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Proposal for Divergence-Free Quantization of Covariant Scalar Fields

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Abstract

Guided by idealized but soluble nonrenormalizable models, a nontraditional proposal for the quantization of covariant scalar field theories is advanced, which achieves a term-by-term, divergence-free perturbation analysis of interacting models expanded about a suitable pseudofree theory (differing from a free theory by an $\hbar$ term). This procedure not only provides acceptable solutions for models for which no acceptable solution currently exists, e.g., φ^4_n, for spacetime dimension $n \geq 4$, but offers a new, divergence-free solution, for less-singular models as well, e.g., φ^4_n, for $n = 2, 3$.

It is common knowledge that divergences arise in the study of covariant quantum field theories, and elaborate efforts are used to nullify the effects of these divergences. In this letter we argue that adopting an appropriate $\hbar$ ambiguity in the quantization procedure can eliminate the divergences that are usually encountered. Although we focus on scalar fields, similar methods may apply for other quantum field theories. As motivation for our approach, we initially analyze how divergences are eliminated in soluble ultralocal models.

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Ultralocal Models

The classical action for the quartic ultralocal model is given by

$$I=\int\{\tfrac{1}{2}[\dot{\phi}(t,\mathbf{x})^2-m_0^2\phi(t,\mathbf{x})^2]-g_0\,\phi(t,\mathbf{x})^4\}\,dt\,d\mathbf{x}\;, \tag{1}$$

where $g_0\geq 0$, $\dot{\phi}(t,\mathbf{x})=\partial\phi(t,\mathbf{x})/\partial t$, $\mathbf{x}\in\mathbb{R}^s$, and $1\leq s<\infty$. With no spatial gradients, the light cone of covariant models collapses to a temporal line reflecting the statistical independence of ultralocal fields at any two distinct spatial points. This vast symmetry ultimately helps determine the quantum theory for such models.

Viewed conventionally, it is hard to imagine a quartic interacting field theory that would cause more trouble in its quantization. On one hand, it is clear that ultralocal models are perturbatively nonrenormalizable for any $s\geq 1$; on the other hand, if viewed nonperturbatively, and limited to mass and coupling constant renormalizations, they lead to free (Gaussian) results based simply on the Central Limit Theorem. Clearly, other methods are required.

Although the quantum theory of these models has been completely solved without introducing cutoffs [1, 2], it is pedagogically useful to study the model as regularized by a hypercubic spacetime lattice with periodic boundary conditions. If $a>0$ denotes the lattice spacing and $L<\infty$ denotes the number of sites on each edge, then the ground-state distribution of the free theory ($g_0\equiv 0$) is described by the characteristic function

$$\begin{aligned}C_f(f)\;&=M\int e^{i\Sigma'_k f_k\,\phi_k\,a^s\,-\,m_0\Sigma'_k\phi_k^2\,a^s}\,\Pi'_k d\phi_k\\ &=e^{-\frac{1}{4}m_0^{-1}\Sigma'_k f_k^2\,a^s}\rightarrow e^{-\frac{1}{4}m_0^{-1}\int f(\mathbf{x})^2\,d\mathbf{x}}\;,\end{aligned} \tag{2}$$

where in the last line we have taken the continuum limit. In this expression $k=(k_1,k_2,\ldots,k_s)$, $k_j\in\mathbb{Z}$, labels the sites in this spatial lattice. It is of interest to calculate mass-like moments in the ground-state distribution as given by

$$I_p(m_0)\equiv M\int[\Sigma'_k\phi_k^2\,a^s]^p\,e^{-m_0\Sigma'_k\phi_k^2\,a^s}\,\Pi'_k d\phi_k=O((N'/m_0)^p)\,I_0(m_0)\;, \tag{3}$$

where $N'\equiv L^s$ is the number of lattice sites in the spatial volume. A perturbation of the mass, with $\Delta\equiv\tilde{m}_0-m_0$, leads to

