ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Boltzmanngasse 9
A-1090 Wien, Austria

Unitary Representations of Unimodular Lie Groups in Bergman Spaces

Giuseppe Della Sala
Joe J Perez

Vienna, Preprint ESI 2267 (2010)

October 13, 2010

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available online at http://www.esi.ac.at

UNITARY REPRESENTATIONS OF UNIMODULAR LIE GROUPS IN BERGMAN SPACES

GIUSEPPE DELLA SALA AND JOE J PEREZ

ABSTRACT. For an arbitrary unimodular Lie group G, we construct strongly continuous unitary representations in the Bergman space of a strongly pseudoconvex neighborhood of G in the complexification of its underlying manifold. In particular, the Bergman spaces of these manifolds are infinite-dimensional.

1. INTRODUCTION

Let M be a complex manifold and denote by $\mathcal{O}(M)$ the space of holomorphic functions on M. Choosing a measure we may analyze the subspace of $\mathcal{O}(M)$ consisting of those members which are square-integrable; this is called the Bergman space of M and we will denote it $L^2\mathcal{O}(M)$. If M admits a free action of a Lie group G, then the measure on M can be chosen to be G-invariant. In this case, the translations induce a strongly continuous unitary representation of G in $L^2\mathcal{O}(M)$. The main goal of this paper is to show that, when G is unimodular, the Bergman space of some naturally constructed manifolds provide nontrivial unitary representations of G.
Our main result is

Theorem 1.1. *If G is a connected unimodular Lie group of dimension n, then there exist a complex manifold M, which is topologically the Cartesian product of G with an n-ball, and a nontrivial strongly continuous unitary representation $\mathcal{R}$ of G in the Bergman space of M such that $\ker \mathcal{R}$ is a compact subgroup of G.*

Corollary 1.2. *If G has no compact subgroups, then the unitary representations constructed here are faithful.*

Date: November 23, 2010.
2000 *Mathematics Subject Classification.* Primary 43A65; 32W05.
GDS is supported by FWF grant Y377, *Biholomorphic Equivalence: Analysis, Algebra and Geometry.*
JJP is supported by FWF grant P19667, *Mapping Problems in Several Complex Variables.*

Remark 1.3. In general, for noncompact M, it is not obvious that $L^2\mathcal{O}(M) \neq \{0\}$. In [GHS], a G-manifold with a nonunimodular structure group is constructed which is Stein but has $L^2\mathcal{O} = \{0\}$. The methods of [P3] may give some insight into manifolds with nonunimodular structure groups.

Let us describe briefly the manifolds involved. Assume G is a Lie group of real dimension n, acting freely and properly by real-analytic transformations on a m-dimensional, C^ω manifold X such that X/G is compact. In [HHK] it is shown that any such G action can be extended to a neighborhood of X in its complexification $X^{\mathbb{C}} \supset X$, and in this neighborhood, the extended transformations can be chosen to be biholomorphisms. Furthermore, the authors construct strongly pseudoconvex G-manifolds M_ϵ, topologically equal to the Cartesian product of X with a real n-dimensional ball of radius ϵ; the group G thus acts freely and properly on M_ϵ by holomorphic transformations with relatively compact quotient. The spaces M_ϵ are called *gauged G-complexifications of X* in [HHK] and elsewhere are frequently called *Grauert tubes*. By construction, the M_ϵ are Stein manifolds (see also [G]) and so possess a rich collection of holomorphic functions $\mathcal{O}(M_\epsilon)$ which is invariant under the induced group action.

Modifying the solution of the Levi problem via $\bar\partial$-Neumann techniques in [K1, K2, FK] somewhat, in [GHS] large spaces of L^2-holomorphic functions were constructed on strongly pseudoconvex regular coverings of compact manifolds; that is, for a discrete group of symmetries. See also [B1, B2, B3, TCM1, TCM2] for treatments of manifolds with discrete symmetries. In [P1], using techniques from [KN, FK, E], an important step in the method of [GHS] was extended to the case of a unimodular Lie group symmetry; that is obtaining the G-Fredholm property for the $\bar\partial$-Neumann Laplacian. Approaching a generalization of the result in [GHS], large spaces of L^2-holomorphic functions were constructed in [P2] on G-manifolds M with compact quotient $\bar{M}/G$ if the following main assumptions are fulfilled. First, we need that the group G be unimodular. Second, the complex manifold M is assumed to be strongly pseudoconvex. Third, and this is the new assumption in [P2] for which there is no analogue in [GHS], roughly speaking, some negative power of a Levi polynomial is required to have the property that convolutions by the group action not smooth its singularities at the boundary of M. This last property was dubbed *amenability* in [P2] and much of the present article will be concerned with demonstrating that it holds for some of the tubes constructed in [HHK]. The main ingredient in the method of [P2] is an application of a generalized

Paley-Wiener theorem from [AL] which combines remarkably well with the G-Fredholm property of the $\bar{\partial}$-Neumann Laplacian.

Our methods are mainly geometric in the present article as the sufficient conditions provided in [P1, P2] are analytic in nature and not clear to check on concrete examples such as the manifolds in [HHK].

The contents of the remainder of this article are as follows. In Sect. 2 we give a fairly precise description of the Levi polynomial at a point of the boundary of a manifold as constructed in [HHK]. It turns out that the M_ϵ naturally constructed from the complexification of the underlying manifold of G do not seem to possess the amenability property of [P2] and so in Sect. 3 we "augment" these manifolds in order to obtain manifolds that do. The point of Sect. 4 is to show that restricting from the augmented manifold back down to M_ϵ yields a nontrivial Bergman space $L^2\mathcal{O}(M_\epsilon)$. Sect. 5 contains a concrete example: the Grauert tube of a Heisenberg group.

2. Preliminaries

2.1. **Local properties of Levi's polynomial.** The main technical material in this paper will concern local details of Levi's polynomial so we will go through a fairly thorough description of this function here. Let M be a complex manifold with nonempty smooth boundary bM, $\bar{M} = M \cup bM$, so that M is the interior of $\bar{M}$, and $\dim_{\mathbb{C}}(M) = n$. We will also assume for simplicity that $\bar{M}$ is a closed subset in $\widetilde{M}$, a complex neighborhood of $\bar{M}$ so that the complex structure on $\widetilde{M}$ extends that of M, and every point of $\bar{M}$ is an interior point of $\widetilde{M}$.

