$\begin{array}{c} {\bf Functorial} \ {\bf A}_{\infty} {\bf -Coproduct} \\ {\bf of} \ {\bf Combinatorial} \ {\bf Simplicial} \ {\bf Chains} \\ {\bf Transferred} \ {\bf to} \ {\bf Itself} \ {\bf under} \ {\bf Barycentric} \ {\bf Subdivision} \end{array}$

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Functorial A_{∞} -coproduct of combinatorial simplicial chains transferred to itself under barycentric subdivision

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ABSTRACT: Let $\overline{C}(M)$ denote simplicial chain complex of standard triangulation of the standard combinatorial simplex builded on vertex set M and $\underline{C}(M)$ denote simplicial chain complex associated with its barycentric subdivision. Over a field of zero characteristic we write explicit formulas for functorial in M strong deformation retraction $\gamma \hookrightarrow \underline{C}(M) \xrightarrow[\sigma]{} \overline{C}(M)$ between these (functorial) chain complexes and show that such retraction is unique up to rescalling of σ , π . It allows to transfer any functorial in M A_{∞}-coproduct on $\overline{C}(M)$ to another functorial A_{∞}-coproduct called the barycentric subdivision of the original one. We argue that there exists a unique functorial A_{∞}coproduct on $\overline{C}(M)$ going to itself under the barycentric subdivision. We prove this conjecture for 1-dimensional simplex and write down an explicit formula for the coproduct in terms of Bernoulli numbers.

KEYWORDS: A-infinity coproducts, combinatorial simplicial chains, barycentric subdivision

AMS SUBJECT CLASSES: 18G55, 55U10, 18G35, 55U15

Contents

	Introduction	2
$\S1$	Recallment on A_{∞} -things	3
$\S2$	Functorial barycentric retraction of combinatorial simplicial chain complexes	10
$\S{3}$	Functorial A_{∞} -coproduct transferred to itself via barycentric retraction	15
$\S4$	Closed formula in dim = 1 case \dots	16
	References	21

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Introduction

Kolmogorov's problem. Given a finite set M let us write $\overline{C} = \overline{C}(M)$ for the complex of oriented simplicial chains of the standard triangulation of combinatorial simplex with the vertex set M (see precise definitions in n° 2.1).

Early in the development of topology, at Moscow's conference in 1935, when multiplicative structures on (co)homologies were just under construction, Kolmogorov asked about functorial w.r.t. inclusions of sets $M \hookrightarrow M'$ co-associative co-product $\delta_2 : \overline{C} \longrightarrow \overline{C} \otimes \overline{C}$.

Note that the functoriality condition forces the coproduct to be equivariant w.r.t. action of symmetric group $\mathfrak{S}_M = \operatorname{Aut}(M)$ and provides the chain complexes of *all* combinatorial simplicial complexes Δ with compatible coalgebra structures such that $\delta_2(\sigma) \subset \mathscr{S}(\sigma) \otimes \mathscr{S}(\sigma)$ for each simplex $\sigma \in \Delta$.

Moreover, Kolmogorov proposed a candidate for such a functorial coproduct. To write it down, let us fix some total ordering $[x_1, x_2, \ldots, x_m]$ on M and write \overline{Y} for an oriented subset $Y \subset M$ whose orientation is induced by this ordering. For two disjoint oriented subsets $\overline{Y} = \overline{y_1 y_2 \ldots y_k}, \overline{Z} = \overline{z_1 z_2 \ldots z_\ell}$ let $\overline{X \cdot Y} \stackrel{\text{def}}{=} \overline{y_1 y_2 \ldots y_k z_1 z_2 \ldots z_\ell}$. Finally, let $\overline{M_i} \stackrel{\text{def}}{=} \overline{M \smallsetminus \{x_i\}}$. In these notations, Kolmogorov's coproduct δ_2^{K} takes \overline{M} to

$$\frac{1}{|M|} \sum_{i=1}^{m} (-1)^{i+1} \sum_{Y \subseteq M_i} \operatorname{sgn}(Y) \binom{|M|-1}{|Y|}^{-1} \overline{Y \cdot \{x_i\}} \otimes \overline{\{x_i\} \cdot (M_i \smallsetminus Y)}$$
(1)

where the second sum runs over all subsets $Y \subset M_i = M \setminus \{x_i\}$ (including \emptyset and M_i), $\binom{*}{*}^{-1}$ stays for inverse binomial coefficient, and $\operatorname{sgn}(Y)$ means the sign of shuffle permutation $\overline{M_i} \leftrightarrow \overline{Y \cdot (M_i \setminus Y)}$. For example:

$$\delta_{2}^{\kappa}(\overline{01}) = \frac{+\left(\overline{10}\otimes\overline{0} + \overline{0}\otimes\overline{01}\right) - \left(\overline{01}\otimes\overline{1} + \overline{1}\otimes\overline{10}\right)}{2} = -\frac{1}{2}\operatorname{ad}_{\overline{01}}\left(\overline{0} + \overline{1}\right)$$

where $b \xrightarrow{\operatorname{ad}_a} a \otimes b - b \otimes a$ is commutation operator in tensor algebra.

Coproduct (1) is well defined¹, functorial w.r.t. inclusions of finite sets $M_1 \longrightarrow M_2$, and compatible with differential $\delta_1 = \overline{\partial}$. However, it turns to be *non-associative*.

At the same time an associative product on cohomologies was constructed by means of much simpler Alexander-Whitney coproduct

$$\delta_2^{\text{AW}}([x_1, x_2, \dots, x_k]) = \sum_{i=1}^k [x_1, x_2, \dots, x_i] \otimes [x_i, x_{i+1}, \dots, x_k],$$

defined for totally ordered chains only and palpable on the level of \overline{C} only up to homotopy. It was enough for solving the homological problem of those years and Kolmogorov's question has fallen outside the mainstream of topology for some time.

However, it comes back in focus when we have to compute, say, higher Massey products needed for recovering the homotopy type of manifold from its cohomologies, not to

¹that is, does depend only on the orientation of chains but not on their total ordering

mention pure combinatorial significance of problem. Another reason for revival the Kolmogorov's problem comes from quantum field theory: the Massey products are nothing but mathematical versions of higher correlators in topological field theories.

In modern setup, the initial Kolmogorov's question could be treated either as: 'How to extend the coproduct (1) to some functorial A_{∞} -coproduct in most symmetric way?' or (in physical cant): 'What should be most symmetric ground theory on \overline{C} producing an effective theory on $H(\overline{C})$ with correlators equal to the Massey products?'.

We attack these questions using barycentric subdivisions and functorial transferring of A_{∞} -coproducts along the barycentric subdivisions.

The paper is organized as follows. In §1 we recall well known basic properties of A_{∞} -coproducts, strong deformation retractions (SDR-data), and explicit formulas for transferring A_{∞} -coproducts by means of SDR-data. We tried to make exposition self contained and as simple as possible.

In §2 we fix the notations concerning combinatorial topological staff and write down precise formulas for *functorial* in M strong deformation retraction between the chain complex $\overline{C}(M)$ of the standard simplex with vertex set M and the chain complex $\overline{C}(B(M))$ of its barycentric subdivision. We also show that up to an obvious rescalling such retraction is unique. Let us note that, in particular, our barycentric SDR-data will be equivariant w.r.t. symmetric group Aut (M) permuting the vertexes.

In §3 we argue that there is unique functorial in M A_{∞}-coproduct

$$\overline{C}^{[M]} \xrightarrow{\delta_{\mathrm{inv}}^{\mathrm{KA}}} \prod_{n \ge 1} \left(\overline{C}^{[M]} \right)^{\otimes n}$$

that goes to itself under the functorial transferring from the barycentric subdivision and write down explicit recursive formulas for it using the summation over oriented planar trees.

In §4 we compute the above sums for 1-dimensional simplex and get precise closet formula for functorial barycentrically stable coproduct of 1-dimensional simplex. It turns out that the the generating function for higher coproducts is closely connected with the Todd series.

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§1. Recallment on A_{∞} -things.

1.1. Grading and sign conventions. In this paper we fix a ground field \Bbbk of zero characteristic and work with graded vector spaces $V = \bigoplus_{n \in \mathbb{Z}} V_n$ over \Bbbk . A linear map

 $V \xrightarrow{\varphi} W$ has degree k, if $\forall i \ \varphi(V_i) \subset W_{i+k}$. Tensor products of operators are composed and applied to tensor products of elements by means of the Koszul sign rule:

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_m) \circ (g_1 \otimes g_2 \otimes \cdots \otimes g_m) = (-1)^{\varepsilon(f;g)} (f_1 \circ g_1) \otimes (f_2 \circ g_2) \otimes \cdots \otimes (f_m \circ g_m) ,$$

$$f_1 \otimes f_2 \otimes \cdots \otimes f_m (v_1 \otimes v_2 \otimes \cdots \otimes v_m) = (-1)^{\varepsilon(f;v)} f_1(v_1) \otimes f_2(v_2) \otimes \cdots \otimes f_m(v_m) ,$$

where the parity ε depends on two ordered collections of degrees as follows:

$$\varepsilon (\alpha_1, \alpha_2, \dots, \alpha_m; \beta_1, \beta_2, \dots, \beta_m) = \deg \alpha_m \cdot (\deg \beta_1 + \dots + \deg \beta_{m-1}) + \deg \alpha_{m-1} \cdot (\deg \beta_1 + \dots + \deg \beta_{m-2}) + \dots + \deg \alpha_2 \cdot \deg \beta_1.$$

We always write $[f,g] \stackrel{\text{def}}{=} fg - (-1)^{\deg f \deg g} gf$ for graded commutator of maps. We consider only degree -1 differentials $V \stackrel{\partial}{\longrightarrow} V$. Note that $[\partial, \partial] = 2 \partial^2 = 0$. Graded vector space V equipped with differential ∂_V is called *a complex*. Shift by *k* takes *V* to V[k] that has

$$V[k]_i = V_{i+k}$$
 and $\partial_{V[k]} = (-1)^k \partial_V$.

