

Comments about Higgs Fields, Noncommutative Geometry and the Standard Model

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Abstract

We make a short review of the formalism that describes Higgs and Yang Mills fields as two particular cases of an appropriate generalization of the notion of connection. We also comment about the several variants of this formalism, their interest, the relations with noncommutative geometry, the existence (or lack of existence) of phenomenological predictions, the relation with Lie super-algebras etc.

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1. Introduction

This paper is written for those who are not willing to become experts in the field of noncommutative geometry but nevertheless want to understand the link between this approach and the usual formulation of the Standard Model of electro-weak interactions. The paper tries to give simple answers to the following questions:

- What has been done in this field so far?
- What are the several approaches?
- What are the results that are going to stay and what are those which are, for the moment, conjectural?
- Is there any relation between this and the formalism of super-connections (and with super-algebras)?

We also make a number of comments that may help the reader to see what is going on in this field.

The construction of the full standard model (with usual quarks and leptons but also with right neutrinos) is carried out by following the simplest possible route (at least the simplest, from the point of view of the present authors) and using an appropriate generalization of the notion of connection. The present paper can be considered as a sequel of [4] but can also be read independently; it should not be considered as an expository lecture on the approach initiated by [1].

2. The meaning of noncommutative geometry

From the point of view of Physics, one can summarize the situation very simply by saying that “commutative geometry” is the collection of mathematical tools describing classical physics whereas “non commutative geometry” is the collection of mathematical tools describing quantum physics.

Commutative geometry (or better “commutative mathematics”) deals with mathematical properties of spaces (measurable, topological, differentiable, riemannian, homogeneous...). For the physicist, these “spaces” provide a mathematical model for the system under study and all the properties of interest can be expressed in terms of an appropriate class of (numerical) functions defined on such spaces. It is a fact – physically obvious but also mathematically rooted – that properties of “spaces” are entirely encoded in terms of properties of algebras of numerical functions (coordinates for example) or of objects themselves defined from numerical functions (forms, tensors etc.) The name “commutative mathematics” comes from the fact that a set of numerical functions defined on a space is a commutative (and associative) algebra for pointwise multiplication and addition of functions.

Non commutative geometry (or better “non commutative mathematics”) deals with mathematical properties of algebras which are not necessarily commutative and generalizes – or tries to generalize – the constructions already known for commutative algebras (i.e. spaces) to non commutative situations (i.e. to operators).

This shows that one should maybe not always speak of “commutative geometry” or of “non commutative geometry” but of “commutative mathematics” or of “non commutative mathematics”. What we have in mind in the present paper is not the use of non commutative

mathematics in physics (because this could include, among many other things, the mathematics of quantum statistical mechanics) but non commutative differential geometry.

From an epistemological point of view, and once the concepts of commutative geometry and/or non commutative geometry have been mathematically studied, one should probably revert the first general statement of the present paragraph and define classical physics itself as a human activity characterized by the wish of understanding what we call "Nature" in terms of commutative mathematics and define quantum physics in the same way but where the models are now expressed in terms of non commutative mathematics. One could even go further and declare that only the choice of the mathematical model (or models) gives a meaning (meanings) to the whole thing (Nature) and that there is no such thing as "reality" per se... but we now abandon these philosophical considerations to return to the differential calculus, commutative or not.

3. Non commutative versus commutative differential calculus

A branch of non commutative differential geometry is non commutative differential calculus. The aim is to be able to consider objects like df or ∇f , i.e. differentials or covariant differentials, and to perform computations with them, assuming that f is no longer a function but an operator acting in some Hilbert space. Such a calculus has been developed in the recent years. There exist several kinds of non commutative differential calculi (for instance [2, 3, 4, 5, 9, 10]) and we do not intend here to describe them all. As a matter of fact, we shall describe none of them. Indeed, it turns out that a very simple by-product of this (these) generalization(s) gives us the necessary tools to understand Higgs fields as generalization of connections (Yang Mills fields). In some cases, for instance the $U(1) \times U(1)$ model studied in [4], this by-product actually belongs to the realm of commutative geometry because it involves only commutative algebras of functions on "spaces"! The point is that it was discovered historically [1] only after the new developpements of non commutative calculus. It is maybe a little bit misleading to call "non commutative" some of these considerations, first because, in simple cases, they are not so, and next because they give the reader the feeling that he should first master all (or most of) the niceties of non commutative differential calculus to understand the constructions. Of course, this is a matter of taste and some people could as well argue that one should always understand the general before going to the particular...

4. Commutative non local differential calculus

In the previous paragraph, we said that the constructions that are at the root of our understanding of Higgs field as generalized connections do not really belong to the realm of non commutative differential geometry (they are a by-product). They however correspond to some commutative - but non local - geometry. Let us see why.

Consider a discrete set $\{L, R\}$ with two elements that we call L and R . Call x the coordinate function $x(L) \doteq 1, x(R) \doteq 0$ and y the coordinate function $y(L) \doteq 0, y(R) \doteq 1$. Notice that $xy = yx = 0, x^2 = x, y^2 = y$ and $x + y = 1$ where 1 is the unit function $1(L) = 1, 1(R) = 1$. An arbitrary element of the associative (and commutative) algebra \mathcal{A} generated by x and y can be written $\lambda x + \mu y$ (where λ and μ are two complex numbers) and can be represented as a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. One can write $\mathcal{A} = \mathbb{C}x \oplus \mathbb{C}y$ and is isomorphic with $\mathbb{C} \oplus \mathbb{C}$.

We now introduce a differential δ satisfying $\delta^2 = 0, \delta 1 = 0$ and the usual Leibnitz rule, along

with formal symbols δx and δy . It is clear that Ω^1 , the space of differentials of degree 1 is generated by the two independent quantities $x\delta x$ and $y\delta y$. Indeed, the relation $x + y = 1$ implies $\delta x + \delta y = 0$, the relations $x^2 = x$ and $y^2 = y$ imply $(\delta x)x + x(\delta x) = (\delta x)$, therefore $(\delta x)x = (1-x)\delta x$ and $(\delta y)y = (1-y)\delta y$. This implies also, for example $\delta x = 1\delta x = x\delta x + y\delta x$, $x\delta x = -x\delta y$, $y\delta x = (1-x)\delta x$, $(\delta x)x = y\delta x = -y\delta y$ etc . More generally, let us call Ω^p , the space of differentials of degree p ; the above relations imply that a base of this vector space is given by $\{x\delta x\delta x \dots \delta x, y\delta y\delta y \dots \delta y\}$. Call $\Omega^0 = \mathcal{A}$ and $\Omega = \bigoplus_p \Omega^p$. This space Ω is an algebra: We can multiply forms freely but one of course has to take into account the Leibnitz rule, for instance $x(\delta x)x(\delta x) = x(1-x)(\delta x)^2$. Since each Ω^p is two dimensional we can easily represent it in terms of matrices. More precisely, we can represent the element $\lambda x(\delta x)^{2p} + \mu y(\delta y)^{2p}$ as the diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ and the the element $\alpha x(\delta x)^{2p+1} + \beta y(\delta y)^{2p+1}$ as the off diagonal matrix $\begin{pmatrix} 0 & i\alpha \\ i\beta & 0 \end{pmatrix}$. In other words we represent even forms by even (i.e. diagonal) matrices and odd forms by odd (i.e. off diagonal) matrices; doing so is not only natural but compulsory if we want the multiplication of matrices to be compatible with the multiplication in Ω . Indeed, the relations

$$\begin{aligned} x(\delta x)^{2p}x &= x(\delta x)^{2p} \\ x(\delta x)^{2p}y &= 0 \\ x(\delta x)^{2p+1}x &= 0 \\ x(\delta x)^{2p+1}y &= x(\delta x)^{2p+1} \end{aligned}$$

imply that the above representation using 2×2 matrices is indeed a homomorphism of \mathbb{Z}_2 -graded algebras from the algebra of universal forms Ω (graded by the parity of p) to the algebra of 2×2 complex matrices (with \mathbb{Z}_2 -grading associated with the decomposition of a matrix into diagonal and non diagonal components). The presence of a factor i in the off diagonal matrices representing odd elements (see above expressions) is necessary for the matrix product to be compatible with the product in Ω . Notice that the algebra Ω is infinite dimensional (since p ranges from 0 to infinity) but if we represent the whole of Ω in terms of 2×2 matrices acting on a fixed 2-dimensional vector space, the p grading is lost and only the \mathbb{Z}_2 grading is left. The differential δ obeys the usual Leibnitz rule when it acts on elements of \mathcal{A} but a graded Leibnitz rule when it acts on elements of Ω , namely $\delta(\omega_1\omega_2) = \delta(\omega_1)\omega_2 + (-1)^{\partial\omega_1}\omega_1\delta(\omega_2)$ where $\partial\omega_1$ denotes 0 or 1 depending if ω_1 is even or odd.