Let us choose a smooth function $\rho : \widetilde{M} \to \mathbb{R}$ so that

$$M = \{z \mid \rho(z) < 0\}, \quad bM = \{z \mid \rho(z) = 0\},$$

and for all $x \in bM$, we have $d\rho(x) \neq 0$. For any $x \in bM$ define the *complex tangent plane* to the boundary at x by

$$T_x^c(bM) = \{w \in \mathbb{C}^n \mid \sum_{k=1}^n \frac{\partial \rho}{\partial z_k}\bigg|_x w_k = 0\}.$$

For $x \in bM$, define the Levi form L_x by

$$L_x(w, \bar{w}) = \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\bigg|_x w_j \bar{w}_k, \quad (w \in T_z^c(bM)).$$

Then M is said to be *strongly pseudoconvex* if for every $x \in bM$, the form L_x is positive definite. Though L_x depends on ρ, its essential features are preserved by biholomorphisms.

Since ρ is real-valued, the Taylor expansion at x of ρ is

$$(1)\quad \rho(z) = \rho(x) + 2\,\mathfrak{Re}\, f(z,x) + L_x(z-x, \bar{z}-\bar{x}) + \mathcal{O}(|z-x|^3), \quad (z \in \mathbb{C}^n)$$

with the *Levi polynomial* f defined by

$$(2)\quad f(z,x) = \sum_{k=1}^{n} \frac{\partial \rho}{\partial z_k}\bigg|_x (z_k - x_k) + \frac{1}{2}\sum_{jk=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}\bigg|_x (z_j - x_j)(z_k - x_k).$$

In our Grauert tubes, the group invariance and the compactness of the quotient guarantee that without loss of generality (replacing ρ by $e^{\lambda\rho} - 1$ with sufficiently large $\lambda > 0$) we may, as in [GHS], choose a defining function of M so that the Levi form $L_x(w, \bar{w})$ is positive for all nonzero $w \in \mathbb{C}^n$ (and not only for $w \in T_x^c(bM)$) and at all points $x \in bM$.

The complex quadric hypersurface $S_x = \{z \mid f(z,x) = 0\}$ has $T_x^c(bM)$ as its tangent plane at x. The strong pseudoconvexity property implies that $\rho(z) > 0$ if $f(z,x) = 0$ and $z \neq x$ is close to x. This means that near x the intersection of S_x with bM contains only x. Since $\rho < 0$ in M, (1) implies that $\mathfrak{Re}\, f(z,x) < 0$ if $x \in bM$ and $z \in M$ is sufficiently close to x. It follows that we can choose a branch of $\log f(z,x)$ so that $z \mapsto \log f(z,x)$ is a holomorphic function in $z \in U_x \cap M$, where U_x is a sufficiently small neighborhood of x in $\widetilde{M}$. Consequently all powers of f are also well-defined and holomorphic in $U_x \cap M$. For $\tau < 0$, the functions $f^\tau : U_x \cap M \to \mathbb{C}$ are holomorphic in a neighborhood of x and blow up only at x.

Lemma 2.1. *Choose local coordinates for which $x \leftrightarrow 0$ and let*

$$a = \frac{\partial \rho}{\partial z_k}\bigg|_0 \quad \text{and} \quad b = \frac{1}{2}\sum_{jk=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}\bigg|_0$$

so that $f(z,0) = a\cdot z + bz\cdot z$. It follows that there are constants $C, D > 0$ so that for z sufficiently near zero in $\bar{M}$ we have

$$C|z|^2 \leq |a\cdot z + bz\cdot z| \leq D|z|.$$

Proof. This is true because

$$\begin{aligned} 2|a\cdot z + bz\cdot z| \geq & -2\,\mathfrak{Re}\,(a\cdot z + bz\cdot z) \\ \geq & \rho(z) - 2\,\mathfrak{Re}\,(a\cdot z + bz\cdot z) = L_0(z,\bar{z}) + \mathcal{O}(|z|^3) \end{aligned}$$

and the Levi form has a smallest eigenvalue $\lambda > 0$, so $L_0(z,\bar{z}) > \lambda|z|^2$. The other estimate is obvious. □

The estimate above yields the following immediately, [P2, Lemma 4.2].

Lemma 2.2. *Let M be a strongly pseudoconvex complex manifold, $n = \dim_{\mathbb{C}} M$, and $\chi \in C_c^\infty(\bar{M})$ with small support near zero. Then $\chi f^{-\tau} \in L^p(M)$ whenever $\tau \in (0, n/p)$.*

2.2. **Grauert tubes.** Let us describe the complexifications from [HHK] in more detail. If X is a real-analytic manifold on which a Lie group G acts freely and properly by real-analytic transformations, then there exists a complexification $X^{\mathbb{C}}$ so that $X \hookrightarrow X^{\mathbb{C}}$ is embedded as a totally real submanifold. Furthermore, the action of G on X extends to a neighborhood V of X in $X^{\mathbb{C}}$ such that the extended transformations are biholomorphisms. In addition, there exists a nonnegative function $\varphi \in C^\omega(V, \mathbb{R})$ with the following properties:

(1) φ is constant along the orbits of G
(2) $\varphi|_X \equiv 0$
(3) φ is strictly plurisubharmonic near X.

For $\epsilon > 0$ sufficiently small, following [HHK], define $M_\epsilon \subset V$ by

$$M_\epsilon = \{\varphi < \epsilon\}.$$

Remark 2.3. In the following work, we can choose any φ satisfying these properties, and not necessarily the one constructed in [HHK]. Incidentally, the Kähler form $i\partial\bar{\partial}\varphi$ provides an invariant Kähler metric on M_ϵ.

Note that G acts freely and properly on M_ϵ by holomorphic transformations and M_ϵ is strongly pseudoconvex as its boundary is the level set of a strictly plurisubharmonic function.

On M_ϵ, it is also true that there is an equivariant retraction $R(t, z)$: $[0, 1] \times M_\epsilon$ onto X; in particular the map $R(1, z)$ is a projection

$$\pi : M_\epsilon \to X \tag{3}$$

commuting with the right G-action.

We will need a concrete description of Levi's polynomial so we compute here the Taylor series of the defining function φ of the tube M_ϵ to second order. The proof uses a part of the argument in [HHK, VIII, Lemma 1].

