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Vienna, Preprint ESI 2260 (2010)

August 3, 2010

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available online at http://www.esi.ac.at

ERGODICITY OF $\mathbb{Z}^2$ EXTENSIONS OF IRRATIONAL ROTATIONS

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ABSTRACT. Let $\mathbf{T} = [0,1)$ be the additive group of real numbers modulo 1, $\alpha \in \mathbf{T}$ be an irrational number and $t \in \mathbf{T}$. We consider skew product extensions of irrational rotations by $\mathbb{Z}^2$ determined by $T\colon \mathbf{T} \times \mathbb{Z}^2 \to \mathbf{T} \times \mathbb{Z}^2$ $T(x, s_1, s_2) = \left(x + \alpha, \quad s_1 + 2\chi_{[0,\frac{1}{2})}(x) - 1, \quad s_2 + 2\chi_{[0,\frac{1}{2})}(x+t) - 1\right)$. We study ergodic components of such extensions and use the results to display irregularities in the uniform distribution of the sequence $\mathbb{Z}\alpha$.

1. INTRODUCTION

The study of irrational rotations of the circle leads to various questions in number theory and ergodic theory. Let $\mathbf{T} = [0,1)$ be the additive group of real numbers modulo 1. Fix an irrational $\alpha \in \mathbf{T}$ and let $t \in \mathbf{T}$ satisfy the condition that neither t nor $t + \frac{1}{2}$ is a multiple of $\alpha \mod 1$. Define a map $f\colon \mathbf{T} \to \mathbb{Z}$ by

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 \le x < \frac{1}{2}; \\ -1 & \text{for} \quad \frac{1}{2} \le x < 1 \end{cases} \tag{1.1}$$

and an irrational rotation T_0 of $\mathbf{T}$ by

$$T_0 x = x + \alpha \mod 1. \tag{1.2}$$

Set $\mathbf{X} = \mathbf{T} \times \mathbb{Z}^2$ and define $T\colon \mathbf{X} \to \mathbf{X}$ by

$$T(x, s_1, s_2) = (x + \alpha, \quad s_1 + f(x), \quad s_2 + f(x+t)). \tag{1.3}$$

T is a skew product extension of irrational rotations on the circle by $\mathbb{Z}^2$ determined by $f(x)$ and t. We study ergodicity of T on $\mathbf{X}$ relative to Haar measure, continuing a theme started by [5], [6] of Schmidt and by [7] of Veech. It is known that such property of skew product extensions of irrational rotations arises from irregularity of distribution of $\mathbb{Z}\alpha$. As for the case of cylinder flows, Oren in [4] gave complete solution to the problem of ergodicity of the map $F\colon \mathbf{T} \times E \to \mathbf{T} \times E$ defined by $F(x, s) = (x + \alpha, s + \mathbf{1}_{[0,\beta)}(x) - \beta)$, where $\beta \in \mathbf{T}$ and E is the closed subgroup of $\mathbb{R}$ generated by 1 and β. Earlier, special cases were done by Schmidt for $\beta = \frac{1}{2}$, $\alpha = \frac{\sqrt{5}-1}{4}$ in [6] and for $\beta = \frac{1}{2}$, α irrational in [5]. Although ergodicity of cylinder flows has been understood thoroughly, due to the fact that $f(x)$ and $f(x+t)$ take on independent values, the situation of $\mathbb{Z}^2$ extensions of irrational rotations appear to be more complicated.

Note that by definition (1.3), we have

$$T^n(x, s_1, s_2) = (x + n\alpha, \quad s_1 + a_n(x), \quad s_2 + a_n(x+t)), \quad \forall n \in \mathbb{Z}, \tag{1.4}$$

2010 *Mathematics Subject Classification.* Primary 37A25; Secondary 11J70.

Key words and phrases. Cocycle, Ergodicity, Irrational rotation.

where

$$a_n(x) = \begin{cases} \sum_{i=0}^{n-1} f(x+i\alpha) = 2\sum_{i=0}^{n-1} \chi_{[0,\frac{1}{2})}(x+i\alpha) - n, & \forall n \geq 1; \\ 0, \quad \text{for} \quad n=0; \\ -a_{-n}(T_0^{-n}x), & \forall n \leq -1. \end{cases} \tag{1.5}$$

$t \in \mathbb{Z}\alpha$ and $t \in \mathbb{Z}\alpha + \frac{1}{2}$ are excluded *a priori*. To see this, note that for nonnegative integer m, $|a_n(x+m\alpha) - a_n(x)|$ is bounded by $2m$ because

(1.6)

$$|a_n(x+m\alpha) - a_n(x)| = \left| \sum_{i=0}^{m-1} f(x+n\alpha+i\alpha) - \sum_{i=0}^{m-1} f(x+i\alpha) \right|$$

$$\leq \sum_{i=0}^{m-1} |f(x+n\alpha+i\alpha)| + \sum_{i=0}^{m-1} |f(x+i\alpha)| \leq 2m, \quad \forall n > m. \tag{1.7}$$

We also have from (1.1) $f(x+\frac{1}{2}) = -f(x)$ and therefore

$$a_n(x+\frac{1}{2}) = -a_n(x), \quad \forall x \in \mathbf{T}, \quad \forall n. \tag{1.8}$$

$|a_n(x+\frac{1}{2}+m\alpha) + a_n(x)|$ is bounded from above by $2m$ thereof.