Thus, the identity map $s : V \longrightarrow V[1]$ has degree -1 and commutes with differentials, i.e. $s \circ \partial_V = -\partial_{V[1]} \circ s$.

1.2. Tensor algebra derivations. We write $\mathsf{T}V \stackrel{\text{def}}{=} \prod_{n \ge 1} V^{\otimes n}$ for completed reduced tensor algebra, that is, the algebra of formal non-commutative power series without constant term:

$$\tau = \sum_{k \ge 1} \tau_k , \quad \tau_k \in V^{\otimes k} .$$

We refer to k as the tensor degree of τ_k , whereas the total degree deg τ_k comes from deg $(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sum deg(v_{\nu})$.

A derivation of TV is k-linear map TV \xrightarrow{D} TV satisfying the Leibnitz rule

$$D \circ \mu = \mu \circ (D \otimes 1 + 1 \otimes D),$$

where $\mathsf{T}V \otimes \mathsf{T}V \xrightarrow{\mu} \mathsf{T}V$ is the tensor multiplication. Being applied to elements, this looks like:

$$D(\omega_1 \otimes \omega_2) = (D\omega_1) \otimes \omega_2 + (-1)^{\deg D \cdot \deg \omega_1} \omega_1 \otimes (D\omega_2)$$

There is k-linear bijection between derivations $\mathsf{T}V \xrightarrow{D} \mathsf{T}V$ and linear maps $V \xrightarrow{\delta} \mathsf{T}V$. It takes D to its restriction

$$\delta_D = D|_V : V \xrightarrow{\delta} \mathsf{T} V \; .$$

Backwards, it extends δ by the Leibnitz rule to the map D_{δ} defined on the whole of TV. Under this bijection the (graded) commutator of derivations $[D_{\delta_1}, D_{\delta_2}]$ turns to the *Gerstenhaber bracket* of maps $\{\delta_1, \delta_2\}$ defined by prescription

$$D_{\{\delta_1,\delta_2\}} \stackrel{\text{def}}{=} [D_{\delta_1}, D_{\delta_2}] = D_{\delta_1} D_{\delta_2} - (-1)^{\deg \delta_1 \cdot \deg \delta_2} D_{\delta_2} D_{\delta_1}$$

1.3. A_{∞} -structures. A_{∞} -coproduct on V is a derivation of degree -1

$$\mathsf{T}(V[1]) \xrightarrow{D} \mathsf{T}(V[1])$$

such that $D^2 = 0$. It defines and is uniquely defined by k-linear map of degree -1

$$\delta = \sum_{n \ge 1} \delta_n = D|_{V[1]} : V[1] \longrightarrow \mathsf{T}(V[1])$$

such that $\{\delta, \delta\} = 0$. In terms of V itself, one can think of the homogeneous components of δ as degree n-2 maps

$$V \xrightarrow{\widetilde{\delta}_n} V^{\otimes n}$$

fitted into commutative diagram:



where $V \xrightarrow{s} V[1]$ is the identity map (recall that deg s = -1 and deg $\delta_n = -1$).

The equation $\{\delta, \delta\} = 0$ can be expanded as a system of quadratic relations on the maps $V \xrightarrow{\widetilde{\delta}_n} V^{\otimes n}$. For $n = 1, 2, 3, \ldots$ they are:

$$\{\delta_1, \delta_1\} = 0 \iff \widetilde{\delta}_1^2 = 0$$

$$\{\delta_1, \delta_2\} + \{\delta_2, \delta_1\} = 0 \iff \widetilde{\delta}_2 \widetilde{\delta}_1 = (\widetilde{\delta}_1 \otimes 1 + 1 \otimes \widetilde{\delta}_1) \widetilde{\delta}_2$$

$$\delta_2^2 + \{\delta_1, \delta_3\} = 0 \iff (\widetilde{\delta}_2 \otimes 1) \otimes \widetilde{\delta}_2 - (1 \otimes \widetilde{\delta}_2) \otimes \widetilde{\delta}_2 =$$

$$(\widetilde{\delta}_1 \otimes 1 \otimes 1 + 1 \otimes \widetilde{\delta}_1 \otimes 1 + 1 \otimes 1 \otimes \widetilde{\delta}_1) \widetilde{\delta}_3 + \widetilde{\delta}_3 \widetilde{\delta}_1$$

....

The first says that $\widetilde{\delta}_1 : V \longrightarrow V$ is a differential on V. The second says that $\widetilde{\delta}_1$ satisfies the co-Leibnitz rule w.r.t. co-multiplication $\widetilde{\delta}_2 : V \longrightarrow V \otimes V$. The third says that the co-associator of this co-multiplication

$$V \xrightarrow{(\widetilde{\delta}_2 \otimes 1) \otimes \widetilde{\delta}_2 - (1 \otimes \widetilde{\delta}_2) \otimes \widetilde{\delta}_2} V^{\otimes 3}$$

is zero homotopic by means of contracting homotopy $\tilde{\delta}_3$, and so on.

Thus, A_{∞} -coproduct with just two non-zero components: $\delta = \delta_1 + \delta_2$ is the standard co-associative DG-coalgebra structure (or just coalgebra structure, if $\delta_1 = 0$).

In terms of dual space, the dual map $\mathsf{T}(V^*[-1]) \xrightarrow{\delta^*} V^*[-1]$ provides V^* with a series of *n*-ary operations

$$\underbrace{V^* \otimes \cdots \otimes V^*}_{n} \xrightarrow{\widetilde{\delta}^*_n} V^* , \quad n = 1, 2, 3, \dots , \qquad (1.1)$$

of degrees deg $\tilde{\delta}_n^* = 2 - n$. They are called *higher multiplications* and satisfy quadratic relations dual to above (they say that $\tilde{\delta}_1^*$ and $\tilde{\delta}_2^*$ provide V^* with the DGA-structure,

possibly non-associative but with the associator homotopic to zero by means of 'triple product' $\tilde{\delta}_3^*$, e.t.c.). These dual data is called A_{∞} -algebra structure on V.

1.3.1. Digression into physical terminology. In physics, higher products (1.1)

$$\langle v_1^*, v_2^*, \dots, v_n^* \rangle \stackrel{\text{def}}{=} \widetilde{\delta}_n^*(v_1^*, v_2^*, \dots, v_n^*)$$

are known as 'correlators'. Typically, they are expressed by some integrals. Quadratic relations between them come from integration tricks (often not well defined). Higher correlators appearing in 'effective theory' attached to some space W usually are computed in terms of appropriate 'ground theory' attached to another space V connected with W by means of some 'reduction'. Mathematically, this is formalized as transferring A_{∞} -structures along deformation retractions.

1.4. SDR-data. Let (V, ∂_V) and (W, ∂_W) be two complexes fitted into diagram

$$\gamma \hookrightarrow V \xrightarrow[\sigma]{\pi} W, \qquad (1.2)$$

where π , σ are morphisms of complexes (that is, commute with the differentials) and $\gamma: V \longrightarrow V$ is k-linear map of degree 1 (that is, a homotopy) such that

$$\pi\sigma = 1_W, \quad \sigma\pi = 1_V + \partial_V\gamma + \gamma\partial_V, \quad \gamma^2 = 0, \quad \pi\gamma = 0, \quad \gamma\sigma = 0.$$
(1.3)

Following [HS] we call these data strong deformation retraction between complexes (V, ∂_V) , (W, ∂_W) or just SDR-data for short. Typical example of SDR-data is as follows.

1.4.1. Retraction onto homology. Let $V = A \oplus B \oplus C$, where $B = \operatorname{im} \partial_V$ is the space of boundaries, $C \subset \ker \partial_V$ is transversal to B inside $\ker \partial_V$, and $A \subset V$ is transversal to $\ker \partial_V$ in V. Thus, ∂_V maps A isomorphically onto B and annihilates $B \oplus C$, i.e. subspace $C \cong H(V)$ represents the homologies. Then SDR-data (1.2) is given by diagram

$$\gamma \hookrightarrow V \xrightarrow[\sigma]{\pi} C,$$

where C is considered as a complex with zero differential, π , σ are predicted by the splitting $V = A \oplus B \oplus C$, and homotopy γ takes B isomorphically onto A via $-\partial_V|_A^{-1}$ and annihilates $C \oplus A$. Relation $\sigma \pi = 1_V + [\partial_V, \gamma]$ holds, because $-[\partial_V, \gamma]$ projects V onto $A \oplus B$ along C. Other relations are evident.