Lemma 2.4. *Let M_ϵ and φ be as before, and let $p \in M_\epsilon$. It follows that there exist local complex coordinates $z_j = x_j + iy_j$, $j = 1, \dots, m$, vanishing at p, such that $\varphi(z) = \sum_j y_j^2 + \mathcal{O}(|z|^3)$.*

Proof. Since $X \subset M_\epsilon$ is totally real, with a holomorphic change of coordinates, we may assume that $X = \{y_j = 0\}$. Since each point of X is a local minimum for φ, we have $\nabla\varphi|_X \equiv 0$, which implies

$\partial^2\varphi/\partial x_k \partial y_j(0) = 0$. Since we also have $\partial^2\varphi/\partial x_k \partial x_j(0) = 0$, we obtain

$$\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}(0) = \frac{\partial^2\varphi}{\partial y_j \partial y_k}(0).$$

Since φ is strictly plurisubharmonic, the form on the left-hand side is positive definite. It follows that the second-order Taylor expansion of φ about $0 \in \mathbb{C}^m$ is a positive definite, purely quadratic polynomial involving only the y_j. The claim is obtained by diagonalizing the real, symmetric form on the right-hand side. □

2.3. **Augmentation of M_ϵ.** From now on we restrict our attention to the case in which M_ϵ as in [HHK, Prop. 4] is the gauged G-complexification of G acting on itself, *i.e.* $X = G$. We have been unable to demonstrate that such M_ϵ possess the amenability property mentioned in the introduction (see Def. 3.7). As a result, we will *augment* M_ϵ in the following way, *cf.* [P2, §4.2].

With z_0 a coordinate on $T = (S^1)^{\mathbb{C}} \cong \mathbb{C}/\mathbb{Z}$, and z' a point in M_δ, define the function

$$\tilde{\varphi}(z) = \tilde{\varphi}(z_0, z') = (\mathfrak{Im}\, z_0)^2 + \varphi(z'), \qquad (z = (z_0, z') \in T \times M_\delta)$$

with $\delta > 0$ small enough so that $\varphi|_{M_\delta}$ is strictly plurisubharmonic. As before, define

$$\tilde{M}_\epsilon = \{\tilde{\varphi} < \epsilon\}$$

for $0 < \epsilon < \delta$. By construction, $\tilde{\varphi}$ is strictly plurisubharmonic, thus $\tilde{M}_\epsilon$ is strongly pseudoconvex for $\epsilon > 0$ sufficiently small. Extending the G action on M_δ to $T \times M_\delta$ by triviality on the T factor, we see that G acts freely and properly on $\tilde{M}_\epsilon$ by holomorphic transformations, and with relatively compact quotient.

We introduce some notation and maps to clarify the geometric situation. With the equivariant projection $\pi : M_\delta \to G$ from (3), define the *slice at* $e \in G$ by

$$S_e := \pi^{-1}(e) \subset M_\delta.$$

By the construction in [HHK], $S_e \cong B^n$, the real n-dimensional ball. For $z = (z_0, z') \in T \times M_\delta$, the point $\tilde{\pi}(z) := (z_0, z' \cdot \pi(z')^{-1})$ belongs to $T \times S_e$ and $\tilde{\varphi}(\tilde{\pi}(z)) = \tilde{\varphi}(z)$ by the invariance of π and $\tilde{\varphi}$. Since the whole tube is in the orbit of S_e, and $\tilde{\varphi}$ is invariant, to know $\tilde{\varphi}$ everywhere, we need only calculate $\tilde{\varphi}|_{S_e}$. To do so, choose coordinates $(\eta_k)_1^n$ on S_e in such a way that $\varphi|_{S_e}(\eta) = \sum_1^n \eta_k^2$. These exist, by [HHK, VIII, Lemma 1] itself, but the η_k this time are not necessarily the imaginary parts of holomorphic coordinates as in Lemma 2.4. We have

$$\{\tilde{\varphi} < \delta\} \cap (T \times S_e) = \{(z_0, z') \in T \times S_e \mid (\mathfrak{Im}\, z_0)^2 + \sum_1^n \eta_k^2 < \delta\}$$
$$= S^1 \times B^{n+1},$$

and we have that $\tilde{\pi}(z) \in \{\tilde{\varphi} < \delta\} \cap (T \times S_e)$ for all $z \in \tilde{M}_\delta$. Thus

$$\tilde{M}_\delta \ni (z_0, z') \longmapsto (\pi(z'), \tilde{\pi}(z_0, z')) \in G \times S^1 \times B^{n+1}$$

is a diffeomorphism and we have

$$\tilde{M}_\epsilon \cong G \times \mathbb{T} \tag{4}$$

where $\mathbb{T} = S^1 \times B^{n+1}$ is a solid torus of $n + 2$ real dimensions.

3. Convolutions of Levi's polynomials

Let us compute the Levi polynomial at the basepoint $(z_0, z') = (i\epsilon, 0)$ of the boundary of $\tilde{M}_\epsilon$ in the local complex coordinates $z = (z_0, z_1, \ldots, z_n)$.

Lemma 3.1. *The Levi polynomial induced by the defining function $\tilde{\varphi}(z)$ at the point $p = (i\epsilon, 0)$ is proportional to*

$$f(z) = 3\epsilon^2 + 2i\epsilon z_0 + z_0^2 + z_1^2 + \ldots + z_n^2.$$

Proof. This follows directly from the form of $\tilde{\varphi}$, Lemma 2.4, and the definition (2). □

We will denote by zt the right-translate of $z \in \tilde{M}_\epsilon$ by $t \in G$.

Lemma 3.2. *In $\tilde{M}_\epsilon$, consider the curve $(0, \epsilon] \ni s \mapsto (is, 0)$ to the basepoint $(i\epsilon, 0)$. There exist coordinates $t = (t_j)_1^n$ in a neighborhood of $e \in G$ such that, for z belonging to the curve, the Levi polynomial at the point zt takes the form*

$$f(zt) = \sigma + t_1^2 + \ldots + t_n^2$$

and $\sigma = \sigma(s) \to 0$ as $s \to \epsilon$.

Proof. As $G = X$, the construction in Lemma 2.4 gives real coordinates in a neighborhood of the identity $e \in G$. Computing $f(zt) = f(z_0, z't)$ along the path, we obtain

$$f((is, e)t) = f(is, t) = 3\epsilon^2 - 2\epsilon s - s^2 + t_1^2 + \ldots + t_n^2$$

from Lemma 3.1. □

Remark 3.3. What we will use of this fact is simply that the Levi polynomial on the orbit of any point in this path takes the form of a constant plus a norm-squared-like quantity on G.