Also note that $a_n(x+t) \equiv a_n(x) \mod 2$. The parity $a_n(x)$ is always the same as that of n from (1.5). Hence T cannot be ergodic on the entire space $\mathbf{X}$. We set $G = \{(s_1, s_2) \in \mathbb{Z}^2 \mid s_1 \equiv s_2 \mod 2\}$. G is cocompact in $\mathbb{Z}^2$.

$a_n(x)$ satisfies the additive cocycle equation

$$a_n(T_0^m x) - a_{n+m}(x) + a_m(x) = 0, \quad \forall m, n \in \mathbb{Z}, \quad \forall x \in \mathbf{T}. \tag{1.9}$$

Following [5, Definition 2.1] we have

Definition 1.1. *$(a,t) : \mathbb{Z} \times \mathbf{T} \to \mathbb{Z}^2$ defined by*

$$(a,t)(n,x) = (a_n(x), \quad a_n(x+t)) \tag{1.10}$$

is called a cocycle for T_0.

[5] showed that ergodicity of T, or equivalently, ergodicity of the cocycle (a,t) is determined by the group $\mathbb{E}^2(a,t)$ of essential values of (a,t). Put $\overline{\mathbb{Z}^2} = \mathbb{Z}^2 \bigcup \{\infty\}$, the one point compactification of $\mathbb{Z}^2$. We have the following definitions of essential values etc.

Definition 1.2. *Let μ be Lebesgue measure on $\mathbf{T}$. An element $(k_1, k_2) \in \overline{\mathbb{Z}^2}$ is called an essential value of (a,t) if for every measurable set $A \subset \mathbf{T}$ with $\mu(A) > 0$, we have*

$$\mu\left(\bigcup_{n\in\mathbb{Z}} \left(A \bigcap T_0^{-n} A \bigcap \{x \mid a_n(x) = k_1\} \bigcap \{x \mid a_n(x+t) = k_2\} \right) \right) > 0, \tag{1.11}$$

We denote the set of essential values of (a,t) by $\overline{\mathbb{E}^2}(a,t)$.

Definition 1.3. *Set $\mathbb{E}^2(a,t) = \overline{\mathbb{E}^2}(a,t) \bigcap \mathbb{Z}^2$. $(k_1,k_2) \in \overline{\mathbb{E}^2}(a,t) \setminus \mathbb{E}^2(a,t)$ only if (k_1,k_2) does not lie in any compact subset of $\mathbb{Z}^2$.*

From [5] we derive the following properties

(1) $\mathbb{E}^2(a,t)$ is a closed subgroup of $\mathbb{Z}^2$ under addition. $(k_1,k_2)\in\mathbb{E}^2(a,t)$ only if $k_1\equiv k_2 \mod 2$.
(2) (a,t) is a coboundary (that is, $a_n(x)=c(T_0^n x)-c(x)$ for a measurable map $c\colon \mathbf{T}\to\mathbb{Z}$) iff $\overline{\mathbb{E}^2}(a,t)=\{(0,0)\}$.

We say that two cocycles $(a,t),(b,t):\mathbb{Z}\times\mathbf{T}\to\mathbb{Z}^2$ are cohomologous if $(a,t)-(b,t)$ is a coboundary. In this case $\overline{\mathbb{E}^2}(a,t)=\overline{\mathbb{E}^2}(b,t)$. Given a cocycle $(a,t):\mathbb{Z}\times\mathbf{T}\to\mathbb{Z}^2$, let $(a,t)^*:\mathbb{Z}\times\mathbf{T}\to\mathbb{Z}^2/\mathbb{E}^2(a,t)$ be the corresponding quotient cocycle. We have the following important result from [5, Lemma 3.10]:

Lemma 1.4. $\mathbb{E}^2(a,t)^*=\{(0,0)\}$.

We say that the cocycle (a,t) is regular if $\overline{\mathbb{E}^2}(a,t)^*=\{(0,0)\}$. (a,t) is called nonregular if $\overline{\mathbb{E}^2}(a,t)^*=\{(0,0),\infty\}$. If (a,t) is regular, then (a,t) is cohomologous to a cocycle $(b,t):\mathbb{Z}\times\mathbf{T}\to\mathbb{E}^2(a,t)$ and the latter is ergodic as a cocycle with values in the closed subgroup $\mathbb{E}^2(a,t)$ (see [5]). In particular, if $\mathbb{E}^2(a,t)$ is cocompact in $\mathbb{Z}^2$ then (a,t) is regular.