1.5. Transferring A_∞-coproducts along SDR-data. Given SDR-data (1.2) and A_∞-coproduct $T(V[1]) \xrightarrow{D_{\delta}} T(V[1])$ on V associated with series

$$\delta = \partial_{V[1]} + \sum_{n \ge 2} \delta_n , \quad V[1] \xrightarrow{\delta_n} V[1]^{\otimes n} ,$$

whose linear term $\delta_1 : V[1] \longrightarrow V[1]$ coincides with the differential $\delta_1 = \partial_{V[1]} = -\partial_V$, then transferred A_{∞} -structure on W is given by derivation

$$D_{\delta_{\mathrm{ind}}} : \mathsf{T}(W[1]) \longrightarrow \mathsf{T}(W[1])$$

associated with power series $\delta_{ind} = \partial_{W[1]} + \sum_{n \ge 2} \delta_{ind,n}$, whose *n*-th degree component

$$W[1] \xrightarrow{\delta_{\mathrm{ind},n}} W[1]^{\otimes n} , \quad n \ge 2 ,$$

admits following explicit description via sum over trees. For each planar tree Γ with one root, n leaves, and internal vertices of valency ≥ 3 consider its natural orientation from the root to the leaves and put the operators



(see fig. 1). Write $\delta_{\Gamma} : W[1] \longrightarrow W[1]^{\otimes n}$ for the composition of these \int operators along the tree Γ . Then

$$\delta_{\mathrm{ind},n} = \sum_{\Gamma} \delta_{\Gamma} \,. \tag{1.4} \qquad \mathbf{Fig. 1. } \delta_{\Gamma}.$$

Further we refer (1.4) as the sum over trees formula. Implication $D_{\delta}^2 = 0 \Rightarrow D_{\delta_{\text{ind}}}^2 = 0$ is natural but not obvious. It was re-proved by many authors under various assumptions¹. For convenience of readers we sketch a proof in the rest of this section. Readers who don't need it may jump now directly to §2.

1.6. What the sum over trees formula comes from. First of all, any SDR-data

$$\gamma \hookrightarrow V \xrightarrow[\sigma]{\pi} W$$

provides tensor algebras of V and W with SDR-data

$$\gamma_{\mathsf{T}} \hookrightarrow \mathsf{T}(V) \xrightarrow[\sigma_{\mathsf{T}}]{\pi_{\mathsf{T}}} \mathsf{T}(W) , \qquad (1.5)$$

where $\mathsf{T}(V)$, $\mathsf{T}(W)$ are equipped with differentials $\partial_{\mathsf{T}(V)}$, $\partial_{\mathsf{T}(W)}$ extending ∂_V , ∂_W by the Leibnitz rule; π_{T} , σ_{T} extend π , σ to the homomorphisms of tensor algebras; and homotopy $\mathsf{T}(V) \xrightarrow{\gamma_{\mathsf{T}}} \mathsf{T}(W)$ extends map $V \xrightarrow{\gamma} V$ to the whole of $\mathsf{T}(V)$ by *twisted Leibnitz rule*

$$\gamma_{\mathsf{T}} \circ \mu = \mu \circ ((f - g) \otimes \gamma_{\mathsf{T}} + \gamma_{\mathsf{T}} \otimes 1)$$

Verification of relations (1.3) is straightforward² (see [EM]).

Further, any map $D: \mathsf{T}(V) \longrightarrow \mathsf{T}(V)$ such that $D^2 = 0$ can be considered as formal perturbation of the differential D_{∂_V} in the SDR-data (1.5). Under some minor technical

¹results of this kind are traced back to Kadeishvili [Kd]; more closed to our subject recent versions can be found in [GL], [HS], [Ma], [Sm] and other papers cited therein

²if two morphisms $V \xrightarrow{f} V$ are homotopic by means of homotopy $V \xrightarrow{\gamma} V$ (i.e. $f - g = [\partial, \gamma]$) then

restrictions, a perturbation of differential in SDR-data can be extended to the perturbation of the whole SDR-data by a simple precise formula. Sufficient for our purposes is following version of this differential perturbation lemma.

1.6.1. LEMMA. Let SDR-data $\gamma \hookrightarrow V \xrightarrow[\sigma]{\sigma} W$ and a map $V \xrightarrow[\sigma]{\varepsilon} V$ satisfy the following two conditions:

- (1) perturbed differential $\partial'_V = \partial_V + \varepsilon$ satisfies ${\partial'_V}^2 = 0$;
- (2) well defined endomorphisms of V are provided by power series¹

$$(1 - \gamma \varepsilon)^{-1} = 1 + \sum_{m \ge 1} (\gamma \varepsilon)^m$$
, $(1 - \varepsilon \gamma)^{-1} = 1 + \sum_{m \ge 1} (\varepsilon \gamma)^m$. (1.6)

Then perturbed diagram $\gamma' \hookrightarrow (V, \partial'_V) \xrightarrow[\sigma']{\pi'} (W, \partial'_W)$ that has

$$\sigma' = (1 - \gamma \varepsilon)^{-1} \sigma = \sigma + \gamma \varepsilon \sigma + \gamma \varepsilon \gamma \varepsilon \sigma + \gamma \varepsilon \gamma \varepsilon \sigma + \cdots,$$

$$\pi' = \pi (1 - \varepsilon \gamma)^{-1} = \pi + \pi \varepsilon \gamma + \pi \varepsilon \gamma \varepsilon \gamma + \pi \varepsilon \gamma \varepsilon \gamma \varepsilon \gamma + \cdots,$$

$$\gamma' = (1 - \gamma \varepsilon)^{-1} \gamma = \gamma (1 - \varepsilon \gamma)^{-1} = \gamma + \gamma \varepsilon \gamma + \gamma \varepsilon \gamma \varepsilon \gamma + \cdots,$$

$$\partial'_W = \partial_W + \varepsilon_{\text{ind}}, \text{ where } \varepsilon_{\text{ind}} = \pi \varepsilon \sigma' = \pi' \varepsilon \sigma = \pi \varepsilon \sigma + \pi \varepsilon \gamma \varepsilon \sigma + \pi \varepsilon \gamma \varepsilon \sigma + \cdots,$$

provides SDR-data between perturbed complexes (V, ∂'_V) and (W, ∂'_W) .

PROOF. Let us start with checking the perturbed version of relations (1.3), which do not include ∂'_W . First of all, $\gamma^2 = 0$ implies

$$(\gamma')^{2} = (1 - \gamma \varepsilon)^{-1} \gamma^{2} (1 - \varepsilon \gamma)^{-1} = 0.$$
 (1.7)

Since $\gamma(1-\gamma\varepsilon)^{-1} = \gamma = (1-\varepsilon\gamma)^{-1}\gamma$, conditions $\pi\gamma = 0$ and $\gamma\sigma = 0$ lead to

$$\pi'\gamma' = \pi(1-\varepsilon\gamma)^{-1}\gamma(1-\varepsilon\gamma)^{-1} = \pi\gamma(1-\varepsilon\gamma)^{-1} = 0,$$

$$\gamma'\sigma' = (1-\gamma\varepsilon)^{-1}\gamma(1-\gamma\varepsilon)^{-1}\sigma = (1-\gamma\varepsilon)^{-1}\gamma\sigma = 0.$$
(1.8)

The same computation shows also that

$$\pi\gamma' = \gamma'\sigma = 0. \tag{1.9}$$

corresponding DG-algebra morphisms $\mathsf{T}(V) \xrightarrow[g_T]{g_T} \mathsf{T}(V)$ are homotopic too by homotopy $\mathsf{T}(V) \xrightarrow{\gamma_T} \mathsf{T}(V)$ extending γ to f-g-derivation of tensor algebra defined by twisted Leibnitz rule $\gamma_T \circ \mu = \mu \circ (f \otimes \gamma_T + \gamma_T \otimes g)$, because $\gamma_T|_{V^{\otimes n}} = \sum_{\alpha+\beta=n-1} f^{\otimes \alpha} \otimes \gamma \otimes g^{\otimes \beta}$ and

$$[\partial_{\mathsf{T}},\gamma_{\mathsf{T}}]|_{V^{\otimes n}} = \sum_{\alpha+\beta=n-1} f^{\otimes \alpha} \otimes (\partial\gamma+\gamma\partial) \otimes g^{\otimes \beta} = \sum_{\alpha+\beta=n-1} f^{\otimes \alpha} \otimes (f-g) \otimes g^{\otimes \beta} = f^{\otimes n} - g^{\otimes n}$$

¹this technical condition holds when $\gamma \varepsilon$ and $\varepsilon \gamma$ are locally nilpotent (say, by reasons of grading — this is the case we deal with); more generally, it holds when $\operatorname{End}(E)$ is complete in some norm such that $||fg|| \leq ||f|| \cdot ||g||$ and $||\varepsilon|| \ll 1$.

