = (\nu - v_{\mathbb{T}}(X^*\nu)) + v_{\mathbb{T}}(X^*\nu)$$

is, on $X(M)$, the decomposition related to the decomposition

$$\mathbb{T}_{X(M)}^*\mathbb{T}M = H_X^o\mathbb{T}M \oplus V_X^o\mathbb{T}M,$$

dual to (3.3). For $\nu = d_{\mathbb{T}}\mu$ we have $v_{\mathbb{T}}(X^*\nu) = v_{\mathbb{T}}(\mathcal{L}_X\mu)$ (Proposition 3.1). Hence, for $Y \in \Phi^1(\tau_{\mathbb{T}M})$ we have

$$\langle \mu, X^*Y \rangle = (\langle d_{\mathbb{T}}\mu, v_{\mathbb{T}}(X^*Y) \rangle) \circ X = (\langle d_{\mathbb{T}}\mu - v_{\mathbb{T}}(\mathcal{L}_X\mu), Y \rangle) \circ X \quad (3.7)$$

It is convenient to introduce an operation

$$X_+: \Phi(\pi_M) \rightarrow \Phi(\pi_{\mathbb{T}M}): \mu \mapsto d_{\mathbb{T}}\mu - v_{\mathbb{T}}(\mathcal{L}_X\mu),$$

which is a $v_{\mathbb{T}}$ -derivation of degree 0.

It follows directly from (3.7) that for $Y \in \Phi^r(\tau_{\mathbb{T}M})$ and $\mu_1, \dots, \mu_r \in \Phi^1(\pi_M)$, $r > 0$, we have

$$\langle (X_+\mu_1) \wedge \dots \wedge (X_+\mu_r), Y \rangle \circ X = \langle \mu_1 \wedge \dots \wedge \mu_r, X^*Y \rangle, \quad (3.8)$$

so the mapping

$$X_*(\mu_1 \wedge \dots \wedge \mu_r) = (X_+\mu_1) \wedge \dots \wedge (X_+\mu_r)$$

regarded as a homomorphism of graded algebras

$$X_*: \Phi(\pi_M) \rightarrow \Phi(\pi_{X(M)}),$$

where

$$\pi_{X(M)}: \mathbb{T}_{X(M)}^*\mathbb{T}M \rightarrow X(M),$$

is dual to X^* :

$$\langle X_*\mu, Y \rangle \circ X = \langle \mu, X^*Y \rangle.$$

For 1-forms we shall write $X_*\mu$ instead of $X_+\mu$.

Proposition 3.3. *Let X be a vector field on M and let λ be a 2-vector field on M . $\mathcal{L}_X\lambda = 0$ if and only if for every pair f, g of functions on M*

$$\langle X_*dg, \widetilde{d\tau\lambda}^1(X_*df) \rangle = 0.$$

Proof. From Theorem 3.2 it follows that $\mathcal{L}_X\lambda = 0$ iff

$$\langle dg, X^*\widetilde{d\tau\lambda}^1(df) \rangle = 0$$

for each pair (f, g) of functions on M . From (3.8) we have then that $\mathcal{L}_X\lambda = 0$ if and only if

$$\langle X_*dg, \widetilde{d\tau\lambda}^1 X_*df \rangle = 0.$$

■

4. Symplectic and Poisson structures.

By the standard definition a symplectic structure on M is a nondegenerate, closed two-form ω on M . The canonical example is the 2-form Ω_M on \mathbb{T}^*M . On the other hand, any 2-form μ on M can be represented by

$$\tilde{\mu}^i: \bigwedge^i \mathbb{T}M \rightarrow \bigwedge^{2-i} \mathbb{T}^*M, \quad i = 0, 1, 2.$$

The standard definition is expressed, in fact, in terms of $\tilde{\mu}^0$. We can also formulate a definition of a symplectic structure in terms of $\tilde{\mu}^2$ making use of the well known formula for the exterior derivative:

Proposition 4.1. *A linear fibre bundle morphism $\nu: \bigwedge^2 \mathbb{T}M \rightarrow \bigwedge^0 \mathbb{T}^*M$ represents a symplectic structure on M if it is nondegenerated (in the obvious meaning) and*

$$X(\nu(Y, Z)) + Y(\nu(Z, X)) + Z(\nu(X, Y)) - \nu([X, Y], Z) - \nu([Y, Z], X) - \nu([Z, X], Y) = 0$$

where $X, Y, Z \in \Phi^1(\tau_M)$.

Now, we provide a definition of a (pre-)symplectic structure in terms of $\tilde{\mu}^1$.

First, we need a condition for a 2-form μ to be closed in terms of $\tilde{\mu}^1$.

Theorem 4.2.

A 2-form μ on M is closed if and only if

$$d_{\mathbb{T}}\mu = (\tilde{\mu}^1)^*\Omega_M$$

Proof. From the formula 2.2 we have

$$di_{\mathbb{T}}\mu = (\tilde{\mu}^1)^*\Omega_M$$

and, consequently,

$$d_{\mathbb{T}}\mu = di_{\mathbb{T}}\mu + i_{\mathbb{T}}d\mu = (\tilde{\mu}^1)^*\Omega_M - i_{\mathbb{T}}d\mu.$$

Since $i_{\mathbb{T}}d\mu = 0$ iff $d\mu = 0$ we get that $d\mu = 0$ if and only if

$$d_{\mathbb{T}}\mu = (\tilde{\mu}^1)^*\Omega_M.$$

■

A condition for a vector bundle morphism $\nu: \mathbb{T}M \rightarrow \mathbb{T}^*M$ to define a symplectic structure on M is given by the following theorem.

Theorem 4.3. *An isomorphism $\nu: \mathbb{T}M \rightarrow \mathbb{T}^*M$ of vector bundles defines a symplectic structure on M if and only if the following diagram is commutative*

$$\begin{array}{ccc}
 \mathbb{T}\mathbb{T}^*M & \xrightarrow{\widetilde{\Omega}_M^1} & \mathbb{T}^*\mathbb{T}^*M \\
 \mathbb{T}\nu \uparrow & & \mathbb{T}^*\nu \downarrow \\
 \mathbb{T}M & & \mathbb{T}^*\mathbb{T}M \\
 \kappa_M \downarrow & & \alpha_M \uparrow \\
 \mathbb{T}M & \xrightarrow{\mathbb{T}\nu} & \mathbb{T}^*\mathbb{T}M
 \end{array} \quad (4.1)$$

Proof. Commutativity of the diagram 4.1 is equivalent to the equality

$$\alpha_M \circ \mathbb{T}\nu \circ \kappa_M = \mathbb{T}^*\nu \circ \widetilde{\Omega}_M^1 \circ \mathbb{T}\nu \quad (4.2)$$

and to the dual (with respect to proper pairings) equality

$$\alpha_M \circ \mathbb{T}\nu^* \circ \kappa_M = \mathbb{T}^*\nu \circ (\widetilde{\Omega}_M^1)^* \circ \mathbb{T}\nu. \quad (4.3)$$

Since $(\widetilde{\Omega}_M^1)^* = -\widetilde{\Omega}_M$ we get from 4.3 that

$$\mathbb{T}\nu = -\mathbb{T}\nu^*$$

and, consequently, that $\nu^* = -\nu$. It proves that ν is skew-symmetric and, consequently, that there exist a 2-form μ on M , such that $\nu = \tilde{\mu}^1$. Equality 4.2 reads now

$$d\mu = (\tilde{\mu}^1)^* \Omega_M.$$

It follows from Theorem 4.2 that μ is closed. ■

As in the case of symplectic structures, a Poisson structure on M can be represented by one of three equivalent objects:

- (1) a bivector field Λ ,
- (2) a homomorphism $\tilde{\Lambda}^1$ of vector bundles,
- (3) a bilinear, skew-symmetric function $\tilde{\Lambda}^2$.

The condition for Λ to be a Poisson structure is vanishing of the Schouten bracket: $[\Lambda, \Lambda] = 0$.

A skew-symmetric function

$$\lambda: \bigwedge^2 \mathbb{T}^*M \rightarrow \bigwedge^0 \mathbb{T}M \simeq M \times \mathbb{R}$$

defines a Poisson structure if the Jacobi identity is fulfilled:

$$\lambda(df, \lambda(dg, dh)) + \lambda(dg, \lambda(dh, df)) + \lambda(dh, \lambda(df, dg)) = 0,$$

where $f, g, h \in \Phi^0(\pi_M)$.

In order to get a definition of a Poisson structure in terms of a vector bundle morphism $\nu: \mathbb{T}^*M \rightarrow \mathbb{T}M$, we need the following theorem.

Theorem 4.4.

A 2-vector field Λ on M defines Poisson structure if and only if

$$(\tilde{\Lambda}^1)_* \Lambda_M \subset d_{\mathbb{T}} \Lambda$$

(one can say that Λ_M and $d_{\mathbb{T}} \Lambda$ are $\tilde{\Lambda}^1$ -related), where $\Lambda_M = (\Omega_M)^{-1}$ is the canonical 2-vector field on \mathbb{T}^*M .