We utilize approach devised in [5], [4] to prove the following theorems:

Theorem 1.5. *For arbitrary irrational $\alpha\in\mathbf{T}$, the group of essential values $\mathbb{E}^2(a,t)$ of the cocycle (a,t) defined in (1.10) is $G=\{(s_1,s_2)\in\mathbb{Z}^2\mid s_1\equiv s_2 \mod 2\}$ for almost all $t\in\mathbf{T}$. In particular, (a,t) is regular for almost all $t\in\mathbf{T}$.*

Theorem 1.6. *If α is badly approximable, then the group of essential values $\mathbb{E}^2(a,t)$ is G if and only if $t\notin\mathbb{Z}\alpha$ and $t\notin\mathbb{Z}\alpha+\frac{1}{2}$.*

2. Period approximating sequences, Partial convergents and other preliminaries

For $x\in\mathbb{R}$ we denote the closest integer to x by $[x]$, denote $x-[x]$ by $\langle x\rangle$ and denote $|x-[x]|$ by $\|x\|$. We assume n to be nonnegative.

According to (1.5) $a_n(x)$ is locally constant except for points of discontinuities of $+2$ at $0,-\alpha,-2\alpha,\dots,-(n-1)\alpha$ and points of discontinuities of -2 at $\frac{1}{2},\frac{1}{2}-\alpha,\frac{1}{2}-2\alpha,\dots,\frac{1}{2}-(n-1)\alpha$. $a_n(x+t)$ is locally constant except for points of discontinuities of $+2$ at $-t,-t-\alpha,-t-2\alpha,\dots,-t-(n-1)\alpha$ and points of discontinuities of -2 at $\frac{1}{2}-t,\frac{1}{2}-t-\alpha,\dots,\frac{1}{2}-t-(n-1)\alpha$.

If we set

$$S_n(x)=\sum_{i=0}^{n-1}\chi_{[0,\frac{1}{2})}(x+i\alpha)=\#\left\{i\mid 0\le i\le n-1;\quad x+i\alpha\in[0,\frac{1}{2})\right\},\tag{2.1}$$

then from (1.5)

$$a_n(x)=2S_n(x)-n.\tag{2.2}$$

The concept of essential values corresponds to that of periods in [4]. We have the following definition:

Definition 2.1. *A period approximating sequence is a sequence $\{(n_l,A_l)\}_{l=1}^{\infty}$ where*
(1) $A_l\subset\mathbf{T}$, each A_l is measurable;
(2) a_{n_l} is constant on both A_l and A_l+t, that is, $a_{n_l}(A_l)=k_1, a_{n_l}(A_l+t)=k_2\quad\forall n_l$;
(3) $\inf_l\mu(A_l)>0$;
(4) $\|n_l\alpha\|\to 0$.

The next lemma shows that a period approximating sequence defines an element in $\mathbb{E}^2(a,t)$.

Lemma 2.2. *If there exists a period approximating sequence* $\{(n_l, A_l)\}_{l=1}^{\infty}$ *such that* $a_{n_l}(A_l) = k_1$, $a_{n_l}(A_l + t) = k_2$, $\forall n_l$, *then* $(k_1, k_2) \in \mathbb{E}^2(a,t)$.

Proof. Set

$$B = \limsup_{l\to\infty} A_l = \bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty} A_i.$$

$\mu(B) > 0$ because $\inf_l \mu(A_l) > 0$ and $\mu(\mathbf{T}) = 1$.

For arbitrary $A \subset \mathbf{T}$ with $\mu(A) > 0$, there exists $m \in \mathbb{Z}$ and $A' \subset A$ such that $\mu(A') > 0$ and $T_0^m A' \subset B$ because the action T_0 is ergodic. Hence

$$\mu\left(B \bigcap T_0^m A'\right) = \mu\left(\bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty}(A_i \bigcap T_0^m A')\right) = \mu\left(T_0^m A'\right) > 0, \tag{2.3}$$

hence there exists a subsequence $\{n_l'\}$ of $\{n_l\}$ such that for each n_l', there exists a measurable set $A'_{n_l'} \subset A'$ with $\mu(A'_{n_l'}) > 0$ and

$$a_{n_l'}(T_0^m x) = k_1, \quad a_{n_l'}(T_0^m x + t) = k_2, \quad \forall x \in A'_{n_l'}. \tag{2.4}$$