A one-form (this will be interpreted as a Higgs field and can be seen to define a generalized connection) is an element of Ω^1 . Take $A \doteq (\varphi x\delta x + \bar{\varphi} y\delta y)$. The matrix representation of A reads therefore

$$A = \begin{pmatrix} 0 & i\varphi \\ i\bar{\varphi} & 0 \end{pmatrix}$$

The corresponding curvature is then $F \doteq \delta A + A^2$, but $A^2 = -\varphi\bar{\varphi}x^2\delta x\delta x - \bar{\varphi}\varphi y^2\delta y\delta y = -\varphi\bar{\varphi}x\delta x\delta x - \bar{\varphi}\varphi y\delta y\delta y$ and $\delta A = \varphi\delta x\delta x + \bar{\varphi}\delta y\delta y = (\bar{\varphi} + \varphi)(x\delta x\delta x + y\delta y\delta y)$, so that the curvature can also be written

$$F = (\varphi + \bar{\varphi} - \varphi\bar{\varphi})(x\delta x\delta x + y\delta y\delta y) = \begin{pmatrix} \varphi + \bar{\varphi} - \varphi\bar{\varphi} & 0 \\ 0 & \varphi + \bar{\varphi} - \varphi\bar{\varphi} \end{pmatrix}$$

We now chose a hermitian product on Ω by declaring the base $x(\delta x)^p, y(\delta y)^q$ to be orthonormal. Then $|F|^2 = \bar{F}F = (\varphi + \bar{\varphi} - \varphi\bar{\varphi})^2$. One recognizes here a (shifted) Higgs potential. The previous calculation (expressed in the language of K -cycles) is already discussed in [1, 2] and can be recognized in [4] where it is written in the language of 2×2 matrices.

The previous construction could of course be generalized. For instance, we could take three points rather than two. It is easy to show that in such a case, Ω^1 is of dimension 6 and Ω^2 of dimension 12. If we take q points the dimension of Ω^p is $q(q-1)^p$. More generally, if we take infinitely many points – take for instance points belonging to a manifold X – it is easy to see that elements of Ω^1 can be defined as functions $A(x, y)$ of two variables on X such that $A(x, x) = 0$ and that elements of Ω^2 can be defined as functions $F(x, y, z)$ of three variables on X such that $F(x, y, y) = F(x, x, y) = 0$.

In the case of the geometry on the discrete set $\{L, R\}$ – that is our main example in the present paper – we recover the fact that an element A of Ω^1 considered as a function of two variables should satisfy the constraints $A(L, L) = A(R, R) = 0$ and can therefore be written as off-diagonal 2×2 matrices indexed by L and R . An element F of Ω^2 considered as a function of three variables should satisfy the constraints $F(L, L, R) = F(R, R, L) = F(L, R, R) = F(R, L, L) = F(R, R, R) = F(L, L, L) = 0$ so that non zero components are $F(L, R, L)$ and $F(R, L, R)$. The fact that $\dim(\Omega^p) = 2$ for all p explains that we can use a representation of fixed dimension (namely 2×2 matrices) for all values of p but one should maybe remember that it would not be so if we were considering a geometry on more than 2 points.

Notice that we are here doing commutative differential calculus (because the associative algebra of functions on a set of 2 elements is just the commutative algebra of diagonal 2 by 2 matrices with real or complex entries) but that we are doing a non local differential calculus because the distance between the two points labelled L and R can not be made infinitesimally small. The reader will have recognized that one can interpret the above results in terms of Higgs fields. This is the subject of our next section.

5. What are the Higgses ? The Yukawa interaction term

Higgs fields (φ) allow left and right fermions (ψ) to communicate. In four dimensional Minkowski space, this is clear from the trilinear Yukawa couplings such as $\overline{\psi}_L \varphi \psi_R + h.c.$ that appear in the Lagrangian density of the Standard Model. This should be contrasted with terms like $\overline{\psi}_L \gamma^\mu A_\mu \psi_L + h.c.$ or $\overline{\psi}_R \gamma^\mu B_\mu \psi_R + h.c.$ where A_μ or B_μ denote usual Yang Mills gauge fields. If we had no Higgs fields, of course we would have no mass term but also no possible communication (interaction) between right and left. There would be no justification for choosing a single connected manifold to modelize our universe. We would have a Minkowski space-time for the right movers and a Minkowski space-time for the left movers. Existence of chirality in four dimensions leads therefore to the conclusion that we live in two parallel universes, one labelled by L and the other by R . Usual connections – Yang Mills fields – connect (infinitesimally) L and L together and R and R together whereas Higgs fields are non local connections that connect L and R and allow us to identify the two copies of our universe.

As explained in all books of particle physics, the scalar interaction (Yukawa) of quarks is a priori of the form

$$\mathcal{L} = (\overline{U}_L \overline{D}_L) \begin{pmatrix} \varphi_+ \\ \varphi_0 \end{pmatrix} \mathcal{M}_D D_R + (\overline{U}_L \overline{D}_L) \begin{pmatrix} \overline{\varphi}_0 \\ -\varphi_- \end{pmatrix} \mathcal{M}_U U_R + h.c.$$

where all quark fields of charge $2/3$ are collected into the multi-spinor field $U = (u c t)^t$ and similarly for quarks of charge $-1/3$ with $D = (d s b)^t$. The 3×3 complex matrices \mathcal{M}_D and \mathcal{M}_U encode all the dimensionless Yukawa coupling constants (here spinor and scalar fields have their usual dimensions, namely $3/2$ and 1 in units of mass). If we expand the previous

expression, we find

$$\mathcal{L} = (\overline{U}_L \mathcal{M}_U \overline{\varphi}_0 U_R + \overline{U}_R \mathcal{M}_U^\dagger \varphi_0 U_L) + (\overline{D}_L \mathcal{M}_D \varphi_0 D_R + \overline{D}_R \mathcal{M}_D^\dagger \overline{\varphi}_0 D_L) + (\overline{U}_L \mathcal{M}_D \varphi_+ D_R + \overline{D}_R \mathcal{M}_D^\dagger \varphi_- U_L) - (\overline{U}_R \mathcal{M}_U^\dagger \varphi_+ D_L + \overline{D}_L \mathcal{M}_U \varphi_- U_R)$$

Let us now collect *all* left-handed quark fields (of charge 2/3 and -1/3) of the standard model into a single spinor $\Psi_L = (U_L D_L)$ and *all* right-handed quark fields into a single spinor $\Psi_R = (U_R D_R)$. Here $U = (u\ c\ t)$ and $D = (d\ s\ b)$. The above Yukawa interaction term reads

$$\mathcal{L} = (\overline{\Psi}_L \quad \overline{\Psi}_R) \begin{pmatrix} 0 & \Phi \\ \Phi^\dagger & 0 \end{pmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$$

where

$$\Phi = \begin{pmatrix} \mathcal{M}_U \overline{\varphi}_0 & \mathcal{M}_D \varphi_+ \\ -\mathcal{M}_U \varphi_- & \mathcal{M}_D \varphi_0 \end{pmatrix}$$

The mass term is obtained by shifting φ_0 and $\overline{\varphi}_0$ by a real constant with dimension of a mass that we call $\nu/\sqrt{2}$. The mass term itself is therefore described by the mass matrix

$$\mathcal{M} = \nu/\sqrt{2} \begin{pmatrix} \mathcal{M}_U & 0 \\ 0 & \mathcal{M}_D \end{pmatrix}$$

Writing $\Psi = (\Psi_L, \Psi_R)^t$, the whole fermionic lagrangian, for quarks, reads $\overline{\Psi}(\mathcal{D} + \mathcal{A})\Psi$ with

$$\mathcal{D} + \mathcal{A} = \begin{pmatrix} i\gamma^\mu \partial_\mu & -\mathcal{M} \\ -\mathcal{M}^\dagger & i\gamma^\mu \partial_\mu \end{pmatrix} + \begin{pmatrix} \gamma^\mu L_\mu & \Phi \\ \Phi^\dagger & \gamma^\mu R_\mu \end{pmatrix}$$