Proof. In local coordinates $\Lambda = \frac{1}{2} \Lambda^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. The 2-vector field Λ defines a Poisson structure if and only if

$$\sum_i (\Lambda^{ij} \frac{\partial}{\partial x^i} \Lambda^{kl} + \Lambda^{ik} \frac{\partial}{\partial x^i} \Lambda^{lj} + \Lambda^{il} \frac{\partial}{\partial x^i} \Lambda^{jk}) = 0. \quad (4.4)$$

The canonical 2-vector field Λ_M on \mathbb{T}^*M is given by

$$\Lambda_M = \sum_i \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_i}.$$

Moreover,

$$d_{\mathbb{T}} \Lambda = \frac{1}{2} \sum_{ijk} \frac{\partial \Lambda^{ij}}{\partial x^k} x^k \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \sum_{ij} \Lambda^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}. \quad (4.5)$$

For $a \in \mathbb{T}^*M$,

$$x^i((\tilde{\Lambda}^1)(a)) = \sum_j \Lambda^{ji} p_i(a)$$

and, consequently,

$$\begin{aligned} (\tilde{\Lambda}^1)_* \left(\frac{\partial}{\partial x^i} \right) &= \frac{\partial}{\partial x^i} + \sum_{j,k} p_j \left(\frac{\partial}{\partial x^i} \Lambda^{jk} \right) \frac{\partial}{\partial x^k} \\ (\tilde{\Lambda}^1)_* \left(\frac{\partial}{\partial p_i} \right) &= \sum_j \Lambda^{ij} \frac{\partial}{\partial x^j}. \end{aligned}$$

Hence

$$(\tilde{\Lambda}^1)_* \Lambda_M = \sum_{ij} \Lambda^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \sum_{ijk} p_l \Lambda^{ij} \left(\frac{\partial}{\partial x^i} \Lambda^{lk} \right) \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^j}. \quad (4.6)$$

It follows from 4.5 and 4.6 that $(\tilde{\Lambda}^1)_* \Lambda_M \subset d_{\mathbb{T}} \Lambda$ if and only if

$$\Lambda^{lk} p_l \frac{\partial}{\partial x^k} \Lambda^{ij} = \Lambda^{kj} \left(\frac{\partial}{\partial x^k} \Lambda^{li} \right) p_l - \Lambda^{ki} \left(\frac{\partial}{\partial x^k} \Lambda^{lj} \right) p_l$$

but this is equivalent to 4.4. ■

Now, we are ready for a proof of an analogue of Theorem 4.3.

Theorem 4.5. A vector bundles morphism $\lambda: \mathbb{T}^*M \rightarrow \mathbb{T}M$ defines a Poisson structure on M if and only if the following diagram is commutative

$$\begin{array}{ccc} \mathbb{T}\mathbb{T}^*M & \xleftarrow{\Lambda_M} & \mathbb{T}^*\mathbb{T}^*M \\ \mathbb{T}\lambda \downarrow & & \mathbb{T}^*\lambda \uparrow \\ \mathbb{T}\mathbb{T}M & & \mathbb{T}_{im(\lambda)}^* \mathbb{T}M \\ \varepsilon'_M \uparrow & & \varepsilon_M \downarrow \\ \mathbb{T}\mathbb{T}M & \xleftarrow{\mathbb{T}\lambda} & \mathbb{T}^*\mathbb{T}M \end{array} \quad (4.7)$$

Proof. Commutativity of the diagram is equivalent to

$$\mathbb{T}\lambda \circ \widetilde{\Lambda}_M^{-1} \circ \mathbb{T}^*\lambda = \varepsilon'_M \circ \mathbb{T}\lambda \circ \varepsilon_M$$

and to the dual equality

$$\mathbb{T}\lambda \circ (\widetilde{\Lambda}_M^{-1})^* \circ \mathbb{T}^*\lambda = \varepsilon'_M \circ \mathbb{T}\lambda^* \circ \varepsilon_M.$$

Since $(\widetilde{\Lambda}_M^{-1})^* = -\widetilde{\Lambda}_M^{-1}$ we get $\mathbb{T}\lambda^* = -\mathbb{T}\lambda$ and, consequently, $\lambda^* = -\lambda$. It follows that there exists a bivector field Λ on M such that $\lambda = \widetilde{\Lambda}^1$. From Theorem 4.4 we have that Λ is a Poisson structure if and only if the diagram 4.7 is commutative. \blacksquare

In the diagrams, $\mathbb{T}^*\lambda$ is a relation (not a mapping) and diagrams are in the category of relations.

In order to illustrate the condition 4.7 let us consider the case of a linear Poisson structure. Let $M = V$ be a vector space. We have obvious identifications:

$$\begin{aligned} \mathbb{T}V &= V \times \mathbb{T}_0V = V \times V, & \mathbb{T}^*V &= V \times V^*, \\ \mathbb{T}\mathbb{T}V &= (V \times V) \times (V \times V), & \mathbb{T}\mathbb{T}^*V &= (V \times V^*) \times (V \times V^*), \\ \mathbb{T}^*\mathbb{T}^*V &= (V \times V^*) \times (V^* \times V). \end{aligned}$$

With these identifications the canonical morphisms $\varepsilon_M, \kappa_M, \widetilde{\Lambda}_M^{-1}$ look like follows:

$$\begin{aligned} \varepsilon_V: \mathbb{T}^*\mathbb{T}V \ni (v, w; a, b) &\longmapsto (v, b; w, a) \in \mathbb{T}\mathbb{T}^*V, \\ \kappa_V: \mathbb{T}\mathbb{T}V \ni (v, w; x, y) &\longmapsto (v, x; w, y) \in \mathbb{T}\mathbb{T}V, \\ \widetilde{\Lambda}_V^{-1}: \mathbb{T}^*\mathbb{T}^*V \ni (v, a; w, b) &\longmapsto (v, a; -w, b) \in \mathbb{T}\mathbb{T}^*V. \end{aligned}$$

A linear Poisson structure Λ is given by a mapping

$$\widetilde{\Lambda}^1: \mathbb{T}^*V \ni (v, a) \longmapsto (v, \lambda(v, a)) \in \mathbb{T}V,$$

where $\lambda: V \times V^* \rightarrow V$ is bilinear. We have for such Λ :

$$\begin{aligned} \mathbb{T}\widetilde{\Lambda}^1: (v, a; w, b) &\longmapsto (v, \lambda(v, a); w, \lambda(v, b) + \lambda(w, a)), \\ \mathbb{T}^*\widetilde{\Lambda}^1: (v, \lambda(v, d); a, b) &\longmapsto (v, d; a + {}^*\lambda(b, d), \lambda^*(v, b)), \end{aligned}$$

where ${}^*\lambda$ and λ^* are conjugate to λ with respect to the left and to the right argument respectively. The condition 4.7 reads as follows:

The mapping $\widetilde{\Lambda}^1$ defines a Poisson structure if and only if the following conditions are satisfied for each v, a, b :

- (1) $\lambda(v, b) = -\lambda^*(v, b)$,
- (2) $\lambda(v, {}^*\lambda(b, a)) = \lambda(\lambda(v, a), b) - \lambda(\lambda(v, b), a)$,

which means that

$${}^*\lambda: V^* \times V^* \rightarrow V^*$$

is a Lie bracket.

5. Tangent symplectic and Poisson structures. Let (M, Λ) be a Poisson manifold, where M is a manifold and $\Lambda \in \Phi^2(\tau_M)$ satisfies $[\Lambda, \Lambda] = 0$. It follows from Theorem 2.5 that $(\mathbb{T}M, d\tau\Lambda)$ is also a Poisson manifold:

$$[d\tau\Lambda, d\tau\Lambda] = d\tau[\Lambda, \Lambda] = 0.$$

We call $d_{\mathbb{T}}\Lambda$ the *tangent Poisson structure*. Let be, in local coordinates,

$$\Lambda = \frac{1}{2}\Lambda^{ij}(x)\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}, \quad \Lambda^{ij} = -\Lambda^{ji},$$

then

$$d_{\mathbb{T}}\Lambda = \Lambda^{ij}(x)\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j} + \frac{1}{2}\frac{\partial\Lambda^{ij}}{\partial x^k}x^k\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}. \quad (5.1)$$

The Poisson structure Λ defines a Poisson bracket $\{f, g\}_{\Lambda} = \Lambda(df \wedge dg)$ which provides the algebra $\Phi^0(\tau_M)$ of smooth functions with a structure of Lie algebra. The tangent Poisson structure defines a Poisson bracket $\{, \}_{d_{\mathbb{T}}\Lambda}$ on $\mathbb{T}M$ which is characterized by the following relations:

$$\begin{aligned} \{d_{\mathbb{T}}f, d_{\mathbb{T}}g\}_{d_{\mathbb{T}}\Lambda} &= d_{\mathbb{T}}\{f, g\}_{\Lambda}, \\ \{d_{\mathbb{T}}f, v_{\mathbb{T}}g\}_{d_{\mathbb{T}}\Lambda} &= \{v_{\mathbb{T}}f, d_{\mathbb{T}}g\} = v_{\mathbb{T}}\{f, g\}_{\Lambda}, \\ \{v_{\mathbb{T}}f, v_{\mathbb{T}}g\}_{d_{\mathbb{T}}\Lambda} &= 0. \end{aligned} \quad (5.2)$$

For a 1-form $\mu \in \Phi^1(\pi_M)$ we put $\Lambda_{\mu} = i_{\mu}\Lambda$. For a function f the vector field Λ_{df} is the *hamiltonian vector field generated by f* .