Note that

$$\begin{aligned}\left|a_{n_l'}(T_0^m x) - a_{n_l'}(x)\right| &= \left|\sum_{i=0}^{n_l'-1} f(x + i\alpha + m\alpha) - \sum_{i=0}^{n_l'-1} f(x + i\alpha)\right| \\ &= \left|\sum_{i=0}^{m-1} f(x + i\alpha + n_l'\alpha) - \sum_{i=0}^{m-1} f(x + i\alpha)\right|,\end{aligned} \tag{2.5}$$

$$\left|a_{n_l'}(T_0^m x + t) - a_{n_l'}(x + t)\right| = \left|\sum_{i=0}^{m-1} f(x + i\alpha + n_l'\alpha + t) - \sum_{i=0}^{m-1} f(x + i\alpha + t)\right|, \tag{2.6}$$

$$\lim \|n_l'\alpha\| = 0, \tag{2.7}$$

as well as the fact that m is fixed and depends on A only, we deduce that there exists some n_l' and $A'' \subset A' \subset A$ with $\mu(A'') > 0$ such that

$$a_{n_l'}(T_0^m x) = a_{n_l'}(x) = k_1, \quad a_{n_l'}(T_0^m x + t) = a_{n_l'}(x + t) = k_2, \quad \forall x \in A''. \tag{2.8}$$

Hence we have

$$\mu\left(A \bigcap T_0^{-n_l'} A \bigcap \left\{x \mid a_{n_l'}(x) = k_1\right\} \bigcap \left\{x \mid a_{n_l'}(x + t) = k_2\right\}\right) > 0. \tag{2.9}$$

$(k_1, k_2) \in \mathbb{E}^2(a,t)$. □

We record the statement of the Denjoy-Koksma inequality [4, Lemma 2] here, which plays a fundamental role in the proof.

Lemma 2.3 (Denjoy-Koksma). *If* $p \in \mathbb{N}, q \in \mathbb{N}$ *satisfy*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2} \quad \text{and} \quad (p,q) = 1,$$

then $|a_q(x)| < 4$, $\forall x \in \mathbf{T}$, *where* $a_q(x)$ *is defined in* (1.5).

It follows from the proof of the above lemma that every interval of the form $\left[\frac{i}{q}, \frac{i+1}{q}\right)$ contains exactly one of the points $j\alpha$ for $0 \le i, j \le q-1$. In other words, the points $j\alpha$ $(0 \le j \le q-1)$ are uniformly distributed on the unit circle.

We rely on numerous facts concerning continued fractions stated in texts such as [1]. A considerable portion of our approach is borrowed from [4]. However, here we need to construct period approximating sequence $\{(n_l, A_l)\}_{l=1}^{\infty}$ such that $a_{n_l}(A_l)$ and $a_{n_l}(A_l + t)$ take on *independent* values whereas predecessors of this paper only deal with cylinder flows.

We denote by $[a_0; a_1, a_2, \dots,]$ the continued fraction of α and call the a_i the partial quotients of α. Denote by $\frac{p_k}{q_k}$ the kth partial convergent of α where $k \ge 0$. It is known from [1] that

$$\frac{p_k}{q_k} = [a_0; a_1, a_2, \dots, a_k]; \tag{2.10}$$

$$\|q_k\alpha\| < \frac{1}{q_{k+1}} < \frac{1}{q_k}; \tag{2.11}$$

$$\min_{q_k \le q < q_{k+1}} \|q\alpha\| = \|q_k\alpha\| > \frac{1}{q_k + q_{k+1}} > \frac{1}{2q_{k+1}}. \tag{2.12}$$

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k. \tag{2.13}$$

Set

$$D(\alpha) = \left\{ q_k \mid \frac{p_k}{q_k} \text{ is a partial convergent of } \alpha \right\}; \tag{2.14}$$

$$q^+ = \min\{q' \in D(\alpha) \mid q' > q\}, \quad \forall q \in D(\alpha). \tag{2.15}$$

Adopting arguments on [5, Page 229-230] we are able to prove the following lemma which constitutes the first step in the entire proof:

Lemma 2.4.

$$\mathbb{E}^2(a,t) \bigcap \{(1,3),(1,-3),(1,1),(1,-1),(3,1),(3,-1),(3,3),(3,-3)\} \neq \emptyset. \tag{2.16}$$

Proof. From (2.13) we derive that there are infinitely many odd $q \in D(\alpha)$. For such $q \in D(\alpha)$, the Denjoy-Koksma inequality applies. In addition, from (1.5) we see that $a_q(x)$ can only be odd, that is, $a_q(x)$ can only be ± 3 or ± 1.

