

## **Convexity of Momentum Maps: A Topological Analysis**

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# Convexity of Momentum Maps: A Topological Analysis

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*Dedicated to Tudor S. Ratiu on the occasion of his 60th birthday and Alan D. Weinstein  
on the occasion of his “retirement” from UC Berkeley*

**Abstract.** We extend the Local-to-Global-Principle used in the proof of convexity theorems for momentum maps to not necessarily closed maps  $f: X \rightarrow Y$  whose target space  $Y$  carries a convexity structure which need not be based on a metric. Using a new factorization of  $f$ , convexity of its image is proved without local fiber connectedness, and for almost arbitrary spaces  $X$ .

## Introduction

Convexity for momentum maps was discovered independently by Atiyah [2] and Guillemin-Sternberg [14] in the case of a Hamiltonian torus action on a compact symplectic<sup>1</sup> manifold  $X$ . It was proved that the image of the momentum map  $\mu$  is a convex polytope, namely, the convex hull of  $\mu(X^T)$ , where  $X^T$  denotes the set of fixed points under the action of the torus  $T$ . In this case,  $\mu$  is open onto its image, and the fibers of  $\mu$  are compact and connected. Two years later, in 1984, Kirwan [20] (see also [15]) extended this result to the action of a compact connected Lie group  $G$ . Here the image of  $\mu: X \rightarrow \text{Lie}(G)^*$  has to be restricted to a closed Weyl chamber in a Cartan subalgebra of  $\text{Lie}(G)$ , i.e. a fundamental domain of  $G$  with respect to its coadjoint action on  $\text{Lie}(G)^*$ . Equivalently, this amounts to a composition of the momentum map  $\mu$  with the projection onto the quotient space  $Y := \text{Lie}(G)^*/G$  modulo the coadjoint action of  $G$ . Up to this time, convexity of  $\mu$  was proved by means of Morse theory, applied to the components of  $\mu$ . This works well as long as  $\mu$  is defined on a compact manifold  $X$ .

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<sup>1</sup>For an introduction to symplectic manifolds, their group actions, and momentum maps, see, e.g., [35, 32, 25, 1]

In 1988, Condevaux, Dazord, and Molino [11] reproved these results in an entirely new fashion. They factor out the connected components of the fibers of  $\mu$  to get a monotone-light factorization  $\mu: X \rightarrow \tilde{X} \rightarrow Y$  (see [23]). If  $\mu$  is proper, i.e. closed and with quasi-compact fibers, the metric of  $Y$  can be lifted to  $\tilde{X}$ . Then a shortest path between two points of  $\tilde{X}$  maps to a straight line in  $Y$ , which proves the convexity of  $\mu(X)$ . Based on this method, Hilgert, Neeb, and Plank [17] extended Kirwan’s result to non-compact connected manifolds  $X$  under the assumption that  $\mu$  is proper.

After this invention, the proof of convexity now splits into two parts: A geometric part where certain local convexity data have to be verified, and a topological part, a kind of “Lokal-global-Prinzip” [17] which proves global convexity à la Condevaux, Dazord, and Molino.

A further step was taken by Birtea, Ortega, and Ratiu [6, 7] who consider a closed, not necessarily proper map  $\mu: X \rightarrow \tilde{X} \rightarrow Y$ , defined on a normal, first countable, arcwise connected Hausdorff space  $X$ . The map  $\mu$  has to be locally open onto its image, locally fiber connected, having local convexity data. Using Vainšteĭn’s Lemma, they prove that the light part  $\tilde{X} \rightarrow Y$  of  $\mu$  is proper. This yields global convexity of  $\mu(X)$  for two almost disjoint kinds of target spaces  $Y$ , either the dual of a Banach space [7] (which implies that the closed unit ball of  $Y$  is weak\* compact), or a complete locally compact length metric space  $Y$  [6]. The second case applies to the cylinder-valued momentum map [28, 29], another invention of Condevaux, Dazord, and Molino [11]: For a symplectic manifold  $(X, \omega)$ , the 2-form  $\omega$  gives rise to a flat connection on the trivial principal fiber bundle  $X \times \text{Lie}(G)^*$  with holonomy group  $H$ . The cylinder-valued momentum map  $\bar{\mu}$  is obtained from  $\mu$  by factoring out  $\bar{H}$  from the target space  $Y$ . The new target space  $\bar{\mu}(X) = Y/\bar{H}$  is a cylinder, hence geodesics on it may differ from shortest paths. The convexity theorem then states that  $\bar{\mu}(X)$  is *weakly convex*, i.e. any two points of  $\bar{\mu}(X)$  are connected by a geodesic arc.

In the present paper, we analyse the topological part of convexity, that is, the passage from local to global convexity. We show that the Lokal-global-Prinzip, as developed thus far, admits a substantial improvement in at least three respects.

Firstly, we replace the monotone-light factorization  $f: X \rightarrow \tilde{X} \rightarrow Y$  that was used for a momentum map  $f = \mu$  by a new factorization

$$f: X \xrightarrow{q^f} X^f \xrightarrow{f^\#} Y$$

of any continuous map  $f: X \rightarrow Y$  which is locally open onto its image. In a sense,  $X^f$  is closer to  $Y$  than the leaf space  $\tilde{X}$  since  $q^f: X \rightarrow X^f$  factors through the monotone part  $X \rightarrow \tilde{X}$  of  $f$ . We show that  $q^f$  is an open surjection, while  $X^f$  admits a basis of open sets  $U$  such that  $f^\#$  maps  $U$  homeomorphically onto a subspace of  $Y$  (Proposition 5). Therefore,  $f^\#$  can take the rôle of the light part of  $f$ , which means that we can drop the assumption that  $f$  (the momentum map) is locally fiber connected.

Secondly, we concentrate on the target space  $Y$  instead of  $X$  to derive the desired properties of  $X^f$ . In this way, the various assumptions on  $X$  boil down to a single one, namely, its connectedness as a topological space. Nevertheless, we need no extra assumptions on the target space  $Y$ .

Thirdly, we merely assume that the map  $f^\#$  is closed, a much weaker condition than the closedness of  $f$ . Even the light part of  $f$  need not be closed. For example,  $f^\#$  is trivial for a local homeomorphism  $f$  - a light map which need not be closed, and with fibers of arbitrary size. Using the properties of  $Y$ , we prove that the fibers of  $f^\#$  are finite (Proposition 10), so that the convexity structure of  $Y$  can be lifted along  $f^\#$  (Theorem 2).

To make the interaction between convexity and topology more visible, we untie the Lokal-global-Prinzip from its metric context by means of a general concept of convexity, which might be of interest in itself. This also unifies the two above mentioned types of target space considered in [6] and [7]. In the linear case [7], the target space  $Y$  may be an arbitrary (not necessarily complete) metrizable locally convex space instead of a dual Banach space. (Metrizability can be weakened by the condition that  $Y$  does not contain a locally convex direct sum  $\mathbb{R}^{(\aleph_0)}$  as a subspace.) In general, geodesics in our (non-linear) target space  $Y$  are one-dimensional continua which need not be metrizable.

In previous versions of the Lokal-global-Prinzip, geodesic arcs or connecting lines between two points of the target space  $Y$  are obtained by a metric on  $Y$ . Without a concept of length, of course, geodesics are no longer available by shortening of arcs in the spirit of the Hopf-Rinow Theorem. Instead, we obtain geodesics by continued *straightening*, using a local convexity structure. In other words, we deal with a “manifold”, that is, a Hausdorff space  $Y$  covered by open subspaces  $U$  with an additional structure of convexity. The axioms of such a *convexity space*  $U$  are very simple: For any pair of points  $x, y \in U$ , there is a minimal connected subset  $C(x, y)$  containing  $x$  and  $y$ , varying continuously with the end points. In a topological vector space,  $C(x, y)$  is just the line segment between  $x$  and  $y$ , while in a uniquely geodesic space,  $C(x, y)$  is the unique shortest path between  $x$  and  $y$ . With respect to the  $C(x, y)$ , there is a natural concept of convexity, and for a convexity space  $U$ , we just require that the  $C(x, y)$  are convex and that  $U$  has a basis of convex open sets (see Definition 1).