Where L_μ and R_μ collectively refer to those components of the gauge fields coupled to the left and right handed sectors and Φ collectively refers as before to Higgs fields couplings. The scalar interaction (Yukawa) of leptons is exactly of the same type. The only possible difference is that, in the minimal Standard Model, one does not usually add right neutrinos. We shall actually add such right neutrinos: They will not be coupled to the gauge fields, of course, but they will give a mass to the different kinds of Dirac neutrinos and will be also coupled between themselves – via mass matrices – and to the Higgses (and also therefore, in the unitary gauge, to the longitudinal part of the gauge bosons). Notice that we do not consider Majorana neutrinos. Introducing right neutrinos, not only allows us to use the same formalism for quarks and leptons (the only difference in the Yukawa interaction term is the replacement of matrices \mathcal{M}_D and \mathcal{M}_U by \mathcal{M}_E and \mathcal{M}_ν respectively) but also, as we shall see later, simplifies our analysis. The Yukawa interaction for leptons is

$$\mathcal{L} = (\overline{\nu}_L \overline{E}_L) \begin{pmatrix} \varphi_+ \\ \varphi_0 \end{pmatrix} \mathcal{M}_E E_R + (\overline{\nu}_L \overline{E}_L) \begin{pmatrix} \overline{\varphi}_0 \\ -\varphi_- \end{pmatrix} \mathcal{M}_\nu \nu_R + h.c.$$

where all leptons fields of charge -1 are collected into the multi-spinor field $E = (e\ \mu\ \tau)^t$ and similarly for the neutrinos $\nu = (\nu_e\ \nu_\mu\ \nu_\tau)^t$. The 3×3 complex matrices \mathcal{M}_E and \mathcal{M}_ν encode all the Yukawa coupling constants. The whole fermionic lagrangian, for leptons, reads as before, but with

$$\Phi = \begin{pmatrix} \mathcal{M}_\nu \overline{\varphi}_0 & \mathcal{M}_E \varphi_+ \\ -\mathcal{M}_\nu \varphi_- & \mathcal{M}_E \varphi_0 \end{pmatrix}$$

In the standard model, one should consider simultaneously not only the three generations of leptons but also three copies (for color) of the three generations of quarks. Taking into account – as above – the presence of three kinds of right neutrinos, we get an interaction

term $\overline{\Psi}(\mathcal{D} + \mathcal{A})\Psi$, with $\Psi = (\Psi_L \Psi_R)$ and where both Ψ_L and Ψ_R are multi-spinor fields – they are column vectors with 24 components (since $24 = 3 + 3 + 3(3 + 3)$), each component being itself a Weyl fermion.

In the spirit of noncommutative geometry, one should think of $\mathcal{D} + \mathcal{A}$ as a generalization of the Dirac operator (it incorporates masses and Yukawa couplings) coupled to an algebraic connection. It should be called the Dirac - Yukawa operator. The first piece in this expression is a generalized differential operator since the mass matrix \mathcal{M} appears as the inverse of a quantity encoding a discrete set of fundamental lengths. The second piece \mathcal{A} is a generalized connection: it incorporates both Yang-Mills and Higgs fields.

6. The bosonic lagrangian

The theory of – usual – connections explains why $F = dA + A^2$ is the natural object (curvature) associated with a Yang-Mills field. The root of the explanation being that the square of the corresponding covariant differential is a linear object whose expression is precisely given by the above formula. In the same way, and as explained (section 4) in a very simple case, the theory of generalized connections shows that $\varphi + \overline{\varphi} + \varphi\overline{\varphi}$ is the natural object (curvature) associated with the Higgs field φ introduced in the section 4 and defined as a non local connection on the discrete set $\{L, R\}$.

Now, we do not have a discrete set but a space $X = M_L \cup M_R$ that is the union of space-time for left-handed movers and space-time for right-handed movers, in other words, we have the product of Minkowski space by a discrete set of two elements called L and R . The generalized curvature \mathcal{F} associated with the generalized connection \mathcal{A} introduced in the previous paragraph is

$$\mathcal{F} = \begin{pmatrix} \mathcal{F}^{LL} & \mathcal{F}^{LR} \\ \mathcal{F}^{RL} & \mathcal{F}^{RR} \end{pmatrix}$$

With

$$\begin{aligned} \mathcal{F}^{LL} &= F^{LL} - ((\Phi + \Phi^\dagger)\nu/\sqrt{2} + \Phi\Phi^\dagger) \\ \mathcal{F}^{RR} &= F^{RR} - ((\Phi + \Phi^\dagger)\nu/\sqrt{2} + \Phi^\dagger\Phi) \\ \mathcal{F}^{LR} &= \nabla\Phi + i(L - R)\nu/\sqrt{2} \\ \mathcal{F}^{RL} &= \nabla\Phi^\dagger - i(L - R)\nu/\sqrt{2} \end{aligned}$$

and

$$\begin{aligned} \nabla\Phi &= \gamma^\mu(\partial_\mu\Phi + i(L_\mu\Phi - \Phi R_\mu)) \\ L &= \gamma^\mu L_\mu \\ F^{LL} &= \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}^L \\ R &= \gamma^\mu R_\mu \\ F^{RR} &= \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}^R \end{aligned}$$

The symbols F^L and F^R denote the usual curvatures of Yang-Mills fields associated with hermitian fields L and R . The expression of matrix elements of \mathcal{F} given before is a non trivial consequence of the formalism of non commutative geometry (or of a non local commutative differential calculus!) and can here be taken as a definition. These expressions can indeed be computed from the theory of general connections (commutative or not). The components of the curvature were obtained first by [1]. Up to different normalization factors and the presence of spurious fields, their expression agrees with the one given just above. This analysis was later improved in [2] (replacement of the so-called algebra Ω by Ω_D). A detailed exposition of the formalism of [2] using K-cycles and Dixmier trace can now be found in several places [3, 16, 17]. The matrix elements of \mathcal{F} given above were obtained by [4, 7] in a simple way

(and using the above notations). Our method is briefly recalled in one of the “comments” of section 8.

Notice that the above expressions for \mathcal{F} have a dimension of a mass squared and that, as a consequence, an arbitrary mass scale $\nu/\sqrt{2}$ appears in the formula. Explicitly, the term $(\Phi + \Phi^\dagger)\nu/\sqrt{2} + \Phi\Phi^\dagger$ and its adjoint can be computed from the expressions of Φ given previously, both in the quark and leptonic sectors. Up to a normalization factors (we shall come back later to this physically important problem) one recognizes that the trace of $\overline{\mathcal{F}}\mathcal{F}$ is nothing else than the lagrangian describing the bosonic sector of the standard model: One obtains directly the expression that usually comes after a shift by $\nu/\sqrt{2}$ in the Higgs fields φ_0 and $\overline{\varphi}_0$ (see [4] for a discussion of this point).

In a sense, the discussion could stop at this point. Indeed, we have seen in section 5 how to re-write the Dirac-Yukawa interaction term of fermions and in this section how to recover the whole bosonic sector of the Standard Model by treating Yang Mills fields together with Higgs fields as different components of a generalized connection. However, there are several claims made in the literature about possible constraints on the parameters of the lagrangian that one could obtain thanks to a formalism of non commutative geometry. Because we want to clarify this point (at least in the present formalism) we shall continue the discussion a little further.

The whole discussion comes actually from our understanding of the notation $Tr\overline{\mathcal{F}}\mathcal{F}$ that should denote a real number. From the one hand, if we decide to introduce, by hand, as many arbitrary constants in the expansion of this quantity (that gives rise to the full bosonic lagrangian of the standard model) as gauge invariance allows, we recover exactly the standard model with the same (unpredictive) relations as usual, namely $M_H = \nu\sqrt{\lambda}$, $M_W = \nu g/2 = gM_H/(2\sqrt{\lambda})$ and $M_Z = M_W/\cos\theta$ where g , θ , ν and λ are undetermined. If, on the other hand, we decide to introduce a *unique* constant $1/g^2$ in front of $Tr\overline{\mathcal{F}}\mathcal{F}$ in order to normalize simultaneously all the gauge fields and Higgs fields, we obtain non trivial relations. The interest of the formalism of non commutative differential geometry is not, for us, tied up with the existence of such relations; it may be, however, that such relations turn out to acquire, some day, a better status. For this reason, and also because the reader may be interested, we shall devote the end of this section to discuss them.

After global multiplication by $1/g^2$, we can rescale gauge fields as usual by $A \rightarrow gA$ and also the Higgs fields by $\Phi \rightarrow g\Phi$. Under identification with the usual lagrangian one obtains immediately $g^2 = \lambda$; this relation is quite natural from a point of view that identifies gauge fields and Yang Mills fields as different components of a generalized connection. In that case, the first general relation giving M_H is not modified but the second relation becomes $M_W = M_H/2$. Moreover, as we shall see below, the value of θ also gets constrained.