Consequently there exists a period approximating sequence $\{(q_l, A_l)\}_{l=1}^{\infty}$ such that $q_l \in D(\alpha)$,

(1) $A_l \subset \mathbf{T}$;
(2) a_{q_l} is constant on both A_l and $A_l + t$, $a_{q_l}(A_l) = k_1, a_{q_l}(A_l + t) = k_2 \quad \forall n_l$;
(3) $\inf_l \mu(A_l) > 0$;
(4) $\|q_l\alpha\| \to 0$

and $(k_1, k_2) \in \{(1,3),(1,-3),(1,1),(1,-1),(3,1),(3,-1),(3,3),(3,-3)\}$
$\bigcup\{-(1,3),-(1,-3),-(1,1),-(1,-1),-(3,1),-(3,-1),-(3,3),-(3,-3)\}$. The proof is complete by noting that $\mathbb{E}^2(a,t)$ is a group under addition. $\square$

A major difficulty to prove Theorem 1.5 is therefore to show that $\mathbb{E}^2(a,t)$ is not isomorphic to $\mathbb{Z}$. We aim to show that $\mathbb{E}^2(a,t)$ is G for almost all t. This is done by using period approximating sequences. We derive from properties of continued fractions the following lemma:

Lemma 2.5. *For any nonzero $q \in D(\alpha)$, we have*

$$\min\left\{\left\|\tfrac{1}{2} - j\alpha\right\| \mid |j| < q\right\} \geq \frac{1}{24q}. \tag{2.17}$$

Proof. We always have

$$\left\|\frac{1}{2} - j\alpha\right\| \geq \frac{\|2(\frac{1}{2} - j\alpha)\|}{2} = \frac{\|2j\alpha\|}{2}. \tag{2.18}$$

We consider five cases separately under the assumption that $0 < |j| < q$.

Case 1: $q^+ \geq 3q$, then since $||2j| - q| < q$ from $0 < |j| < q$, we have $\|(|2j| - q)\alpha\| > \frac{1}{2q}$ from (2.12) and

$$\|2j\alpha\| = \|(|2j| - q)\alpha + q\alpha\| \geq \|(|2j| - q)\alpha\| - \|q\alpha\| > \frac{1}{2q} - \frac{1}{q^+} > \frac{1}{2q} - \frac{1}{3q} = \frac{1}{6q}. \tag{2.19}$$

Here we also used the inequality $\|q\alpha\| < \dfrac{1}{q^+}$ from (2.11).

Case 2: If $q^+ < 3q$ and $q^{++} < 3q$, then since $|2j| < 2q \leq q^{++}$, we have from (2.12)

$$\|2j\alpha\| \geq \|q^+\alpha\| \geq \frac{1}{2q^{++}} > \frac{1}{6q}. \tag{2.20}$$

Case 3: If $q^+ < 3q$, $q^{++} \geq 3q$ and $|q^+ - |2j|| < q$, then we have $\|(|2j| - q^+)\alpha\| > \frac{1}{2q}$ from (2.12) and

$$\|2j\alpha\| = \|(|2j| - q^+)\alpha + q^+\alpha\| \geq \|(|2j| - q^+)\alpha\| - \|q^+\alpha\| > \frac{1}{2q} - \frac{1}{q^{++}} > \frac{1}{2q} - \frac{1}{3q} = \frac{1}{6q}. \tag{2.21}$$

Case 4: If $q^+ < 3q$, $q^{++} \geq 3q$, $|q^+ - |2j|| \geq q$ and $|2j| \leq q$, then from (2.12) we get

$$\|2j\alpha\| \geq \|q\alpha\| > \frac{1}{2q^+} \geq \frac{1}{6q}. \tag{2.22}$$

Case 5: If $q^+ < 3q$, $q^{++} \geq 3q$, $|q^+ - |2j|| \geq q$ and $|2j| > q$, then

$$q^+ - |4j| < 3q - 2q = q, \quad 2q - q^+ > 2q - 3q = -q; \tag{2.23}$$

$$|2j| \leq q^+ - q \to q^+ - |4j| \geq q^+ - 2(q^+ - q) = 2q - q^+ > -q; \tag{2.24}$$

hence $|q^+ - |4j|| < q$ and from (2.12)

$$\|4j\alpha\| = \|(q^+ - |4j|)\alpha - q^+\alpha\| \geq \|(q^+ - |4j|)\alpha\| - \|q^+\alpha\| > \frac{1}{2q} - \frac{1}{q^{++}} \geq \frac{1}{6q}; \tag{2.25}$$

and $\|2j\alpha\| \geq \dfrac{\|4j\alpha\|}{2}$. The inequality is established. □

3. Proof of main theorems

Following [4] we set for each $q \in D(\alpha)$

$$\begin{aligned} \epsilon(q) &= q \cdot \min\left\{\|-t - j\alpha\| \mid |j| < q\right\}; \\ \theta(q) &= q \cdot \min\left\{\left\|\tfrac{1}{2} - t - j\alpha\right\| \mid |j| < q\right\}. \end{aligned} \tag{3.1}$$

We immediately derive that $\epsilon(q) < 1$ and $\theta(q) < 1$ from the proof of the Denjoy-Koksma inequality.