If convexity is given by a metric, straightening and shortening of arcs leads to the same result, namely, a geodesic of minimal length. For a non-metrizable arc  $A$  between two points  $x$  and  $y$ , there is a substitute for the length of  $A$ , namely, the closed convex hull  $\overline{C(A)}$  which is diminished by straightening. As a first step, an inscribed “line path”  $L$  (in a geodesic sense) satisfies  $\overline{C(L)} \subset \overline{C(A)}$ , and  $\overline{C(L)}$  is the closed convex hull of the finitely many extreme points of  $L$ . For a given line path  $L$  between  $x$  and  $y$ , assume that the closed convex hull  $\overline{C(L)}$  is compact. Using Zorn’s Lemma, we minimize the connected set  $\overline{C(L)}$  to a compact convex set  $C$  with  $x, y \in C$ . In contrast to the Hopf-Rinow situation, where the shortening of

$L$  is achieved via the Arzelà-Ascoli Theorem, the straightening method needs the compactness of  $\overline{C(L)}$  to show that connectedness carries over to  $C$ . By the local convexity structure, it then follows that  $C$  contains a line path  $L_0$  between  $x$  and  $y$ . Thus if  $C = L_0$ , the line path  $L_0$  must be a geodesic.

So we require two properties to get the straightening process work: First, the closed convex hull of a finite set must be compact; second, a minimal compact connected convex set  $C$  containing  $x$  and  $y$  has to be a geodesic.

To establish a Lokal-global-Prinzip for continuous maps  $X \rightarrow Y$ , possible self-intersections of the arcs to be straightened have to be taken into account. Precisely, this means that closed convex subsets of  $Y$  have to be replaced by *étale* maps, i.e. closed locally convex maps  $e: C \rightarrow Y$ , such that the connected space  $C$  admits a covering by open sets mapped homeomorphically onto convex subsets of  $Y$ . We call  $Y$  a *geodesic manifold* if the above two properties hold with an adaption to étale maps  $e: C \rightarrow Y$ , that is, the second property now states that if  $C$  is compact and minimal with respect to  $x, y \in C$ , then  $e$  can be regarded as a geodesic with possible self-intersections. (Such a geodesic is transversal if and only if  $e = e^\#$ .) If the charts  $U$  of  $Y$  are regular Hausdorff spaces which satisfy a certain finiteness condition (see Definition 2) which holds, for example, if  $U$  is either locally compact or first countable, we call  $Y$  a *geodesic  $q$ -manifold* (the “ $q$ ” refers to the finiteness condition). Obvious examples of geodesic  $q$ -manifolds are complete locally compact length metric spaces, or metrizable locally convex topological linear spaces (Examples 6 and 7). Our main result consists in the following

**Lokal-global-Prinzip.** *Let  $f: X \rightarrow Y$  be a locally convex continuous map from a connected topological space  $X$  to a geodesic  $q$ -manifold  $Y$ . Assume that  $f^\#$  is closed. Then any two points of  $f(X)$  are connected by a geodesic arc.*

For an inclusion map  $f: C \hookrightarrow Y$ , the conditions on  $f$  turn into the assumptions of the Tietze-Nakajima Theorem (see [27]), i.e. the subset  $C$  is closed, connected, and locally convex. Thus in case of a locally convex topological vector space  $Y$ , the result for  $C \hookrightarrow Y$  yields Klee’s Convexity Theorem [21], while for a complete Riemannian manifold  $Y$ , we get a Theorem of Bangert [3].

## 1 Convexity spaces

Let  $X$  be a Hausdorff space. We endow the power set  $\mathfrak{P}(X)$  with a topology as follows. For any open set  $U$  of  $X$ , define

$$\tilde{U} := \{C \in \mathfrak{P}(X) \mid C \subset U\}. \quad (1)$$

The collection  $\mathfrak{B}$  of sets (1) is closed under finite intersection. We take  $\mathfrak{B}$  as a basis of open sets for the topology of  $\mathfrak{P}(X)$ .

**Definition 1.** Let  $X$  be a Hausdorff space together with a continuous map

$$C: X \times X \rightarrow \mathfrak{P}(X). \quad (2)$$

We call a subset  $A \subset X$  *convex* if  $C(x, y) \subset A$  holds for all  $x, y \in A$ . We say that  $X$  is a *convexity space* with respect to a map (2) if the following are satisfied.

- (C1) The  $C(x, y)$  are convex for all  $x, y \in X$ .
- (C2) The  $C(x, y)$  are minimal among the connected sets  $C \subset X$  with  $x, y \in C$ .
- (C3)  $X$  has a basis of convex open sets.

Note that (C1) implies that  $C(y, x) \subset C(x, y)$ . Hence  $C$  is symmetric:

$$C(x, y) = C(y, x). \quad (3)$$

From (C2) we infer that

$$C(x, x) = \{x\}. \quad (4)$$

Moreover, (C2) implies that every convexity space  $X$  is connected. The restriction of the map (2) to a convex subset  $A \subset X$  makes  $A$  into a convexity space. Hence (C3) implies that  $X$  is locally connected.

**Lemma 1.** *Let  $X$  be a convexity space. For  $x, y \in X$ , the set  $C(x, y) \setminus \{y\}$  is connected.*

*Proof.* Let  $A$  be the connected component of  $x$  in  $C(x, y) \setminus \{y\}$ . Since  $\{y\}$  is closed, every  $z \in C(x, y) \setminus \{y\}$  admits a convex neighbourhood  $U$  with  $y \notin U$ . Hence  $C(x, y) \setminus \{y\}$  is locally connected, and thus  $A$  is open in  $C(x, y)$ . Since  $C(x, y)$  is connected, it follows that  $A$  cannot be closed in  $C(x, y)$ . Thus  $y \in \overline{A}$ , which shows that  $A \cup \{y\}$  is connected. By (C2), this gives  $A \cup \{y\} = C(x, y)$ , whence  $A = C(x, y) \setminus \{y\}$ .  $\square$

As a consequence, the  $C(x, y)$  can be equipped with a natural ordering.

**Proposition 1.** *Let  $X$  be a convexity space. For  $x, y \in X$ , the set  $C(x, y)$  is linearly ordered by*

$$z \leq t : \Longleftrightarrow z \in C(x, t) \Longleftrightarrow t \in C(z, y) \quad (5)$$

for  $z, t \in C(x, y)$ .

*Proof.* For any  $z \in C(x, y)$ , the set  $C(x, z) \cup C(z, y)$  is connected. Therefore, (C1) and (C2) give

$$C(x, y) = C(x, z) \cup C(z, y). \quad (6)$$

To verify the second equivalence in (5), it suffices to show that

$$z \in C(x, t) \Rightarrow t \in C(z, y)$$

holds for  $z, t \in C(x, y)$ . By Eq. (6), it is enough to prove the implication

$$z \in C(x, t) \setminus \{t\} \Rightarrow t \notin C(x, z). \quad (7)$$

Assume that  $z \in C(x, t) \setminus \{t\}$ . Then Eq. (4) gives  $x \in C(x, t) \setminus \{t\}$ . Hence Lemma 1 and (C2) yield  $C(x, z) \subset C(x, t) \setminus \{t\}$ , which proves (7). Clearly, the relation (5) is reflexive and transitive. By (7), it is a partial order. Furthermore, (5) and (6) imply that it is a linear order.  $\square$

Note that the ordering of  $C(x, y)$  depends on the pair  $(x, y)$  which determines the initial choice  $x \leq y$ . Thus as an ordered set,  $C(y, x)$  is dual to  $C(x, y)$ .