On our tubes, augmented and unaugmented, we have smooth, free, right actions of G with relatively compact quotients. Choosing a biinvariant measure dt on G, and fixing smooth measures dq on B^n and dQ on $\mathbb{T}$, the tensor product measures on the tubes allow us to decompose $L^2(M_\epsilon)$ and $L^2(\tilde{M}_\epsilon)$ as follows

$$\begin{aligned} L^2(M_\epsilon, dt \otimes dq) &\cong L^2(G, dt) \otimes L^2(B^n, dq) \\ L^2(\tilde{M}_\epsilon, dt \otimes dQ) &\cong L^2(G, dt) \otimes L^2(\mathbb{T}, dQ), \end{aligned} \tag{5}$$

which also present the Hilbert spaces as free Hilbert G-modules, [P1].

Choosing an $r > 0$ for which M_r is defined, fix a measure dq on M_r/G. For every $\epsilon < r$, the measure we will take on $M_\epsilon \subset M_r$ will always be simply the restriction of $dt \otimes dq$. We will choose dQ later, but for now we think of it as being an arbitrary smooth measure.

On all these manifolds, the global right G-action and the Haar measure dt combine to define convolution operators in L^2 which we will write

$$(R_\Delta u)(z) = \int_G dt\, \Delta(t) u(zt),$$

for example, for $\Delta \in L^1(G)$, $u \in L^2$.

Let us begin our analysis of the asymptotics of convolutions of powers of f. For U_x a coordinate neighborhood at the boundary of $\tilde{M}_\epsilon$ choose a cut-off function $\chi \in C_c^\infty(U_x)$, so that $\chi = 1$ in a neighborhood of x.

Lemma 3.4. [P2, Rem. 4.3] *Let M be a strongly pseudoconvex G-manifold and let f be a Levi polynomial on M. Then for all $x \in \bar{M}/G$, $\|\chi f^{-\tau}(\cdot, x)\|_{L^1(G)} < \infty$ as long as $2\tau < \dim_{\mathbb{R}} G$.*

We will also need the following fact regarding convolutions of powers of Levi's polynomial.

Lemma 3.5. *Let $n = \dim_{\mathbb{R}} G$, $\tau \in (0, n/2)$ and $\Delta \in L^2(G)$. Then $R_\Delta \chi f^{-\tau} \in L^2(M)$.*

Proof. Since the quotient measure of M/G is finite, Jensen's inequality holds there, thus

$$\begin{aligned} \|R_\Delta h\|^2_{L^2(M)} &= \int_{M/G} dx \int_G dt \left| \int_G ds\ \Delta(s) h(ts, x) \right|^2 \\ &\leq \|\Delta\|^2_{L^2(G)} \int_{M/G} dx\ \|h(\cdot, x)\|^2_{L^1(G)} \\ &\lesssim \|\Delta\|^2_{L^2(G)} \left| \int_{M/G} dx\ \|h(\cdot, x)\|_{L^1(G)} \right|^2 \\ &= \|\Delta\|^2_{L^2(G)} \|h\|^2_{L^1(M)}, \end{aligned} \tag{6}$$

where we have also used Young's inequality. Lemmata 2.2 and 3.4 provide that $h = \chi f^{-\tau} \in L^1(M)$, which gives the result. □

Remark 3.6. Note that in the expression (6), with $h = \chi f^{-\tau}$, the G-integral is over a compact neighborhood U of the identity. This neighborhood can be chosen as small as we like, choosing χ accordingly.

For M a strongly pseudoconvex, connected unimodular G-manifold with relatively compact quotient, in [P2] a sufficient condition is given for $L^2\mathcal{O}(M)$ be large in the sense of von Neumann's G-dimension. It is the following

Definition 3.7. Let $G \to M \xrightarrow{p} Y$ be a principal G-bundle and let $\xi : \bar{Y} \to M$ be a piecewise continuous section so that $\xi|_{p(\text{supp}\chi)}$ is continuous. The action of G on M is called *amenable* if there exist an $x \in bM$ and $\tau > 0$ so that if f is a Levi polynomial at x, then 1) $\chi f^{-\tau} \in L^2(M)$, 2) $\|\chi f^{-\tau}(\cdot, \xi)\|_{L^1(G)} < \infty$ for all $\xi \in p(\text{supp}\chi)$, and 3) for any nonzero $\Delta \in C^\infty(G)$, we have $R_\Delta \chi f^{-\tau} \notin C^\infty(\bar{M})$.

Remark 3.8. There is nothing special about the Levi polynomial in the analysis in [P2]; it is just a convenient choice. We would do just as well with any function, singular at a point $p \in bM$ and holomorphic in some neighborhood of p in M, with the integrability conditions required by the definition.

In the following, without loss of generality by the invariance of all the structures involved, we will assume that Δ be nonzero in a neighborhood of the identity of G, translating if necessary. Similarly, if necessary, multiplying by a constant, we can assume that $\Delta(e) > 0$.

Proposition 3.9. *The G action on the manifold $\tilde{M}_\epsilon$ is amenable.*

Proof. First note that if $\dim_{\mathbb{R}} G = n$, then $\dim_{\mathbb{C}} \tilde{M}_\epsilon = n + 1$, thus

(1) $\chi f^{-\tau} \in L^2(M)$ if $\tau \in [0, \frac{n+1}{2})$

(2) $\|\chi f^{-\tau}(\cdot, \xi)\|_{L^1(G)} < \infty$ for all $\xi \in Y$ if $2\tau < \dim_{\mathbb{R}} G = n$.

by Lemmata 2.2 and 3.4, respectively. So let us choose $\tau < n/2$ and estimate the convolution along a path to the basepoint in $b\tilde{M}_\epsilon$, using Lemma 3.2:

$$(R_\Delta \chi f^{-\tau})(is, 0) = \int_U dt\ \Delta(t)\chi(s,t) \left[3\epsilon^2 - 2\epsilon s - s^2 + |t|^2\right]^{-\tau} \tag{7}$$

where we have rewritten χ's dependences and the region of integration is the neighborhood of the identity in G given in Rem. 3.6. Now partition U into subregions U_1, U_2 such that U_1 is a neighborhood of e for which $\Delta \approx \Delta(e)$. It follows that the integral over U_2 is a smooth function of s. To verify that the convolution not be smooth at the basepoint, it thus suffices to analyze the U_1-integral.