Theorem 5.1. *The tangent Poisson structure $d_{\mathbb{T}}\Lambda$ is linear with respect to the vector bundle structure in $\tau_M: \mathbb{T}M \rightarrow M$. Moreover, for $\mu, \vartheta \in \Phi^1(\pi)$ we have the following formula:*

$$\{i_{\mathbb{T}}\mu, i_{\mathbb{T}}\vartheta\}_{d_{\mathbb{T}}\Lambda} = i_{\mathbb{T}}(d\langle\Lambda, \mu \wedge \vartheta\rangle) + i_{\Lambda_{\mu}}d\vartheta - i_{\Lambda_{\vartheta}}d\mu. \quad (5.3)$$

Proof. It enough to proof the formula 5.3. We have

$$\begin{aligned} \{i_{\mathbb{T}}\mu, i_{\mathbb{T}}\vartheta\}_{d_{\mathbb{T}}\Lambda} &= \langle d_{\mathbb{T}}\Lambda, di_{\mathbb{T}}\mu \wedge di_{\mathbb{T}}\vartheta \rangle = \\ &= \langle d_{\mathbb{T}}\Lambda, d_{\mathbb{T}}\mu \wedge d_{\mathbb{T}}\vartheta - d_{\mathbb{T}}\mu \wedge i_{\mathbb{T}}d\vartheta - i_{\mathbb{T}}d\mu \wedge d_{\mathbb{T}}\vartheta + i_{\mathbb{T}}d\mu \wedge i_{\mathbb{T}}d\vartheta \rangle. \end{aligned} \quad (5.4)$$

Since the 1-forms $i_{\mathbb{T}}d\mu$ and $i_{\mathbb{T}}d\vartheta$ are vertical, we have

$$\langle d_{\mathbb{T}}\Lambda, i_{\mathbb{T}}d\mu \wedge i_{\mathbb{T}}d\vartheta \rangle = 0.$$

Moreover,

$$\langle d_{\mathbb{T}}\Lambda, d_{\mathbb{T}}\mu \wedge d_{\mathbb{T}}\vartheta \rangle = i_{d_{\mathbb{T}}\vartheta}i_{d_{\mathbb{T}}\mu}d_{\mathbb{T}}\Lambda = i_{d_{\mathbb{T}}\vartheta}d_{\mathbb{T}}(i_{\mu}\Lambda) = d_{\mathbb{T}}(\langle\Lambda, \mu \wedge \vartheta\rangle)$$

and

$$\begin{aligned} \langle d_{\mathbb{T}}\Lambda, d_{\mathbb{T}}\mu \wedge i_{\mathbb{T}}d\vartheta \rangle &= \langle d_{\mathbb{T}}(\Lambda_{\mu}), i_{\mathbb{T}}d\vartheta \rangle = i_{d_{\mathbb{T}}(\Lambda_{\mu})}i_{\Gamma}v_{\mathbb{T}}(d\vartheta) = \\ &= -i_{\Gamma}i_{d_{\mathbb{T}}\Lambda_{\mu}}v_{\mathbb{T}}d\vartheta = -i_{\Gamma}v_{\mathbb{T}}(i_{\Lambda_{\mu}}d\vartheta) = -i_{\mathbb{T}}(i_{\Lambda_{\mu}}d\vartheta), \end{aligned}$$

where Γ is a second order vector field (see Section 2). Similarly,

$$\langle d_{\mathbb{T}}\Lambda, i_{\mathbb{T}}d\mu \wedge d_{\mathbb{T}}\vartheta \rangle = i_{\mathbb{T}}(i_{\Lambda_{\vartheta}}\mu)$$

and the theorem follows. ■

Since $d_{\mathbb{T}}\Lambda$ is a linear Poisson structure, it defines an algebroid structure in the dual vector bundle \mathbb{T}^*M . The theorem provides an explicit formula for the Lie bracket in $\Phi^1(\pi_M)$:

$$\begin{aligned} \{\mu, \vartheta\}_{\Lambda} &= d\langle\Lambda, \mu \wedge \vartheta\rangle + i_{\Lambda_{\mu}}\vartheta - i_{\Lambda_{\vartheta}}\mu \\ &= \mathcal{L}_{\Lambda_{\mu}}\vartheta - \mathcal{L}_{\Lambda_{\vartheta}}\mu - d\langle\Lambda, \mu \wedge \vartheta\rangle. \end{aligned}$$

This is exactly the well-known extension of a Poisson bracket to 1-forms.

6. Canonical vector fields. It is well known [Tu2] that for a symplectic manifold (M, ω) the tangent structure $(\mathbb{T}, d_{\mathbb{T}}\omega)$ is also a symplectic manifold. We use Proposition 3.1 to get a simple proof that canonical vector fields on a symplectic manifold are Lagrangian submanifolds with respect to the tangent symplectic structure.

Proposition 6.1. *Let $X: M \rightarrow \mathbb{T}M$ be a vector field on a symplectic manifold (M, ω) . The vector field X is canonical, i. e., $\mathcal{L}_X\omega = 0$ if and only if $X(M)$ is a Lagrangian submanifold of $(\mathbb{T}M, d_{\mathbb{T}}\omega)$.*

Proof. From the Proposition 3.1, $\mathcal{L}_X\omega = X^*d_{\mathbb{T}}\omega$. Hence $\mathcal{L}_X\omega = 0$ if and only if $X^*d_{\mathbb{T}}\omega = 0$, but the last means that $X(M)$ is isotropic in $(\mathbb{T}M, d_{\mathbb{T}}\omega)$ and, consequently, Lagrangian (the dimension of $X(M)$ is equal to the dimension of M). ■

We have an analogue of this theorem for Poisson manifolds. To formulate it we need the definition of a Lagrangian submanifold of a Poisson manifold.

Definition. *Let (M, Λ) be a Poisson manifold. A submanifold $N \subset M$ is Lagrangian if for each $m \in N$*

$$\tilde{\Lambda}^1(\mathbb{T}_m N)^\circ = \tilde{\Lambda}^1(\mathbb{T}_m^* M) \cap \mathbb{T}_m N \quad (6.1)$$

Theorem 6.2. *Let $X: M \rightarrow \mathbb{T}M$ be a vector field on a Poisson manifold (M, Λ) . X is canonical, i. e., $\mathcal{L}_X\Lambda = 0$ if and only if $X(M)$ is a Lagrangian submanifold of $(\mathbb{T}M, d_{\mathbb{T}}\Lambda)$.*

Proof.

From Proposition 3.3, $\mathcal{L}_X\Lambda = 0$ if and only if, for $f, g \in C^\infty(M)$,

$$\langle \widetilde{d_{\mathbb{T}}\Lambda}^1(X_*df), X_*dg \rangle = 0.$$

On the other hand, $X(M)$ is Lagrangian if for each $v \in X(M)$

$$\{\widetilde{d_{\mathbb{T}}\Lambda}^1(X_*df)(v) | f \in C^\infty(M)\} = \mathbb{T}_v X(M) \cap \widetilde{d_{\mathbb{T}}\Lambda}^1(\mathbb{T}_v^* \mathbb{T}M). \quad (6.2)$$

Since $w \in \mathbb{T}_v X(M)$ is characterized by $X_*df(w) = 0$, the inclusion " \subset " is equivalent to

$$\langle \widetilde{d_{\mathbb{T}}\Lambda}^1(X_*df)(v), X_*dg \rangle = 0.$$

Thus, if $X(M)$ is Lagrangian then X is canonical. In order to prove that for a canonical X the submanifold $X(M)$ is Lagrangian, it is enough to show that the inclusion " \supset " holds. Let be $a \in \mathbb{T}_v^* \mathbb{T}M$. There are functions f, g on M such that $a = d_v(d_{\mathbb{T}}f) + d_v(v_{\mathbb{T}}g)$. We have then (2.9)

$$\widetilde{d_{\mathbb{T}}\Lambda}^1(a) = d_{\mathbb{T}}(\tilde{\Lambda}^1 df) + v_{\mathbb{T}}(\tilde{\Lambda}^1 dg). \quad (6.3)$$