**Example 1.** Let  $\Omega$  be a linearly ordered set. A subset  $I$  of  $\Omega$  is said to be an *interval* if  $a \leq c \leq b$  with  $a, b \in I$  implies that  $c \in I$ . The intervals  $\{c \in \Omega \mid c < b\}$  and  $\{c \in \Omega \mid c > a\}$  with  $a, b \in \Omega$  form a sub-basis for the *order topology* of  $\Omega$ . Note that an open set of  $\Omega$  is a disjoint union of open intervals. Therefore,  $\Omega$  is connected if and only if it is a *linear continuum*, i.e. if every partition  $\Omega = I \sqcup J$  into non-empty intervals  $I, J$  determines a unique element between  $I$  and  $J$ . With the order topology, a linear continuum  $\Omega$  is a locally compact convexity space with

$$C(x, y) = \{z \in \Omega \mid x \leq z \leq y\} \quad (8)$$

in case that  $x \leq y$ . Here the convex sets of  $\Omega$  are just the connected sets of  $\Omega$ .

**Example 2.** More generally, we define a *tree continuum* to be a Hausdorff space  $X$  for which every two points  $x, y \in X$  are contained in a smallest connected set  $C(x, y)$  such that each  $C(x, y)$  is a linear continuum, and  $X$  carries the finest topology for which the inclusions  $C(x, y) \hookrightarrow X$  are continuous. Thus  $U \subset X$  is open if and only if every  $x \in U$  is an “algebraically inner” point (see [22], §16.2), i.e. if for each  $y \in X \setminus \{x\}$ , there exists some  $z \in C(x, y) \setminus \{x\}$  with  $C(x, z) \setminus \{z\} \subset U$ . Then  $X$  is a convexity space. For example, every one-dimensional CW-complex without cycles is of this type.

**Example 3.** In the Euclidean plane  $\mathbb{R}^2$ , consider the solution curves  $c: \mathbb{R} \rightarrow \mathbb{R}^2$  of the differential equation  $y' = 3y^{\frac{3}{2}}$  (including the singular solution  $c: x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$ ). With the finest topology making the solution curves continuous,  $\mathbb{R}^2$  becomes a tree continuum. Here every point of the singular line is a branching point of order 4.

The following lemma is well-known (see [36], Theorem 26.15).

**Lemma 2.** *Let  $X$  be a connected topological space with an open covering  $\mathfrak{U}$ . For any pair of points  $x, y \in X$ , there is a finite sequence  $U_1, \dots, U_n \in \mathfrak{U}$  with  $x \in U_1$ ,  $y \in U_n$ , and  $U_i \cap U_{i+1} \neq \emptyset$  for  $i < n$ .*

**Proposition 2.** *Let  $X$  be a convexity space. For  $x, y \in X$ , the subspace  $C(x, y)$  is compact and carries the order topology.*

*Proof.* Let  $C(x, y) = \bigcup \mathcal{U}$  be a covering by convex open sets. By Lemma 2, there is a finite sequence  $U_1, \dots, U_n \in \mathcal{U}$  with  $x \in U_1, y \in U_n$ , and  $U_i \cap U_{i+1} \neq \emptyset$  for  $i < n$ . Hence  $C(x, y) = U_1 \cup \dots \cup U_n$ , which shows that  $C(x, y)$  is compact.

For  $u < v$  in  $C(x, y)$ , the sets  $C(x, u)$  and  $C(v, y)$  are compact, hence closed in  $C(x, y)$ . So the open intervals of  $C(x, y)$  are open sets in  $C(x, y)$ . Conversely, a convex open set in  $C(x, y)$  is an interval which must be an open interval since  $C(x, y)$  is connected.  $\square$

Up to here, we have not used the continuity of the map (2) in Definition 1.

**Proposition 3.** *Let  $X$  be a convexity space. The closure of any convex set  $A \subset X$  is convex.*

*Proof.* Let  $A \subset X$  be a convex set, and let  $x, y \in \overline{A}$  be given. For any  $z \in C(x, y)$ , we have to show that  $z \in \overline{A}$ . Suppose that there is a convex neighbourhood  $W$  of  $z$  with  $W \cap A = \emptyset$ . Then  $z \neq x, y$ . By Proposition 2, there exist  $u, v \in W \cap C(x, y)$  with  $u < z < v$ . Since  $C(x, u)$  and  $C(v, y)$  are compact, there are disjoint open sets  $U, V$  in  $X$  with  $C(x, u) \subset U$  and  $C(v, y) \subset V$  (see, e.g., [19], chap. V, Theorem 8). Hence  $C(x, y) \subset U \cup V \cup W$ . So there are neighbourhoods  $U' \subset U$  of  $x$  and  $V' \subset V$  of  $y$  with  $C(x', y') \subset U \cup V \cup W$  for all  $x' \in U'$  and  $y' \in V'$ . Choose  $x', y' \in A$ . Then  $C(x', y') \subset A$ , which yields  $C(x', y') \subset U \cup V$ , where  $x' \in U' \subset U$  and  $y' \in V' \subset V$ , contrary to the connectedness of  $C(x', y')$ .  $\square$

**Definition 2.** Let  $X$  be a convexity space. Define a *star* in  $X$  with *center*  $x \in X$  and *end set*  $E \subset X \setminus \{x\}$  to be a subspace  $S(x, E) := \bigcup \{C(x, z) \mid z \in E\}$  with  $C(x, z) \cap C(x, z') = \{x\}$  for different  $z, z' \in E$  such that  $S(x, E)$  carries the finest topology which makes the embeddings  $C(x, z) \hookrightarrow S(x, E)$  continuous for all  $z \in E$ . We call  $X$  *star-finite* if every closed star in  $X$  has a finite end set.

Thus every star is a tree continuum (Example 2). Recall that a topological space  $X$  is said to be a *q-space* [24] if every point of  $X$  has a sequence  $(U_n)_{n \in \mathbb{N}}$  of neighbourhoods such that every sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x_n \in U_n$  admits an accumulation point. For example, every locally compact space, and every first countable space  $X$  is a *q-space*.

**Proposition 4.** *Let  $X$  be a convexity space which is a q-space. Then  $X$  is star-finite.*

*Proof.* Let  $S(x, E)$  be a closed star in  $X$ , and let  $(U_n)_{n \in \mathbb{N}}$  be a sequence of neighbourhoods of  $x$  such that every sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x_n \in U_n$  has an accumulation point. Suppose that  $E$  is infinite. Since  $U_n \cap C(x, z) \neq \{x\}$  for all  $n \in \mathbb{N}$  and  $z \in E$ , we find a subset  $\{z_n \mid n \in \mathbb{N}\}$  of  $E$  and a sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x \neq x_n \in C(x, z_n) \cap U_n$ .



Thus  $(x_n)_{n \in \mathbb{N}}$  has an accumulation point  $z$ . Because of the star-topology,  $z$  cannot belong to  $S(x, E)$ , contrary to the assumption that  $S(x, E)$  is closed.  $\square$

**Example 4.** A topological vector space  $X$  is a convexity space with respect to straight line segments if and only if  $X$  is locally convex. Moreover, a locally convex space  $X$  is star-finite if and only if  $X$  does not contain a locally convex direct sum  $\mathbb{R}^{(\aleph_0)}$  as a subspace. In fact, every subspace  $\bigoplus_{x \in E} \mathbb{R}x$  of  $X$  with  $|E| = \aleph_0$  is complete ([31], II.6.2) and gives rise to a closed star  $S(0, E)$ . Conversely, let  $S(x, E) \subset X$  be a closed star with  $E$  infinite. Since finite dimensional subspaces of  $X$  are star-finite by Proposition 4, we can assume that  $E$  is linearly independent and  $x = 0$ . Then the subspace  $\bigoplus_{x \in E} \mathbb{R}x$  of  $X$  is a locally convex direct sum.

Note that every metrizable locally convex space  $X$  is first countable ([31], I, Theorem 6.1), hence star-finite by Proposition 4.