Rather than writing again in full the well known bosonic lagrangian of the Standard Model, we shall examine several of the terms, as they appear here. First of all, notice that one can identify the two sides of

$$\lambda\left\{(\varphi + \overline{\varphi})\frac{\nu}{\sqrt{2}} + \varphi\overline{\varphi}\right\}^2 \equiv \mu^2\left(\varphi + \frac{\nu}{\sqrt{2}}\right)^2 + \lambda\left(\varphi + \frac{\nu}{\sqrt{2}}\right)^4 - \frac{\mu^4}{4\lambda}$$

provided $\nu^2 \equiv -\mu^2/\lambda$. The mass value for the Higgs particle coming from this usual expression is $M_H = \nu\sqrt{\lambda}$. Notice also that the left hand side contains no additive constant (absence of cosmological term).

In our case, the Higgs potential itself coming from $\frac{1}{g^2}Tr(\overline{\mathcal{F}}\mathcal{F})$ reads,

$$V(\Phi) = \frac{2}{g^2}Tr((\Phi + \Phi^\dagger)\nu/\sqrt{2} + \Phi\Phi^\dagger)((\Phi + \Phi^\dagger)\nu/\sqrt{2} + \Phi\Phi^\dagger)^\dagger$$

If we now express Φ in terms of the component Higgs fields and in terms of the matrices of Yukawa couplings then remove the factor $1/g^2$, in front, by rescaling the fields, we see that $V(\Phi)$ contains a term equal to $\nu^2 \varphi_0 \bar{\varphi}_0 \text{Tr}(M_U M_U^\dagger + M_D M_D^\dagger)$ but the term $\text{Tr} \nabla \Phi \nabla \Phi^\dagger$ leads to a kinetic term for ϕ_0 equal to $2 \nabla \varphi_0 \nabla \varphi_0 (\text{Tr} M_U M_U^\dagger + \text{Tr} M_D M_D^\dagger)$ so that the mass of the Higgs field does not depend on the mass of fermions and stays undetermined (remember that ν is a free parameter). Other authors [17], using a different formalism find quite stringent constraints relating M_H to the fermionic masses.

The full bosonic interaction contains also a term $\mathcal{F}^{LR^\dagger} \mathcal{F}^{RL}$; using the previous expression for \mathcal{F}^{LR} implies that the field $L - R$ becomes massive, as it should. Indeed it corresponds to the Z and W bosons. One may adopt the point of view that the present formalism dictates a particular value for the Weinberg angle; this value turns out to depend upon the fermionic content of the theory. Indeed, the gauge fields L and R consist of three copies of

$$\begin{aligned} L &= \frac{i}{\sqrt{2}} \vec{\tau} \vec{W} + \frac{i}{\sqrt{x}} \begin{pmatrix} y & \\ & y \end{pmatrix} B \\ R &= 0 + \frac{i}{\sqrt{x}} \begin{pmatrix} y+1 & \\ & y-1 \end{pmatrix} B \end{aligned}$$

Here $y = 1/3$ for quarks since their weak hypercharge is equal to $(y = \frac{1}{3}, y = \frac{1}{3}; y+1 = \frac{4}{3}, y-1 = -\frac{2}{3})$ and $y = 0$ for leptons since their weak hypercharge is equal to $(y = -1, y = -1; y+1 = 0, y-1 = -2)$. We are introducing here right neutrinos that are isospin singlets and for which $y = 0$.

For colourless quarks alone, the normalization $\left(\frac{1}{\sqrt{x}}\right)^2 \left[\frac{1}{3}^2 + \frac{1}{3}^2 + \frac{4}{3}^2 + \left(-\frac{2}{3}\right)^2\right] = 1$

would lead to $x = 22/9$ and $tg\theta = 3/\sqrt{11}$

For leptons alone, the normalization $\left(\frac{1}{\sqrt{x}}\right)^2 \left[(-1)^2 + (-1)^2 + 0^2 + (-2)^2\right] = 1$

would lead to $x = 6$ and $tg\theta = 1/\sqrt{3}$

More generally, if one uses an arbitrary representation and normalize fields L and R to 1 as above, one finds

$$tg^2\theta = 4\text{Tr}I_3^2/\text{Tr}Y^2$$

which, in the case of three families of quarks (with color) and leptons, gives $tg^2\theta = 3/5$ (or $\sin^2\theta = 3/8$) as it is in the unified $SU(5)$ theory. This would be therefore the ‘‘predicted’’ value for the Weinberg angle. However, in the usual approach, and even without $SU(5)$ unification, one would obtain exactly the same value by postulating that the gauge group is not an arbitrary group isomorphic with $SU(3) \times SU(2) \times U(1)$ but a group *metrically* isomorphic with the $SU(3) \times SU(2) \times U(1)/(\mathbb{Z}_2 \times \mathbb{Z}_3)$ subgroup of $SU(5)$. In absence of a principle based on the ideas of group symmetries (or a generalization of such a principle), one could then ask on which grounds one should postulate such a property. The same argument (or objection) holds here. Indeed gauge invariance alone allows for the introduction of arbitrary constants in front of the individual components of the gauge group. The conclusion is therefore that, although the value $tg^2\theta = 3/5$ appears quite ‘‘naturally’’ in this formalism, it should not be taken as an unescapable consequence of the construction.

A last possible ‘‘constraint’’ concerns the mass of the W (or Z) particle. Indeed, from the expression of \mathcal{F} we obtain a term $\frac{1}{4}(\frac{\nu}{\sqrt{2}})^2 g^2 \text{Tr}(L - R)^2$ that gives a mass to the W and the

Z. The trace itself reads

$$\begin{aligned} L - R &= \left(\frac{i}{\sqrt{2}}(\tau_1 W_1 + \tau_2 W_2) + \frac{i}{\sqrt{2}} W_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i}{\sqrt{x}} B \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}\right) - \frac{i}{\sqrt{x}} B \begin{pmatrix} y+1 & 0 \\ 0 & y-1 \end{pmatrix} \\ &= \frac{i}{\sqrt{2}}(\tau_1 W_1 + \tau_2 W_2) + \frac{i}{\sqrt{2}} \left(W_3 - \sqrt{\frac{2}{x}} B\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

so

$$\text{Tr}(L - R)^2 = W_1^2 + W_2^2 + \left(W_3 - \sqrt{\frac{2}{x}} B\right)^2 = 2W_+ W_- + \frac{1}{\cos^2 \theta} \cdot Z^2$$

This gives the relation $M_W = \nu g/2$ which is well known in the standard model. In general, we have $M_W = gM_H/(2\sqrt{\lambda})$ and this becomes only a constraint (namely $M_H = 2M_W$) if we set λ to the “natural” value $\lambda = g^2$ as discussed before. One could hope that such relations could hold at a scale where the previous value for θ is experimentally satisfied (maybe at some grand unification scale). Notice that other authors [17], using a different formalism (relying upon the choice of another differential algebra), obtain another type of relations. Of course, we cannot (and will not) pretend that other approaches should, or not, lead to the same “numerical” relations. Existence of constraints such as the above ones can anyway be criticized since gauge invariance alone allows us to multiply terms of the bosonic lagrangian by arbitrary constants; this possibility can be related to the choice of particular scalar products in the space of forms [6]) and there are no compelling reasons to set such constants equal to one (although it may look quite natural in this formalism).

The main conclusion of this section is that the structure of the whole bosonic lagrangian of the Standard Model can be obtained from the formalism of non commutative geometry. Whether or not one should look for constraints and take them seriously is another matter. Our opinion is that, before reaching any conclusion on this line, one should wait till we have a full understanding of the fully quantized field theory in terms of non commutative geometry.

7. Higgs fields and super-algebras

The space where Ψ lives is naturally \mathbb{Z}_2 graded by L and R , i.e. Ψ can be decomposed into a left and a right part. Therefore transformations that map Ψ fields to themselves fall naturally into 2 kinds: those mapping L to L (and R to R) – we call them “even”– and those mapping L to R (and conversely) – we call them “odd”. Mathematically speaking, the space of these transformations can be considered as an associative \mathbb{Z}_2 graded matrix algebra whose corresponding Lie super-algebra is usually denoted by $GL(p|q)$ where p (resp. q) is the number of left Weyl (resp. right) fermions entering the Lagrangian. The usual Yang-Mills fields can be decomposed onto the even part whereas the Higgs fields can be decomposed onto the odd part of this algebra. This is a rather trivial remark since any Yang-Mills theory (and not only the Standard Model) defined on an even dimensional space-time can be analysed along the same lines. Another way to express the same idea is to say that any Yang Mills theory with p left Weyl fermions and q right Weyl fermions can be formulated in terms of representation theory of *some* super Lie algebra possessing a representation on a graded vector space of dimension $p + q$. In the case of the Standard Model (with right neutrinos), and because all the fermionic species are coupled to the *same* gauge and Higgs bosons, the matrix describing this interaction can be decomposed on a subset of the generators of $GL(24|24)$. Since we have only 4 gauge bosons and 4 Higgs bosons, we need only to use 8 generators (4 even and 4 odd ones); in other words we only need to use (or to recognize) the Lie superalgebra $Sl(2|1)$. The physical representations of interest (namely leptons, quarks and possibly right neutrinos)