$$I_1(\tilde{m}_0)=I_1(m_0)-\Delta\,I_2(m_0)+\tfrac{1}{2}\Delta^2\,I_3(m_0)-\cdots\;, \tag{4}$$

which, assuming $m_0 = O(1)$, exhibits increasingly divergent contributions in the continuum limit in which $a \rightarrow 0$ and $L \rightarrow \infty$ such that $N' a^s = (La)^s$ remains large but finite. The origin of these divergences is exposed if we pass to hyperspherical coordinates, where $\phi_k \equiv \kappa \eta_k$, $\Sigma'_k \phi_k^2 \equiv \kappa^2$, and $\Sigma'_k \eta_k^2 \equiv 1$, for which (3) becomes

$$I_p(m_0) = 2M \int \kappa^{2p} a^{sp} \, e^{-m_0 \kappa^2 a^s} \, \kappa^{(N'-1)} \, d\kappa \, \delta(1 - \Sigma'_k \eta_k^2) \, \Pi'_k d\eta_k \ , \tag{5}$$

which not only reveals the source of the divergences as the factor N' in the measure factor $\kappa^{(N'-1)}$, but also confirms the approximate evaluation of (3) by a steepest descent analysis of the κ integration. If we could somehow change the power of κ in the measure of (5) to $\kappa^{(R-1)}$, where R is a finite factor, these divergences would be eliminated!

The theory of infinite divisibility [3] ensures us that besides the Gaussian ground-state distributions there are only Poisson ground-state distributions that respect the ultralocal symmetry of the model, and they are described by characteristic functions of the form

$$C(f) = \exp\{-\textstyle\int d\mathbf{x} \int [1 - \cos(f(\mathbf{x})\lambda)] \, c(\lambda)^2 \, d\lambda\} \ , \tag{6}$$

where $\int [\lambda^2/(1+\lambda^2)] \, c(\lambda)^2 \, d\lambda < \infty$, but $\int c(\lambda)^2 \, d\lambda = \infty$ (to ensure the smeared field operator only has a continuous spectrum). As an important example, let us assume that $c(\lambda)^2 = b \exp(-bm\lambda^2)/|\lambda|$, where b is a positive constant with dimensions (Length)$^{-s}$, and m is a mass parameter. For this example, it follows that

$$\begin{aligned} M' \int e^{i\Sigma'_k f_k \phi_k a^s \, - \, m_0 \Sigma'_k \phi_k^2 a^s} \, \Pi'_k [|\phi|^{(1-2ba^s)}]^{-1} \, \Pi'_k d\phi_k \\ = \Pi'_k \{1 - (ba^s) \textstyle\int [1 - \cos(f_k \lambda)] \, e^{-bm\lambda^2} \, d\lambda/|\lambda|^{(1-2ba^s)} \} \\ \rightarrow \exp\{-b \textstyle\int d\mathbf{x} \int [1 - \cos(f(\mathbf{x})\lambda)] \, e^{-bm\lambda^2} \, d\lambda/|\lambda|\} \ ; \end{aligned} \tag{7}$$

here we have set $m_0 = ba^s m$, $\lambda = \phi \, a^s$, and used the fact that to leading order $M' = (ba^s)^{N'}$, which holds because

$$(ba^s) \textstyle\int e^{-bm\lambda^2} \, d\lambda/|\lambda|^{(1-2ba^s)} \simeq 2(ba^s) \int_0^B d\lambda/\lambda^{(1-2ba^s)} = B^{2ba^s} \rightarrow 1 \ , \tag{8}$$

provided that $0 < B < \infty$.

Observe that the lattice ground-state distribution for this example is

$$\frac{(ba^s)^{N'}\, e^{-m_0\Sigma'_k\phi_k^2\, a^s}}{\Pi'_k|\phi_k|^{(1-2ba^s)}} = \frac{(ba^s)^{N'}\, e^{-m_0\kappa^2\, a^s}}{\kappa^{(N'-2ba^s\,N')}\,\Pi'_k|\eta_k|^{(1-2ba^s)}}\ , \tag{9}$$

which has exactly the right factor to change the κ measure from $\kappa^{(N'-1)}$ to $\kappa^{(R-1)}$, where in the present case $R = 2ba^sN'$ [a finite number chosen in order to ensure a meaningful continuum limit for (7)]. If we adopt (9) as the appropriate "pseudofree" ground-state distribution, then all divergences due to integration over κ will disappear!