To compute compositions $\pi'\sigma'$ and $\sigma'\pi'$, we note that

$$(1 - \gamma \varepsilon)^{-1} = 1 + \gamma' \varepsilon$$
$$(1 - \varepsilon \gamma)^{-1} = 1 + \varepsilon \gamma'$$

and rewrite π' and σ' as $\pi' = \pi(1 + \varepsilon \gamma')$, $\sigma' = (1 + \gamma' \varepsilon)\sigma$. Then, by (1.7), (1.9)

$$\pi'\sigma' = \pi(1 + \varepsilon\gamma')(1 + \gamma'\varepsilon)\sigma = \pi\sigma = 1$$

For the reverse composition we use $\sigma \pi = 1 + \partial_V \gamma + \gamma \partial_V$ and $\varepsilon^2 = -\partial_V \varepsilon - \varepsilon \partial_V$:

$$\sigma'\pi' = (1+\gamma'\varepsilon)\sigma\pi(1+\varepsilon\gamma') = (1+\gamma'\varepsilon)(1+\partial_V\gamma+\gamma\partial_V)(1+\varepsilon\gamma') =$$

= 1+ \gamma'\varepsilon + \varepsilon'\gamma'\varepsilon + \varepsilon'\varepsilon'\varepsilon + \varepsilon'\varepsilon'\varepsilon + \varepsilon'\varepsilon + \varep

Since $\gamma \varepsilon \gamma' = \gamma' \varepsilon \gamma = \gamma' - \gamma$, latter two summands in (1.10) can be rewritten as

$$\begin{aligned} (1+\gamma'\varepsilon)\partial_V\gamma(1+\varepsilon\gamma') + (1+\gamma'\varepsilon)\gamma\partial_V(1+\varepsilon\gamma') &= \\ &= (1+\gamma'\varepsilon)\partial_V\gamma' + \gamma'\partial_V(1+\varepsilon\gamma') = \partial_V\gamma' + \gamma'\partial_V + \gamma'\varepsilon\partial_V\gamma' + \gamma'\partial_V\varepsilon\gamma' \,. \end{aligned}$$

This cancels minus terms of (1.10) and gives the required identity

$$\sigma'\pi' = 1 + \gamma'\varepsilon + \varepsilon\gamma' + \gamma'\partial_V + \partial_V\gamma' = 1 + [\partial'_V, \gamma'].$$
(1.11)

To check the relations containing ∂'_W , we note that $\partial'_W = \partial_W + \varepsilon'$ can be equivalently rewritten as $\partial'_W = \pi \partial'_V \sigma' = \pi' \partial'_V \sigma$. Now we use (1.11) to verify the commutation relation $\partial'_W \pi' = \pi' \partial'_V$:

$$\begin{aligned} \partial'_W \pi' &= \pi \partial'_V \sigma' \pi' = \pi \partial'_V (1 + \partial'_V \gamma' + \gamma' \partial'_V) = & \text{(because of } {\partial'_V}^2 = 0) \\ &= \pi (1 + \partial'_V \gamma') \partial'_V = \pi (1 + \varepsilon \gamma' + \partial_V \gamma') \partial'_V = & \text{(because of } \pi \partial'_V = \partial'_W \pi) \\ &= \pi (1 + \varepsilon \gamma') \partial'_V + \partial_W \pi \gamma' \partial'_V = & \text{(because of } \pi \gamma' = 0) \\ &= \pi' \partial'_V \end{aligned}$$

Similar computation shows that $\sigma' \partial'_W = \partial'_V \sigma'$. All these commutation relations imply the last required identity ${\partial'_W}^2 = {\partial'_W} \pi' {\partial'_V} \sigma = \pi' {\partial'_V} \partial'_V \sigma = 0.$

1.6.2. Proof of 'sum over trees' formula. Assume we are given with SDR-data

$$\gamma \hookrightarrow (V, \partial_V) \xrightarrow[\sigma]{\pi} (W, \partial)$$

and A_{∞} -coproduct $\mathsf{T}(V[1]) \xrightarrow{D_{\delta}} \mathsf{T}(V[1])$, whose linear component coincides with the differential $\delta_1 : V[1] \xrightarrow{-\partial_V} V[1]$, that is, perturbs differential $\partial_{\mathsf{T}(V[1])}$ in SDR-data

$$\gamma_{\mathsf{T}}[1] \hookrightarrow \left(\mathsf{T}(V[1], \partial_{\mathsf{T}(V[1])}) \xrightarrow[\sigma_{\mathsf{T}}[1]]{} (\mathsf{T}(W[1], \partial_{\mathsf{T}(W[1])}) \right)$$
(1.12)

induced by the initial SDR-data (as it was explained in n° 1.6) and shifted by 1. Write this perturbation as $D = \partial_{\mathsf{T}(V[1])} + D_{\delta} = D_{\partial_{V[1]}+\delta}$ and extend it by the differential perturbation lemma n° 1.6.1 to the perturbation of the whole of SDR-data (1.12). The resulting perturbed data

$$\gamma'_{\mathsf{T}} \hookrightarrow (\mathsf{T}(V[1], D) \xrightarrow[\sigma']{\pi'} (\mathsf{T}(W[1], D_{\delta_{\mathrm{ind}}})$$

contain the required A_{∞} -coproduct $D_{\delta_{\text{ind}}} = D_{\partial_{\mathsf{T}(W[1])}} + D_{\delta_{\text{ind}}} = D_{\partial_{W[1]} + \delta_{\text{ind}}}$ associated with map $\delta_{\text{ind}} = \partial_{W[1]} + \sum_{n \ge 2} \delta_{\text{ind},n}$. It has

$$\widetilde{\delta}_{\mathrm{ind},n} = \pi^{\otimes n} \circ (\delta + \delta \gamma_{\mathsf{T}} \delta + \delta \gamma_{\mathsf{T}} \delta \gamma_{\mathsf{T}} \delta + \delta \gamma_{\mathsf{T}} \delta \gamma_{\mathsf{T}} \delta \gamma_{\mathsf{T}} \delta + \cdots)_{n,1} \circ \sigma \,,$$

where $(*)_{n,1} : V \longrightarrow V^{\otimes n}$ means those component of bracketed operator that sends $V \subset \mathsf{T}(V)$ to $V^{\otimes n} \subset \mathsf{T}(V)$. It equals the sum of all oriented trees with one input and n outputs composed from corollas



with the right ones allowed only as the last elements of the composition. Since total number of γ 's in $\delta \gamma_{\mathsf{T}} \delta \ldots \gamma_{\mathsf{T}} \delta$ equals the number of internal edges in the corresponding trees and outgoing γ 's are killed by the final $\pi^{\otimes n}$ applied to all leaves, we get what we want.

§2. Functorial barycentric retraction of combinatorial simplicial chain complexes.

2.1. Combinatorial simplicial complexes. Informally, a combinatorial simplicial complex is a topological space properly triangulated by simpleces in such a way that each simplex is uniquely determined by its vertexes (see fig. 2). Formally, combinatorial simplicial complex is defined by set of vertices M and set of simpleces A, which is a subset in the set of all subsets in M such that Δ contains all elements of M and all subsets of each $\sigma \in \Delta$.

We write $\underline{x_1 x_2 \dots x_k}$ for simpleces $\{x_1, x_2, \dots, x_k\} \in \Delta$ considered as non-ordered non-oriented sets. They form a full subcategory in category $\mathscr{S}(M)$ of all subsets in M with inclusions as the morphisms. The complex



 $\Delta = \mathscr{S}(M)$ exhausting the whole of this category is called the standard combinatorial simplex with vertex set M.

We always write $[X_1, X_2, \ldots, X_k]$ for totally ordered collections of objects (of any nature).

An oriented simplex $\overline{x_1 x_2 \dots x_k}$ is the orbit of $[x_1, x_2, \dots, x_k]$ under the action of alternating subgroup $\mathfrak{A}_k \subset \mathfrak{S}_k$ by the permutations of x_{ν} 's. Thus, each simplex produces two oriented simpleces. We write $\overline{C}_k(\Delta)$ for vector space spanned by oriented simpleces $\overline{x_1 x_2 \dots x_k}$ of cardinality k in simplicial complex Δ modulo the relation saying that the sum of two oriented simpleces obtained from the same non-oriented simplex equals zero. So, for the standard combinatorial simplex $\overline{C}_k(\mathscr{S}(M)) = \Lambda^k(\mathbb{k}^M)$ is k-th exterior power of vector space with basis M.

The spaces of oriented simpleces are naturally organized in chain complex

$$0 \longrightarrow \overline{C}_m \xrightarrow{\overline{\partial}} \overline{C}_{m-1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \overline{C}_2 \xrightarrow{\overline{\partial}} \overline{C}_1 \longrightarrow 0$$

(where m is the total number of vertices) with differential

$$\overline{\partial} : \overline{x_1 \, x_2 \, \dots \, x_k} \longmapsto \sum_{\nu=1}^k (-1)^{\nu} \cdot \overline{x_1 \, \dots \, x_{\nu-1} \, x_{\nu+1} \, \dots \, x_k} \,.$$

We call it a chain complex of combinatorial simplicial complex and denote by \overline{C} . Let us stress that simplicial chain complex commonly used in topology is $\overline{C}[1]$, because our \overline{C} is graded by means of *combinatorial* degree, which equals the cardinality $|\sigma|$ of simplex σ (considered as a subset of M). This combinatorial degree should not be confused with topological degree equal to dimension dim $\sigma = |\sigma| - 1$.

For example, A_{∞} -coproduct of *topological* chains is given by degree -1 map

$$\overline{C}[2] \xrightarrow{\delta} \mathsf{T}(\overline{C}[2])$$

whose homogeneous components $\overline{C} \xrightarrow{\widetilde{\delta}_n} \overline{C}^{\otimes n}$ have combinatorial degree 2n - 3. In particular, in combinatorial grading they are *odd* and satisfy sign-less quadratic relations.

2.2. Flags and barycentric subdivisions A flag of length k in M is a chain of strictly embedded non-oriented sets

$$F_1 \subset F_2 \subset \cdots \subset F_k \tag{F}$$

Flag (F) defines and is uniquely defined by ordered collection of graded components $[G_1, G_2, \ldots, G_k], G_1 \stackrel{\text{def}}{=} F_1, G_i \stackrel{\text{def}}{=} F_i \setminus F_{i-1} \text{ for } i \ge 2$. We often use $[G_1, G_2, \ldots, G_k]$ as alternative notation for flag (F).