## 2 Local openness onto the image

For a topological space  $X$ , the infinitesimal structure at a point  $x$  is given by the set  $\mathfrak{D}_x$  of filters on  $X$  which converge to  $x$ . Let  $\mathfrak{F}(X)$  denote the set of all filters on  $X$ . We make  $\mathfrak{F}(X)$  into a topological space with a basis of open sets

$$\tilde{U} := \{\alpha \in \mathfrak{F}(X) \mid U \in \alpha\}, \quad (9)$$

where  $U$  runs through the class of open sets in  $X$ . Every continuous map  $f: X \rightarrow Y$  induces a map  $\mathfrak{F}(f): \mathfrak{F}(X) \rightarrow \mathfrak{F}(Y)$ . For an open set  $V$  in  $Y$ , we have

$$\mathfrak{F}(f)^{-1}(\tilde{V}) = \widetilde{f^{-1}(V)}, \quad (10)$$

which shows that  $\mathfrak{F}(f)$  is continuous. Consider the subspace

$$\mathfrak{D}(X) := \{(x, \alpha) \in X \times \mathfrak{F}(X) \mid \alpha \in \mathfrak{D}_x\} \quad (11)$$

of  $X \times \mathfrak{F}(X)$ . Note that for every  $x \in X$ , the neighbourhood filter  $\mathcal{U}(x)$  of  $x$  is the coarsest filter in  $\mathfrak{D}_x$ . Thus, regarding  $\mathfrak{D}_x$  as a subset of  $\mathfrak{D}(X)$ , we get a pair of continuous maps

$$X \xrightarrow{\mathcal{U}} \mathfrak{D}(X) \xrightarrow{\lim} X \quad (12)$$

with  $\lim(x, \alpha) := x$  and  $\lim \circ \mathcal{U} = 1_X$ . In particular,  $\mathfrak{D}_x = \lim^{-1}(x)$ .

For a continuous map  $f: X \rightarrow Y$ , the local behaviour at  $x \in X$  is given by the induced map  $\mathfrak{D}_x f: \mathfrak{D}_x \rightarrow \mathfrak{D}_{f(x)}$ . Thus we get an endofunctor  $\mathfrak{D}: \mathbf{Top} \rightarrow \mathbf{Top}$  of the category  $\mathbf{Top}$  of topological spaces with continuous maps as morphisms. The functor  $\mathfrak{D}$  is augmented by the natural transformation  $\lim: \mathfrak{D} \rightarrow 1$ . On the other hand, the equation  $\mathcal{U} \circ f = \mathfrak{D}(f) \circ \mathcal{U}$  holds if and only if  $f$  is open.

**Definition 3.** A continuous map  $f: X \rightarrow Y$  between topological spaces is said to be *locally open onto its image* [4] if every  $x \in X$  admits an open neighbourhood  $U$  such that the induced map  $U \rightarrow f(U)$  is open onto the subspace  $f(U)$  of  $Y$ . We call  $f$  *filtered* if  $f$  is locally open onto its image and  $\mathfrak{D}(f) \circ \mathcal{U}$  is injective.

For example, the identity map  $1_X: X \rightarrow X$  is filtered if and only if every point of  $X$  is determined by its neighbourhood filter, i.e. if  $X$  is a  $T_0$ -space. The following structure theorem holds for continuous maps which are locally open onto their image.

**Proposition 5.** *Let  $f: X \rightarrow Y$  be a continuous map which is locally open onto its image. Up to isomorphism, there is a unique factorization  $f = pq$  in **Top** into an open surjection  $q$  and a filtered map  $p$ . If  $f$  is filtered, then every point  $x \in X$  has an open neighbourhood which is mapped homeomorphically onto a subspace of  $Y$ .*

*Proof.* Consider the following commutative diagram

$$\begin{array}{ccccc} 1: X & \xrightarrow{\mathcal{U}} & \mathfrak{D}(X) & \xrightarrow{\lim} & X \\ \downarrow q^f & & \downarrow \mathfrak{D}(f) & & \downarrow f \\ f^\#: X^f & \xrightarrow{e} & \mathfrak{D}(Y) & \xrightarrow{\lim} & Y, \end{array}$$

where  $X^f$  is the image of  $\mathfrak{D}(f) \circ \mathcal{U}$ , regarded as a quotient space of  $X$ , and  $f^\# := \lim \circ e$ . We will prove that  $f = f^\# \circ q^f$  gives the desired factorization. Let us show first that  $q^f$  is open. Thus let  $U$  be an open set of  $X$ . We have to verify that  $(q^f)^{-1}q^f(U)$  is open in  $X$ . Since  $f$  is locally open onto its image, we can assume that the induced map  $U \rightarrow f(U)$  is open. Let  $x \in (q^f)^{-1}q^f(U)$  be given. Then  $q^f(x) \in q^f(U)$ . So there exists some  $y \in U$  with  $q^f(x) = q^f(y)$ , i.e.  $f(x) = f(y)$  and  $f(\mathcal{U}(x)) = f(\mathcal{U}(y))$ . Hence there is an open neighbourhood  $V \in \mathcal{U}(x)$  with  $f(V) \subset f(U)$ . Again, we can assume that the induced map  $V \rightarrow f(V)$  is open. Furthermore, there is an open neighbourhood  $U' \subset U$  of  $y$  with  $f(U') \subset f(V)$ , and  $f(U')$  is open in  $f(U)$ , hence in  $f(V)$ . Therefore,  $V' := V \cap f^{-1}(f(U'))$  is an open neighbourhood of  $x$  with  $f(V') = f(U')$ .

For any  $x' \in V'$ , there is a point  $y' \in U'$  with  $f(x') = f(y')$ . So the continuity of  $f$  implies that  $f(\mathcal{U}(x')) = f(\mathcal{U}(y'))$ , which gives  $q^f(x') = q^f(y')$ , and thus  $V' \subset (q^f)^{-1}q^f(U') \subset (q^f)^{-1}q^f(U)$ . This proves that  $q^f$  is open. Consequently,  $f^\#$  is locally open onto its image.

Since  $q^f$  is open, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{q^f} & X^f \\ \downarrow \mathcal{U} & & \downarrow \mathcal{U} \\ \mathfrak{D}(X) & \xrightarrow{\mathfrak{D}(q^f)} & \mathfrak{D}(X^f). \end{array}$$

Hence  $\mathfrak{D}(f^\#) \circ \mathcal{U} \circ q^f = \mathfrak{D}(f^\#) \circ \mathfrak{D}(q^f) \circ \mathcal{U} = \mathfrak{D}(f) \circ \mathcal{U} = e \circ q^f$ . Therefore,  $\mathfrak{D}(f^\#) \circ \mathcal{U} = e$ , which implies that  $f^\#$  is filtered.

Now let  $f = pq = p'q'$  be two factorizations with  $p, p'$  filtered and  $q, q'$  open. Then  $\mathfrak{D}(p') \circ \mathcal{U} \circ q' = \mathfrak{D}(p') \circ \mathfrak{D}(q') \circ \mathcal{U} = \mathfrak{D}(p) \circ \mathfrak{D}(q) \circ \mathcal{U} = \mathfrak{D}(p) \circ \mathcal{U} \circ q$ . Since  $\mathfrak{D}(p') \circ \mathcal{U}$  is injective, there exists a map  $e: E \rightarrow E'$  with  $q' = eq$ . Since  $q$  is open, the map  $e$  is continuous. So we get a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{q} & E & \xrightarrow{p} & Y \\ \parallel & & \downarrow e & & \parallel \\ X & \xrightarrow{q'} & E' & \xrightarrow{p'} & Y \end{array}$$

in **Top**. By symmetry, we find a continuous map  $e': E' \rightarrow E$  with  $q = e'q'$  and  $p' = pe'$ . Since  $q$  and  $q'$  are surjective,  $e$  must be a homeomorphism. This proves the uniqueness of the factorization.