It follows that

$$\mathbb{T}\tau_M(\widetilde{d_{\mathbb{T}}\Lambda}^1 a) = \mathbb{T}\tau_M(d_{\mathbb{T}}(\tilde{\Lambda}^1 df)) = \tilde{\Lambda}^1 d_m f, \quad (6.4)$$

where $m = \tau_M v$. Thus the tangent projections of vectors in $\mathbb{T}X(M) \cap \widetilde{d_{\mathbb{T}}\Lambda}^1(\mathbb{T}^* \mathbb{T}M)$ are in the image of $\tilde{\Lambda}^1$. For a vector $w \in \tilde{\Lambda}^1(\mathbb{T}^* M)$ there is only one vector $v \in \mathbb{T}X(M)$ such that $\mathbb{T}\tau_M v = w$, namely, $v = \mathbb{T}X(w)$. Let be $w = \tilde{\Lambda}^1 d_m f$. The inclusion " \subset " implies that $\widetilde{d_{\mathbb{T}}\Lambda}^1(d_{X(m)}(d_{\mathbb{T}}f - v_{\mathbb{T}}(Xf)))$ is tangent to $X(M)$. Since (6.3 and 6.4)

$$\mathbb{T}\tau_M \widetilde{d_{\mathbb{T}}\Lambda}^1(d(d_{\mathbb{T}}f - v_{\mathbb{T}}(Xf))(X(m))) = \tilde{\Lambda}^1 d_m f = w,$$

we have $\widetilde{d_{\mathbb{T}}\Lambda}^1(X_*df)(X(m)) = \mathbb{T}X(w)$. ■

7. Tangent Poisson-Lie structures. Let M, N be differentiable manifolds. Since there is a canonical identification $\mathbb{T}(M \times N) \simeq \mathbb{T}M \times \mathbb{T}N$ we have also canonical inclusions $\bigwedge^2 \mathbb{T}M \times \bigwedge^2 \mathbb{T}N \subset \bigwedge^2 \mathbb{T}(M \times N)$ and $\Phi^2(\tau_M) \oplus \Phi^2(\tau_N) \subset \Phi^2(\tau_{N \times M})$. It is trivial task to verify, that if (M, Λ) and (N, Π) are Poisson manifolds then $(M \times N, \Lambda \oplus \Pi)$ is also a Poisson manifold. $\Lambda \oplus \Pi$ is called the *product Poisson structure*. Finally, we recall the notion of a *Poisson map*. A smooth mapping $\varphi: M \rightarrow N$ is a Poisson map if

$$\bigwedge^2 \mathbb{T}\varphi \circ \Lambda = \Pi \circ \varphi$$

or, equivalently,

$$\mathbb{T}\varphi \circ \tilde{\Lambda} \circ \mathbb{T}^*\varphi = \tilde{\Pi},$$

where $\tilde{\Lambda} = \tilde{\Lambda}^1, \tilde{\Pi} = \tilde{\Pi}^1$.

Let (G, m) be a Lie group and let (G, Λ) be a Poisson manifold. We say that (G, m, Λ) is a *Poisson-Lie group* if the group multiplication m is a Poisson mapping

$$m: (G \times G, \Lambda \oplus \Lambda) \rightarrow (G, \Lambda)$$

or, equivalently,

$$\mathbb{T}m \circ (\tilde{\Lambda} \times \tilde{\Lambda}) \circ \mathbb{T}^*m = \tilde{\Lambda}. \quad (7.1)$$

We say in this case that Λ is *multiplicative*.

Applying the tangent functor to 7.1, we get

$$\mathbb{T}\mathbb{T}m \circ (\mathbb{T}\tilde{\Lambda} \times \mathbb{T}\tilde{\Lambda}) \circ \mathbb{T}\mathbb{T}^*m = \mathbb{T}\tilde{\Lambda}. \quad (7.2)$$

It is well known that $(\mathbb{T}G, \mathbb{T}m)$ is a Lie group.

Theorem 7.1. *Let (G, m, Λ) be a Poisson-Lie group. Then $(\mathbb{T}G, \mathbb{T}m, d_{\mathbb{T}}\Lambda)$ is also a Poisson-Lie group.*

Proof. We have to show that $d_{\mathbb{T}}\Lambda$ is multiplicative with respect to $\mathbb{T}m$, i. e., that

$$\mathbb{T}\mathbb{T}m \circ (\widetilde{d_{\mathbb{T}}\Lambda} \times \widetilde{d_{\mathbb{T}}\Lambda}) \circ \mathbb{T}^*\mathbb{T}m = \widetilde{d_{\mathbb{T}}\Lambda}.$$

Since, due to (2.6), $\widetilde{d_{\mathbb{T}}\Lambda} = \kappa_G \circ \mathbb{T}\tilde{\Lambda} \circ \varepsilon_G$, it follows from functorial properties of κ and ε that

$$\begin{aligned} \mathbb{T}\mathbb{T}m \circ (\widetilde{d_{\mathbb{T}}\Lambda} \times \widetilde{d_{\mathbb{T}}\Lambda}) \circ \mathbb{T}^*\mathbb{T}m &= \mathbb{T}\mathbb{T}m \circ \kappa_{G \times G} \circ (\mathbb{T}\tilde{\Lambda} \times \mathbb{T}\tilde{\Lambda}) \circ \varepsilon_{G \times G} \circ \mathbb{T}^*\mathbb{T}m \\ &= \kappa_G \circ \mathbb{T}\mathbb{T}m \circ (\mathbb{T}\tilde{\Lambda} \times \tilde{\Lambda}) \circ \mathbb{T}\mathbb{T}^*m \circ \varepsilon_G \end{aligned}$$

and by (7.2) the required identity follows. ■

Let \mathfrak{g} be the Lie algebra of the Lie group G . The tangent bundle $\mathbb{T}G$ can be trivialized by right or left translations:

$$(K^r, \tau_G): \mathbb{T}G \rightarrow \mathfrak{g} \times G$$

or

$$(\tau_G, K^l): \mathbb{T}G \rightarrow G \times \mathfrak{g},$$

where $K^r(K^l)$ is the right-invariant (left-invariant) Maurer-Cartan form. The group structure in $\mathbb{T}G$ is given by the formula

$$(X, g) \cdot (Y, h) = (X + \text{Ad}(g)Y, gh)$$

in the right trivialization and

$$(g, X) \cdot (h, Y) = (gh, X + \text{Ad}(h^{-1})Y)$$

in the left trivialization. The neutral element e^T is represented by $(0, e)$ in the right trivialization and by $(e, 0)$ in the left trivialization.

The Lie algebra $\mathbb{T}\mathfrak{g}$ of $\mathbb{T}G$ is isomorphic as a linear space to $\mathfrak{g} \times \mathfrak{g}$. This isomorphism is implemented by the zero section $j_G: G \rightarrow \mathbb{T}G$ and the obvious embedding $j_{\mathfrak{g}}: \mathfrak{g} \rightarrow \mathbb{T}_e G \subset \mathbb{T}G$ and is given by the following formula

$$\mathfrak{g} \times \mathfrak{g} \ni (X, Y) \rightarrow \mathbb{T}_e j_G(X) + \mathbb{T}_0 j_{\mathfrak{g}}(Y) \in \mathbb{T}_{e^T} \mathbb{T}G. \quad (7.3)$$

From now on we shall denote $\mathbb{T}_e j_G(X)$ by \dot{X} and $\mathbb{T}_0 j_{\mathfrak{g}}(Y)$ by \widehat{Y} . We have the following commutation rules:

$$\begin{aligned} [\dot{X}, \dot{Y}]_{\mathbb{T}\mathfrak{g}} &= [X, Y]_{\mathfrak{g}}, \\ [\dot{X}, \widehat{Y}]_{\mathbb{T}\mathfrak{g}} &= [\widehat{X}, \dot{Y}]_{\mathbb{T}\mathfrak{g}} = \widehat{[X, Y]}_{\mathfrak{g}}, \\ [\widehat{X}, \widehat{Y}]_{\mathbb{T}\mathfrak{g}} &= 0. \end{aligned}$$

It follows that the Lie algebra $\mathbb{T}\mathfrak{g}$ is a semidirect product $\mathfrak{g} \ltimes \mathfrak{g}$ with respect to the adjoint representation in \mathfrak{g} of the Lie algebra \mathfrak{g} , i. e.,

$$[(X, Y), (X', Y')]_{\mathfrak{g} \ltimes \mathfrak{g}} = ([X, X']_{\mathfrak{g}}, [X, Y']_{\mathfrak{g}} + [Y, X']_{\mathfrak{g}}).$$

Theorem 7.2. *Let be $X \in \mathfrak{g}$ and let X_G^l be the corresponding left invariant vector field on G . Then*

$$\widehat{X}_{\mathbb{T}G}^l = (X_G^l)^v \quad \text{and} \quad \dot{X}_{\mathbb{T}G}^l = (X_G^l)^c,$$

i. e., the corresponding to \widehat{X} and \dot{X} left invariant vector fields on $\mathbb{T}G$ are the vertical and complete lifts of X_G^l respectively. An analogous statement is true for right invariant vector fields.

Proof. The group multiplication $\mathbb{T}m$ in $\mathbb{T}G$ is the tangent lift of the group multiplication m in G . It follows that

$$\mathbb{T}m(u_g, v_h) = \mathbb{T}L_g(v_h) + \mathbb{T}R_h(u_g)$$

where L_g and R_h are left and right translations by g and h respectively. The left translation $L_{u_g}^{\mathbb{T}}$ by u_g in $\mathbb{T}G$ is therefore given by

$$L_{u_g}^{\mathbb{T}}(v_h) = \mathbb{T}L_g(v_h) + \mathbb{T}R_h(u_g).$$

It is easy to verify that $(X_G^l)^c(e^T) = \dot{X}$ and $(X_G^l)^v(e^T) = \widehat{X}$. Since the mapping (7.3) is a linear isomorphism, it is enough to show that vector fields $(X_G^l)^c$ and $(X_G^l)^v$ are left-invariant on $\mathbb{T}G$. In other words, we have to show that flows they generate commute with left translations.