Flag (F) is called *saturated*, if all its graded components have cardinality one. Saturated flags (F) stay in bijection with total orderings on F_k .

Recall that barycentric subdivision B(M), of a simplex with vertex set M, is a simplicial complex whose vertexes are non-empty subsets of M and k-vertex simpleces are the flags of length k in M. Thus, vertexes of B(M) are the objects of the category $\mathscr{S}(M)$ (subsets of M with inclusions as the morphisms), oriented edges of B(M) are the morphisms $F_1 \subset F_2$ in $\mathscr{S}(M)$, 2-dimensional faces are the pairs of consequent morphisms $F_1 \subset F_2 \subset F_3$ in $\mathscr{S}(M)$, e.t.c.



In mechanical terms, vertex $F \subset M$ depicts the centre of mass for the points of F. Geometrically, we put new vertex in the barycenter of each face and join it with all other vertexes of that face.

2.2.1. Barycentric chain complex $\underline{C}^{[M]} = \overline{C}^{[B(M)]}$ is the chain complex of simplicial complex B(M). It looks like

$$0 \longrightarrow \underline{C}_{m+1} \xrightarrow{\underline{\partial}} \underline{C}_m \xrightarrow{\underline{\partial}} \cdots \xrightarrow{\underline{\partial}} \underline{C}_2 \xrightarrow{\underline{\partial}} \underline{C}_1 \longrightarrow 0.$$

Basis of vector space \underline{C}_k consists of length k flags $F_1 \subset F_2 \subset \cdots \subset F_k$ and differential $\underline{C}_k \xrightarrow{\partial} \underline{C}_{k-1}$ takes this flag to

$$\sum_{\nu=1}^{k} (-1)^{\nu} F_1 \subset \cdots \subset F_{\nu-1} \subset F_{\nu+1} \subset \cdots \subset F_k$$

(ν th term of the filtration is omitted for $\nu = 1, 2, ..., k$). In terms of graded components this is written as

$$\sum_{\nu=1}^{k-1} (-1)^{\nu} [G_1, \ldots, G_{\nu-1}, G_{\nu} \sqcup G_{\nu+1}, G_{\nu+2}, \cdots, G_k]$$

(ν th comma is replaced by union).

In the next section we construct functorial in M SDR-data between chain complex $\overline{C}^{[M]}$ of standard simplex $\mathscr{S}(M)$ and chain complex $\overline{C}^{[B(M)]}$ of its barycentric subdivision B(M)

$$\gamma \hookrightarrow \overline{C}^{[B(M)]} \xrightarrow[\sigma]{\pi} \overline{C}^{[M]}$$
(BR)

This *barycentric retraction* is essentially unique. Each functorial in M A_{∞}-coproduct $\delta^{[M]}$ on $\overline{C}^{[M]}$, by functoriality, provides $\overline{C}^{[B(M)]}$ with A_{∞}-coproduct $\delta^{[B(M)]}$, which can be transferred along (BR) back to $\overline{C}^{[M]}$. Thus, we get new functorial in M A_{∞}-coproduct $\delta^{[M]}_{ind}$ on $\overline{C}^{[M]}$.

We say that functorial A_{∞} -coproduct of combinatorial simplicial chains is *barycentri*cally stable, if it is transferred to itself under this procedure.

In the next sections we explain why such a product δ^{bs} should exist, be unique up to a constant factor, and have $\delta_1^{\text{bs}} = \overline{\partial}$ and $\delta_2^{\text{bs}} = \delta^{\kappa}$ from (1). We write explicit recursive formula expressing δ_k^{bs} through $\delta_{< k}^{\text{bs}}$ and deduce nice closed formula for $\delta^{\text{bs}}(\overline{01})$. **2.3. Functorial barycentric retraction of combinatorial simplex** Let \Bbbk be a field of zero characteristics. In this section we write down explicit formulas for functorial w.r.t. inclusions of finite sets $M_1 \hookrightarrow M_2$ strong deformation retraction

$$\gamma \hookrightarrow \overline{C}^{[B(M)]} \xrightarrow[\sigma]{\pi} \overline{C}^{[M]}$$

and show that it is unique up to rescalling $\sigma \mapsto t\sigma$, $\pi \mapsto t^{-1}\pi$ by some $t \in \mathbb{k}$.

Note that functoriality means existence of the same retraction for any combinatorial simplicial complex $\Delta \subset \mathscr{S}(M)$. It also forces maps σ , π , γ to be equivariant w.r.t. the action of the permutation group Aut (M).

Actually, in a presence of this equivariance, the SDR-data relations

$$\pi\sigma = 1_W$$
, $\sigma\pi = 1_V + \partial_V\gamma + \gamma\partial_V$, $\gamma^2 = 0$, $\pi\gamma = 0$, $\gamma\sigma = 0$

produce quite over-determined system of linear equations and this is a kind of luck that it turns to be solvable at all. The solution is given by precise formulas presented below.

There is unique up to rescalling intertwiner $\overline{C} \subset \stackrel{\sigma}{\longrightarrow} \underline{C}$ from the sign representation of symmetric group Aut (M) on the oriented faces of M to the tabloid representation on the flags. Geometrically, it takes each simplex to the oriented chain formed by all its barycentric pieces. In combinatorial terms,

$$\sigma\left(\overline{x_1\,x_2\,\ldots\,x_k}\right) = \sum_{g \in \mathfrak{S}_k} \operatorname{sgn}(g)[x_{g(1)}\,x_{g(2)}\,\ldots\,x_{g(k)}]$$

(alternated sum of saturated flags built from $\{x_1, x_2, \ldots, x_k\}$). For example:

 $\sigma(\overline{012}) = \left([0,1,2] + [1,2,0] + [2,1,0] \right) - \left([0,2,1] + [2,1,0] + [1,0,2] \right) \,,$

Note that $\overline{C} \xrightarrow{\sigma} \underline{C}$ obviously commutes with differentials.

Formula for functorial projection $\underline{C} \xrightarrow{\pi} \overline{C}$ is less obvious and requires combinatorial denominator

$$\pi \left(F_1 \subset F_2 \subset \cdots \subset F_k \right) = \frac{1}{\prod\limits_{\nu=1}^k |F_{\nu}|} \cdot \sum_{\substack{(x_1, x_2, \dots, x_k) \\ \text{running through} \\ G_1 \times G_2 \times \cdots \times G_k}} \overline{x_1 x_2 \dots x_k}.$$

Thus, we sum oriented simpleces $\overline{x_1 x_2 \dots x_k}$ for all possible choices of $x_i \in F_i \setminus F_{i-1}$ and divide the result by the product of cardinalities of the flag sets $|F_i|$. For example: $\pi([\underline{01}]) = \frac{1}{2} (\overline{0} + \overline{1}), \pi([0, 1]) = \frac{1}{2} \overline{01}, \pi([\underline{01}, 2]) = \frac{1}{6} (\overline{02} + \overline{12}), \pi([0, \underline{12}]) = \frac{1}{3} (\overline{01} + \overline{02}).$

Most complicated ingredient of our SDR data is the functorial homotopy $\gamma : \underline{C} \longrightarrow \underline{C}$ (of degree 1). By the definition, it annihilates saturated flags and sends non-saturated flag $F_1 \subset F_2 \subset \cdots \subset F_k$ to

$$\sum_{\substack{i:\\F_i|>i}} (-1)^i \cdot \prod_{\nu=1}^i |F_{\nu}|^{-1} \cdot \sum_{\substack{(x_1, x_2, \dots, x_i)\\ \text{running through}\\G_1 \times G_2 \times \dots \times G_i}} \sum_{g \in \mathfrak{S}_i} \operatorname{sgn}(g) \cdot F^{(i)}(x, g) \,,$$

where $F^{(i)}(x,g)$ denotes the following flag of length (k+1)

$$\underline{x_{g(1)}} \subset \underline{x_{g(1)}x_{g(2)}} \subset \cdots \subset \underline{x_{g(1)}x_{g(2)}} \ldots \underline{x_{g(i)}} \subset F_i \subset F_{i+1} \subset \cdots \subset F_k.$$

Thus, for each i such that sub-flag

$$F_1 \subset F_2 \subset \cdots \subset F_i$$

is not saturated we consider all ordered collections $(x_1, x_2, \ldots, x_i) \in G_1 \times G_2 \times \cdots \times G_i$ and form of alternated sum of all saturated flags built of them; then we extend each saturated flag to the right side by $\subset F_i \subset F_{i+1} \subset \cdots \subset F_k$ and divide the sum by $(-1)^i \prod_{\nu=1}^i |F_{\nu}|$; finally, we add together these weighted sums coming from all *i*'s.