Finally, let  $f: X \rightarrow Y$  be filtered. For a given  $x \in X$ , let  $U$  be an open neighbourhood such that the induced map  $r: U \rightarrow f(U)$  is open. Since  $i: U \hookrightarrow X$  is open, we have a commutative diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{\mathcal{U}} & \mathfrak{D}(X) \\ & \swarrow f & \uparrow i & & \uparrow \mathfrak{D}(i) \\ Y & & U & \xrightarrow{\mathcal{U}} & \mathfrak{D}(U) \\ & \searrow j & \downarrow r & & \downarrow \mathfrak{D}(r) \\ & & f(U) & \xrightarrow{\mathcal{U}} & \mathfrak{D}(f(U)) \end{array}$$

which shows that  $\mathfrak{D}(j) \circ \mathcal{U} \circ r = \mathfrak{D}(f) \circ \mathcal{U} \circ i$  is injective. Hence  $r$  is injective.  $\square$

In the sequel, we keep the notation of Proposition 5 and write

$$f: X \xrightarrow{q^f} X^f \xrightarrow{f^\#} Y \quad (13)$$

for the factorization of a map  $f$  which is locally open onto its image.

**Remarks. 1.** Although the factorization (13) is unique up to isomorphism, it does not give rise to a factorization system [12, 10], i.e. a pair  $(\mathcal{E}, \mathcal{M})$  of subcategories such that every commutative square

$$\begin{array}{ccc} E_1 & \xrightarrow{f_1} & M_1 \\ e \downarrow & \nearrow d & \downarrow m \\ E_0 & \xrightarrow{f_0} & M_0 \end{array} \quad (14)$$

with  $e \in \mathcal{E}$  and  $m \in \mathcal{M}$  admits a unique diagonal  $d$  with  $f_1 = de$  and  $f_0 = md$  (see [16], Proposition 1.4). Apart from the fact that local openness onto the image is not closed under composition (consider the maps  $\mathbb{R} \xrightarrow{i} \mathbb{R}^2 \xrightarrow{p} \mathbb{R}$  with  $i(x) = \begin{pmatrix} x \\ x^3 - 3x \end{pmatrix}$  and  $p: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto y$ ), there cannot be a factorization system since open surjections are not stable under pushout (take, e.g., the pushout of the open surjection  $\mathbb{R} \twoheadrightarrow \{0\}$  and the inclusion  $\mathbb{R} \hookrightarrow \mathbb{R}^2$ ).

**2.** If  $f: X \rightarrow Y$  is locally open onto its image and locally fiber connected [4, 17], the Lemma of Benoist ([4], Lemma 3.7) states that the monotone part  $\pi$  of the monotone-light factorization  $f = \tilde{f} \circ \pi$  is open. Here the local fiber-connectedness of  $f$  implies that  $\pi$  is locally open onto its image. Hence  $\pi = q^\pi$  is open by Proposition 5. In general,  $q^f$  always factors through  $\pi$ , but the two factorizations need not be isomorphic. For example, a local homeomorphism  $f: X \twoheadrightarrow Y$  is open, but its fibers are discrete.

### 3 Convexity of maps

In this brief section, we introduce local convexity and extend this concept from subsets to continuous maps (cf. [5] for a notion of convex maps in terms of paths).

**Definition 4.** Let  $X$  be a topological space. We define a *local convexity structure* on  $X$  to be an open covering  $X = \bigcup \mathfrak{U}$  by convexity spaces  $U \in \mathfrak{U}$  (with the induced topology) such that for any  $U \in \mathfrak{U}$ , every convex open subspace of  $U$  belongs to  $\mathfrak{U}$  (as a convexity space). We call a subset  $C \subset X$  *convex* if  $C \cap U$  is convex for all  $U \in \mathfrak{U}$ . We say that  $C$  is *locally convex* if every  $z \in C$  admits a neighbourhood  $U \in \mathfrak{U}$  such that  $C \cap U$  is convex.

The covering  $\mathfrak{U}$  will be referred to as the *atlas* of the local convexity structure. In the special case  $X \in \mathfrak{U}$ , the atlas  $\mathfrak{U}$  just consists of the convex open sets of a convexity space  $X$ .

In contrast to local convexity, our concept of convexity refers to all sets in  $\mathfrak{U}$ . So the intersection of convex sets is convex, and every subset  $A \subset X$  admits a *convex hull*  $C(A)$ , that is, a smallest convex set  $C \supset A$ . The next proposition generalizes Proposition 3.

**Proposition 6.** *Let  $X$  be a topological space with a local convexity structure  $\mathfrak{U}$ . The closure of any convex set  $A \subset X$  is convex.*

*Proof.* For every  $U \in \mathfrak{U}$ , we have  $\overline{A} \cap U = \overline{A \cap U} \cap U$ . This set is convex by Proposition 3. Hence  $\overline{A}$  is convex.  $\square$

Definition 4 admits a natural extension to continuous maps.

**Definition 5.** Let  $f: X \rightarrow Y$  be a continuous map between topological spaces, where  $Y$  has a local convexity structure  $\mathfrak{V}$ . We call  $f$  *locally convex* if every  $x \in X$  admits an open neighbourhood  $U$  such that the induced map  $U \rightarrow f(U)$  is open, and  $f(U)$  is a convex subspace of some  $V \in \mathfrak{V}$ .

**Remarks. 1.** A subset  $A \subset Y$  is locally convex if and only if the inclusion map  $A \hookrightarrow Y$  is locally convex.

**2.** The open neighbourhood  $U$  of  $x$  in Definition 5 can be chosen arbitrarily small. In fact, let  $U' \subset U$  be any smaller open neighbourhood of  $x$ . Then  $f(U')$  is an open subset of  $f(U)$ . Hence there exists some  $V' \in \mathfrak{V}$  with  $f(x) \in V' \cap f(U) \subset f(U')$ . Thus  $U'' := U' \cap f^{-1}(V')$  is an open neighbourhood of  $x$  with  $f(U'') = V' \cap f(U') = V' \cap f(U)$ , which is a convex subspace of  $V'$ .

**3.** If  $X$  is a connected Hausdorff space and  $Y$  a length metric space [9, 13], a continuous map  $f: X \rightarrow Y$  is locally convex if and only if  $f$  is locally open onto its image and has local convexity data in the sense of [6].

**Proposition 7.** Let  $f: X \rightarrow Y$  be a continuous map between topological spaces, where  $Y$  has a local convexity structure  $\mathfrak{V}$ . If  $f$  is locally convex, then  $f^\#$  is locally convex.

*Proof.* Assume that  $f$  is locally convex, and let  $U$  be an open neighbourhood of  $x \in X$  such that the induced map  $U \rightarrow f(U)$  is open onto a convex subspace of some  $V \in \mathfrak{V}$ . Since  $q^f$  is open by Proposition 5, this property of  $U$  carries over to the neighbourhood  $q^f(U)$  of  $q^f(x)$ . Hence  $f^\#$  is locally convex.  $\square$

## 4 Geodesic manifolds

In this section, we introduce a general concept of geodesic which does not refer to any kind of metric.

**Definition 6.** Let  $Y$  be a topological space with a local convexity structure  $\mathfrak{V}$ , and let  $e: C \rightarrow Y$  be a continuous map with a connected topological space  $C$ . By  $\mathfrak{V}_e$  we denote the set of all open sets  $U$  in  $C$  which are mapped homeomorphically onto a convex subspace of some  $V \in \mathfrak{V}$ . We call  $e$  *étale* if  $e$  is closed and  $\mathfrak{V}_e$  covers  $C$ . We say that  $e: C \rightarrow Y$  is *generated* by a subset  $F \subset C$  if there is no closed connected subspace  $A \subsetneq C$  with  $F \subset A$  such that  $e(U \cap A)$  is convex for all  $U \in \mathfrak{V}_e$ .