$$\begin{aligned}(R_\Delta \chi f^{-\tau})(is,0) &\sim \int_{U_1} dt\ \Delta(t)\chi(s,t)\left[3\epsilon^2 - 2\epsilon s - s^2 + |t|^2\right]^{-\tau} \\ &\sim \int_0^\delta \frac{r^{n-1}dr}{[3\epsilon^2 - 2\epsilon s - s^2 + r^2]^\tau} \\ &\sim \int_0^\delta \frac{r^{n-1}dr}{[\sigma + r^2]^\tau}.\end{aligned}$$

Here, $\sigma(s) = 3\epsilon^2 - 2\epsilon s - s^2 \to 0$ as $s \to \epsilon^-$ and the symbol $\sim$ means that the functions have identical singular support. Away from the path of the singularity, the convolution is smooth, thus we have

$$\lim_{s\to\epsilon^-} \frac{\partial^k}{\partial s^k}(R_\Delta \chi f^{-\tau})(is,0) \longrightarrow \infty$$

for $\tau > 0$, $\tau + k > n$. Taking any $\tau \in (0, n/2)$ suffices. □

Remark 3.10. In the situation above, the group action avoids the "bad" direction (*i.e.* $T^c(bM)^\perp = (T(bM)\cap JT(bM))^\perp \subset T(bM)$) in the boundary. Estimates of convolutions in substantially greater generality suggest that the amenability property is satisfied whenever this is the case. For example, in the language of [FS], subgroups of the Heisenberg group of the form $\{(z,0) \mid z \in \mathbb{C}^n\}$ lead to amenable actions while those containing $\{(0,t) \mid t \in \mathbb{R}\}$ do not. This tempts us to conjecture that in the gauged G-complexification M of a G-manifold X for which $\dim_{\mathbb{R}} X > \dim_{\mathbb{R}} G$, the extended G-action on M is amenable. Similarly, the following assertion suggests that groups for which factors split off can be treated without augmenting.

Theorem 3.11. $\dim_G L^2\mathcal{O}(\tilde{M}_\epsilon) = \infty$.

Proof. By Prop. 3.9, $\tilde{M}_\epsilon$ satisfies the requirements of [P2, Thm. 5.2]. □

Remark 3.12. The positivity of the G-dimension is sufficient to obtain the infinite-dimensionality of the Bergman space of $\tilde{M}_\epsilon$ over the complex numbers.

4. Bergman representation spaces

4.1. **Preparatory geometric considerations.** Any G-invariant Riemannian metric on $\bar{M}$ is complete in the following sense. For any point $x_0 \in \bar{M}$ and for any $R > 0$ the ball $B(t,R) = \{s \in \bar{M} : \mathrm{dist}(t,s) < R\}$

of the corresponding geodesic metric is relatively compact in $\bar{M}$, [GHS, Lemma 1.1].

We will need the following topological lemma. For $K \subset G$ and $t \in G$, denote by Kt the set $\{kt : k \in K\} \subset G$.

Lemma 4.1. *Let G be a non-compact Lie group, $K \subset G$ a compact subset containing the identity, and $L \subset G$ an unbounded sequence. It follows that there exists a $t \in L$ such that $K \cap Kt = \emptyset$.*

Proof. Choose a right-invariant Riemannian metric d on the group G. By the observation above, G's closed and d-bounded subsets are compact. For K as in the statement, choose $R > 0$ such that $K \subset B(e, R)$. Suppose, by contradiction, that $K \cap Kt \neq \emptyset$ for all $t \in L$, and choose $k_t \in K$ such that $k_t t \in K$. Then for any $p \in Kt$ ($p = p't$ with $p' \in K$) we have

$$d(e, p) \leq d(e, k_t t) + d(k_t t, p) = d(e, k_t t) + d(k_t, p')$$

since, by invariance of d, we have $d(k_t t, p) = d(k_t, p')$. We continue, noting that

$$\cdots \leq d(e, k_t t) + d(k_t, e) + d(e, p') \leq 3R,$$

thus $Kt \subset B(e, 3R)$ for all $t \in L$. It follows that $KL = \{kt : t \in L, k \in K\} \subset B(e, 3R)$, so that $L \subset KL$ is bounded and hence relatively compact, a contradiction. □

4.2. **Restrictions to the Grauert tubes of G.** Denoting the real n-ball by B^n, topologically, the Grauert tubes M_ϵ of $X = G$ embed in $\tilde{M}_\delta$ in the following way:

$$M_\epsilon \cong G \times B^n \hookrightarrow G \times \mathbb{T} \cong \tilde{M}_\delta. \tag{8}$$

More specifically, in the notation of Sect. 2.3, for every fixed $x_0 + iy_0 = z_0 \in T$ with $y_0^2 < \delta$, the map $M_{\delta - y_0^2} \ni p \to (x_0, y_0, p) \in \tilde{M}_\delta$ is an embedding of $M_{\delta - y_0^2}$ as a complex submanifold of $\tilde{M}_\delta$ with $\operatorname{codim}_{\mathbb{C}} M_\delta = 1$. Thus the restriction to the copy of M_ϵ at z_0, denoted $M_\epsilon^{z_0}$, is a map

$$\mathcal{O}(\tilde{M}_\delta) \ni u \longmapsto u(z_0, \cdot) \in \mathcal{O}(M_\epsilon^{z_0})$$

and so elements of $L^2\mathcal{O}(\tilde{M}_\delta)$ restrict to elements of $\mathcal{O}M_\epsilon^{z_0}$ as long as $y_0^2 + \epsilon \leq \delta$.

The holomorphic functions constructed in Thm. 3.11 are, by construction, smooth in the closure of $\tilde{M}_\delta$ except along the orbit of the basepoint of Levi's polynomial. Thus the restrictions of these functions are smooth in the closures of the $M_\epsilon^{z_0}$.

In the decompositions (5), choose the measure Q on $\mathbb{T} = \tilde{M}_\delta / G$ as a tensor product $dq \otimes d\lambda$ where $d\lambda$ is the Lebesgue measure on the cylinder T and where dq is the fixed, invariant measure on $M_\delta^{z_0} \cong M_\delta$ that we defined in Sect. 3.

For $h \in L^2(M_\epsilon)$, we will denote by t_*h the function

$$t_*h : z' \longmapsto h(z't).$$

This function is in $L^2(M_\epsilon)$ by the G-invariance of the measure on M_ϵ.