Free and Pseudofree Theories

What exactly do we mean by free and pseudofree theories? An elementary example of a theory that involves pseudofree behavior is given by the anharmonic oscillator with the classical action

$$I = \textstyle\int\{\frac{1}{2}[\dot{x}(t)^2 - x(t)^2] - g_0 x(t)^{-4}\}\, dt\ , \tag{10}$$

where $g_0 \geq 0$. The free theory ($g_0 \equiv 0$) has solutions $A\cos(t+\gamma)$ that freely cross $x = 0$; when $g_0 > 0$, however, *no* solution can cross $x = 0$, and the limit of the interacting solutions as $g_0 \to 0$ becomes $\pm|A\cos(t+\gamma)|$. This latter behavior describes the classical pseudofree model, i.e., the model continuously connected to the interacting models as $g_0 \to 0$. Quantum mechanically, the imaginary-time propagator for the free theory is given by

$$K_f(x'', T; x', 0) = \textstyle\sum_{n=0}^{\infty} h_n(x'')\, h_n(x')\, e^{-(n+1/2)T}\ , \tag{11}$$

where $h_n(x)$ denotes the nth Hermite function. However, for the interacting quantum theories, as the coupling $g_0 \to 0$, the imaginary-time propagator converges to

$$K_{pf}(x'', T; x', 0) = \theta(x''x')\textstyle\sum_{n=0}^{\infty} h_n(x'')\, [h_n(x') - h_n(-x')]\, e^{-(n+1/2)T}\ , \tag{12}$$

where $\theta(y) = 1$ if $y > 0$ and $\theta(y) = 0$ if $y < 0$, which characterizes the quantum pseudofree model. This behavior has arisen because within a functional integral the interaction acts partially as a hard core projecting out certain histories that would otherwise be allowed by the free theory; any perturbation analysis of the interaction term clearly must take place about the pseudofree theory and not about the free theory. The field theory models are more involved, but the basic ideas are essentially the same.

Lessons from Ultralocal Models

Observe for the classical ultralocal models that when $g_0 > 0$ it is necessary that $\int \phi(t, \mathbf{x})^4 dt\, d\mathbf{x} < \infty$ to derive the equations of motion, but when $g_0 = 0$ this restriction is absent. Thus the set of classical solutions for $g_0 > 0$ does *not* reduce as $g_0 \rightarrow 0$ to the set of classical solutions of the free theory; instead, the set of classical solutions for $g_0 > 0$ passes by continuity to a set of classical solutions of the free theory that also incorporates the hard-core consequences of the condition $\int \phi(t, \mathbf{x})^4 dt\, d\mathbf{x} < \infty$. An interacting classical theory that is not continuously connected to its own free classical theory is likely to be associated with an interacting quantum theory that is not continuously connected to its own free quantum theory. This situation is easy to see for the ultralocal models. The characteristic function of the ground-state distribution has either a Gaussian or a Poisson form as indicated, and there is no continuous, reversible path between the two varieties. If one seeks nontriviality, then the interacting theory must be of the Poisson type; and as the coupling constant vanishes, the continuous limit must also be a Poisson distribution, namely the pseudofree model as chacterized by (7).

To complete the ultralocal story, we observe that the ground-state distribution for interacting models is also of the Poisson form, where

$$c(\lambda)^2 = b\, \exp[-y(\lambda)]/|\lambda| \tag{13}$$

for suitable functions $y(\lambda)$. Each such distribution leads to a lattice Hamiltonian and thereby a lattice action for a full (Euclidean) lattice spacetime functional integral formulation. The pseudofree model has the lattice action of a traditional free theory augmented by a local counterterm proportional to $\hbar^2$ and (surprise!) *inversely* proportional to the field squared, so that it accounts for the denominator factor, which has been central to an overall divergence-free formulation. The form of the nontraditional counterterm is implicitly given in the next section, and since these models have been extensively discussed elsewhere [1, 2], we do not pursue them further.