In particular, étale maps are locally convex. Furthermore, every étale map  $e: C \rightarrow Y$  induces a local convexity structure  $\mathfrak{V}_e$  on  $C$ . So the condition (Definition 6) that

$e(U \cap A)$  is convex for all  $U \in \mathfrak{V}_e$  just states that  $A$  is convex with respect to  $\mathfrak{V}_e$ . If  $F \subset C$  is connected, then  $C(F)$  is connected. Therefore, an étale map  $e: C \rightarrow Y$  is generated by a connected set  $F$  if and only if  $\overline{C(F)} = C$ . Note that the composition of étale maps is étale.

**Definition 7.** Let  $Y$  be a Hausdorff space with a local convexity structure  $\mathfrak{V}$ . We call  $Y$  a *geodesic manifold* if the following are satisfied.

- (G1) For a finite set  $F \subset Y$ , the closure of  $C(F)$  is compact.
- (G2) If an étale map  $e: C \rightarrow Y$  with  $C$  compact is generated by  $\{x, y\} \subset C$ , then every connected set  $A \subset C$  with  $x, y \in A$  coincides with  $C$ .

If, in addition, every  $V \in \mathfrak{V}$  is star-finite and regular (as a topological space), we call  $Y$  a *geodesic  $q$ -manifold*.

The letter “ $q$ ” is reminiscent of Proposition 4. Since a geodesic manifold  $Y$  is locally connected, [8], chap. I, 11.6, Proposition 11, implies that  $Y$  is the topological sum of its connected components.

**Definition 8.** Let  $Y$  be a geodesic manifold. We define a *geodesic* in  $Y$  to be an étale map  $e: C \rightarrow Y$ , generated by  $\{x, y\} \subset C$ , where  $C$  is compact. The points  $e(x)$  and  $e(y)$  will be called the *end points* of the geodesic.

More generally, we define a *line path* in  $Y$  to be a continuous map  $e: L \rightarrow Y$ , where  $L$  is a linear continuum (Example 1) with end points  $x_0$  and  $x_n$  and a sequence of intermediate points  $x_0 < x_1 < \dots < x_n$  such that for  $i < n$ , the restriction of  $e$  to the interval  $[x_i, x_{i+1}]$  is an inclusion which identifies  $[x_i, x_{i+1}]$  with  $C(e(x_i), e(x_{i+1})) \subset U_i$  for some  $U_i$  in the atlas of  $Y$ . If  $e$  is an inclusion, we speak of a *simple line path* and identify it with the subset  $L \subset Y$ . A subset  $A \subset Y$  will be called *line-connected* if every pair of points  $x, y \in A$  is connected by a simple line path  $L \subset A$ .

**Proposition 8.** Let  $Y$  be a geodesic manifold with atlas  $\mathfrak{V}$ , and let  $e: C \rightarrow Y$  be an étale map. Then  $C$  is line-connected.

*Proof.* Let  $x, y \in C$  be given. By Lemma 2, there is a sequence  $U_1, \dots, U_n \in \mathfrak{V}_e$  with  $x \in U_1, y \in U_n$ , and  $U_i \cap U_{i+1} \neq \emptyset$  for  $i < n$ . Choose  $x_i \in U_i \cap U_{i+1}$  for  $i < n$ . With  $x_0 := x$  and  $x_n := y$ , the  $C(x_i, x_{i+1})$  constitute a line path  $e: L \rightarrow Y$  in  $C$  which connects  $x$  and  $y$ . Assume that the interval  $[x, x_i] \subset L$  maps onto a simple line path  $L'$ . If  $C(x_i, x_{i+1})$  intersects  $L'$  in a point  $\neq x_i$ , there is a largest  $z \in C(x_i, x_{i+1})$  with property. Thus, if  $z'$  denotes the corresponding point on  $L'$ , we can replace the interval  $[z', z]$  by  $\{z\}$  and attach the segment  $C(z, x_{i+1})$ . After less than  $n$  modifications, we get a simple line path between  $x$  and  $y$ .  $\square$

By (G2), we have the following

**Corollary.** *Let  $Y$  be a geodesic manifold. Every geodesic with end points  $x, y \in Y$  is a line path.*

In particular, a simple geodesic with end points  $x, y \in Y$  is just a minimal connected set  $C \subset Y$  with  $x, y \in C$  which is locally convex.

Let  $Y$  be a geodesic manifold. For  $x, y \in Y$ , we define a *simple arc* between  $x$  and  $y$  to be a subspace  $A \subset Y$  which is a linear continuum with end points  $x$  and  $y$ . We fix a linear order on  $A$  such that  $x$  becomes the smallest element and denote the set of all such  $A$  by  $\Omega_Y(x, y)$ . In particular, every simple line path between  $x$  and  $y$  belongs to  $\Omega_Y(x, y)$ . Clearly, every  $A \in \Omega_Y(x, y)$  admits an inscribed line path  $L$  between  $x$  and  $y$ . Although there is no concept of length at our disposal, the intuition that  $L$  is “shorter” than  $A$  can be expressed by the inclusion  $\overline{C(L)} \subset \overline{C(A)}$ . Thus it is natural to define a preordering on  $\Omega_Y(x, y)$  by

$$A \prec B : \Longleftrightarrow \overline{C(A)} \subset \overline{C(B)}. \quad (15)$$

If  $A \prec B$  holds for a pair  $A, B \in \Omega_Y(x, y)$ , we say that  $A$  is a *straightening* of  $B$ . Define  $B \in \Omega_Y(x, y)$  to be *minimal* if  $A \prec B$  implies  $B \prec A$  for all  $A \in \Omega_Y(x, y)$ . We have the following straightening theorem which justifies the term “geodesic” manifold in Definition 7.

**Theorem 1.** *Let  $Y$  be a geodesic manifold. Every simple arc  $A \in \Omega_Y(x, y)$  in  $Y$  can be straightened to a minimal  $C \in \Omega_Y(x, y)$ . A simple arc  $A \in \Omega_Y(x, y)$  is minimal if and only if  $A$  is a convex simple geodesic.*

*Proof.* Let  $A \in \Omega_Y(x, y)$  be given. Since  $C(A)$  is connected,  $\overline{C(A)}$  is connected. Proposition 6 implies that  $\overline{C(A)}$  is convex. So the inclusion  $\overline{C(A)} \hookrightarrow Y$  is étale. By Proposition 8, there exists a simple line path  $L \subset \overline{C(A)}$  between  $x$  and  $y$ . Hence  $L \prec A$ . As  $L$  belongs to the convex hull of a finite set, (G1) implies that  $\overline{C(L)}$  is compact. We have to verify that  $\overline{C(L)}$  contains a minimal  $C \in \Omega_Y(x, y)$ . Let  $\mathcal{C}$  be a chain of compact convex connected sets  $C \subset \overline{C(L)}$  with  $x, y \in C$ . Then  $D := \bigcap \mathcal{C}$  is compact and convex, and  $x, y \in D$ . We show first that every open set  $V$  of  $Y$  with  $D \subset V$  contains some  $C \in \mathcal{C}$ . In fact, the set  $\overline{C(L)}$  is compact, and  $\bigcap_{C \in \mathcal{C}} (C \setminus V) = \emptyset$ . Hence  $C \setminus V = \emptyset$  for some  $C \in \mathcal{C}$ .

Next we show that  $D$  is connected. Suppose that there is a disjoint union  $D = D_1 \sqcup D_2$  with non-empty compact sets  $D_1$  and  $D_2$ . Then we can find open sets  $U_1$  and  $U_2$  in  $Y$  with  $D_i \subset U_i$  such that  $U_1 \cap U_2 = \emptyset$  (see, e.g., [19], chap. V, Theorem 8). Hence  $D \subset U_1 \sqcup U_2$ , which yields  $C \subset U_1 \sqcup U_2$  for some  $C \in \mathcal{C}$ . Since  $C$  is connected, we can assume that  $C \subset U_1$ . This gives  $D_2 \subset U_1 \cap U_2 = \emptyset$ , a contradiction. Thus  $D$  is connected. By Zorn’s Lemma, it follows that there exists a minimal compact convex connected set  $C$  with  $x, y \in C$ . Hence  $C \hookrightarrow Y$  is an

étale map generated by  $\{x, y\}$ . Therefore, (G2) implies that  $C$  admits no connected proper subset  $C' \subset C$  with  $x, y \in C'$ . By Proposition 8, it follows that  $C$  is a simple line path, whence  $C \in \Omega_Y(x, y)$ , and  $C$  is minimal.