The flow ψ^t of the vertical lift is given by

$$\psi^t(v_h) = v_h + tX_G^l(h),$$

we have then

$$\begin{aligned} L_{u_g}^{\mathbb{T}} \circ \psi^t(v_h) &= \mathbb{T}L_g(v_h + tX_G^l(h)) + \mathbb{T}R_h(u_g) \\ &= \mathbb{T}L_g(v_h) + \mathbb{T}R_h(u_g) + t\mathbb{T}L_g(X_G^l(h)) \\ &= \mathbb{T}L_g(v_h) + \mathbb{T}R_h(u_g) + tX_G^l(gh) = \psi^t \circ L_{u_g}^{\mathbb{T}}(v_h). \end{aligned}$$

We made use of the left-invariance of X_G^l .

The flow of the complete lift $(X_G^l)^c$ is the tangent lift of the flow φ^t of X_G^l . We have

$$\begin{aligned} L_{u_g}^\top \circ \mathbb{T}\varphi^t(v_h) &= \mathbb{T}L_g \circ \mathbb{T}\varphi^t(v_h) + \mathbb{T}R_{\varphi^t(h)}(u_g) \\ &= \mathbb{T}(L_g \circ \varphi^t)(v_h) + \mathbb{T}R_{\varphi^t(h)}(u_g) \end{aligned}$$

and, since φ^t is the flow of a left-invariant vector field, i. e.,

$$L_g \circ \varphi^t = \varphi^t \circ L_g$$

and

$$\varphi^t \circ R_h(g) = \varphi^t(gh) = L_g(\varphi^t(h)) = R_{\varphi^t(h)}(g),$$

we get

$$\mathbb{T}(L_g \circ \varphi^t)(v_h) + \mathbb{T}R_{\varphi^t(h)}(u_g) = \mathbb{T}(\varphi^t \circ L_g)(v_h) + \mathbb{T}(\varphi^t(h) \circ R_h)(u_g) = \mathbb{T}\varphi^t \circ L_{u_g}^\top(v_h).$$

■

8. Tangent Poisson-Lie algebras of Poisson-Lie groups. The Lie algebra of a Poisson-Lie group inherits a Poisson structure. We recall here a standard construction. More natural and more geometric one will be given in Section 10 (Proposition 10.4).

Let (G, m, Λ) be a Poisson-Lie group. A Poisson structure Λ on a Lie group can be regarded, using the right trivialization of $\mathbb{T}G$, as a mapping $\bar{\Lambda}: G \rightarrow \mathfrak{g} \wedge \mathfrak{g}$. The Poisson structure Λ is multiplicative if and only if $\bar{\Lambda}$ is a 1-cocycle of G with respect to the adjoint representation of G in $\mathfrak{g} \wedge \mathfrak{g}$. In particular, we have $\bar{\Lambda}(e) = 0$. The tangent mapping

$$\lambda = \mathbb{T}_e \bar{\Lambda}: \mathbb{T}_e G = \mathfrak{g} \rightarrow \mathbb{T}_0(\mathfrak{g} \wedge \mathfrak{g}) = \mathfrak{g} \wedge \mathfrak{g},$$

being a 1-cocycle of \mathfrak{g} , defines a cobracket (Poisson bracket on \mathfrak{g}). The pair (\mathfrak{g}, λ) is called the *tangent Poisson-Lie algebra* of the Poisson-Lie group (G, m, Λ) .

Let (X_1, \dots, X_n) be a basis of the Lie algebra \mathfrak{g} . We can write

$$\Lambda = \sum \lambda^{ij} X_i^r \wedge X_j^r,$$

where λ^{ij} are smooth functions on G and X_i^r are the corresponding right invariant vector fields. We have then

$$\bar{\Lambda}(g) = \sum \lambda^{ij}(g) X_i \wedge X_j$$

and

$$\lambda(X_k) = \langle X_k, d\lambda^{ij}(e) \rangle X_i \wedge X_j = \frac{1}{2} c_k^{ij} X_i \wedge X_j,$$

where c_k^{ij} are structure constants of the Lie algebra \mathfrak{g} . The cobracket $\lambda: \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ may be regarded as a bivector field on \mathfrak{g} which defines a linear Poisson structure on \mathfrak{g} .

Definition. A Poisson-Lie algebra (\mathfrak{g}, δ) is a Lie algebra \mathfrak{g} and a 1-cocycle (cobracket) $\delta: \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ (with respect to the adjoint representation of \mathfrak{g} in $\mathfrak{g} \wedge \mathfrak{g}$) such that it defines a Poisson structure on \mathfrak{g} or, equivalently, that the dual mapping $\delta^*: \mathfrak{g}^* \wedge \mathfrak{g}^*$ is a Lie bracket on \mathfrak{g}^* .

The tangent Poisson-Lie algebra is an example of an abstract Poisson-Lie algebra. For simple connected Lie groups there is one-to-one correspondence between Poisson-Lie structures on G and Poisson-Lie algebra structures on \mathfrak{g} . The Poisson-Lie algebra (\mathfrak{g}, λ) can be seen as a quadruple $(\mathfrak{g}, \mathfrak{g}^*, [\cdot, \cdot], [\cdot, \cdot]^*)$ and, for this reason, it is called sometimes a *Lie bialgebra*.

Theorem 8.1. *Let $\lambda: \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ be the cobracket of the tangent Poisson-Lie algebra of a Poisson-Lie group (G, m, Λ) . Then*

$$d_{\mathbb{T}}\lambda: \mathbb{T}\mathfrak{g} \rightarrow \mathbb{T}\mathfrak{g} \wedge \mathbb{T}\mathfrak{g}$$

is the cobracket of the tangent Poisson-Lie algebra of the tangent Poisson-Lie group $(\mathbb{T}G, \mathbb{T}m, d_{\mathbb{T}}\lambda)$. The dual mapping

$$(d_{\mathbb{T}}\lambda)^*: (\mathbb{T}\mathfrak{g})^* \wedge (\mathbb{T}\mathfrak{g})^* \rightarrow (\mathbb{T}\mathfrak{g})^*$$

is the Lie bracket on the tangent Lie algebra of $(\mathfrak{g}^, \lambda^*)$.*

Proof. For any $X \in \Phi^r(\tau_M)$ and for any smooth function $f \in \Phi^0(\pi_M)$ we have

$$d_{\mathbb{T}}(fX) = \tau_M^*(f)d_{\mathbb{T}}X + d_{\mathbb{T}}f \cdot v_{\mathbb{T}}(X).$$

Moreover, $Y^v(\tau_M^*f) = 0$, $Y^v(d_{\mathbb{T}}f) = Y^c(\tau_M^*f) = \tau_M^*(Y(f))$ and $Y^c(d_{\mathbb{T}}f) = d_{\mathbb{T}}(Y(f))$ for any vector field $Y \in \Phi^1(\tau_M)$.

Now, let $\Lambda = \Lambda^{ij}X_i^r \wedge X_j^r$ and $\lambda(X_k) = \frac{1}{2}c_k^{ij}X_i \wedge X_j$ for a basis (X_1, \dots, X_n) in \mathfrak{g} . We have

$$\begin{aligned} d_{\mathbb{T}}\Lambda &= 2\tau_G^*(\Lambda^{ij})(X_i^r)^c \wedge (X_j^r)^v + d_{\mathbb{T}}(\Lambda^{ij})(X_i^r)^v \wedge (X_j^r)^v \\ &= 2\tau_G^*(\Lambda^{ij})(\dot{X}_i)^r \wedge (\hat{X}_j)^r + d_{\mathbb{T}}(\Lambda^{ij})(\hat{X}_i)^r \wedge (\hat{X}_j)^r. \end{aligned}$$

The cobracket δ on the Lie algebra of $\mathbb{T}G$ is given by the formulae

$$\begin{aligned} \delta(\hat{X}_k) &= (X_k^r)^v (2\tau_G^*(\Lambda^{ij}))(e^T)\dot{X}_i \wedge \hat{X}_j + (X_k^r)^v (d_{\mathbb{T}}(\Lambda^{ij}))(e^T)\hat{X}_i \wedge \hat{X}_j \\ &= \tau_G^*(X_k^r(\Lambda^{ij}))(e^T)\hat{X}_i \wedge \hat{X}_j = X_k^r(\Lambda^{ij})(e)\hat{X}_i \wedge \hat{X}_j \\ &= \langle X_k, d\Lambda^{ij} \rangle(e)\hat{X}_i \wedge \hat{X}_j = \frac{1}{2}c_k^{ij}\hat{X}_i \wedge \hat{X}_j \end{aligned}$$

and

$$\begin{aligned} \delta(\dot{X}_k) &= (X_k^r)^c (2\tau_G^*(\Lambda^{ij}))(e^T)\dot{X}_i \wedge \hat{X}_j + (X_k^r)^c (d_{\mathbb{T}}(\Lambda^{ij}))(e^T)\hat{X}_i \wedge \hat{X}_j \\ &= 2X_k^r(\Lambda^{ij})(e)\dot{X}_i \wedge \hat{X}_j + d_{\mathbb{T}}(X_j^r(\Lambda^{ij}))(e^T)\hat{X}_i \wedge \hat{X}_j \\ &= c_k^{ij}\dot{X}_i \wedge \hat{X}_j \end{aligned}$$