Proposition 3.1. *If*

$$\limsup_{\substack{q\in D(\alpha)\\ q\to\infty}} \min\{\epsilon(q),\theta(q)\}>0, \tag{3.2}$$

then $\mathbb{E}^2(a,t)=\left\{(k_1,k_2)\in\mathbb{Z}^2 \mid k_1\equiv k_2 \mod 2\right\}=G$.

Proof. Let $\{q_n\}_{n=1}^{\infty}\subset D(\alpha)$ be such that $\min\{\epsilon(q_n),\theta(q_n)\}>\delta>0$, $\forall n$.

Recall $a_{q_n}(x)$ as set in (1.5) is locally constant except for points of discontinuities of $+2$ at $0,-\alpha,-2\alpha,\dots,-(q_n-1)\alpha$ and points of discontinuities of -2 at $\frac{1}{2},\frac{1}{2}-\alpha,\frac{1}{2}-2\alpha,\dots,\frac{1}{2}-(q_n-1)\alpha$. $a_{q_n}(x+t)$ is locally constant except for points of discontinuities of $+2$ at $-t,-t-\alpha,-t-2\alpha,\dots,-t-(q_n-1)\alpha$ and points of discontinuities of -2 at $\frac{1}{2}-t,\frac{1}{2}-t-\alpha,\dots,\frac{1}{2}-t-(q_n-1)\alpha$.

For fixed n, let $I_1,I_2,\dots,I_{4q_n}$ denote the intervals of constancy of both $a_{q_n}(x)$ and $a_{q_n}(x+t)$ in cyclic order. Since $a_{q_n}(\cdot)$ takes on at most four values by Lemma 2.3, there exists a union of intervals, A_n, such that $a_{q_n}(x)$ and $a_{q_n}(x+t)$ are constant on A_n and $\mu(A_n)\geq\frac{1}{16}$. Let A_n' be the union of intervals proximal on the right to those of A_n. Note that the distance between any discontinuities of $a_{q_n}(x)$ and $a_{q_n}(x+t)$ is given by $\|(i-j)\alpha\|$ or $\|\frac{1}{2}+(i-j)\alpha\|$ or $\|-t+(i-j)\alpha\|$ or $\|\frac{1}{2}-t+(i-j)\alpha\|$ for $0\leq i,j\leq q_n-1$. From (2.12), Lemma 2.5 and (3.2), we have that $\min\left\{\frac{1}{24q_n},\frac{\epsilon(q_n)}{q_n},\frac{\theta(q_n)}{q_n}\right\}$ is a lower bound for the lengths $|I_i|$, $i=1,2,\dots,4q_n$. Since every interval of length $\frac{2}{q_n}$ must contain a $+2$ discontinuity by discussion following Lemma 2.3, we have $|I_i|<\frac{2}{q_n}$. Therefore we have

$$\frac{|I_i|}{|I_j|}>\frac{1}{2}\min\left\{\frac{1}{24},\epsilon(q_n),\theta(q_n)\right\},\quad 1\leq i,j\leq 4q_n. \tag{3.3}$$

By setting $\epsilon=\min\left\{\frac{1}{24},\delta\right\}$, we thus have $\mu(A_n')\geq\frac{1}{2}\epsilon\mu(A_n)\geq\frac{1}{32}\epsilon$, $\forall n$. $(a,t)(q_n,x)=(a_{q_n}(x),a_{q_n}(x+t))$ can take on A_n' only the values $(a_{q_n}(A_n)\pm 2,a_{q_n}(A_n+t))$ or $(a_{q_n}(A_n),a_{q_n}(A_n+t)\pm 2)$ since each interval of A_n' is proximal on the right to one of A_n. We can find $A_n''\subset A_n'$ such that $a_{q_n}(x)$ and $a_{q_n}(x+t)$ are both constant on A_n'', $\mu(A_n'')\geq\frac{1}{128}\epsilon$ and

$$(a_{q_n}(A_n''),a_{q_n}(A_n''+t))=(a_{q_n}(A_n)\pm 2,a_{q_n}(A_n+t)) \text{ or } (a_{q_n}(A_n),a_{q_n}(A_n+t)\pm 2)\,. \tag{3.4}$$

We assume that $a_{q_n}(A_n)=1$ and $a_{q_n}(A_n+t)=3$, that is $(1,3)$ lies in $\mathbb{E}^2(a,t)$. We prove both $(2,0)$ and $(0,2)$ lie in $\mathbb{E}^2(a,t)$. Other possibilities can be treated analogously.

Case 1:

Suppose we have $(3,3)$ and $(1,3)$ both lie in $\mathbb{E}^2(a,t)$ as a result of the above arguments. $(\pm 2,0)$ lies in $\mathbb{E}^2(a,t)$ because $\mathbb{E}^2(a,t)$ is a subgroup of $\mathbb{Z}^2$.