In particular, we have shown that if  $A \in \Omega_Y(x, y)$  is minimal, then  $A$  is a convex simple geodesic between  $x$  and  $y$ . Conversely, if  $A \in \Omega_Y(x, y)$  is a convex simple geodesic, then  $A = \overline{C(A)}$ , and thus  $A$  is minimal.  $\square$

We conclude this section with some typical examples.

**Example 5.** Let  $Y$  be a geodesic manifold with atlas  $\mathfrak{V}$ , and let  $Z$  be a closed locally convex subspace. Then  $Z \hookrightarrow Y$  is étale. Every finite set  $F$  in  $Z$  is contained in a compact convex set  $C$  in  $Y$ . Hence  $C \cap Z$  is compact and convex in  $Z$ . Thus  $Z$  satisfies (G1). As (G2) trivially carries over to  $Z$ , it follows that  $Z$  is a geodesic manifold. If  $Y$  is a geodesic  $q$ -manifold, then so is  $Z$ .

**Example 6.** Let  $Y$  be a complete locally compact length metric space [9, 13]. By the Hopf-Rinow Theorem ([9], Proposition I.3.7), the closed metric balls in  $Y$  are compact, and any two points in  $Y$  are connected by a shortest path. It is natural to assume that  $Y$  admits a basis of convex open sets where shortest paths are unique. This provides  $Y$  with a local convexity structure  $\mathfrak{V}$  which satisfies (G1). Note that by [9], I.3.12, the map (2) is continuous where it is defined.

Now let  $e: C \rightarrow Y$  be an étale map generated by  $\{x, y\} \subset C$ , where  $C$  is compact. Similar to the case of a covering of length metric spaces ([9], Proposition I.3.25), the length metric  $d_Y$  of  $Y$  can be lifted to a length metric  $d_C$  of  $C$  such that  $d_C(u, v) \geq d_Y(e(u), e(v))$  for all  $u, v \in C$ . (If  $d_C(u, v) = 0$  with  $u \neq v$ , a neighbourhood  $U \in \mathfrak{V}_e$  of  $u$  cannot contain  $v$ . As  $U$  contains a closed neighbourhood of  $u$  in  $C$ , we get  $d_C(u, v) > 0$ .) Since  $C$  is compact, the Hopf-Rinow Theorem, applied to  $C$ , yields a shortest path  $L \subset C$  between  $x$  and  $y$ . Hence  $C = L$ , which proves (G2). By Proposition 4,  $Y$  is a geodesic  $q$ -manifold.

**Example 7.** Let  $Y$  be a locally convex topological vector space. For  $x, y \in Y$ , we set  $C(x, y) := \{\lambda x + (1 - \lambda)y \mid 0 \leq \lambda \leq 1\}$  to make  $Y$  into a convexity space. For a finite set  $F \subset Y$ , the closed convex hull  $\overline{C(F)}$  of  $F$  is contained in a finite dimensional subspace of  $Y$ . Hence  $\overline{C(F)}$  is compact. Thus  $Y$  satisfies (G1). Let  $e: C \rightarrow Y$  be an étale map generated by  $\{x, y\} \subset C$ , where  $C$  is compact. By Proposition 8,  $e$  is generated by a simple line path in  $C$ . Hence  $e(C)$  is contained in a finite dimensional subspace of  $Y$ . So Example 6 applies, which proves (G2). Thus  $Y$  is a geodesic manifold. Moreover, Example 4 shows that  $Y$  is a geodesic  $q$ -manifold if and only if  $Y$  does not contain a locally convex direct sum  $\mathbb{R}^{(\aleph_0)}$  as a subspace.



## 5 The Lokal-global-Prinzip

With respect to convex neighbourhoods, étale maps have the following disjointness property.

**Proposition 9.** *Let  $Y$  be a geodesic manifold with atlas  $\mathfrak{V}$ , and let  $e: C \rightarrow Y$  be an étale map. Assume that  $U, U' \in \mathfrak{V}_e$ . If  $e|_{U \cup U'}$  is not injective, then  $U \cap U' = \emptyset$ .*

*Proof.* If  $e|_{U \cup U'}$  is not injective, there exist  $x \in U$  and  $x' \in U'$  with  $e(x) = e(x')$ . Suppose that there is some  $z \in U \cap U'$ . Then  $x \neq z$ , and  $U \cap U' \cap C(x, z)$  is a convex open subset of  $C(x, z) \setminus \{x\}$ . Hence there is a point  $t \in C(x, z)$  with  $(U \setminus U') \cap C(x, z) = C(x, t)$ . So the homeomorphisms  $C(x, z) \cong C(e(x), e(z)) \cong C(x', z)$  give rise to a point  $t' \in U'$  with  $e(t) = e(t')$  and  $(U' \setminus U) \cap C(x', z) = C(x', t')$ . Moreover,  $D := C(t, z) \cup C(t', z) = C(t, z) \cup \{t'\}$  since  $e|_U$  is injective. Therefore,  $D$  is not a minimally connected superset of  $\{t, z\}$ . On the other hand,  $D$  is compact with open subsets  $C(t, z)$  and  $C(t', z)$ . Hence  $e|_D: D \rightarrow Y$  is an étale map generated by  $\{t, z\}$ , contrary to (G2).  $\square$

As an immediate consequence, the fibers of an étale map can be separated by pairwise disjoint neighbourhoods.

**Corollary 1.** *Let  $Y$  be a geodesic manifold, and let  $e: C \rightarrow Y$  be an étale map. For a given  $y \in Y$ , choose a neighbourhood  $U_x \in \mathfrak{V}_e$  of each  $x \in e^{-1}(y)$ . Then the  $U_x$  are pairwise disjoint.*

**Corollary 2.** *Let  $Y$  be a geodesic manifold, and let  $e: C \rightarrow Y$  be an étale map. Then  $C$  is a Hausdorff space.*

*Proof.* Let  $x, x' \in C$  be given. If  $e(x) \neq e(x')$ , there are disjoint neighbourhoods of  $e(x)$  and  $e(x')$ , and their inverse images give disjoint neighbourhoods of  $x$  and  $x'$ . So we can assume that  $e(x) = e(x')$ . Choose  $U, U' \in \mathfrak{V}_e$  with  $x \in U$  and  $x' \in U'$ . By Proposition 9,  $U \cap U' = \emptyset$ . Thus  $C$  is Hausdorff.  $\square$

If the geodesic manifold is regular, the fibers are even discrete, which leads to the following finiteness result.

**Proposition 10.** *Let  $e: C \rightarrow Y$  be an étale map into a geodesic  $q$ -manifold  $Y$ . Then the fibers of  $e$  are finite.*

*Proof.* Let  $\mathfrak{V}$  denote the atlas of  $Y$ , and let  $y \in Y$  be given. For each  $x \in e^{-1}(y)$ , we choose a neighbourhood  $U_x \in \mathfrak{V}_e$  such that the images  $e(U_x)$  are contained in a fixed  $V' \in \mathfrak{V}$ . By the Corollary 1, these neighbourhoods are pairwise disjoint.

