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Vienna, Preprint ESI 229 (1995)

May 26, 1995

Supported by Federal Ministry of Science and Research, Austria
Available via http://www.esi.ac.at

UAHEP 952
May 1995

Lie-Poisson Deformation of the Poincaré Algebra

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ABSTRACT

We find a one parameter family of quadratic Poisson structures on $\mathbf{R}^4 \times SL(2,C)$ which satisfies the property *a)* that it is preserved under the Lie-Poisson action of the Lorentz group, as well as *b)* that it reduces to the standard Poincaré algebra for a particular limiting value of the parameter. (The Lie-Poisson transformations reduce to canonical ones in that limit, which we therefore refer to as the 'canonical limit'.) Like with the Poincaré algebra, our deformed Poincaré algebra has two Casimir functions which we associate with 'mass' and 'spin'. We parametrize the symplectic leaves of $\mathbf{R}^4 \times SL(2,C)$ with space-time coordinates, momenta and spin, thereby obtaining realizations of the deformed algebra for the cases of a spinless and a spinning particle. The formalism can be applied for finding a one parameter family of canonically inequivalent descriptions of the photon.

1 Introduction

A number of authors have examined deformations of the Poincaré algebra in quantum theory, in addition to investigating the effects of deforming the usual space-time symmetry.[1-7] Of course there exists no unique procedure for carrying out such deformations. The proposals made so far are generally within the framework of Hopf algebras, and they often rely upon making a contraction of the quantum de Sitter algebra.

Another technique is to deform the Poincaré algebra already at the classical level.[8] There the deformed algebra is to be realized in terms of Poisson brackets rather than commutation relations, and the construction should be made within the framework of Poisson-Lie groups in order to make connection with Hopf algebras in quantum theory. This is the approach we shall follow here.

The classical analysis is considerably simpler than its quantum counterpart for a number of reasons. One reason is of courses that the elements of the algebra, i.e. the classical observables, are commuting variables. In addition to this, to check the consistency of the algebra we essentially only need to verify the Jacobi identity (although this may not always be so easy). Furthermore, in the classical theory symmetries are associated with ordinary Lie groups, and not quantum groups.

With regard to the classical symmetries, which we denote by $\mathcal{S}$, we shall here only be concerned with the Lorentz group [or actually, $SL(2,C)$], which we here regard as a Poisson-Lie group. Thus unlike in canonical theories, the symmetry group carries a Poisson structure, which by definition is consistent with left or right group multiplication. Such structures are well known. Our choice of Poisson brackets $\{,\}_{\mathcal{S}}$ corresponds to the classical analogue of the defining relations for $SL_q(2,C)$.[9]

With regard to the space of classical observables, which we denote by $\mathcal{O}$, we examine a one parameter family of algebras defined on $\mathbf{R}^4 \times SL(2,C)$. These algebras are preserved under the Lie-Poisson action of $\mathcal{S}$. Furthermore, they reduce to the Poincaré algebra for a particular limiting value of the parameter. We refer to the limiting value as the 'canonical limit'. This is because $\mathcal{S}$ has a trivial Poisson bracket in the limit, i.e. $\{,\}_{\mathcal{S}} \rightarrow 0$, and hence the Lie-Poisson action of $\mathcal{S}$ on $\mathcal{O}$ simplifies to the canonical action in the limit.

For all values of the deformation parameter, $\mathcal{S}$ will act on $\mathcal{O}$ in the standard way, i.e. momentum transforms as a Lorentz vector, while angular momentum transforms with the adjoint representation of $SL(2,C)$. This defines a map, $\sigma : \mathcal{S} \times \mathcal{O} \rightarrow \mathcal{O}$. Because $\mathcal{S}$ induces the Lie-Poisson action on $\mathcal{O}$, σ must be a Poisson map, which means that if f_1 and f_2 are functions on $\mathcal{O}$, then

$$\{f_1, f_2\}_{\mathcal{O}} \circ \sigma = \{f_1 \circ \sigma, f_2 \circ \sigma\}_{\mathcal{O}\times\mathcal{S}} \; , \tag{1}$$

where the product Poisson structure is assumed on $\mathcal{O}\times\mathcal{S}$ (which means that the symmetries have zero Poisson brackets with the classical observables).