correspond to direct sums of $Sl(2|1)$ representations of dimension $3 = 2 + 1$, $4 = 2 + 2$ or 1 . This fact was actually observed long ago [27, 28] and sometimes perceived as a kind of “miracle”; for us, we consider this property as almost tautological. The emergence of Lie superalgebras could lead people to think that one should try to enlarge the formalism of gauge theory to accommodate Lie superalgebras... Such attempts have been investigated in the past and shown to lead to serious problems and have, in any case, nothing to do with the Standard Model itself and even less with the non commutative geometry presentation of the Standard Model. In order to stress this point, let us consider the following analogy: one can observe that Dirac spinors form a representation of the Clifford algebra (the Dirac algebra of γ -matrices); this is well known; as a consequence it is also true that the spinors with four complex components also provide a representation for the (non simple) Lie algebra generated by taking commutators of *arbitrary* products of γ matrices; this does not mean that the lagrangian of quantum electrodynamics should be invariant (globally or locally) under such transformations. The fact that an algebra (like the full algebra of γ matrices) is not directly related with an invariance of the lagrangian does not make it useless (the spin group and its Lie algebra can of course be expressed in terms of the γ 's but the Clifford algebra itself is much bigger). Not all algebras related to the mathematics of a physical model need to describe “invariances” or “symmetries”; the fact that they do not does not make them useless! The same thing is also true here for the super-algebra along the representation of which one can decompose the matrices acting on the vector space spanned by the multi-component spinor fields $\Psi = (\Psi_L \Psi_R)$. This useful algebra is spanned by 8 generators. The first four are matrices that, in the interaction term of the lagrangian describing interaction between fermions and gauge bosons, appear as coefficients of the Yang-Mills fields W_{\pm} , W_3 and B ; they are denoted, as usual, by I_{\pm} , I_3 and Y . The last four are matrices that appear as coefficients of the Higgs fields φ_+ , φ_- , φ_0 and $\bar{\varphi}_0$; they give rise (after having added the hermitian conjugate) to the Yukawa and mass interaction term. We call them $-\Omega'_+$, $-\Omega_-$, Ω'_- and Ω_+ . More precisely, consider the following (block) matrices:

$$\begin{aligned} \Omega_+ &= \begin{pmatrix} 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix} & \Omega'_- &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \Omega_- &= \begin{pmatrix} 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -b & 0 & 0 & 0 \end{pmatrix} & \Omega'_+ &= \begin{pmatrix} 0 & 0 & 0 & e \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

where a, b, g, e are themselves square matrices, for example of size 3×3 if we consider only quarks coming in 3 families. In this case, we decide to label the basis as follows: $\Psi = (u, c, t)_L (d, s, b)_L (u, c, t)_R (d, s, b)_R$. Let us define $\sqrt{2}I_{\pm} = \{\Omega_{\pm}, \Omega'_{\pm}\}$, $2I_3 = \{\Omega_+, \Omega'_-\} - \{\Omega_-, \Omega'_+\}$ and $Y = \{\Omega_+, \Omega'_-\} + \{\Omega_-, \Omega'_+\}$. The electric charge is $Q \doteq I_3 + Y/2 = \Omega_+, \Omega'_-$. Then, provided matrices e, b, g, a satisfy the relation $eb + ga = 1$, one can show (it is straightforward but cumbersome) that the Ω matrices satisfy the relations

$$\begin{aligned} [I_3, I_{\pm}] &= \pm I_{\pm} & [I_+, I_-] &= I_3 & [Y, I_{\pm}] &= [Y, I_3] = 0 \\ [Y, \Omega_{\pm}] &= -\Omega_{\pm} & [Y, \Omega'_{\pm}] &= \Omega'_{\pm} & [I_3, \Omega_{\pm}] &= \pm \frac{1}{2} \Omega_{\pm} \\ [I_3, \Omega'_{\pm}] &= \pm \frac{1}{2} \Omega'_{\pm} & [I_{\pm}, \Omega_{\mp}] &= \frac{1}{\sqrt{2}} \Omega_{\pm} & [I_{\pm}, \Omega'_{\mp}] &= \mp \frac{1}{\sqrt{2}} \Omega'_{\pm} \\ [I_{\pm}, \Omega_{\pm}] &= [I_{\pm}, \Omega'_{\pm}] = 0 & \{\Omega_{\pm}, \Omega_{\pm}\} &= \{\Omega_{\pm}, \Omega_{\mp}\} = 0 & \{\Omega'_{\pm}, \Omega'_{\pm}\} &= \{\Omega'_{\pm}, \Omega'_{\mp}\} = 0 \\ \{\Omega_{\pm}, \Omega'_{\pm}\} &= \sqrt{2} I_{\pm} & \{\Omega_{\pm}, \Omega'_{\mp}\} &= \pm I_3 + Y/2 \end{aligned}$$

One recognizes here the usual relations defining the Lie super algebra of $SL(2|1)$. In the case of quarks, one furthermore impose the following constraints for the hypercharge generator:

$Y = (Y_L^U Y_L^D Y_R^U Y_R^D)$ with $Y_L^U = Y_L^D = 1/3 \mathbf{1}_{3 \times 3}$, $Y_R^U = 4/3 \mathbf{1}_{3 \times 3}$ and $Y_R^D = -2/3 \mathbf{1}_{3 \times 3}$. These constraints are satisfied if and only if, on top of the relation $eb + ga = \mathbf{1}$, the matrices e, b, g, a satisfy also the relations $-eb + ga = (1/3)\mathbf{1}$, $2ag = (4/3)\mathbf{1}$ and $-2be = -(2/3)\mathbf{1}$. Indeed, one finds $Y_L^U = Y_L^D = -eb + ga$, $Y_R^U = 2ag$ and $Y_R^D = -2be$. This imply in particular $ga = ag$ and $eb = be$. One can then check that matrices I_{\pm} and I_3 are then automatically what they should be.

One may notice that the above expressions for Omega matrices describing the gauge and Yukawa couplings of the quark family define a Lie superalgebra representation which is equivalent to the sum of (three) irreducible representations (each irreducible itself splits into the direct sum of a doublet and two singlets under the branching to the Lie algebra of $SU(2) \times U(1)$).

Define now $\phi = \bar{\varphi}_0 \Omega_+ + \varphi_0 \Omega'_- - \varphi_+ \Omega'_+ - \varphi_- \Omega_-$ and write $\mathcal{C} = \bar{\Psi} \phi \Psi$. This expression can not be real, indeed $\mathcal{C} = \mathcal{C}^\dagger$ would imply $g = a^\dagger$, $b = -e^\dagger$, but the other constraints would lead to a contradiction ($ee^\dagger = -(1/3)\mathbf{1}$). To obtain a real expression, one has to add \mathcal{C} and \mathcal{C}^\dagger . Writing $\mathcal{L} = \mathcal{C} + \mathcal{C}^\dagger$ gives

$$\mathcal{L} = \frac{(\bar{U}_L(g + a^\dagger)\bar{\varphi}_0 U_R + \bar{U}_R(g^\dagger + a)\varphi_0 U_L) + (\bar{D}_L(-e + b^\dagger)\varphi_0 D_R + \bar{D}_R(-e^\dagger + b)\bar{\varphi}_0 D_L) - (\bar{U}_L(e - b^\dagger)\varphi_+ D_R - \bar{D}_R(e^\dagger - b)\varphi_- U_L) - (\bar{U}_R(g^\dagger + a)\varphi_+ D_L - \bar{D}_L(g + a^\dagger)\varphi_- U_R)}{}$$

and we recognize the expression of \mathcal{L} given in section 5, with the identification $\mathcal{M}_U = g + a^\dagger$ and $\mathcal{M}_D = -e + b^\dagger$. Warning: The matrix ϕ defined previously in terms of the Ω matrices is not equal to the matrix Φ defined in section 5; in order to compare the two expressions, one has to first add the conjugated expressions \mathcal{C} and \mathcal{C}^\dagger . Taking into account the constraints on blocks a, g, e and b , one obtains the relations: $\mathcal{M}_U = g + (2/3)(g^{-1})^\dagger$ and $\mathcal{M}_D = -e + (1/3)(e^{-1})^\dagger$. These relations do not imply any “new” constraints on mass matrices \mathcal{M}_U and \mathcal{M}_D since g and e are themselves arbitrary. The main interest of those formulae is to provided a new parametrisation for mass matrices or matrices of Yukawa couplings. This could, in turn, suggest new phenomenological ansatz for them and may even give us more insight into the structure of fermionic mass matrices. Such an ansatz was analysed in [6] in the case of two families and lead to a phenomenological expression of the Cabibbo angle in terms of quark masses.