Without loss of generality, we can assume that  $|C| > 1$ . Since  $C$  is a connected Hausdorff space by Corollary 2, this implies that  $C$  has no isolated points. As  $e$  is closed, the complement of  $\bigcup\{U_x \mid x \in e^{-1}(y)\}$  is mapped to a closed set  $A \subset Y$  with  $y \notin A$ . So there exists an open neighbourhood  $W \subset V'$  of  $y$  with  $e^{-1}(W) \subset \bigcup\{U_x \mid x \in e^{-1}(y)\}$ . By the regularity of  $Y$ , we find a convex open neighbourhood  $V$  of  $y$  with  $\overline{V} \subset W$ .

For any  $x \in e^{-1}(y)$ , the set  $U_x \cap e^{-1}(V)$  is an open neighbourhood of  $x$ , hence not a singleton. Therefore, the  $V_x := e(U_x \cap e^{-1}(V))$  are convex subsets of  $V$  with  $|V_x| > 1$  and  $y \in V_x$ . Choose arbitrary  $z_x \in U_x \cap e^{-1}(V)$  with  $y_x := e(z_x) \neq y$  for all  $x \in e^{-1}(y)$ . Now let  $Z \subset \bigcup\{C(x, z_x) \mid x \in e^{-1}(y)\}$  be such that  $Z \cap C(x, z_x)$  is closed in  $U_x \cap e^{-1}(V)$  for every  $x \in e^{-1}(y)$ . We claim that  $Z$  is closed. Thus let  $z \in \overline{Z}$  be given. Then  $e(z) \in e(\overline{Z}) \subset \overline{V} \subset W$ . Hence  $z \in e^{-1}(W) \subset \bigcup\{U_x \mid x \in e^{-1}(y)\}$ , which yields  $z \in Z$ . Thus  $Z$  is closed. Since  $e$  is closed, this implies that  $S(y) := \bigcup\{C(y, y_x) \mid x \in e^{-1}(y)\}$  is closed and carries the finest topology such that the maps  $C(y, y_x) \hookrightarrow S(y)$  are continuous for all  $x \in e^{-1}(y)$ .

Suppose that  $e^{-1}(y)$  is infinite. By Ramsey's Theorem [30], there must be an infinite subset  $E$  of  $e^{-1}(y)$  such that either  $C(y, y_u) \cap C(y, y_v) = \{y\}$  for all pairs of different  $u, v \in E$ , or  $C(y, y_u) \cap C(y, y_v) \neq \{y\}$  for different  $u, v \in E$ . The first case is impossible since  $V$  is star-finite by Definition 7. Otherwise, there is a point  $y' \in V \setminus \{y\}$  and a set  $Z \subset \bigcup\{C(x, z_x) \mid x \in e^{-1}(y)\}$  with  $|Z \cap C(x, z_x)| = 1$  for all  $x \in E$  such that  $e(Z)$  is an infinite non-closed subset of  $C(y, y')$ . Since  $Z$  is closed, this gives a contradiction.  $\square$

As a consequence, the geodesic structure of a geodesic  $q$ -manifold can be lifted along étale maps.

**Theorem 2.** *Let  $e: C \rightarrow Y$  be an étale map into a geodesic  $q$ -manifold  $Y$  with atlas  $\mathfrak{V}$ . Then  $C$  is a geodesic  $q$ -manifold with atlas  $\mathfrak{V}_e$ .*

*Proof.* By Corollary 2 of Proposition 9,  $C$  is a Hausdorff space. We show first that  $C$  is regular. Let  $U_x \in \mathfrak{V}_e$  be a neighbourhood of  $x \in C$ . We choose neighbourhoods  $U_z \in \mathfrak{V}_e$  for all  $z$  in the fiber of  $y := e(x)$ . By Corollary 1 of Proposition 9, the  $U_z$  are pairwise disjoint. Since  $Y$  is regular and  $e$  closed, there is a closed neighbourhood  $V$  of  $y$  with  $e^{-1}(V) \subset \bigcup\{U_z \mid z \in e^{-1}(y)\}$ . Hence

$$U_x \cap e^{-1}(V) = e^{-1}(V) \setminus \bigcup\{U_z \mid z \in e^{-1}(y) \setminus \{x\}\}$$

is a closed neighbourhood of  $x$ . Thus  $C$  is regular.

Let  $F \subset C$  be finite. Then  $\overline{C(e(F))}$  is compact. By Proposition 10, the fibers of  $e$  are compact. Hence  $e^{-1}(\overline{C(e(F))})$  is compact by [8], chap. I.10, Proposition 6. Furthermore,  $e^{-1}(\overline{C(e(F))})$  is convex with respect to  $\mathfrak{V}_e$ . Therefore, the closed subset  $\overline{C(F)}$  of  $e^{-1}(\overline{C(e(F))})$  is compact. This proves (G1) for  $C$ .

Next let  $e': C' \rightarrow C$  be an étale map with  $C'$  compact which is generated by  $\{x, y\} \subset C'$ . Then  $ee'$  is étale and generated by  $\{x, y\}$ . Hence  $C'$  is minimal among the connected sets  $B \subset C'$  with  $x, y \in B$ . Thus  $C$  satisfies (G2).

Finally, let  $S(x, E) := \bigcup \{C(x, z) \mid z \in E\}$  be a closed star in some  $U \in \mathfrak{V}_e$ . Since  $C$  is regular, we find a closed convex neighbourhood  $U' \subset U$  of  $x$ . By Proposition 3, this implies that  $S(x, E) \cap U'$  is a star in  $U$  which is closed in  $C$ . Therefore,  $e(S(x, E) \cap U')$  is a closed star in some  $V \in \mathfrak{V}$ . So  $E$  is finite, which proves that  $C$  is a geodesic  $q$ -manifold.  $\square$

Now we are ready to prove our main result which essentially states that the image of an étale map is weakly convex in the following sense (cf. [6], Definition 2.16).

**Definition 9.** Let  $Y$  be a geodesic manifold. We call a subset  $A \subset Y$  *weakly convex* if every pair of points  $x, y \in A$  can be connected by a geodesic.

The following theorem extends previous versions of the Lokal-global-Prinzip for convexity of maps (see [11, 17, 6, 7]).

**Theorem 3.** *Let  $f: X \rightarrow Y$  be a locally convex continuous map from a connected topological space  $X$  to a geodesic  $q$ -manifold  $Y$ . Assume that  $f^\#$  is closed. Then  $f(X)$  is weakly convex.*

*Proof.* Let  $\mathfrak{V}$  be the atlas of  $Y$ . By Proposition 7, the map  $f^\#$  again is locally convex, and Proposition 5 implies that  $f^\#$  is étale. By Theorem 2, it follows that  $X^f$  is a geodesic manifold. For  $z, z' \in X^f$ , Proposition 8 shows that there is a connecting simple line path  $L$  between  $z$  and  $z'$ . Theorem 1 shows that  $L$  can be straightened to a convex simple geodesic  $C$ . Thus  $f^\#|_C: C \rightarrow Y$  is a geodesic between  $f^\#(z)$  and  $f^\#(z')$ . Hence  $f(X)$  is weakly convex.  $\square$

In the special case where  $f$  is an inclusion  $X \hookrightarrow Y$ , the preceding proof yields

**Corollary.** *Let  $C$  be a closed connected locally convex subset of a geodesic manifold  $Y$ . Then  $C$  is weakly convex.*

*Proof.* By Example 5,  $C$  is a geodesic manifold, and  $C \hookrightarrow Y$  is étale. As in the proof of Theorem 3, this implies that  $C$  is weakly convex.  $\square$

**Remarks. 1.** If  $f$  is closed, then  $f^\#$  is closed. However, the latter condition is much weaker. For example, if  $f$  is a local homeomorphism, then  $f^\#$  is identical, but  $f$  need not be closed.

**2.** The preceding corollary extends Klee's generalization of a classical result due to Tietze [33] and Nakajima (Matsumura) [26]. Klee's Theorem [21] states that

the above corollary holds in a locally convex topological vector space  $Y$ . Note that the usual proof of Klee's Theorem rests on the linear structure of  $Y$ , while the corollary of Theorem 3 merely depends on a local convexity structure in the sense of Definition 4.

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