However, we do take from the ultralocal model *the central principle of our analysis*, which we dub "measure mashing". In particular, in extending our analysis to covariant models, we adopt the "slick trick" that worked so well for the ultralocal models, namely, choosing a pseudofree model that changes the measure factor $\kappa^{(N'-1)}$ for the hyperspherical radius to the form $\kappa^{(R-1)}$, where R is a suitable finite factor for the model in question.

Covariant Models

We restrict our initial attention to models with the classical action given by

$$A = \int (\tfrac{1}{2}\{\dot\phi(x)^2 - [\nabla\phi(x)]^2 - m_0^2\phi(x)^2\} - g_0\,\phi(x)^4)\, d^n\!x\ , \tag{14}$$

where $x = (t = x_0, x_1, x_2, \ldots, x_s) \in \mathbb{R}^n$, $n = s+1 \geq 5$, $g_0 \geq 0$, $\dot\phi(x) = \partial\phi(x)/\partial t$, and $[\nabla\phi(x)]^2 \equiv \Sigma_{j=1}^s(\partial\phi(x)/\partial x_j)^2$. It is not obvious, but for the spacetime dimensions in question, the interaction term imposes a restriction on the free action as follows from the multiplicative inequality [4, 2]

$$\{\int \phi(x)^4\, d^n\!x\}^{1/2} \leq C \int \{\dot\phi(x)^2 + [\nabla\phi(x)]^2 + \phi^2\}\, d^n\!x\ , \tag{15}$$

where for $n \leq 4$ (the renormalizable cases), $C = 4/3$ is satisfactory, while for $n \geq 5$ (the nonrenormalizable cases), $C = \infty$ meaning that there are fields for which the left side diverges while the right side is finite (e.g., $\phi_{singular}(x) = |x|^{-p}\, e^{-x^2}$, where $n/4 \leq p < n/2 - 1$). As a consequence, for $n \geq 5$ the set of solutions of the interacting classical theory do *not* reduce to the set of solutions of the free classical theory as the coupling constant $g_0 \rightarrow 0$. We now examine the quantum theory in the light of this knowledge, and we initially focus on finding a suitable pseudofree model for covariant theories.

Choosing the Covariant Pseudofree Model

For covariant scalar fields, the lattice version of a free, nearly massless, quantum theory has a characteristic functional for the ground-state distribution given by

$$C_f(f) = M' \int e^{i\Sigma'_k f_k \phi_k\, a^s\, -\, \Sigma'_{k,l}\phi_k\, A_{k-l}\, \phi_l\, a^{2s}}\, \Pi'_k d\phi_k\ , \tag{16}$$

where A_{k-l} accounts for the derivatives and a small, well-chosen, artificial mass-like contribution. The quantum Hamiltonian for this ground state (restoring $\hbar$) becomes

$$\mathcal{H}_f = -\tfrac{1}{2}\hbar^2 a^{-s} {\textstyle\sum'_k} \frac{\partial^2}{\partial\phi_k^2} + \tfrac{1}{2}{\textstyle\sum'_{k,l}}\phi_k\, A^2_{k-l}\,\phi_l\, a^{3s} - E_0\ , \tag{17}$$

where E_0 is a constant ground state energy and

$$A^2_{k-l} \equiv \Sigma'_p A_{k-p} A_{p-l} \equiv \Sigma_{j=1}^s [\,2\,\delta_{k,l} - \delta_{k+\delta_j,l} - \delta_{k-\delta_j,l}\,]\, a^{-(2s+2)} \\ + s\, L^{-2s}\, a^{-(2s+2)}\ , \tag{18}$$

where $k \pm \delta_j \equiv (k_1, k_2, \ldots, k_j \pm 1, \ldots, k_s)$, and the last factor is a small, artificial mass-like term (introduced to deal with the zero mode $\phi_k \rightarrow \phi_k + \xi$). The true mass term will be introduced later along with the quartic interaction when we discuss the final model.