Lemma 4.2. *Let $h \in L^2(M_\epsilon)$ with $\|h\|_{L^2} = 1$. Let $L \subset G$ be a sequence, chosen in such a way that no subsequence $(t_k)_k \subset L$ has compact closure. It follows that the complex vector space $\langle h \rangle_L$ generated by $\{t_*h : t \in L\}$ is infinite-dimensional over $\mathbb{C}$.*

Proof. Suppose, by contradiction, that $\{t_*h \mid t \in L\}$ is contained in a finite-dimensional space. Since $\|t_*h\|_{L^2} = 1$ for all $t \in L$, there exists a sequence $(t_k)_k \subset L$ such that $(t_{k*}h)_k$ is convergent. Denote by $h_0 \in L^2(M_\epsilon)$ the limit of this sequence and note that $\|h_0\|_{L^2} = 1$.

Let $K \subset G$ be a fixed, arbitrary compact set containing the identity. We will obtain a contradiction to the assertion that $\|h_0\|_{L^2} = 1$ by showing that $h_0|_{\pi^{-1}(K)} \equiv 0$, where π is the G-equivariant projection map $\pi : M_\epsilon \to G$, *cf.* Sect. 2.2.

Fix $\epsilon_0 > 0$, and let $K_0 \supset K$ be a compact set such that the function h from the statement satisfies

$$\int_{M_\epsilon \setminus \pi^{-1}(K_0)} |h|^2 dt \otimes dq < \epsilon_0.$$

By a repeated application of Lemma 4.1, we obtain a sequence $(t_k)_k$ in L such that $K_0 \cap K_0 t_k = \emptyset$, thus

$$\pi^{-1}(K_0) \cap \left[\pi^{-1}(K_0)t_k\right] = \pi^{-1}(K_0 \cap K_0 t_k) = \emptyset \tag{9}$$

for all k, by the equivariance of π. It follows that

$$\int_{\pi^{-1}(K_0)} |t_{k*}h|^2 = \int_{\pi^{-1}(K_0)t_k} |h|^2 \le \int_{M_\epsilon \setminus \pi^{-1}(K_0)} |h|^2 < \epsilon_0,$$

the first inequality coming from (9). Since $\pi^{-1}(K) \subset \pi^{-1}(K_0)$, we have $\int_{\pi^{-1}(K)} |t_{k*}h|^2 < \epsilon_0$ and so $\|h_0\|_{L^2(\pi^{-1}(K))} \le \epsilon_0$ since h_0 is the limit of $(t_{k*}h)_k$. By letting $\epsilon_0 \to 0$ we obtain $h_0|_{\pi^{-1}(K)} \equiv 0$. □

Proposition 4.3. *For $\epsilon > 0$ sufficiently small, let M_ϵ be the Grauert tube of G. It follows that the space $L^2\mathcal{O}(M_\epsilon)$ is infinite-dimensional as a vector space over $\mathbb{C}$.*

Proof. Let $u \in L^2\mathcal{O}(\tilde{M}_\delta)$ be a non-trivial function provided by Thm. 3.11. From Fubini's theorem follows

$$\infty > \int_{\tilde{M}_\delta} |u|^2 dt \otimes dQ = \int_{\tilde{M}_\delta} |u|^2 dt \otimes dq \otimes d\lambda =$$

$$= \int_{\{y_0^2<\delta\}} d\lambda(z_0) \int_{M_\epsilon^{z_0}} |u(z_0, z')|^2 (dt \otimes dq)(z')$$

where at each height y_0 we have chosen the radius $\epsilon = \epsilon(y_0) = \delta - y_0^2$. Thus the integral of $|u(\zeta, z')|^2$ over $\tilde{M}_\delta \cap \{z_0 = \zeta\} = M^\zeta_{\delta-(\mathfrak{Im}\,\zeta)^2}$ is finite for almost every $\zeta \in \{(\mathfrak{Im}\, z_0)^2 < \delta\}$; in other words, for ϵ in a subset of positive measure in $(0, \delta)$, we obtain that $L^2\mathcal{O}(M_\epsilon) \neq \{0\}$.

For such an ϵ, choose a nontrivial $h \in L^2\mathcal{O}(M_\epsilon)$. Any sequence $L \subset G$ as in Lemma 4.2 satisfies $\langle h\rangle_L \subset L^2\mathcal{O}(M_\epsilon)$. Lemma 4.2 implies that $\dim_{\mathbb{C}} L^2\mathcal{O}(M_\epsilon) = \infty$. □

Remark 4.4. By analyticity, compactness and the necessity that a set of positive measure be infinite, we have in fact that $L^2\mathcal{O}(M_\epsilon) \neq \{0\}$ for all $\epsilon < \delta$ and not only for a set of positive measure.

In the subspace of the Bergman space that we have constructed, we have a faithful unitary representation of G "modulo compact subgroups." That is the content of the following

Theorem 4.5. *Let G be a connected unimodular group. Then, there is a representation $\mathcal{R}$ of G in $L^2\mathcal{O}(M_\epsilon)$, where M_ϵ is a neighborhood of G in its complexification, such that* $\ker \mathcal{R}$ *is a compact subgroup of G. In particular, if G does not have compact subgroups then $\mathcal{R}$ is a faithful representation.*

Proof. By Prop. 4.3, $L^2\mathcal{O}(M_\epsilon)$ is nontrivial. The representation $\mathcal{R}$ is given by

$$\mathcal{R}(t)v(z) = t_*v(z') = v(z't), \qquad (t \in G,\ v \in L^2\mathcal{O}(M_\epsilon)).$$

Let H be the kernel of $\mathcal{R}$, and choose a function $u \in L^2\mathcal{O}(M_\epsilon)$, $u \neq 0$; then $t_*u = u$ for all $t \in H$. In particular, the space generated by the translates of u by elements of H is 1-dimensional. By the same arguments as in Lemma 4.2, H must be relatively compact because otherwise, we could find a discrete subset $L \subset H$ with the property that every infinite subsequence of L not be relatively compact, contradicting $\dim_{\mathbb{C}}\langle u\rangle_H = 1$. □

5. The tube of the Heisenberg group $\mathbb{H}_3(\mathbb{R}) \hookrightarrow \mathbb{H}_3(\mathbb{C})$

We describe a simple method of constructing examples of our G-manifolds, applied here to the Heisenberg group in three dimensions. For $\mathbb{K} = \mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$, define

$$\mathbb{H}_3(\mathbb{K}) = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_k \in \mathbb{K} \right\}, \quad \mathfrak{h}_3(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & \theta_1 & \theta_3 \\ 0 & 0 & \theta_2 \\ 0 & 0 & 0 \end{pmatrix} \mid \theta_k \in \mathbb{R} \right\}.$$