Remark: The quantity \mathcal{C} may be thought as the contribution to the lagrangian of a particular representation of $Sl(2|1)$. One can think of \mathcal{C}^\dagger as the contribution of the antiquark representation to the lagrangian. However this identification is a little bit tricky and may lead to possible mistakes of interpretation; indeed, \mathcal{C} is not hermitian but \mathcal{C}^\dagger is not the charge conjugate representation (in any case weak interactions usually violate charge conjugation and one should not build a lagrangian that would be C -even !). Given Ω_{\pm} and Ω'_{\pm} as before, one can define the following “hatted” Ω matrices: $\hat{\Omega}'_- = \Omega_+^\dagger$, $\hat{\Omega}'_+ = \Omega_-^\dagger$, $\hat{\Omega}_+ = \Omega_-^\dagger$ and $\hat{\Omega}_- = \Omega_+^\dagger$. It is then straightforward to check that these hatted Ω matrices generate (thanks to the same commutation relations) matrices \hat{Y} , \hat{I}_{\pm} and \hat{I}_3 , with, for example $\hat{Y} = \text{Diag}(a^\dagger g^\dagger - b^\dagger e^\dagger, a^\dagger g^\dagger - b^\dagger e^\dagger, 2g^\dagger a^\dagger, -2e^\dagger b^\dagger)$. We obtain in this way a new representation (the relation $b^\dagger e^\dagger + e^\dagger b^\dagger = \mathbf{1}$ being automatically satisfied since $eb + be = \mathbf{1}$). With \mathcal{C} as before, we can rewrite \mathcal{C}^\dagger as $\mathcal{C}^\dagger = \bar{\Psi} \phi^\dagger \Psi$ and ϕ^\dagger as

$$\begin{aligned} \phi^\dagger &= \varphi_0 \Omega_+^\dagger + \bar{\varphi}_0 \Omega_-^\dagger - \varphi_- \Omega_+^\dagger - \varphi_+ \Omega_-^\dagger \\ &= \varphi_0 \Omega'_- + \bar{\varphi}_0 \Omega'_+ - \varphi_- \Omega_- - \varphi_+ \Omega'_+ \end{aligned}$$

so that ϕ^\dagger itself appears as the contribution associated with the “hatted” representation. If one wishes to use \mathcal{C}^\dagger in terms of a contribution of antiparticles, for instance \bar{d}_{RSL} as $-\bar{s}_R^c d_L^c$, one can do it, modulo proper care, but it may be misleading.

For leptons, the idea is the same as for the quarks and, in order to straighten even more the analogy, we add right neutrinos to the Standard Model (they will turn out to be iso-singlets, as they should be). We shall order the basis as follows: $\Psi = (\Psi_L \Psi_R)$ with $\Psi_L = (\nu_e, \nu_\mu, \nu_\tau)_L$ (e, μ, τ) $_L$, $\Psi_R = (\nu_e, \nu_\mu, \nu_\tau)_R$ (e, μ, τ) $_R$ and define matrices Omega as previously, in terms of new 3×3 block matrices e, b, g, a . However, in the case of leptons, the constraints for the hypercharge generator are different. Indeed, $Y = (Y_L^\nu Y_L^E Y_R^\nu Y_R^E)$ with $Y_L^\nu = Y_L^E = -1$, $Y_R^\nu = 0$ and $Y_R^E = -2$. These constraints are satisfied if and only if, on top of the relation $eb + ga = \mathbf{1}$ (which ensures that commutation relations for $SL(2|1)$ hold), the matrices e, b, g, a satisfy also the relations $eb = \mathbf{1}$ and $ag = 0$. With these constraints, one can then check that matrices I_\pm, I_3 and Y defined as before in terms of the matrices Ω are then automatically what they should be.

One may notice that the above expressions for Omega matrices describing the gauge and Yukawa couplings of the lepton family (including right neutrinos) define a Lie superalgebra representation which is equivalent to the sum of (three) reducible indecomposable representations (each of them splits into the direct sum of a doublet, a singlet, and the trivial representation under the branching to the Lie algebra of $SU(2) \times U(1)$).

Again, we define $\phi = \bar{\varphi}_0 \Omega_+ + \varphi_0 \Omega'_- - \varphi_+ \Omega'_+ - \varphi_- \Omega_-$ and write $\mathcal{C} = \bar{\Psi} \phi \Psi$. This expression can not be real, and, in order to obtain a real expression, one has to add, as before, \mathcal{C} and \mathcal{C}^\dagger . Writing $\mathcal{L} = \mathcal{C} + \mathcal{C}^\dagger$ gives

$$\begin{aligned} \mathcal{L} = & (\bar{\nu}_L(g + a^\dagger)\bar{\varphi}_0\nu_R + \bar{\nu}_R(g^\dagger + a)\varphi_0\nu_L) + (\bar{E}_L(-e + b^\dagger)\varphi_0 E_R + \bar{E}_R(-e^\dagger + b)\bar{\varphi}_0 E_L) - \\ & (\bar{\nu}_L(e - b^\dagger)\varphi_+ E_R - \bar{E}_R(e^\dagger - b)\varphi_- \nu_L) - (\bar{\nu}_R(g^\dagger + a)\varphi_+ E_L - \bar{E}_L(g + a^\dagger)\varphi_- \nu_R) \end{aligned}$$

We recognize the expression of \mathcal{L} given in section 5, with the identification $\mathcal{M}_\nu = g + a^\dagger$ and $\mathcal{M}_E = -e + b^\dagger$ but the matrices a, g, e and b are not totally arbitrary since they should here satisfy the constraints $eb = \mathbf{1}$ and $ag = 0$. These relations do not imply any constraints on mass matrices \mathcal{M}_ν and \mathcal{M}_E but provided a new parametrisation for them. This parametrization in terms of a, g, e, b may, in turn, suggest new phenomenological ansatz (for instance one can see what happens if these matrices a, g, e, b have particularly simple forms). Such ansatz should then be considered as educated guesses but not as "predictions".

Before ending this section, we would like to notice that there exists still another interesting family of parametrizations for matrices a, g, e and b . The reader can indeed check that, if we chose arbitrary (3×3) matrices N_L, N_{UR}, N_{DR} and choose a, g, e and b in such a way that $2ga = (4/3)\mathbf{1} + N_L$, $2ag = (4/3)\mathbf{1} + N_{UR}$, $2eb = (2/3)\mathbf{1} - N_L$ and $2be = (2/3)\mathbf{1} - N_{DR}$, then, all commutation relations for Ω matrices are still satisfied. The generators I_3, I_+ and I_- obtained from them are also equal to what they should be. However, the obtained hypercharge generator Y is not diagonal (and not necessarily hermitian) but equal to $(1/3)\mathbf{1} + N_L, (1/3)\mathbf{1} + N_L, (4/3)\mathbf{1} + N_{UR}, -(2/3)\mathbf{1} + N_{DR}$. In other words, this describes a family of quarks-like objects which are not eigenstates of hypercharge (hence of charge). The Lie superalgebra specialist may relate this possibility to the existence of reducible indecomposable representations of $SL(2|1)$ with non diagonal Cartan subalgebra [14] (take N_L, N_{UR}, N_{DR} nilpotent matrices). Relation between family mixing and existence of such representations was suggested in [6] but was leading to difficulties (emergence of flavour changing neutral currents in the quark sector) that could only be cured by a rather *ad hoc* treatment of the definition of charge conjugacy. Here, we just notice that, after having defined ϕ and \mathcal{C} as before and added the (usual) complex conjugate, one obtain a real expression and one can choose to diagonalize simultaneously I_3 and Y . The rotated quark-like objects become now hypercharge (and charge) eigenstates, but the values of their charges are not standard and deviate from their usual values by corrections encoded in matrices N_L, N_{UR}, N_{DR} . This last

family of parametrization leads therefore to something that deviates from the Standard Model and we shall not elaborate more on this topic.

The \mathbb{Z}_2 -graded algebra discussed in this section is not usually mentioned in textbooks explaining the construction of the Standard Model. However, if one decides to rewrite the lagrangian in terms of multicomponent spinor fields $\Psi = (\Psi_L \Psi_R)$ gathering all left and right fermionic species in this way, this algebra (or better representations of it) appears naturally. It plays a role very similar to the (Clifford) Dirac algebra itself. We suggest to call it the “Yukawa algebra”. Again, one should not consider this algebra as a “symmetry” of the model and it is probably better to avoid the word “symmetry” in this context in order to avoid possible misunderstandings.