We next modify the free ground-state distribution in order to suggest a suitable characteristic functional for the pseudofree ground-state distribution by the expression

$$C_{pf}(f) = M'' \int e^{i\Sigma'_k f_k \phi_k \, a^s - \Sigma'_{k,l} \phi_k \, A_{k-l} \, \phi_l \, a^{2s} - W(\phi \, a^{(s-1)/2}/\hbar^{1/2})} \times \{\Pi'_k [\Sigma'_l J_{k,l} \, \phi_l^2]\}^{-(1-R/N')/2} \, \Pi'_k d\phi_k \ , \qquad (19)$$

where the constants $J_{k,l} \equiv 1/(2s+1)$ for the $(2s+1)$ points that include $l = k$ and all the $2s$ spatially nearest neighbors to k; $J_{k,l} \equiv 0$ for all other points. Stated otherwise, the term $\Sigma'_l J_{k,l} \phi_l^2$ is *an average of field-squared values* at $l = k$ and the $2s$ spatially nearest neighbors to k. Note well, that this term leads to a factor of $\kappa^{-(N'-R)}$ that, in effect, replaces the hyperspherical radius variable measure term $\kappa^{(N'-1)}$ by the factor $\kappa^{(R-1)}$ (i.e., mashing the measure), and since R is finite, this choice eliminates any divergences caused by integrations over the variable κ. Indeed, hereafter, we choose the finite factor $R = 1$ in an initial effort to find suitable pseudofree models for the covariant theories. The factor A_{k-l} is the same as introduced for the free theory, while the function W is implicitly defined below.

The Hamiltonian for the Covariant Pseudofree Model

The Hamiltonian follows from the proposed ground state wave function contained in (19) . To understand the role played by W, let us first assume that $W = 0$. Then, in taking the necessary second-order derivatives, there will be a contribution when one derivative acts on the A_{k-l} factor in the exponent and the other derivative acts on the denominator factor involving $J_{k,l}$. The result will be a cross term that exhibits a long-range interaction that would cause difficulty for causality in the continuum limit. Instead, at this point, we focus on the Hamiltonian itself as primary (rather than the ground state), and adopt the Hamiltonian for the pseudofree model as

$$\mathcal{H}_{pf} = -\tfrac{1}{2}\hbar^2 a^{-s} \textstyle\sum'_k \dfrac{\partial^2}{\partial \phi_k^2} + \tfrac{1}{2} \textstyle\sum'_k (\phi_k^* - \phi_k)^2 a^{s-2} + \tfrac{1}{2} s (L^{-2s} a^{-2}) \textstyle\sum'_k \phi_k^2 a^s + \tfrac{1}{2}\hbar^2 \textstyle\sum'_k \mathcal{F}_k(\phi) \, a^s - E_{pf} \ , \qquad (20)$$

where k^* represents a spatially nearest neighbor to k in the positive sense, implicitly summed over all s spatial directions, and the counterterm $\mathcal{F}_k(\phi)$, which follows from both derivatives acting on the $J_{k,l}$ factor, is given by

$$\begin{aligned}\mathcal{F}_k(\phi) \equiv \frac{1}{4}\Big(\frac{N'-1}{N'}\Big)^2 a^{-2s} \textstyle\sum'_{r,t} \displaystyle\frac{J_{r,k}\,J_{t,k}\,\phi_k^2}{[\Sigma'_l\,J_{r,l}\,\phi_l^2]\,[\Sigma'_m\,J_{t,m}\,\phi_m^2]}\\ -\frac{1}{2}\Big(\frac{N'-1}{N'}\Big)\,a^{-2s} \textstyle\sum'_t \displaystyle\frac{J_{t,k}}{[\Sigma'_m\,J_{t,m}\,\phi_m^2]}\\ +\Big(\frac{N'-1}{N'}\Big)\,a^{-2s} \textstyle\sum'_t \displaystyle\frac{J_{t,k}^2\,\phi_k^2}{[\Sigma'_m\,J_{t,m}\,\phi_m^2]^2}\;. \end{aligned} \tag{21}$$

We observe that this form for the counterterm leads to a local potential in the continuum limit even though it is a rather unfamiliar one. (**Remark:** If $J_{k,l}$ is taken as $\delta_{k,l}$ and $N'-1$ is replaced by $N'-2ba^sN'$, the resultant counterterm is that appropriate to the ultralocal models.)