For example, γ acts on two combinatorially dif-



ferent internal edges of the barycentric subdivision of the triangle as follows:

$$\begin{split} \gamma([\underline{2},\underline{01}] &= -\frac{1}{3} \left([\underline{2},\underline{0},\underline{1}] - [\underline{0},\underline{2},\underline{1}] + [\underline{2},\underline{1},\underline{0}] - [\underline{1},\underline{2},\underline{0}] \right) \\ \gamma([\underline{01},\underline{2}] &= \frac{1}{2} \left([\underline{0},\underline{1},\underline{2}] + [\underline{1},\underline{0},\underline{2}] \right) \\ &- \frac{1}{6} \left([\underline{0},\underline{2},\underline{1}] - [\underline{2},\underline{0},\underline{1}] + [\underline{1},\underline{2},\underline{0}] - [\underline{2},\underline{1},\underline{0}] \right) \end{split}$$

Next formulas show how does γ act on points, edges and triangles (all flags containing non-proper inclusions are declared to be zeros):

$$\begin{split} \gamma(X) &= \frac{1}{|X|} \sum_{x \in X} (\underline{x} \subset X) \\ \gamma(X \subset Y) &= \frac{1}{|X|} \sum_{x \in X} (\underline{x} \subset X \subset Y) \\ &\quad - \frac{1}{|X||Y|} \sum_{\substack{x \in X \\ y \in Y \smallsetminus X}} \left[(\underline{x} \subset \underline{xy} \subset Y) - (\underline{y} \subset \underline{xy} \subset Y) \right] \\ \gamma(X \subset Y \subset Z) &= \frac{1}{|X|} \sum_{x \in X} (\underline{x} \subset X \subset Y \subset Z) \\ &\quad - \frac{1}{|X||Y|} \sum_{\substack{x \in X \\ y \in Y \smallsetminus X}} \left[(\underline{x} \subset \underline{xy} \subset Y \subset Z) - (\underline{y} \subset \underline{xy} \subset Y \subset Z) \right] \\ &\quad + \frac{1}{|X||Y||Z|} \sum_{\substack{x \in X \\ y \in Y \smallsetminus X}} \left[(\underline{x} \subset \underline{xy} \subset \underline{xyz} \subset Z) - (\underline{y} \subset \underline{xyz} \subset Z) \\ &\quad + (\underline{z} \subset \underline{xz} \subset \underline{xyz} \subset Z) - (\underline{x} \subset \underline{xyz} \subset Z) \\ &\quad + (\underline{z} \subset \underline{xz} \subset \underline{xyz} \subset Z) - (\underline{x} \subset \underline{xyz} \subset Z) \right] \end{split}$$

§3. Functorial A_{∞} -coproduct transferred to itself via barycentric retraction.

3.1. Analysis of 'sum over trees formula'. Let $\overline{C}[2] \xrightarrow{\delta^{\text{bs}}} \mathsf{T}(\overline{C}[2])$ provide topological chain complex $\overline{C}[1]$ with functorial A_{∞} -coproduct transferred to itself along functorial SDR-data.

$$\gamma \hookrightarrow \overline{C}^{[B(M)]} \xrightarrow[\sigma]{\pi} \overline{C}^{[M]}.$$

Then each homogeneous component $\widetilde{\delta}_n^{\mathrm{bs}} : \overline{C} \longrightarrow \overline{C}^{\otimes n}$ satisfies

$$\widetilde{\delta}^{\mathrm{bs}}_n = \sum_{\Gamma} \widetilde{\delta}^{\mathrm{bs}}_{\Gamma} = A(\widetilde{\delta}^{\mathrm{bs}}_n) + B(\widetilde{\delta}^{\mathrm{bs}}_{< n})$$

The middle sum runs over trees with one input and n output slots decorated by σ and π 's, internal edges decorated by γ 's, and vertexes decorated $\tilde{\delta}_i^{\text{bs}}$.

$$A(\widetilde{\delta}_n^{\mathrm{bs}}) \stackrel{\mathrm{def}}{=} \pi^{\otimes n} \circ \widetilde{\delta}^{\mathrm{bs}} \circ \sigma$$

stays for the summand contributed by one vertex tree (corolla with 1 root and *n* leaves). It is the only summand depending on $\tilde{\delta}_n^{\text{bs}}$ and this dependence is *linear*. The sum of all other terms is denoted by $B(\tilde{\delta}_{\leq n}^{\text{bs}})$. Thus, barycentric stability of functorial A_{∞} -coproduct δ^{bs} implies

 $(1-A) \cdot \widetilde{\delta}_n^{\text{bs}} = B(\widetilde{\delta}_{< n}^{\text{bs}}) \text{ for all } n \ge 3.$ (BS)

For n = 1, 2 the barycentric stability is expressed by equations

$$\widetilde{\delta}_1^{\mathrm{bs}} = \pi \circ \widetilde{\delta}_1^{\mathrm{bs}} \circ \sigma \ , \quad \widetilde{\delta}_2^{\mathrm{bs}} = (\pi \otimes \pi) \circ \widetilde{\delta}_2^{\mathrm{bs}} \circ \sigma$$

3.1.1. PROPOSITION. Pairs of compatible tensors $\overline{C}^{[M]} \xrightarrow{\widetilde{\delta}_1^{[M]}} \overline{C}^{[M]}, \overline{C}^{[M]} \xrightarrow{\widetilde{\delta}_2^{[M]}} \overline{C}^{[M]}^{[M] \otimes 2}$ of combinatorial degrees -1 and +1 that are functorial in M and go to itself via linear transformations $\delta_1^{[M]} \mapsto \pi \circ \delta_1^{[B(M)]} \circ \sigma$ and $\delta_2^{[M]} \mapsto \pi^{\otimes 2} \circ \delta_2^{B([M])} \circ \sigma$ form 1-dimensional subspace spanned by the simplicial chain differential $\overline{\partial}$ and the Kolmogorov co-product (1)

3.2. Recursive formula for δ^{bs} . One could try to recover the whole of barycentrically stable A_{∞} -coproduct δ^{bs} from its starting terms δ_1^{bs} , δ_2^{bs} by solving (BS) w.r.t. δ_n^{bs} .

3.2.1. CONJECTURE.. Eigenvalues of linear operator $A : \delta_n^{[M]} \mapsto \pi^{\otimes n} \circ \delta_n^{B([M])} \circ \sigma$, which acts on functorial in M tensors $\overline{C}^{[M]} \xrightarrow{\delta_n^{[M]}} \overline{C}^{[M]^{\otimes n}}$ of combinatorial degree 2n - 2, never are equal to 1 for $n \ge 3$ and decrease exponentially as $n \to +\infty$. Thus, barycentrically stable functorial A_{∞} -coproduct of combinatorial simplicial chains is unique up to rescalling and can be computed by recursion:

$$\delta_n^{bs} = (1 - A)^{-1} \cdot B(\delta_{< n}^{bs}), \qquad (3.1)$$

where $B(\delta_{\leq n}^{\text{bs}})$ is the sum over planar trees with one incoming slot, n outgoing slots, and internal vertexes of valency $3 \leq v \leq (n-1)$ oriented from the input to the outputs and decorated by σ on input, π 's on outputs, γ 's on internal edges and δ_{ν} 's on internal vertexes.

3.3. Open questions. It would be very interesting to compute eigenvectors and eigenvalues of linear operator

$$A_n : \widetilde{\delta}_n^{[M]} \mapsto \pi^{\otimes n} \circ \widetilde{\delta}_n^{B([M])} \circ \sigma$$

acting on functorial in M tensors $\overline{C}^{[M]} \xrightarrow{\widetilde{\delta}_n^{[M]}} \overline{C}^{[M]^{\otimes n}}$ of combinatorial degree 2n-3.

We expect elegant generating series (over n) for such eigentensors. Conjecturally, they should be closely connected with quasi-symmetric functions and Malvenuto–Reutenauer Hopf algebra of permutations as well as with its partner — non-commutative symmetric functions investigated extensively by Gelfand, Lascoux, Retakh, and others.

Computational experiments made by Shamil Shakirov corroborate the above conjecture as far as he can compute eigenvalues by Maple. For example, in 2-dimensional case they decrease as 6^{-n} .

But the most interesting problem is to get closed formula for the whole barycentrically stable functorial A_{∞} -coproduct

$$\delta^{\mathrm{bs}}: \overline{C}[2] \longrightarrow \mathsf{T}(\overline{C}[2])$$

in all higher dimensions.

It follows from general Koszul duality for operads that the image of δ^{bs} lies in subalgebra of Lie power series. Thus, we expect close connections between δ_n^{bs} 's and projectors onto the subspaces of Lie polynomials. A closed formula for δ^{bs} costs, probably, the same prise as the Kampbell–Hausdorf formula.

§4. Closed formula in dim = 1 case.

4.1. Starting remarks. Since the combinatorial degree of $\delta_n : \overline{C} \longrightarrow \overline{C}^{\otimes n}$ is 2n-3, the co-product $\delta(\overline{0})$ should have just one non-zero component. Namely, up to constant factor

$$\widetilde{\delta}(\overline{0}) = \widetilde{\delta}_2(\overline{0}) = \overline{0} \otimes \overline{0} \tag{4.1}$$

We fix this factor to be 1 and define the functorial coproduct of point by formula (4.1).