($d_{\mathbb{T}}(f)$ is zero on the zero section). It follows that $\delta = d_{\mathbb{T}}\lambda$. ■

Let (G, m, Λ) be a Poisson-Lie group and let $(\mathfrak{g}, [\cdot, \cdot], \mathfrak{p})$ be its Lie bialgebra. Let us suppose that \mathfrak{p} is a coboundary (e. g. G is semisimple), i. e., that

$$\mathfrak{p}(X) = [X, r] = r^{ij}([X, X_i] \wedge X_j + X_i \wedge [X, X_j])$$

for some $r = r^{ij}X_i \wedge X_j \in \mathfrak{g} \wedge \mathfrak{g}$. It is known [Dr] that in this case the Poisson structure Λ on G can be written in the form

$$\Lambda = r^l - r^r,$$

where r^l and r^r are the left- and right-invariant 2-vector fields corresponding to r . Since Λ is a Poisson structure, r must satisfy the generalized classical Yang-Baxter equation

$$\text{ad}_X[r, r] = 0$$

for every $X \in \mathfrak{g}$. The bracket $[r, r]$ is the algebraic Schouten bracket. An element of $\mathfrak{g} \wedge \mathfrak{g}$ which satisfies this equation is called a *generalized r -matrix* and the corresponding Poisson structure Λ is called *quasitriangular*.

Theorem 8.2. *Let $\Lambda = r^l - r^r$ be a quasitriangular Poisson-Lie structure on a Lie group G with the r -matrix $r = r^{ij} X_i \wedge X_j$, $r^{ij} = -r^{ji}$. Then $d_{\mathbb{T}}\Lambda$ is a quasitriangular Poisson-Lie structure on $\mathbb{T}G$ with the r -matrix $d_{\mathbb{T}}r = 2r^{ij} \hat{X}_i \wedge \hat{X}_j$.*

Proof. Since $\Lambda = r^{ij} (X_i^l \wedge X_j^l - X_i^r \wedge X_j^r)$, we have

$$\begin{aligned} d_{\mathbb{T}}\Lambda &= 2r^{ij} ((X_i^l)^c \wedge (X_j^l)^v - (X_i^r)^c \wedge (X_j^r)^v) \\ &= 2r^{ij} ((\hat{X}_i)^l \wedge (\hat{X}_j)^l - (\hat{X}_i)^r \wedge (\hat{X}_j)^r) = (d_{\mathbb{T}}r)^l - (d_{\mathbb{T}}r)^r \end{aligned}$$

and it is easy to check that $d_{\mathbb{T}}r$ is really an r -matrix. ■

9. Tangent lifts of generalized foliations.

Definition. *A generalized distribution on a manifold M is a subset $S \subset \mathbb{T}M$ such that $S(x) = S \cap \mathbb{T}_x M$ is a linear subspace for each point $x \in M$. S is said to be smooth if it is generated by the family*

$$\mathcal{X}(S) = \{X \in \Phi^1(\tau_M) : \forall x \in M \quad X(x) \in S(x)\}$$

of smooth vector fields, i. e., $S(x)$ is spanned by $\{X(x) : X \in \mathcal{X}(S)\}$.

A smooth distribution is completely integrable if for every point $x \in M$ there exists an integral submanifold of S , everywhere of maximal dimension, which contains x .

The maximal integral submanifolds of a completely integrable distribution S form a partition of M , called the generalized foliation of M defined by S .

Let us notice that for a completely integrable distribution S the family $\mathcal{X}(S)$ is a Lie subalgebra of the Lie algebra of vector fields on M . The following theorem is due to H. Sussmann [Sus].

Theorem. (Sussmann) *Let S be a smooth distribution on M and let $\mathcal{D} \subset \mathcal{X}(S)$ be a family of vector fields such that \mathcal{D} spans S . The following properties are equivalent*

- (1) *S is completely integrable,*
- (2) *S is invariant with respect to flows $\exp(tX)$ of vector fields $X \in \mathcal{D}$,*
- (3) *flows of vector fields from $\mathcal{X}(S)$ preserve S .*

Theorem 9.1. *Let S be a completely integrable generalized distribution. Then the distribution $S^{\mathbb{T}}$ generated by the family $\{X^v, X^c : X \in \mathcal{X}(S)\}$ of vector fields on $\mathbb{T}M$ is completely integrable*

Proof. We have $\exp(tX^v)(v) = v + tX(\tau_M v)$ and $\exp(tX^c) = \mathbb{T}\exp(tX)$. Due to the formulae

$$(\exp(tX^v))_* Y^v = Y^v, \quad (\exp(tX^v))_* Y^c = Y^c + t[X, Y]^v$$

and

$$(\exp(tX^c))_* Y^v = ((\exp(tX))_* Y)^v, \quad (\exp(tX^c))_* Y^c = ((\exp(tX))_* Y)^c,$$

it follows that $S^{\mathbb{T}}$ is invariant with respect to flows of X^c and X^v . From the theorem of Sussmann, $S^{\mathbb{T}}$ is integrable. ■

Definition. *The tangent foliation $\mathcal{F}^{\mathbb{T}}$ of a generalized foliation \mathcal{F} defined by S is the foliation defined by $S^{\mathbb{T}}$.*

Proposition 9.2. *If a 1-form μ annihilates a completely integrable distribution S then $v_{\mathbb{T}}(\mu)$ and $d_{\mathbb{T}}(\mu)$ annihilate $S^{\mathbb{T}}$.*

Proof. Let $X \in \mathcal{X}(S)$. We have from Proposition 2.6 that

$$\langle v_{\mathbb{T}}(\mu), X^v \rangle = 0, \quad \langle d_{\mathbb{T}}(\mu), X^c \rangle = d_{\mathbb{T}}\langle \mu, X \rangle = 0$$

and

$$\langle v_{\mathbb{T}}(\mu), X^c \rangle = \langle d_{\mathbb{T}}(\mu), X^v \rangle = v_{\mathbb{T}}(\langle \mu, X \rangle) = 0.$$

■

Proposition 9.3. *If a submanifold $N \subset M$ is a union of leaves of the foliation \mathcal{F} (N is \mathcal{F} -foliated) then $\mathbb{T}_N M = \tau_M^{-1}(N)$ is $\mathcal{F}^\mathbb{T}$ -foliated.*

Proof. It is enough to prove Proposition in the case of N being a single leaf. Let F be a leaf of $\mathcal{F}^\mathbb{T}$. Since $\mathbb{T}\tau_M(X^v) = 0$ and $\mathbb{T}\tau_M(X^c) = X$ for any vector field X on M , the tangent projection $\tau_M(F)$ of F is contained in a leaf of \mathcal{F} . It follows that $\mathbb{T}_N M$ is $\mathcal{F}^\mathbb{T}$ -foliated. ■

Proposition 9.4. *If F is a leaf of \mathcal{F} then $\mathbb{T}F$ is a leaf of $\mathcal{F}^\mathbb{T}$.*

Proof. It is obvious that $\mathbb{T}F \subset S^\mathbb{T}$ where S is the generalized distribution related to \mathcal{F} . We have to show that $\mathbb{T}F$ is maximal. Since F is maximal, $\mathbb{T}_x F = S(x)$ and $S(F)$ is spanned by vector fields tangent to F . On the other hand it is easy to see that if X is a vector field on M , tangent to F on F , then $d_\mathbb{T}X$ and $v_\mathbb{T}X$ are tangent to $\mathbb{T}F$ on $\mathbb{T}F$. Since $S^\mathbb{T}$ is generated by the family $\{X^v, X^c: X \in \mathcal{X}(S)\}$, $S^\mathbb{T}(\mathbb{T}F) = \mathbb{T}\mathbb{T}F$, i. e., $\mathbb{T}F$ is an integral submanifold of $S^\mathbb{T}$ which is clearly maximal. ■

10. Symplectic foliations of Poisson manifolds.

Let (M, Λ) be a Poisson manifold. The characteristic distribution S of Λ is generated by hamiltonian vector fields. It is well known that S is completely integrable and that Λ defines symplectic structures on leaves of S .

Proposition 10.1. *$S^\mathbb{T}$ is the characteristic distribution of $d_\mathbb{T}\Lambda$.*

Proof. Since the vertical and tangent lifts of 1-forms on M generate the module of 1-forms on $\mathbb{T}M$ it is enough to notice that (2.9)

$$\langle v_\mathbb{T}(\mu), d_\mathbb{T}\Lambda \rangle = (i_\mu \Lambda)^v \quad \text{and} \quad \langle d_\mathbb{T}\mu, d_\mathbb{T}\Lambda \rangle = (i_\mu \Lambda)^c.$$

and that, consequently, the characteristic distribution of $d_\mathbb{T}\Lambda$ is generated by the complete and vertical lifts of hamiltonian vector fields on (M, Λ) , i. e., of vector field from $\mathcal{X}(S)$. ■

The following theorem by Weinstein [We] describes the local structure of a Poisson manifold.