With this involved counterterm, the pseudofree Hamiltonian is completely defined, and we define the implicitly given expression for the pseudofree ground state to be the ground state $\Psi_{pf}(\phi)$ for this Hamiltonian. For large ϕ values the A_{k-l} term well represents the solution, and for small ϕ values the denominator term involving the $J_{k,l}$ term also well represents the solution, The role of the (unknown) function W and E_{pf} is to fine tune the solution so that it satisfies the equation $\mathcal{H}_{pf}\,\Psi_{pf}(\phi)=0$. The manner in which both a and $\hbar$ appear in $\mathcal{H}_{pf}$ dictates how they appear in W as $W(\phi\,a^{(s-1)/2}/\hbar^{1/2})$.

Final Form of Lattice Hamiltonian and Lattice Action

It is but a small step to propose expressions for the lattice Hamiltonian and lattice action in the presence of the proper mass term and the quartic interaction. The lattice Hamiltonian is given by

$$\begin{aligned}\mathcal{H} = -\tfrac{1}{2}\hbar^2\,a^{-s}\textstyle\sum'_k \displaystyle\frac{\partial^2}{\partial\phi_k^2} + \tfrac{1}{2}\textstyle\sum'_k(\phi_{k^*}-\phi_k)^2\,a^{s-2} + \tfrac{1}{2}s(L^{-2s}\,a^{-2})\textstyle\sum'_k\phi_k^2\,a^s\\ +\tfrac{1}{2}m_0^2\textstyle\sum'_k\phi_k^2\,a^s + \lambda_0\textstyle\sum'_k\phi_k^4\,a^s + \tfrac{1}{2}\hbar^2\textstyle\sum'_k\mathcal{F}_k(\phi)\,a^s - E\;,\end{aligned} \tag{22}$$

and the Euclidean lattice action reads

$$\begin{aligned}I(\phi,a,\hbar) = +\tfrac{1}{2}\textstyle\sum_k\sum_{k^*}(\phi_{k^*}-\phi_k)^2\,a^{n-2} + \tfrac{1}{2}s(L^{-2s}\,a^{-2})\textstyle\sum_k\phi_k^2\,a^n\\ +\tfrac{1}{2}m_0^2\textstyle\sum_k\phi_k^2\,a^n + \lambda_0\textstyle\sum_k\phi_k^4\,a^n + \tfrac{1}{2}\hbar^2\textstyle\sum_k\mathcal{F}_k(\phi)\,a^n\;,\end{aligned} \tag{23}$$

where the last sum on k^*, here made explicit, is a sum over all n lattice directions in a positive sense from the site k, and in both expressions the counterterm $\mathcal{F}_k(\phi)$ is given in (21). When one studies the full action, as in a Monte Carlo analysis, then the small, artificial mass-like term can be omitted.

The generating functional for Euclidean lattice spacetime averages is given, as usual, by

$$\langle e^{Z^{-1/2}\Sigma_k h_k \phi_k a^n/\hbar}\rangle \equiv \widetilde{M} \int e^{Z^{-1/2}\Sigma_k h_k \phi_k a^n/\hbar - I(\phi,a,\hbar)/\hbar}\, \Pi_k d\phi_k \ , \quad (24)$$