8. Comments

- The roads towards noncommutative geometry and the standard model: All approaches use three ingredients: an associative algebra A describing “space”, a module over this algebra describing “matter” and, finally, a differential algebra where generalized differentials live. In the very first papers on the subject, A. Connes uses the algebra ΩA of universal forms. He then introduced [2] one of its quotients, called it $\Omega_D A$ and used it in the sequel; this approach was followed in particular by [16, 17, 18, 23]. In [9, 13] a differential algebra $\Omega_{Der} A$ was built from A -valued derivations of the algebra A and subsequently used in various papers [10, 11, 12]. The formalism of A. Connes is very general and should be able to handle many problems of differential calculus in non commutative geometry. The formalism proposed in [4] is not that general, but if the purpose is to consider Yang-Mills fields and Higgs fields as different components of the same mathematical object (an algebraic connection), and to recover the lagrangian of the standard model in a very simple way, this formalism is sufficient and does not require any mathematics beyond an elementary knowledge of $n \times n$ matrices. The remark at the origin of [4, 6, 7] is that it is not necessary to use a \mathbb{Z} -graded differential algebra to perform the analysis. The situation is similar to what happens with ordinary differential forms: one studies usually the theory of connections (Yang-Mills fields) and covariant derivatives in terms of (usual) differential forms like $A_\mu dx^\mu$; however, in order to write a lagrangian describing the interactions with spinor fields, it is enough to represent these forms (in particular Yang Mills potential and curvature) in terms of γ matrices like $A_\mu \gamma^\mu$ (which build a \mathbb{Z}_2 -graded algebra). The same thing is true here: rather than using a \mathbb{Z} -graded algebra of generalized differential forms, we only use its representation on the \mathbb{Z}_2 graded vector spanned by left and right spinor fields. This is why we only use \mathbb{Z}_2 graded algebras in [4] (not \mathbb{Z} graded ones). We find this approach more familiar to physicists, and rich enough to illustrate the idea of considering Higgs fields as gauge bosons associated with the gauging of discrete directions (namely the jump between left and right chiralities).
- The general Connes’ formalism involves three steps: One starts with a piece of data containing the (generalized) Dirac operator \mathcal{D} – actually the Dirac-Yukawa operator – acting on a Hilbert space \mathcal{H} , together with an associative algebra \mathcal{A} describing “space” and also acting on \mathcal{H} . In the case of a pure abelian theory, \mathcal{A} would just be equal to the algebra of complex functions on space-time. Actually, the formalism of [1] takes space-time as euclidean and the Lorentz signature is recovered only at the end, thanks to a Wick rotation. In the case of a $U(1) \times U(1)$ theory with symmetry breaking, \mathcal{A} is equal

to the direct sum of two algebras of complex functions on space-time (one is labelled by “left” and the other by “right”); this algebra is still commutative and can be written as the space of 2×2 diagonal matrices with entries that are functions on space-time. In a more complicated setting where the gauge group is not abelian, one has just to replace the previous \mathcal{A} by its tensor product with an appropriate matrix algebra. In the second step one has to construct a differential algebra Ω_D (whose definition relies on the choice of \mathcal{D}) out of which one defines the generalized connections and curvatures. The third step is the construction of the Yang-Mills (or generalized Yang-Mills) Lagrangian itself and involves the so-called Dixmier trace as a substitute for integration. The triple $(\mathcal{A}, \mathcal{H}, D)$ is called a K -cycle or a spectral triple. The formalism presented in [4] starts with the same Dirac-Yukawa operator but does not require the construction of the algebra Ω_D and the use of the Dixmier trace. Its clear advantage is simplicity but it is lacking the character of generality expressed in [2, 3]. The only quantity to be computed is the expression for the generalized curvature \mathcal{F} in terms of the generalized connection \mathcal{A} , but this can be done once and for all.

- In order to build any example of non commutative geometry, one needs an associative algebra (replacing “space”), an algebra of generalized forms and a module (the space of matter fields). However, for a given “space” and a given kind of matter fields, the choice of the differential algebra Ω is not unique (all possible choices can ultimately be seen as quotients of a so-called “universal” one). The choice used by A. Connes in [1] is not the same as the choice made by the same author in [2] and none of them coincide either with the differential algebra introduced by [9] or with ours. The choice described in [5] is a differential algebra Ξ equal to the tensor product of usual differential forms times the differential algebra built in section 4 for the space with 2 elements $\{L, R\}$. One takes \mathcal{A} as 1-form and, using an appropriate d -operator defines \mathcal{F} as the 2-form $d\mathcal{A} + \mathcal{A}\mathcal{A}$. The expression previously given at the beginning of section 6 is nothing else than the representation of \mathcal{F} in the fermionic space. The calculations can be done very simply at the representation level (this is recalled in section 8) and this is why we skip the discussion concerning the actual choice of the \mathbb{Z} graded differential algebra. Our point of view is that this freedom in the choice of the differential algebra does not matter (in the case of the physical system studied here) because all such choices lead *essentially* to the same result, namely to the expression of \mathcal{F} similar to the one given in section 6 and to the usual lagrangian of the standard model. “*Essentially*” means that all the results agree, up to factors of normalization. For instance, the expression of the Higgs potential is exactly the same but the overall coefficient may change. For instance, in the Connes’ approach (where one has to divide Ω by “junk forms” to obtain Ω_D), the kinetic term $\overline{D\varphi}D\varphi$ is proportional to $\text{tr}(MM^\dagger)$ whereas the Higgs potential is proportional to $\text{tr}[(MM^\dagger)_-]^2$, with $(MM^\dagger)_- = MM^\dagger - \frac{1}{n}\text{tr}MM^\dagger$, where M is a $n \times n$ fermionic mass matrix. This potential vanishes whenever MM^\dagger is proportional to the unit matrix. In our case, this was not so (see section 6). The coefficients appearing in front of the kinetic term for scalar fields and of the Yukawa potential depend upon the specific choice of the differential algebra. Spaces $\Omega_D^0(A)$, $\Omega_D^1(A)$ and $\Omega_D^2(A)$ for $A = C(M) \oplus C(M)$ have been computed in [2] and it happens that (when $MM^\dagger \neq \mathbf{1}$) they are respectively isomorphic, as vector spaces, to the Ξ^0 , Ξ^1 and Ξ^2 described in [5] but the algebraic structure differs. More generally, the structure of Ω_D , when A is an arbitrary tensor product of algebras was investigated in [19, 15, 21]. Introducing an extra arbitrary constant in front of the whole potential amounts to disregard possible mass relations. In such a case the several approaches become completely equivalent, as far as physics and the standard model of electroweak interactions are concerned.

- Calculation of the curvature: In [4, 6] the calculation of the coefficients of the generalized curvature is performed in the representation space (where only the \mathbb{Z}_2 -grading matters). The whole \mathbb{Z} -graded differential algebra ΩA of universal forms is therefore mapped onto an algebra that can be taken as the Z_2 -graded tensor product of the usual algebra of differential forms times another Z_2 graded differential algebra built in terms of even dimensional square matrices. Here we forget about the \mathbb{Z} -grading of forms and remember only their parity, or equivalently, we represent forms by using γ matrices belonging to the Clifford algebra. The product is $(a \otimes B) \odot (a' \otimes B') \doteq (-1)^{\partial B \partial a'} (aa') \otimes (B \wedge B')$ where a is a $2n \times 2n$ matrix and B is a form. The differential is

$$d(a \otimes B) \doteq da \otimes B + (-1)^{\partial a} a \otimes dB$$

If we take \mathcal{A} as in section 5 and define $\mathcal{F} \doteq d\mathcal{A} + \mathcal{A} \odot \mathcal{A}$, we obtain the curvature \mathcal{F} given at the beginning of section 6. The reader should consult [4] for a simple exposition of this calculation. The definition and calculation of \mathcal{A} and \mathcal{F} in the Connes' formalism is given in [2, 3, 16].