The same reasons of degree show, the restriction of map $\overline{C} \xrightarrow{\widetilde{\delta}_n} \overline{C}^{\otimes n}$ onto cardinality 2 simplex $\overline{01}$ lie in the linear span of maps

$$\begin{array}{l} \overline{01}\longmapsto \overline{01}^{\otimes\alpha}\otimes \overline{0}\otimes \overline{01}^{\otimes\beta} \\ \overline{01}\longmapsto \overline{01}^{\otimes\alpha}\otimes \overline{1}\otimes \overline{01}^{\otimes\beta} \end{array} \quad (\text{where } \alpha,\beta \ge 0, \ \alpha+\beta=n-1) \end{array}$$

Among these maps, the functorial ones (that is, commuting with transposition $0 \leftrightarrow 1$) are spanned by

$$\delta_n^{\alpha} : \overline{01} \longmapsto \overline{01}^{\otimes \alpha} \otimes \left(\overline{0} + (-1)^n \cdot \overline{1}\right) \otimes \overline{01}^{\otimes \beta} .$$

$$(4.2)$$

At the same time, tensors (4.2) are precisely the eigenvectors of linear operator

$$A_n: \ \delta_n^{[\overline{01}]} \mapsto \pi^{\otimes n} \circ \delta_n^{[B(\overline{01})]} \circ \sigma$$

and have eigenvalues $(1/2)^{n-1}$ for odd n and $(1/2)^{n-2}$ for even n. Indeed,

$$\begin{split} \delta_n^{\alpha} \circ \sigma \left(\overline{01} \right) &= \delta_n^{\alpha} ([0,1] - [1,0]) = \\ &= [0,1]^{\otimes \alpha} \otimes \left(\underline{0} + (-1)^n \cdot \underline{01} \right) \otimes [0,1]^{\otimes \beta} - [1,0]^{\otimes \alpha} \otimes \left(\underline{1} + (-1)^n \cdot \underline{01} \right) \otimes [1,0]^{\otimes \beta} \end{split}$$

and, applying $\pi^{\otimes n}$, we get

$$\begin{aligned} \pi^{\otimes n} \circ \delta_n^{\alpha} \circ \sigma \left(\overline{01}\right) &= \\ \frac{1}{2^{n-1}} \cdot \overline{01}^{\otimes \alpha} \otimes \left(\overline{0} + (-1)^n \cdot \frac{\overline{0} + \overline{1}}{2}\right) \otimes \overline{01}^{\otimes \beta} - \frac{1}{2^{n-1}} \cdot \overline{10}^{\otimes \alpha} \otimes \left(\overline{1} + (-1)^n \cdot \frac{\overline{0} + \overline{1}}{2}\right) \otimes \overline{10}^{\otimes \beta} \\ &= \frac{3 + (-1)^n}{2^n} \cdot \overline{01}^{\otimes \alpha} \otimes \left(\overline{0} + (-1)^n \cdot \overline{1}\right) \otimes \overline{01}^{\otimes \beta} = \frac{3 + (-1)^n}{2^n} \cdot \delta_n^{\alpha} \left(\overline{01}\right) \,. \end{aligned}$$

This agrees with the above claim that $\widetilde{\delta}_1^{bs} = \delta_1^1 = \overline{\partial} : \overline{0}_1 \longrightarrow \overline{0} - \overline{1}$ is the only functorial A-invariant differential and forces $\widetilde{\delta}_2^{\text{bs}}$ to take

$$\overline{01} \longmapsto x \cdot \left(\overline{0} + \overline{1}\right) \otimes \overline{01} + y \cdot \overline{01} \otimes \left(\overline{0} + \overline{1}\right)$$

Evaluating $\widetilde{\delta}_2^{\text{bs}} \circ \widetilde{\delta}_1^{\text{bs}} + \left(1 \otimes \widetilde{\delta}_1^{\text{bs}} + \widetilde{\delta}_1^{\text{bs}} \otimes 1\right) \circ \widetilde{\delta}_2^{\text{bs}} = 0$ at $\overline{01}$ and using formula (4.1) for the coproduct of point, we get

$$0 = \overline{0} \otimes \overline{0} - \overline{1} \otimes \overline{1} - x \cdot (\overline{0} + \overline{1}) \otimes (\overline{0} - \overline{1}) + y \cdot (\overline{0} - \overline{1}) \otimes (\overline{0} + \overline{1}) =$$

= $(1 - x + y) \cdot (\overline{0} \otimes \overline{0} - \overline{1} \otimes \overline{1}) + (x + y) \cdot (\overline{1} \otimes \overline{0} - \overline{0} \otimes \overline{1})$

Thus $\widetilde{\delta}_2^{\mathrm{bs}}(\overline{01}) = \frac{1}{2} \left((\overline{0} + \overline{1}) \otimes \overline{01} - \overline{01} \otimes (\overline{0} + \overline{1}) \right) = -\frac{1}{2} \cdot \mathrm{ad}_{\overline{01}}(\overline{0} + \overline{1})$. Since for $n \ge 3$ the eigenvalues of A_n never equal 1, all higher components of δ^{bs} are uniquely recovered from $\widetilde{\delta}_1^{\text{bs}}$ and $\widetilde{\delta}_2^{\text{bs}}$ by means of recursive formula (3.1)

$$\widetilde{\delta}_n^{\mathrm{bs}} = (1 - A)^{-1} \cdot B(\widetilde{\delta}_{< n}^{\mathrm{bs}}),$$

where $B(\widetilde{\delta}_{< n}^{bs})$ is the sum over oriented planar trees with one root, n leaves, internal vertexes of valencies $3 \leq v \leq (n-1)$, and decorated by σ on root, π 's on leaves, γ 's on edges, and $\widetilde{\delta}_{\nu}^{\rm bs}$'s on vertexes.

4.2. Computation of $\widetilde{\delta}_3^{\text{bs}}$. For n = 3 there are totally 2 trees in the sum. Both grow from the root corolla $\widetilde{\delta}_2^{\mathrm{bs}} \circ \sigma$ (see fig. 4), which takes



$$\overline{01} \xrightarrow{\sigma} [0,1] - [1,0] \xrightarrow{\widetilde{\delta}_2^{\mathrm{bs}}} \frac{\mathrm{ad}_{[1,0]} \left(\underline{1} + \underline{01} \right) - \mathrm{ad}_{[0,1]} \left(\underline{0} + \underline{01} \right)}{2}.$$

Homotopy γ annihilates everything except for the only non-saturated flag <u>01</u>. This allows to forget about <u>0</u>, <u>1</u> and forces to apply π to all factors [0, 1], [1, 0] and to replace <u>01</u> by $\widetilde{\delta}_2^{\text{bs}} \circ \gamma(\underline{01})$, which takes

$$\underline{01} \xrightarrow{\gamma} \underbrace{[0,1] + [1,0]}{2} \xrightarrow{\widetilde{\delta}_2^{\text{bs}}} - \frac{\operatorname{ad}_{[0,1]}\left(\underline{0} + \underline{01}\right) + \operatorname{ad}_{[1,0]}\left(\underline{1} + \underline{01}\right)}{2}.$$

It is productive to think of $\widetilde{\delta}_3^{\rm bs}$ as composition of two 'propagators'

$$\overline{C} \xrightarrow{[\widetilde{\delta}_2^{\mathrm{bs}} \circ \sigma]} \overline{C} \otimes \underline{C} \oplus \underline{C} \otimes \overline{C} \xrightarrow{[\widetilde{\delta}_2^{\mathrm{bs}} \circ \gamma]} \overline{C}^{\otimes 3}$$

The first $\overline{C} \xrightarrow{[\widetilde{\delta}_2^{\mathrm{bs}} \circ \sigma]} \overline{C} \otimes \underline{C} \oplus \underline{C} \otimes \overline{C}$ is obtained from $\widetilde{\delta}_2^{\mathrm{bs}} \circ \sigma$ by removing from the result all occurrences of $\underline{0}, \underline{1} \in \ker \gamma$ and replacing all [0, 1], [1, 0] by $\pi([0, 1]) = \overline{01}/2, \pi([1, 0]) = -\overline{01}/2$. It takes

$$\overline{01} \xrightarrow{[\widetilde{\delta}_{2}^{\mathrm{bs}} \circ \sigma]} \stackrel{1}{\longrightarrow} \frac{1}{4} \left(\operatorname{ad}_{\overline{10}} \left(\underline{01} \right) - \operatorname{ad}_{\overline{01}} \left(\underline{01} \right) \right) = -\frac{1}{2} \operatorname{ad}_{\overline{01}} \left(\underline{01} \right) \,.$$

Then the second $\overline{C} \otimes \underline{C} \oplus \underline{C} \otimes \overline{C} \xrightarrow{[\widetilde{\delta}_2^{\text{bs}} \circ \gamma]} \xrightarrow{\overline{C}} \overline{C}^{\otimes 3}$ replaces each <u>01</u> by

$$(\pi \otimes \pi) \circ \widetilde{\delta}_{2}^{\mathrm{bs}} \circ \gamma (\underline{01}) = -\frac{1}{2} \pi \otimes \pi \left(\operatorname{ad}_{[0,1]} \left(\underline{0} + \underline{01} \right) + \operatorname{ad}_{[1,0]} \left(\underline{1} + \underline{01} \right) \right) = -\frac{1}{8} \operatorname{ad}_{\overline{01}} \left(\overline{0} - \overline{1} \right) .$$

Thus, the sum over trees sends $\frac{1}{16}\overline{01} \longrightarrow \operatorname{ad}_{\overline{01}}^2(\overline{0}-\overline{1})$, then $(1-A)^{-1}$ multiplies the result by its eigenvalue $(1-\frac{1}{4})^{-1} = \frac{4}{3}$, and we get finally

$$\widetilde{\delta}_{3}^{\mathrm{bs}}(\overline{01}) = \frac{1}{12} \cdot \mathrm{ad}_{\overline{01}}^{2} \left(\overline{0} - \overline{1}\right)$$

4.2.1. THEOREM. For all $n \ge 3$

$$\widetilde{\delta}_{n}^{bs}\left(\overline{01}\right) = \frac{B_{n-1}}{(n-1)!} \cdot \operatorname{ad}_{\overline{01}}^{n-1}\left(\overline{0}-\overline{1}\right) = \\ = \frac{B_{n-1}}{(n-1)!} \cdot \sum_{\beta=0}^{n-1} (-1)^{\beta} \binom{n-1}{\beta} \cdot \overline{01}^{\otimes(n-1-\beta)} \otimes (\overline{0}-\overline{1}) \otimes \overline{01}^{\otimes\beta} \quad (4.3)$$

where B_{n-1} is the Bernoulli number and $ad_a : b \mapsto a \otimes b - b \otimes a$ is the commutation operator in the tensor algebra.