For $\mathbb{H}_3(\mathbb{R}) \hookrightarrow \mathbb{H}_3(\mathbb{C})$, close to the origin of $\mathbb{H}_3(\mathbb{C})$, a slice through e, $S_e \subset \mathbb{H}_3(\mathbb{C})$, can be obtained as the exponential of the normal vectors at the identity in the natural inclusion. In other words, denoting by $T_e\mathbb{H}_3(\mathbb{R})^{\perp}$ the orthogonal complement of $T_e\mathbb{H}_3(\mathbb{R})$ in $T_e\mathbb{H}_3(\mathbb{C})$, we have

$$S_e = \exp\left[T_e\mathbb{H}_3(\mathbb{R})^{\perp}\right] \subset \mathbb{H}_3(\mathbb{C}). \tag{10}$$

It follows that $S_e \subset \mathbb{H}_3(\mathbb{C})$ consists of matrices of the form $\exp[i\Theta]$ with $\Theta = (\theta_{jk})_{jk}$ in $\mathfrak{h}_3(\mathbb{R})$. In [HHK], an invariant strictly plurisubharmonic function φ is constructed abstractly; here we give a concrete description.

Expressing an arbitrary element Z in the form $Z = \exp[i\Theta]t$, with $t \in \mathbb{H}_3(\mathbb{R})$ and $\Theta \in \mathfrak{h}_3(\mathbb{R})$, as in the description (10), the definition of φ given in [HHK] becomes

$$\varphi(\exp[i\Theta]t) = \sum_{jk} \theta_{jk}^2. \tag{11}$$

Note that this definition makes the right-$\mathbb{H}_3(\mathbb{R})$ invariance of φ manifest. So, given $Z \in \mathbb{H}_3(\mathbb{C})$, consider the factorization

$$Z = \exp[i\Theta]t \tag{12}$$

with $t \in \mathbb{H}_3(\mathbb{R})$ and Θ a real matrix. Defining $X = \mathfrak{Re}\, Z$ and $Y = \mathfrak{Im}\, Z$, the identity $\exp[i\Theta] = \cos[\Theta] + i\sin[\Theta]$, provides

$$Z = \mathfrak{Re}\, Z + i\mathfrak{Im}\, Z = X + iY = (\cos[\Theta] + i\sin[\Theta])t.$$

Equating real and imaginary parts,

$$X = \cos[\Theta]t, \qquad Y = \sin[\Theta]t,$$

thus

$$t = [\cos\Theta]^{-1}X = [\sin\Theta]^{-1}Y$$

and so

$$\Theta = \tan^{-1}(YX^{-1}) = YX^{-1} - \frac{(YX^{-1})^3}{3} + \frac{(YX^{-1})^5}{5} - \dots. \tag{13}$$

We easily compute $\exp[i\Theta]$ for $\Theta \in \mathfrak{h}_3(\mathbb{R})$;

$$\exp\left[i\begin{pmatrix} 0 & \theta_1 & \theta_3 \\ 0 & 0 & \theta_2 \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & i\theta_1 & i\theta_3 - \theta_1\theta_2/2 \\ 0 & 1 & i\theta_2 \\ 0 & 0 & 1 \end{pmatrix},$$

thus the factorization (12) is explicitly

$$\begin{aligned} Zt &= \exp[i\Theta]t = Xt + iYt \\ &= \begin{pmatrix} 1 & i\theta_1 & i\theta_3 - (1/2)\theta_1\theta_2 \\ 0 & 1 & i\theta_2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t_1 & t_3 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{14}$$

and any element of $\mathbb{H}_3(\mathbb{C})$ has a unique decomposition of this form. For a general $Z \in \mathbb{H}_3(\mathbb{C})$, computing YX^{-1}, which we need in (13), we get

$$YX^{-1} = \begin{pmatrix} 0 & y_1 & y_3 - x_2 y_1 \\ 0 & 0 & y_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{15}$$

where $z_k = x_k + iy_k$. The next step, to compute the arctangent, is easy in our case because of the nilpotence, but in general the equality

$$\Theta = \tan^{-1}(YX^{-1}) = YX^{-1}$$

is a good, invariant approximation in a neighborhood of the group. Denoting by Θ^T the transpose of Θ, the sum of the squares of the elements of Θ as in Eq. (11) can be written $\operatorname{tr}(\Theta^T\Theta)$ in general, but for our purposes, it is enough to read off

$$\begin{aligned} \varphi(Z) &= \operatorname{tr}(\tan^{-1}(YX^{-1})^T \tan^{-1}(YX^{-1})) = \operatorname{tr}((YX^{-1})^T YX^{-1}) \\ &= (\mathfrak{Im}\, z_1)^2 + (\mathfrak{Im}\, z_2)^2 + (\mathfrak{Im}\, z_3 - \mathfrak{Re}\, z_2\, \mathfrak{Im}\, z_1)^2, \end{aligned} \tag{16}$$

from (15). This is an invariant, strictly plurisubharmonic function in a neighborhood of $\mathbb{H}_3(\mathbb{R})$ in $\mathbb{H}_3(\mathbb{C})$. Note that φ in these coordinates is already of the form of Lemma 2.4.

Now, as in Sect. 2.3, consider $(S^1 \times \mathbb{H}_3(\mathbb{R}))^{\mathbb{C}} \cong \mathbb{C}/\mathbb{Z} \times \mathbb{H}_3(\mathbb{C})$ with $\mathbb{H}_3(\mathbb{R})$ acting from the right, and trivially on the first factor:

$$\mathbb{H}_3(\mathbb{R}) \ni t : (z_0; Z) \longmapsto (z_0, Zt).$$

Defining a new function by $\tilde{\varphi}(z_0, Z) = (\mathfrak{Im}\, z_0)^2 + \varphi(Z)$, an easy calculation shows that $\tilde{M}_\epsilon = \{\tilde{\varphi} < \epsilon\}$ is strongly pseudoconvex as long as $\epsilon < 1$. With the definition (2), at $z^0 = (z_0, z_1, z_2, z_3) = (i/2, 0, 0, 0) \in \{\tilde{\varphi} = 1/4\}$, $\tilde{\varphi}$ induces a Levi polynomial proportional to

$$f(z) := 3/4 + iz_0 + z_0^2 + z_1^2 + z_2^2 + z_3^2,$$

as in Lemma 3.1. The action of $t \in \mathbb{H}_3(\mathbb{R})$ on f is given by

$$f(zt) = 3/4 + iz_0 + z_0^2 + (z_1 + t_1)^2 + (z_2 + t_2)^2 + (z_3 + z_1 t_2 + t_3)^2$$