Theorem. (Weinstein) *Let (M, Λ) be a Poisson manifold of rank $2k$ at $x_0 \in M$. Then there is an open neighbourhood U of x_0 such that $(U, \lambda|_U)$ is isomorphic to a product $(N \times V, \Lambda_N \times \Lambda_V)$ of Poisson manifolds where (N, Λ_N) is a symplectic manifold of dimension $2k$ and the rank of (V, Λ_V) is zero at z_0 , $x_0 = (y_0, z_0)$.*

The theorem of Weinstein shows that while analyzing only local properties of Poisson manifolds it is enough to consider two cases:

- (1) Λ is regular,
- (2) Λ vanishes at a point.

Theorem 10.2. *Let (M, Λ) be a Poisson manifold.*

- (1) *If Λ is a regular Poisson structure of rank $2k$ then $d_\mathbb{T}\Lambda$ is regular of rank $4k$.*
- (2) *If Λ is of rank 0 at $x_0 \in M$ then $\mathbb{T}_{x_0}M$ is a Poisson submanifold of $(\mathbb{T}M, d_\mathbb{T}\Lambda)$ and $d_\mathbb{T}\Lambda$ defines on $\mathbb{T}_{x_0}M$ a linear Poisson structure (Kostant-Kirillov-Souriau structure). It induces then a Lie algebra structure on $\mathbb{T}_{x_0}^*$.*

Proof.

- (1) It follows from the theorem of Weinstein that we can choose local coordinates on M such that

$$\Lambda = \sum_{j,i=1}^r \Lambda^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \quad \Lambda^{ij} = -\Lambda^{ji}, \quad \det(\Lambda^{ij}) \neq 0$$

where Λ^{ij} are constant. Hence, in the adopted coordinate system,

$$d_\mathbb{T}\Lambda = 2 \sum_{j,i=1}^r \Lambda^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

(2) We have locally

$$\Lambda(x) = \sum_{j,i} \Lambda^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \quad \Lambda^{ij} = -\Lambda^{ji}, \quad \Lambda^{ij}(x_0) = 0$$

and

$$d_{\mathbb{T}}\Lambda(x_0, \dot{x}) = \sum_{j,i,k} \frac{\partial \Lambda^{ij}}{\partial x^k}(x_0) \dot{x}^k \frac{\partial}{\partial \dot{x}^i} \wedge \frac{\partial}{\partial \dot{x}^j}.$$

Hence $\mathbb{T}_{x_0}M$ is a Poisson submanifold with the Poisson bracket

$$\{\dot{x}^i, \dot{x}^j\} = 2 \sum_k \frac{\partial \Lambda^{ij}}{\partial x^k}(x_0) \dot{x}^k.$$

■

Corollary. *If Λ is of rank 0 at x_0 then the Poisson structure $d_{\mathbb{T}}\Lambda$ on \mathbb{T}_{x_0} is given by the formula*

$$d_{\mathbb{T}}\Lambda(v) = v_{\mathbb{T}}(\mathcal{L}_{\tilde{v}}\Lambda)(v), \quad v \in \mathbb{T}_{x_0}M,$$

where $\tilde{v} \in \Phi^1(\tau_M)$ is such that $\tilde{v}(x_0) = v$.

Proof. Since $\Lambda(x_0) = 0$ it follows that $d_{\mathbb{T}}\Lambda(v)$ is vertical and, consequently,

$$v_{\mathbb{T}}(\tilde{v}^* d_{\mathbb{T}}\Lambda)(v) = d_{\mathbb{T}}\Lambda(v)$$

(see Section 3). It follows from Theorem 3.2 that

$$d_{\mathbb{T}}\Lambda(v) = v_{\mathbb{T}}(\mathcal{L}_{\tilde{v}}\Lambda)(v).$$

■

The following two propositions complete our discussion on the structure of the tangent Poisson manifold.

Proposition 10.3. *Let f be a local Casimir of Λ , i. e., $\Lambda(df, \cdot) = 0$. Then $v_{\mathbb{T}}(f)$ and $d_{\mathbb{T}}f$ are local Casimirs of $d_{\mathbb{T}}\Lambda$.*

Moreover, if Λ is regular at $x \in M$ with symplectic leaves determined by Casimirs (f_1, \dots, f_n) , then Λ is regular at $v \in \mathbb{T}_xM$ with symplectic leaves determined by Casimirs $(\tau_M^(f_1), \dots, \tau_M^*(f_n), d_{\mathbb{T}}f_1, \dots, d_{\mathbb{T}}f_n)$.*

Proof. It follows from (2.9) that if f generates a zero-hamiltonian field then also τ_M^*f and $d_{\mathbb{T}}f$ generate the zero-hamiltonian field on $\mathbb{T}M$, i. e., they are Casimirs.

Let $x \in M$ be a regular point of Λ with the symplectic foliation in the neighbourhood of x defined by Casimirs (f_1, \dots, f_n) . We may assume that df_1, \dots, df_n are linearly independent at x . It follows that $(d\tau_M^*(f_1), \dots, d\tau_M^*(f_n), dd_{\mathbb{T}}f_1, \dots, dd_{\mathbb{T}}f_n)$ are linearly independent at $v \in \mathbb{T}_xM$. Since the rank of $d_{\mathbb{T}}\Lambda$ is $2(\dim M - n)$ (Theorem 10.2) the theorem follows. ■

If (G, Λ) is a Poisson-Lie group then $\Lambda(e) = 0$ and identifying \mathfrak{g} with \mathbb{T}_eG we get a Poisson structure on \mathfrak{g} induced by $d_{\mathbb{T}}\Lambda$.

Proposition 10.4. *The Poisson structure on \mathfrak{g} induced by $d_{\mathbb{T}}\Lambda$ is equal to the Poisson structure defined by the cobracket of the tangent Poisson-Lie algebra.*

Proof. Let (x^i) be a coordinate system on G centered at e , i. e., $x^i(e) = 0$. It defines a basis (X_i) of the Lie algebra \mathfrak{g} , $X_i = \frac{\partial}{\partial x^i}(e)$. We have then

$$\Lambda = \Lambda^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} = \lambda^{ij} X_i^r \wedge X_j^r$$

and

$$\Lambda^{ij} = \lambda^{ij} + \alpha^{ij}.$$

The induced cobracket λ is given by

$$\lambda(X_k) = \frac{\partial \lambda^{ij}}{\partial x_k}(0) X_i \wedge X_j.$$

On the other hand,

$$d_{\mathbb{T}}|_{\mathbb{T}_\epsilon G} = \frac{\partial \Lambda^{ij}}{\partial x_k}(0) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \frac{\partial \Lambda^{ij}}{\partial x_k}(0) X_i \wedge X_j$$

and

$$\begin{aligned} \Lambda^{ij}(0) &= \lambda^{ij}(0) = 0, \\ \frac{\partial}{\partial x_i} - X_i^r &= a_i^k X_i^r, \end{aligned}$$

where $a_i^k(0) = 0$. It follows that

$$\lambda^{ij} = \Lambda^{ij} + \Lambda^{il} a_l^j - \Lambda^{jl} a_l^i + \frac{1}{2}(a_k^i a_l^j - a_k^j a_l^i) \Lambda^{kl}.$$

Hence

$$\frac{\partial \lambda^{ij}}{\partial x_k}(0) = \frac{\partial \Lambda^{ij}}{\partial x_k}(0).$$

■

11. Examples.

Example 1

On $\mathfrak{su}(2)^* \simeq \mathbb{R}^3$ consider the linear Poisson structure

$$\Lambda = z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}.$$

The symplectic foliation of \mathbb{R}^3 is the union of 2-dimensional spheres $x^2 + y^2 + z^2 = r > 0$ and the origin $(0, 0, 0)$ as a 0-dimensional leaf. It is regular outside the origin and is defined by the Casimir $f(x, y, z) = x^2 + y^2 + z^2$.

The tangent Poisson structure is given by the formula

$$\begin{aligned} d_{\mathbb{T}}\Lambda &= z \left(\frac{\partial}{\partial \dot{x}} \wedge \frac{\partial}{\partial \dot{y}} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \dot{y}} \right) + x \left(\frac{\partial}{\partial \dot{y}} \wedge \frac{\partial}{\partial \dot{z}} + \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial \dot{z}} \right) + y \left(\frac{\partial}{\partial \dot{z}} \wedge \frac{\partial}{\partial \dot{x}} + \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \dot{x}} \right) \\ &\quad + z \frac{\partial}{\partial \dot{x}} \wedge \frac{\partial}{\partial \dot{y}} + x \frac{\partial}{\partial \dot{y}} \wedge \frac{\partial}{\partial \dot{z}} + y \frac{\partial}{\partial \dot{z}} \wedge \frac{\partial}{\partial \dot{x}}. \end{aligned}$$

The symplectic foliation of $\mathbb{T}\mathbb{R}^3$ is regular outside $\mathbb{T}_0\mathbb{R}^3$ and it is determined by two Casimirs

$$f_0(x, y, z, \dot{x}, \dot{y}, \dot{z}) = x^2 + y^2 + z^2, \quad f_1(x, y, z, \dot{x}, \dot{y}, \dot{z}) = x\dot{x} + y\dot{y} + z\dot{z}.$$

The tangent space $\mathbb{T}_0\mathbb{R}^3 \simeq \mathbb{R}^3$ has a linear Poisson structure

$$z \frac{\partial}{\partial \dot{x}} \wedge \frac{\partial}{\partial \dot{y}} + x \frac{\partial}{\partial \dot{y}} \wedge \frac{\partial}{\partial \dot{z}} + y \frac{\partial}{\partial \dot{z}} \wedge \frac{\partial}{\partial \dot{x}}.$$

which is equal to Λ .