Moreover, there exists a period approximating sequence $\{(q_n,A_n)\}_{n=1}^{\infty}$ which defines $(1,3)\in\mathbb{E}^2(a,t)$. Namely we have

(1) $A_n\subset\mathbf{T}$;
(2) a_{q_n} is constant on both A_n and A_n+t, $a_{q_n}(A_n)=1, a_{q_n}(A_n+t)=3,\quad\forall n$;
(3) $\inf_n\mu(A_n)>0$;
(4) $\|q_n\alpha\|\to 0$.

Therefore there exists a period approximating sequence $\{(q_n',B_n')\}_{n=1}^{\infty}$ which defines $(k,1)\in\mathbb{E}^2(a,t)$ for some $k\in\{\pm1,\pm3\}$. Namely we have

(1) $\{q_n'\}$ is a subsequence of $\{q_n\}$, $B_n'+t\subset A_n'$, $\mu(B_n')\geq\frac{1}{4}\mu(A_n')$;

(2) $a_{q'_n}$ is constant on both B'_n and B'_n+t, $a_{q'_n}(B'_n)=k$,
$a_{q'_n}(B'_n+t)=a_{q'_n}(A'_n)=1, \quad \forall n'$;
(3) $\inf_{n'}\mu(B'_n)>0$;
(4) $\|q'_n\alpha\|\to 0$.

$$(3,3)\in\mathbb{E}^2(a,t) \text{ and } (2,0)\in\mathbb{E}^2(a,t)\to(k,3)\in\mathbb{E}^2(a,t); \tag{3.5}$$

$$(k,1)\in\mathbb{E}^2(a,t) \text{ and } (k,3)\in\mathbb{E}^2(a,t)\to(0,2)\in\mathbb{E}^2(a,t). \tag{3.6}$$

Consequently both $(2,0)$ and $(0,2)$ lie in $\mathbb{E}^2(a,t)$.

Case 2:

Suppose we have $(-1,3)$ and $(1,3)$ both lie in $\mathbb{E}^2(a,t)$. $(\pm 2,0)$ lies in $\mathbb{E}^2(a,t)$ because $\mathbb{E}^2(a,t)$ is a subgroup of $\mathbb{Z}^2$.

Moreover, there exists a period approximating sequence $\{(q_n,A_n)\}_{n=1}^{\infty}$ which defines $(1,3)\in\mathbb{E}^2(a,t)$. Namely we have

(1) $A_n\subset\mathbf{T}$;
(2) a_{q_n} is constant on both A_n and A_n+t, $a_{q_n}(A_n)=1, a_{q_n}(A_n+t)=3, \quad \forall n$;
(3) $\inf_n\mu(A_n)>0$;
(4) $\|q_n\alpha\|\to 0$.

Therefore there exists a period approximating sequence $\{(q'_n,B'_n)\}_{n=1}^{\infty}$ which defines $(k,1)\in\mathbb{E}^2(a,t)$ for some $k\in\{\pm1,\pm3\}$. Namely we have

(1) $\{q'_n\}$ is a subsequence of $\{q_n\}$, $B'_n+t\subset A'_n$, $\mu(B'_n)\geq\frac{1}{4}\mu(A'_n)$;
(2) $a_{q'_n}$ is constant on both B'_n and B'_n+t, $a_{q'_n}(B'_n)=k$,
$a_{q'_n}(B'_n+t)=a_{q'_n}(A'_n)=1, \quad \forall n'$;
(3) $\inf_{n'}\mu(B'_n)>0$;
(4) $\|q'_n\alpha\|\to 0$.

$$(1,3)\in\mathbb{E}^2(a,t) \text{ and } (2,0)\in\mathbb{E}^2(a,t)\to(k,3)\in\mathbb{E}^2(a,t); \tag{3.7}$$

$$(k,1)\in\mathbb{E}^2(a,t) \text{ and } (k,3)\in\mathbb{E}^2(a,t)\to(0,2)\in\mathbb{E}^2(a,t). \tag{3.8}$$

Consequently both $(2,0)$ and $(0,2)$ lie in $\mathbb{E}^2(a,t)$.

Case 3:

Suppose we have $(1,1)$ and $(1,3)$ both lie in $\mathbb{E}^2(a,t)$. $(0,2)$ lies in $\mathbb{E}^2(a,t)$. $(2,2)$ also lies in $\mathbb{E}^2(a,t)$ and therefore $(2,0)$ lies in $\mathbb{E}^2(a,t)$.