where Z is the field strength renormalization constant and Σ/Π (without primes) denotes a sum/product over the full spacetime lattice now with $k = (k_0, k_1, k_2, \ldots, k_s)$, where k_0 denotes the imaginary-time direction. Elsewhere [5, 6], we have studied the perturbation analysis of (24) and have determined that: (i) the proper *field strength renormalization* is given by $Z = N'^{-2}(qa)^{1-s}$, (ii) the proper *mass renormalization* is given by $m_0^2 = N'(qa)^{-1}m^2$, and (iii) the proper *coupling constant renormalization* is given by $g_0 = N'^3(qa)^{s-2}g$. Here, q denotes a positive constant with dimensions $(\text{Length})^{-1}$, and m and g represent finite physical factors. It is noteworthy that $Zm_0^2 = m^2/[N'(qa)^s]$ and $Z^2 g_0 = g/[N'(qa)^2]$. The characterization of the model is now complete. (**Remark:** Although we have confined attention to models with quartic interactions, measure mashing also enables higher powers, e.g., φ_n^{44}, φ_n^{444}, etc., to be handled just as well [5].)

Much has changed by passing from a free model to a pseudofree model as the center of focus. Traditionally, when forming local products from free field operators, normal ordering is used. On the contrary, after measure mashing, the pseudofree field operators satisfy multiplicative renormalization, and no normal ordering is involved. Indeed, the very coefficients m_0^2 and g_0 act partially as multiplicative renormalization factors for the associated products involved. To say that there are no divergences means, for example, that the expression $m_0^2\Sigma'_k\phi_k^2 a^s$ is a well defined, and this fact is established by ensuring that $m_0^2\Sigma'_k\langle\phi_k^2\rangle a^s \propto N'a^s < \infty$. The same holds true for $g_0\Sigma'_k\phi_k^4 a^s$, which is shown to be well defined by noting that $g_0\Sigma'_k\langle\phi_k^4\rangle a^s \propto N'a^s < \infty$. These quantities remain bounded even in the continuum limit.

Extension to Less Singular Scalar Models

Let us take up the extension of measure mashing to other models such as φ^4_n, for $n \leq 4$. Although the classical pseudofree theory is identical to the free theory in these cases, this fact does not prevent us from suggesting the consideration of mashing the measure for such less singular models in an effort to eliminate divergences that arise in those cases. For $n = 2$, it is well known that normal ordering removes all divergences, but it is also well known that normal ordering is a rather strange rule to define local products. In particular, if we rewrite the product of two field operators as

$$\varphi(x)\,\varphi(y) = \langle 0|\,\varphi(x)\,\varphi(y)\,|0\rangle + :\varphi(x)\,\varphi(y): \,, \tag{25}$$

then, as $y \rightarrow x$, the most singular term is the first term, but since it is a multiple of unity, the *second and less singular term* is chosen to define the local product, $\varphi(x)^2_{Renormalized} =: \varphi(x)^2 :$. In sharp contrast, in the operator product expansion, schematically given by

$$\varphi(x)\,\varphi(y) = c_1(x,y)\,\zeta_1(\tfrac{1}{2}(x+y)) + c_2(x,y)\,\zeta_2(\tfrac{1}{2}(x+y)) + \cdots \,, \tag{26}$$

the local product, as $y \rightarrow x$, is defined as that operator, say $\varphi(x)^2_{Renormalized} = \zeta_1(x)$, for which the associated c-number coefficient $c_1(x,y)$ is the *most singular* as $y \rightarrow x$; this is a very reasonable rule, but it differs markedly from the rule of normal ordering. To adopt measure mashing for ϕ^4_2 would introduce the operator product expansion and thereby a more natural local product definition. This same feature also applies to ϕ^4_3 and ϕ^4_4, and moreover it would eliminate divergences that appear in those models. It could even offer a nontrivial proposal for ϕ^4_4 which is widely believed to be trivial when quantized conventionally.

Is all this a physically realistic proposal? Presumably, the answer would depend on the application, so it is too soon to expect a firm answer to this question. Nevertheless, it would seem there is progress already just to have a possible solution to nonrenormalizable models rather than the unsatisfactory results obtained by conventional techniques.

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