4.2.2. Remark. Since $B_k = 0$ for all odd $k \ge 3$, all the components of even tensor degree do vanish except for

$$\widetilde{\delta}_2^{\mathrm{bs}}(\overline{01}) = B_1 \cdot \mathrm{ad}_{\overline{01}}(\overline{0} + \overline{1}) \;.$$

All the other components can be combined into one operator

$$\left(1 + \sum_{k \ge 2} \frac{B_k}{k!} \operatorname{ad}_{\overline{01}}^k\right) \circ \overline{\partial}$$

4.3. Proof of theorem n^{\circ} 4.2.1 uses induction over *n* **and consists of two steps:**

(1) The non-zero contribution to the sum over trees in the recursive formula for δ_n^{bs} comes only from one trunk trees with γ 's staying along the trunk. The contribution of such a tree can be written as composition of propagators

$$\underline{C} \longrightarrow \underline{C} \cdot \overline{C}^{\otimes k} \stackrel{\text{def}}{=} \bigoplus_{\mu + \nu = k - 1} \overline{C}^{\otimes \mu} \otimes \underline{C} \otimes \overline{C}^{\otimes \nu}$$

completely similar to ones used in the computation of $\widetilde{\delta}_3^{\text{bs}}$ in n° 4.2. In n° 4.3.1 we use the inductive assumptions on $\widetilde{\delta}_{< n}^{\text{bs}}$ to compute the contribution of all these propagators in terms of Bernoulli numbers.

(2) In n° 4.3.2 we show that the precise formula for propagators obtained at the first step reproduces the required value for δ_n^{bs} after summation over trees. The key argument here is the following recursive formula for Bernoulli numbers

$$\frac{B_m}{m!} = \frac{1}{16} \cdot \left(1 - \frac{1}{2^m}\right)^{-1} \sum_{\substack{m-2=\\k_1+\dots+k_i}} \left(\frac{-B_{k_1}}{2^{k_1}k_1!}\right) \cdot \dots \cdot \left(\frac{-B_{k_i}}{2^{k_i}k_i!}\right)$$
(4.4)

where summation runs over all compositions of (m-2) into a sum of numbered positive even integers. We verify (4.4) in the last n° 4.3.3.

4.3.1. Contribution of propagators. Contribution of each tree is a composition of propagators analogous to ones used in n° 4.2. The first applied to $\overline{01}$ is the root propagator

$$\overline{C} \xrightarrow{[\widetilde{\delta}_r^{\mathrm{bs}} \circ \sigma]} \bullet \underline{C} \cdot \overline{C}^{\otimes (r-1)}$$

It takes firstly

$$\overline{01} \xrightarrow{\widetilde{\delta}_r^{\mathrm{bs}} \circ \sigma} \xrightarrow{B_{r-1}} \left(\operatorname{ad}_{[0,1]}^{r-1}(\underline{0} - \underline{01}) - \operatorname{ad}_{[1,0]}^{r-1}(\underline{1} - \underline{01}) \right) ,$$

then applies π to all tensor factors [0, 1], [1, 0] and reduces the remaining factor modulo ker γ . Since for odd r this gives

$$\operatorname{ad}_{\overline{01}}^{r-1}(\underline{0}-\underline{1}) \equiv 0 \pmod{\ker \gamma},$$

the root propagator necessary has even tensor degree, which is forced to be equal 2, because of inductive assumption and vanishing of the Bernoulli numbers B_{2k+1} for $k \ge 1$. Thus, the root propagator takes

$$\overline{01} \xleftarrow{[\widetilde{\delta}_2^{\mathrm{bs}} \circ \sigma]} - \frac{\mathrm{ad}_{\overline{01}}(\underline{01})}{2} \pmod{\ker \gamma}.$$

The root propagator is followed by the trunk propagators

$$\underline{C} \xrightarrow{[\widetilde{\delta}_k^{\mathrm{bs}} \circ \gamma]} \underline{C} \cdot \overline{C}^{\otimes (k-1)}.$$

Each of them takes firstly

$$\underline{01} \xrightarrow{\widetilde{\delta}_k^{\mathrm{bs}} \circ \gamma} \underbrace{B_{k-1}}_{(k-1)!} \cdot \frac{\mathrm{ad}_{[0,1]}^{k-1}(\underline{0} - \underline{01}) + \mathrm{ad}_{[1,0]}^{k-1}(\underline{1} - \underline{01})}{2}$$

and then applies π to all [0, 1]'s and [1, 0]'s. This gives

$$\frac{B_{k-1}}{2^k(k-1)!} \cdot \begin{vmatrix} \operatorname{ad}_{\overline{01}}^{k-1} \left(\underline{0} + \underline{1} - 2 \cdot \underline{01} \right) & \text{(for odd } k) \\ \operatorname{ad}_{\overline{01}}^{k-1} \left(\underline{0} - \underline{1} \right) & \text{(for even } k) \end{vmatrix}$$

Since $\underline{0} - \underline{1} \in \ker \gamma$, each trunk propagator except for the last one has odd tensor degree k and sends

$$\underline{01} \xrightarrow{[\delta_k^{\mathrm{bs}} \circ \gamma]} \underline{-B_{k-1}}_{2^{k-1}(k-1)!} \cdot \mathrm{ad}_{\overline{01}}^{k-1}(\underline{01}) .$$

Since $\underline{0} + \underline{1} - 2 \cdot \underline{01} \in \ker \pi$, the last trunk propagator has even tensor degree, that is 2, and takes

$$\underline{01} \stackrel{(\pi \otimes \pi) \circ \widetilde{\delta}_2^{\mathrm{bs}} \circ \gamma}{\longrightarrow} -\frac{1}{8} \cdot \operatorname{ad}_{\overline{01}} \left(\overline{0} - \overline{1} \right)$$

4.3.2. Inductive step. It follows from the above computations that for even $n \ge 4$ the sum over threes vanishes and $\delta_n^{\text{bs}} = 0$. For odd *n* the sum over threes equals

$$\frac{1}{16} \sum_{\substack{n-3=\\k_1+\dots+k_i}} \left(\frac{-B_{k_1}}{2^{k_1}k_1!}\right) \cdot \dots \cdot \left(\frac{-B_{k_i}}{2^{k_i}k_i!}\right) \cdot \operatorname{ad}_{\overline{01}}^{n-1}(\overline{0}-\overline{1})$$

where the sum runs over all distributions of n-3 valences between interior (neither the root nor the last) trunk propagators. Since the eigenvalue of A on this eigenvector is 2^{1-n} , it follows from recursion (4.4) that

$$\delta_{n}^{\mathrm{bs}}\left(\overline{01}\right) = \frac{1}{16} \left(1 - \frac{1}{2^{n-1}}\right)^{-1} \sum_{\substack{n-3=\\k_{1}+\dots+k_{i}}} (-1)^{i} \prod_{\nu=1}^{i} \frac{B_{k_{\nu}}}{2^{k_{\nu}}k_{\nu}!} \cdot \mathrm{ad}_{\overline{01}}^{n-1}(\overline{0} - \overline{1}) = \frac{B_{n-1}}{(n-1)!} \cdot \mathrm{ad}_{\overline{01}}^{n-1}(\overline{0} - \overline{1})$$

To complete the proof it remains to verify the recursion (4.4) for Bernoulli numbers.

4.3.3. Prof of recursion (4.4). The Bernoulli numbers B_i with $i \ge 3$ can be defined by means of 'cotangensum'

$$(t/2) \cdot \operatorname{cth}(t/2) = 1 + \sum_{k \ge 3} (B_k/k!) \cdot t^k.$$

Obvious relation $\operatorname{cth}(t) = \frac{1}{2} \left(\operatorname{cth}(t/2) + \operatorname{th}(t/2) \right)$ implies the identity

$$t \cdot \operatorname{cth}(t) - (t/2) \cdot \operatorname{cth}(t/2) = (t^2/4) \cdot ((t/2) \cdot \operatorname{cth}(t/2))^{-1}$$

Expanding $(1 + \sum (B_k/k!)t^k)^{-1}$ as the geometric progression and comparing coefficients at t^m , we get recursive formula

$$(2^m - 1) \cdot \frac{B_m}{m!} = \frac{1}{4} \sum_{\substack{m-2=\\k_1 + \dots + k_i}} (-1)^i \prod_{\nu=1}^i \frac{B_{k_\nu}}{k_\nu!} \,,$$

It remains to multiply both sides by

$$(2^m - 1)^{-1} = \left(1 - \frac{1}{2^m}\right)^{-1} \cdot \frac{1}{2^{k_1}} \cdot \dots \cdot \frac{1}{2^{k_i}} \cdot \frac{1}{4}$$

This completes the proof of theorem $n^{\circ} 4.2.1$.

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