In all cases we have shown both $(2,0)$ and $(0,2)$ lie in $\mathbb{E}^2(a,t)$. Along with the assumption that $(1,3)$ lies in $\mathbb{E}^2(a,t)$, we derive that $\mathbb{E}^2(a,t)=G$ as desired. □

Remark 3.2. *For arbitrary α the set of t satisfying* (3.2) *has full Lebesgue measure. Therefore for almost all $t\in\mathbf{T}$, we have $\mathbb{E}^2(a,t)=G$ and Theorem 1.5 is established.*

Next we prove Theorem 1.6. Note that α is badly approximable if and only if its partial quotients are bounded.

Proposition 3.3. *If α is badly approximable and*

$$\lim_{\substack{q\in D(\alpha)\\ q\to\infty}}\min\{\epsilon(q),\theta(q)\}=0, \tag{3.9}$$

then $t\in\mathbb{Z}\alpha$ or $t\in\mathbb{Z}\alpha+\frac{1}{2}$.

Proof. For each $q \in D(\alpha)$, let $|i_q| < q, |j_q| < q$ be such that
$$\epsilon(q) = q \left\|-t - i_q\alpha\right\|, \quad \theta(q) = q \left\|\tfrac{1}{2} - t - j_q\alpha\right\|.$$
Then we have from the assumption of the proposition
$$\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} \min\left\{q \left\|-t - i_q\alpha\right\|, \quad q \left\|\tfrac{1}{2} - t - j_q\alpha\right\|\right\} = 0.$$
Because α is badly approximable, $\frac{q^+}{q}$ and $\frac{q^{++}}{q}$ have a uniform upper bound and
$$\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} \min\left\{q^{++} \left\|-t - i_q\alpha\right\|, \quad q^{++} \left\|\tfrac{1}{2} - t - j_q\alpha\right\|\right\} = 0.$$
Also we have for arbitrary n_1 and n_2 the following inequalities:
$$\left\|n_1\alpha - n_2\alpha\right\| \leq \left\|-t - n_1\alpha\right\| + \left\|-t - n_2\alpha\right\|, \tag{3.10}$$
$$\left\|\tfrac{1}{2} + n_1\alpha - n_2\alpha\right\| \leq \left\|\tfrac{1}{2} - t - n_1\alpha\right\| + \left\|-t - n_2\alpha\right\|. \tag{3.11}$$
If we have $q^{++} \left\|-t - i_{q^+}\alpha\right\| < \frac{1}{100}$ and $q^{++} \left\|\frac{1}{2} - t - j_q\alpha\right\| < \frac{1}{100}$, then by (3.11)
$$q^{++} \left\|\tfrac{1}{2} + i_{q^+}\alpha - j_q\alpha\right\| < \frac{1}{50}.$$
Because
$$|i_{q^+} - j_q| \leq |i_{q^+}| + |j_q| < q^+ + q \leq q^{++},$$
this contradicts Lemma 2.5, which asserts that $q^{++} \left\|\frac{1}{2} + i_{q^+}\alpha - j_q\alpha\right\| \geq \dfrac{1}{24}$. Hence we have
$$\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} q \left\|-t - i_q\alpha\right\| = 0 \quad \text{or} \quad \lim_{\substack{q\in D(\alpha)\\ q\to\infty}} q \left\|\tfrac{1}{2} - t - j_q\alpha\right\| = 0.$$
Suppose we have $\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} q \left\|-t - i_q\alpha\right\| = 0$, then by (3.10)
$$\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} q^{++} \left\|i_{q^+}\alpha - i_q\alpha\right\| = 0.$$
From (2.12) we derive that for q large enough $i_{q^+} = i_q$, that is, i_q is constant. Hence $t \in \mathbb{Z}\alpha$.

Suppose we have $\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} q \left\|\frac{1}{2} - t - j_q\alpha\right\| = 0$, then
$$\lim_{\substack{q\in D(\alpha)\\ q\to\infty}} q^{++} \left\|j_{q^+}\alpha - j_q\alpha\right\| = 0.$$
From (2.12) we derive that for q large enough $j_{q^+} = j_q$, that is, j_q is constant. Hence $t \in \mathbb{Z}\alpha + \frac{1}{2}$. □

Remark 3.4. *When α is not badly approximable, Merrill* [3] *showed that if t belongs to an uncountable set of zero measure containing numbers well approximable by multiples of α, the cocycle $v = \chi_{[0,t)} - \chi_{[\frac{1}{2},\frac{1}{2}+t)}$ is a coboundary. This implies $\mathbb{E}^2(a,t) = \{(k,k) \mid k \in \mathbb{Z}\}$. Similarly, If $t + \frac{1}{2}$ belongs to an uncountable set of zero measure containing numbers well approximable by multiples of α, then $\mathbb{E}^2(a,t) = \{(k,-k) \mid k \in \mathbb{Z}\}$.*

Acknowledgements. The author is supported by Austrian Science Fund (FWF) Grant NFN S9613. The author thanks Professor Klaus Schmidt for helpful discussions and the ESI for hospitality and partial support.

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