Example 2

In this example $M = \mathbb{R}^4$ and Λ is the following quadratic Poisson structure:

$$\begin{aligned} \Lambda = x_1 x_3 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} + x_1 x_4 \frac{\partial}{\partial x_4} \wedge \frac{\partial}{\partial x_2} + x_2 x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} \\ + x_2 x_4 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4} + (x_3^2 + x_4^2) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1}. \end{aligned}$$

This structure is degenerated at points of the linear subspace $x_3 = x_4 = 0$ and regular outside it. The symplectic foliation is the intersection of the „book” foliation generated by vector fields $x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}$, $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$, consisting of 3-dimensional half-spaces sewed up along the 2-dimensional edge $x_3 = x_4 = 0$ of 0-dimensional leaves, and the spherical foliation.

The unit sphere $S^3 = \{x : \sum x_i^2 = 1\}$ we identify with the $SU(2)$ group by

$$(x_1, x_2, x_3, x_4) \longmapsto \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

With this identification, the tensor Λ restricted to S^3 defines a Poisson-Lie structure on $SU(2)$. This structure is quasitriangular with the generalized r-matrix

$$r = \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \in \mathfrak{su}(2) \wedge \mathfrak{su}(2).$$

Here we used identifications

$$\mathfrak{su}(2) = T_e SU(2) \simeq T_{(1,0,0,0)} S^3,$$

so in the matrix form,

$$r = X \wedge Y \quad \text{with} \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathfrak{su}(2).$$

It is worthy noticing that this Poisson-Lie structure is related to the quantum $SU(2)$ group of Woronowicz (cf. [Gr]).

Singular points on S^3 form a 1-dimensional circle $x_1^2 + x_2^2 = 1$, $x_3 = x_4 = 0$ which corresponds to the Cartan subgroup (maximal torus) of $SU(2)$. At a singular point $x = (\cos \varphi, \sin \varphi, 0, 0)$, the tangent space $T_x S^3$ carries a linear Poisson structure induced by $d_T \Lambda$

$$d_T \Lambda|_{T_x S^3} = \cos(\varphi) (\dot{x}_4 \frac{\partial}{\partial \dot{x}_4} \wedge \frac{\partial}{\partial \dot{x}_2} + \dot{x}_3 \frac{\partial}{\partial \dot{x}_1} \wedge \frac{\partial}{\partial \dot{x}_2}) + \sin(\varphi) (\dot{x}_3 \frac{\partial}{\partial \dot{x}_1} \wedge \frac{\partial}{\partial \dot{x}_3} + \dot{x}_4 \frac{\partial}{\partial \dot{x}_1} \wedge \frac{\partial}{\partial \dot{x}_4})$$

associated with the Lie bracket

$$\begin{aligned} \{\dot{x}_1, \dot{x}_3\} &= \sin(\varphi) \dot{x}_3, \\ \{\dot{x}_1, \dot{x}_4\} &= \sin(\varphi) \dot{x}_4, \\ \{\dot{x}_2, \dot{x}_3\} &= -\cos(\varphi) \dot{x}_3, \\ \{\dot{x}_2, \dot{x}_4\} &= -\cos(\varphi) \dot{x}_4. \end{aligned}$$

In particular, on $T_e SU(2) = T_{(1,0,0,0)} S^3$ we have

$$\dot{x}_4 \frac{\partial}{\partial \dot{x}_4} \wedge \frac{\partial}{\partial \dot{x}_2} + \dot{x}_3 \frac{\partial}{\partial \dot{x}_3} \wedge \frac{\partial}{\partial \dot{x}_2}.$$

It follows that the cobracket λ of the tangent bialgebra of the Poisson-Lie group $SU(2)$ is given by

$$\lambda\left(\frac{\partial}{\partial \dot{x}_4}\right) = \frac{\partial}{\partial \dot{x}_4} \wedge \frac{\partial}{\partial \dot{x}_2}, \quad \lambda\left(\frac{\partial}{\partial \dot{x}_3}\right) = \frac{\partial}{\partial \dot{x}_3} \wedge \frac{\partial}{\partial \dot{x}_2}, \quad \lambda\left(\frac{\partial}{\partial \dot{x}_2}\right) = 0$$

and the associated Lie bracket on $\mathfrak{su}(2)^*$ is given by

$$\{\dot{x}_4, \dot{x}_2\} = \dot{x}_4, \quad \{\dot{x}_3, \dot{x}_2\} = \dot{x}_3, \quad \{\dot{x}_4, \dot{x}_3\} = 0.$$

We recognize this structure as the structure of the Lie algebra $\mathfrak{sb}(2, \mathbb{C})$ of 2×2 traceless, upper triangular complex matrices with real diagonal elements.

The introduced Poisson-Lie structure on $SU(2)$ defines then the group $SB(2, \mathbb{C})$ as the dual group. It is not difficult to verify that the corresponding double group can be identified with $SL(2, \mathbb{C})$ with $SU(2)$ and $SB(2, \mathbb{C})$ canonically embedded as subgroups.

12. Conclusions. In this paper we tried to show the importance of the concept of tangent structures. The next step should be the analysis of multitangent constructions, important for classical field theory (multiphase approach) and classical mechanics, of extended objects. As we have seen, the conditions discussed in Section 4 are, in fact, compatibility conditions of tangent and canonical structures. This idea can be applied in more general situations like in nonrelativistic, time dependent mechanics, where the structure needed is more general than Poisson or symplectic one ([Ur]).

REFERENCES

- [By] G. Byrnes, *A complete set of Bianchi identities for tensor fields along the tangent bundle projection*, J. Phys. A: Math. Gen. **27** (1994), 6617–6632.
- [Co1] T. Courant, *Tangent Dirac structures*, J. Phys. A: Math. Gen. **23** (1990), 5153–5168.
- [Co2] T. Courant, *Tangent Lie Algebroids*, J. Phys. A: Math. Gen. **27(13)** (1994), 4527–4536.
- [Co3] T. Courant, *Tangent Poisson Structure*, unpublished.
- [Dr] V. G. Drinfeld, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation*, Soviet Math. Dokl. **27(1)** (1983), 68–71.
- [Gr] J. Grabowski, *Quantum $SU(2)$ group of Woronowicz and Poisson structures*, Differential Geometry and Its Applications (J. Janyška and D. Krupka, eds.), Proc. Conf. Brno (1989), World Scientific, 1990, pp. 313–322.
- [KMS] I. Kolář, P. W. Michor and J. Slovák, *Natural Operations in Differential Geometry*, Springer-Verlag, 1993.
- [MCS1] E. Martínez, J. F. Cariñena and W. Sarlet, *Derivations of differential forms along the tangent bundle projection*, Diff. Geom. Applic. **2** (1992), 17–43.
- [MCS2] E. Martínez, J. F. Cariñena and W. Sarlet, *Derivations of differential forms along the tangent bundle projection II*, Diff. Geom. Applic. **3** (1993), 1–29.
- [MCS3] E. Martínez, J. F. Cariñena and W. Sarlet, *Geometric characterization of separable second-order differential equations*, Math. Proc. Camb. Phil. Soc. **113** (1993), 205.
- [Mi] P. W. Michor, *Remarks on the Schouten-Nijenhuis bracket*, Suppl. Rend. Circ. Mat. Palermo, Serie II **16** (1987), 207–215.
- [MFLMR] *The inverse problem in the calculus of variations and the geometry of the tangent bundle*, Physics Reports **188** (1990), 147–284.
- [PiTu] G. Pidello and W. M. Tulczyjew, *Derivations of differential forms on jet bundles*, Ann. Mat. Pura Appl. **147** (1987), 249–265.
- [Sus] H. J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973), 171–188.
- [Tu1] W. M. Tulczyjew, *Hamiltonian systems, Lagrangian systems, and the Legendre transformation*, Ann. Inst. H. Poincaré **27** (1977), 101–114.
- [Tu2] W. M. Tulczyjew, *A symplectic formulation of relativistic particle dynamics*, Acta Phys. Pol. **B 8** (1977), 431–447.
- [Ur] P. Urbański, *Affine Poisson structures in analytical mechanics*, Quantization and Infinite Dimensional Systems (J.-P. Antoine and A. Odziejewicz, eds.), Plenum Press, New York and London, 1994, pp. 123–129.
- [We] A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geometry **22** (1983), 523–557.

E-mail address: jagrab@mimuw.edu.pl

DIVISION OF MATHEMATICAL METHODS IN PHYSICS, UNIVERSITY OF WARSAW, HOŻA 74, 00-682
WARSAW, POLAND

E-mail address: urbanski@fuw.edu.pl