- The reader remembers that, in case of spontaneous symmetry breaking, nature (or the formalism!) chooses a vacuum and that there are usually infinitely many equivalently such vacua. Since the formalism of non commutative geometry brings us a formalism where the translation in the Higgs field is done automatically, one may wonder about what has happened with the previous freedom for choosing a vacuum. The answer is that the freedom of choice is encoded in the definition of the differential δ . This δ is not unique and it can be seen that different choices for it amount to choose different minima for the Higgs potential (see [4]).
- In the formalism used by [2], the algebra of quaternions plays a very special role (it acts in particular on the algebra of generalized differential forms). It may help the reader to notice that quaternions are already present in the usual formalism of the standard model (also in ours!) but this fact is not necessarily used or recognized. The simplest way to see quaternions acting here is to look at the Yukawa interaction term given in section 5 or, equivalently, to put together in a square 2×2 matrix the two Higgs doublets coupling the left fermionic doublets to the fermionic right and left singlets. Discarding Yukawa coupling constants we have

$$\Phi = \begin{pmatrix} \bar{\varphi}_0 & \varphi_+ \\ -\varphi_- & \varphi_0 \end{pmatrix}$$

One recognizes here a quaternion. Indeed, using Pauli matrices σ_i , set $\gamma_1 = i\sigma_1$, $\gamma_2 = i\sigma_2$, $\gamma_3 = -i\sigma_3$, then $\gamma_i^2 = -1$ and $\gamma_i \gamma_j = \epsilon_{ijk} \gamma_k$. Moreover

$$a_0 \mathbf{1} + a_1 \gamma_1 + a_2 \gamma_2 + a_3 \gamma_3 = \begin{pmatrix} a_0 - ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 + ia_3 \end{pmatrix}$$

This expression can obviously be identified with Φ .

- On the nature and value of the constants appearing in the standard model: In any renormalizable quantum field theory, some parameters appear in the classical lagrangian. These constants may be free or may be related by some kind of gauge invariance property that one wants to enforce at the renormalized level. The free parameters have to be then fixed by an (arbitrary) renormalization prescription. We want now to stress the following. First of all, since these parameters are free, it is obvious that any kind of

new constraint will be superimposed to the usual formalism. To be made widely acceptable, such a constraint should satisfy two criteria. The first is that it should physically work, the next is that it should come from a kind of aesthetical construction (usually encompassing or generalizing an inherited formalism).

Our next comment concerns stability by renormalization. It has been shown [24] that some proposed constraints among masses of particles of the standard model may not be stable with respect to the renormalization group. This comment should be properly understood and maybe taken with a grain of salt. Indeed, the free parameters of the standard model are... free. Therefore one can renormalize them at will (at a given scale) and one can, in particular renormalize them in such a way that any relation between them is satisfied. Of course it is absolutely true that a renormalization prescription involves the choice of a scale (this “subtraction point” is usually chosen equal to some value of q^2 where q is a four-momentum) and that a numerical relation between renormalized parameters may be deformed by a change of scale if the relation is not invariant under the renormalization group. But this fact does not mean that the relation itself is physically meaningless. For example, the relation $m_{muon} = 206 m_{electron}$ which is valid in the on shell renormalization scheme of quantum electrodynamics can be imposed and is actually imposed (because it is experimentally true on shell). However this last relation is not invariant under the renormalization group equations of QED! The third and last comment to be made about these problems of relations between constants of the standard model was already made in the text (section 6) but we repeat it here. Descriptions of the Standard Model based on non commutative geometry supplemented by the choice of specific scalar products in the space of fields seem to lead, at the classical level, to relations between the – otherwise arbitrary – constants of the model. Gauge invariance alone allows for more freedom; in absence of a full description of (spontaneously broken) quantum gauge field theories in terms of non commutative geometry, such constraints should be considered, in our opinion, as educated guesses. The reader may refer to [17, 18] for a detailed analysis of these constraints in the Connes’ framework. For us, the true power of the non commutative geometry description of the standard model (and of quantum physics in general) is not tied up with the relevance of such constraints.

- Right neutrinos, simplicity and non trivial bundles (projectors).

In its simplest “version”, the standard model does not incorporate right neutrinos. From the point of view of non commutative geometry and if one restricts oneself to the leptonic sector (take for instance the example of one family), lacking right neutrinos is a little bit of a nuisance. Indeed, in such a case, and in the language of A. Connes [1], the bundle of leptonic species is non trivial and one needs to introduce a projector in the formalism, projector whose curvature itself enters the final expression. In the formalism explained in [4], the same phenomena appears because our approach (based on the tensorization of two by two complex matrices by arbitrary ones) leads to even dimensional square matrices. In order to accomodate an odd number of Weyl fermions (for instance e_L, e_R and ν_L) one has to embed the odd dimensional matrix describing the connection into a even dimensional one by adding line and columns of zeros. But then action of the d operator creates non zero entries in such places. The curvature is then not equal to $F = dA + A^2$ but to $F = p(dA + A^2)p + pdpdp$ where p projects on the odd dimensional subspace spanned by the Weyl fermions. The result is, as it should, the usual standard model without right neutrinos. However, introducing right neutrinos in the game (like in [6] and like in section 5 of the present paper) simplifies considerably

the formalism because one does not have to introduce such a projector. One cannot not say that “Non commutative geometry predicts that the neutrino has a mass” but it is clear that, from our perspective, the formalism is much simpler with a right neutrino than without. Let us remind the reader that such neutrinos are absolutely compatible with present experimental data since the ν_R that one introduces for each family is not coupled to the (transverse part of the) gauge fields. Its main interest is to give a mass to the corresponding particle (hence a Dirac spinor) and to introduce mixing between fermionic families via a fermionic analogue of the Kobayashi-Maskawa matrix. Introduction of right neutrinos in the Connes’ formalism was recently investigated in [25].

- In [18], the authors raise the question: “Is any Connes-Lott model a Yang-Mills-Higgs model and conversely?” Their answer is clearly “No”. However, one should not assimilate the approach initiated in [1] and further discussed and detailed in the last chapter of [3] with non commutative geometry in general or even with non commutatively inspired models. Other avenues are possible. Our belief is that it should be possible to distort sufficiently enough the formalism proposed initially by [1] (or [4]) to accomodate many cases of classical gauge field theory with symmetry breaking... For instance it was shown in [22] how to introduce “symmetries” into the Higgs fields of a non commutative model of the kind [1] in order to be able to accomodate scalar fields belonging to representations a priori forbidden in the initial framework. Such a distortion of the formalism allows of course the construction of more general models but at the expense of aesthetics.
- The formalism introduced in [4] could induce one to think that this approach is a particular case of the formalism of super connections introduced and discussed in [26]. This is actually not so. The algebraic connections defined by [1] are clearly not super connections since they are degree one forms in a \mathbb{Z} – *graded* differential algebra. Of course this algebra is also \mathbb{Z}_2 graded since forms are even or odd but generalized Yang Mills potentials are defined here as one-forms and not only as odd forms. The same is true in our case, see the discussion in [5]. Notice that when the differential algebra is represented on the space of spinors, the \mathbb{Z} grading is lost and only the \mathbb{Z}_2 grading is left. Contrarily to what happens in the standard model, where the generalized gauge field incorporates only Higgs and usual spin one gauge fields, an algebraic super connection would also incorporate other kinds of fields. Of course, one can consider a connection as some kind of “truncated” super connection, but such a terminology becomes then pointless and can bring confusion.
- The ubiquitous number 24 (joke). With three families of quarks, three colors, three families of leptons including right neutrinos, we have 24 elementary right handed Weyl fermions and 24 elementary left handed Weyl fermions. The number of arbitrary parameters of the model is also equal to 24 (the gauge coupling constants g, g' , the 6 quark masses, the 6 leptonic masses, the $8 = 2 \times 4$ parameters of the leptonic and quark mixing matrices, the mass of the W and the mass of the Higgs). We shall stop here these numerological remarks and suggest the reader interested in the beautiful properties of the number 24 to dive into almost any book of arithmetics, modular function theory or higher dimensional cristallography.
- In the present formalism, the square of the generalized curvature gives us the whole bosonic classical lagrangian of the standard model. One may wander about the possibility of maintaining this unification at all orders of perturbative Q.F.T. In other words,

is it possible to write the effective lagrangian at one loop (at two loops etc) of the theory in terms of the generalized curvature \mathcal{F} ? Such a possibility is by no means ruled out but has not been proven yet. If true, such a property would pave the way to the next (ultimate?) goal of non commutative geometry (as far as physics of electroweak interactions are concerned): a non perturbative formulation of a fully quantized gauge field theory. The completion of such an ambitious program would really be a truly non commutative achievement.

- Final comment. Higgs fields and Yang-Mills fields can now be thought of as two particular components of the same object \mathcal{A} and that the square of the corresponding generalized curvature gives us the bosonic lagrangian of the standard model. This unification is a little bit like the unification of electric and magnetic fields taken as independent components of the Faraday tensor F . By itself, such a unification is not a new theory in the sense that it was “already there”. After all one can very well work with electromagnetism without using a manifestly covariant formalism! Also, it does not bring necessarily any new numerical prediction. However, a unification of that kind is important at the conceptual level. Moreover it usually leads to generalizations or to new ideas that could be hardly thought of in a less unified framework. Unification of Higgs and Yang-Mills fields is, for us the most important success of non commutative differential geometry in physics. This success is certainly going to stay.

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