The convolution of χf^{-1} by a convolution kernel $\Delta \in C^\infty(\mathbb{H}_3(\mathbb{R}))$ is approximated in a neighborhood of the basepoint z^0 by

$$(R_\Delta \chi f^{-1})(z) \sim \int dt\ \ [f(zt)]^{-1}$$

with the integral over a small ball at zero in $\mathbb{R}^3$. Now consider the path from inside the manifold to the basepoint $(i/2, 0, 0, 0)$ given by $s \mapsto$

$(is/2, 0, 0, 0)$, $s \to 1^-$. Along this path, a multiple of the convolution simplifies to

$$(R_\Delta \chi f^{-1})(is/2, 0) \sim \int dt\ \left[3 - 2s - s^2 + 4t_1^2 + 4t_2^2 + 4t_3^2\right]^{-1}$$

$$\sim \int_0^\delta \frac{r^2 dr}{[\sigma + 4r^2]^\tau} \tag{17}$$

with $\sigma \to 0$ as $s \to 1^-$, as in Prop. 3.9.

Note that since $\mathbb{H}_3(\mathbb{R})$ has a cocompact discrete subgroup, $\mathbb{H}_3(\mathbb{Z})$, the construction of L^2 holomorphic functions in [GHS] yields other elements of $L^2\mathcal{O}(\{\varphi < \epsilon\})$, these having local peak points in the boundary (with no augmenting). Of course, in such a concrete example, one could also construct elements of $L^2\mathcal{O}$ by hand. Also note that a generic unimodular Lie group possesses no discrete cocompact subgroups, [M].

We end this example with a question: Is it possible to relate the representations we obtain here and those in [GHS] to the classified representations of $\mathbb{H}_3(\mathbb{R})$ in [F, T]?

Acknowledgments. The authors thank Frank Kutzschebauch for providing the impetus to do this work and for pointing out errors in a previous version. It is also a pleasure to thank Bernhard Lamel for helpful conversations and the Erwin Schrödinger Institute for its generous hospitality.

References

[AL] Arnal, D. Ludwig, J.: Q.U.P. and Paley-Wiener properties of unimodular, especially nilpotent, Lie groups, *Proc. of the AMS*, **125** (1997) 1071–1080

[B1] Brudnyi, A.: Representation of holomorphic functions on coverings of pseudoconvex domains in Stein manifolds via integral formulas on these domains. *J. Funct. Anal.* **231** (2006) 418–437

[B2] Brudnyi, A.: On holomorphic L^2 functions on coverings of strongly pseudoconvex manifolds. *Publ. Res. Inst. Math. Sci.* **43** (2007) 963–976

[B3] Brudnyi, A.: On holomorphic functions of slow growth on coverings of strongly pseudoconvex manifolds. *J. Funct. Anal.* **249** (2007) 354–371

[E] Engliš, M.: Pseudolocal estimates for $\bar\partial$ on general pseudoconvex domains, *Indiana Univ. Math. J.* **50** (2001) 1593–1607

[F] Folland, G.B.: Compact Heisenberg manifolds as CR manifolds, *J. Geom. Anal.* (2004) **14** 521–532

[FK] Folland, G.B., Kohn J.J.: The Neumann Problem for the Cauchy-Riemann Complex, *Ann. Math. Studies*, No. 75 Princeton University Press, Princeton, N.J. 1972

[FS] Folland, G.B., Stein, E.M.: Estimates for the $\bar\partial_b$-complex and analysis on the Heisenberg group, *Comm. Pure Appl. Math.* **27** (1974) 429–522.

[G] Grauert, H.: Bemerkenswerte pseudokonvexe Manninfaltigkeiten, *Math. Z.* **81** (1963) 377–391

[GHS] Gromov, M., Henkin, G., Shubin, M.: Holomorphic L^2 functions on coverings of pseudoconvex manifolds, *Geom. Funct. Anal.* **8** (1998) 552–585

[HHK] Heinzner, P., Huckleberry, A.T., Kutzschebauch, F.: Abels' theorem in the real analytic case and applications to complexifications. In: *Complex Analysis and Geometry*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker 1995, 229–273

[K1] Kohn, J.J.: Harmonic Integrals on strongly pseudoconvex manifolds, I, *Ann. of Math.* **78** (1963) 112–148

[K2] Kohn, J.J.: Harmonic Integrals on strongly pseudoconvex manifolds, II, *Ann. of Math.* **79** (1964) 450–472

[KN] Kohn, J.J.: Nirenberg, L.: Non-coercive boundary value problems *Comm. Pure Appl. Math.* **18** (1965) 443–492

[M] Margulis, G.A.: *Discrete Subgroups of Semisimple Lie Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 17, Springer-Verlag, Berlin, 1991

[P1] Perez, J.J.: The G-Fredholm property of the $\bar{\partial}$-Neumann problem, *J. Geom. Anal.* (2009) **19**, 87–106

[P2] Perez, J.J.: The Levi problem on strongly pseudoconvex G-bundles, *Ann. Glob. Anal. Geom.* (2010) **37**, 1–20

[P3] Perez, J.J.: A transversal Fredholm property for the $\bar{\partial}$-Neumann problem on G-bundles, `http://arxiv.org/abs/0912.4287` *to appear in Contemp. Math.*

[T] Thangavelu, S.: *Harmonic analysis on the Heisenberg group* Birkhäuser Boston; 1ed (1998)

[TCM1] Todor, R., Chiose, I., Marinescu, G.: L^2 holomorphic sections of bundles over weakly pseudoconvex coverings. *Geom. Dedicata* **91** (2002) 23–43

[TCM2] Todor, R., Chiose, I., Marinescu, G.: Morse inequalities for covering manifolds, *Nagoya Math. J.* **163** (2001) 145–165

Fakultät für Mathematik, Universität Wien, Vienna, Austria
E-mail address: `beppe.dellasala@gmail.com`

Fakultät für Mathematik, Universität Wien, Vienna, Austria
E-mail address: `joe_j_perez@yahoo.com`