Our deformed Poincaré algebra on $\mathcal{O}$ can be completely specified by four quadratic Poisson bracket relations $\{,\}_{\mathcal{O}}$ which we give below. The brackets are evaluated between variables, i.e. 'momenta' and 'angular momenta', spanning $\mathbf{R}^4 \times SL(2,C)$. The momenta will be expressed in terms of a 2×2 hermitean matrix $\tilde{p}$, while the angular momenta are contained in a 2×2 complex unimodular matrix γ. The four Poisson bracket relations can be written in terms of a classical $r-$matrix (and its hermitean conjugate $r^\dagger$) which is assumed to satisfy the (modified) classical Yang-Baxter equations. Using tensor product notation, the four relations are:

$$\{\underset{1}{\tilde p},\underset{2}{\tilde p}\} \;=\; r\underset{1}{\tilde p}\,\underset{2}{\tilde p} + \underset{1}{\tilde p}\,\underset{2}{\tilde p}\,r^\dagger - \underset{2}{\tilde p}\,r^\dagger\underset{1}{\tilde p} - \underset{1}{\tilde p}\,r\,\underset{2}{\tilde p}\;, \tag{2}$$

$$\{\underset{1}{\gamma},\underset{2}{\gamma}\} \;=\; r^\dagger\underset{1}{\gamma}\,\underset{2}{\gamma} + \underset{1}{\gamma}\,\underset{2}{\gamma}\,r - \underset{2}{\gamma}\,r\,\underset{1}{\gamma} - \underset{1}{\gamma}\,r^\dagger\underset{2}{\gamma}\;, \tag{3}$$

$$\{\underset{1}{\gamma},\underset{2}{\bar\gamma}\} \;=\; r\underset{1}{\gamma}\,\underset{2}{\bar\gamma} + \underset{1}{\gamma}\,\underset{2}{\bar\gamma}\,r - \underset{2}{\bar\gamma}\,r\,\underset{1}{\gamma} - \underset{1}{\gamma}\,r\,\underset{2}{\bar\gamma}\;, \tag{4}$$

$$\{\underset{1}{\tilde p},\underset{2}{\gamma}\} \;=\; r^\dagger\underset{1}{\tilde p}\,\underset{2}{\gamma} + \underset{1}{\tilde p}\,\underset{2}{\gamma}\,r - \underset{2}{\gamma}\,r^\dagger\underset{1}{\tilde p} - \underset{1}{\tilde p}\,r^\dagger\underset{2}{\gamma}\;, \tag{5}$$

where $\bar\gamma = \gamma^{\dagger^{-1}}$. The 1 and 2 labels refer to two separate vector spaces, with $\underset{1}{\tilde p} = \tilde p \otimes \mathbb{1}$, $\underset{2}{\tilde p} = \mathbb{1} \otimes \tilde p$, $\underset{1}{\gamma} = \gamma \otimes \mathbb{1}$ and $\underset{2}{\gamma} = \mathbb{1} \otimes \gamma$, $\mathbb{1}$ being the unit operator acting on the vector spaces. r acts nontrivially on both vector spaces. Here we shall utilize the following 4×4 matrix realization for r:

$$r = \frac{i\lambda}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & 4 & -1 & \\ & & & 1 \end{pmatrix}\quad, \tag{6}$$

λ being a real parameter.

Eqs. (2-5) give a one parameter family of quadratic Poisson structures on $\mathcal{O}$. Jacobi identities involving $\tilde p$, γ and $\bar\gamma$ are satisfied in part due to r satisfying the (modified) classical Yang-Baxter equations. (We however found it more convenient to use algebraic manipulation packages to check them.) It can also be checked that $\det(\gamma)$ has vanishing brackets with all observables and hence eqs. (2-5) are consistent with the unimodularity condition. Furthermore, the relations (2) and (3) define skew symmetric brackets, the former being invariant under hermitean conjugation.

In Sec. 2, we shall show that the quadratic algebra on $\mathcal{O}$ defined by eqs. (2-5) is preserved under the Lie-Poisson action of the symmetries $\mathcal{S}$. There we shall also show that it is a deformation of the standard Poincaré algebra, the latter being recovered when $\lambda \to 0$, and in that limit, (the canonical limit) the Lie-Poisson transformations reduce to canonical transformations. Like with the Poincaré algebra, the algebra described by eqs. (2-5) has two Casimir invariants. One of the Casimirs is the square of the momenta, while the other is the square of a vector which is a deformation of the Pauli-Lubanski vector. We can therefore associate the two Casimirs with "mass" and "spin".

In Secs. 3 and 4, we parametrize the symplectic leaves of $\mathcal{O}$ with variables which we associate with space-time coordinates x, momenta $\tilde p$ and spin γ_s. We give a realization of the algebra (2-5) in terms of x and $\tilde p$ in Sec. 3. The Poisson structure for these variables was already written down in ref. [11]. There it was shown to be preserved under the Lie-Poisson action of $\mathcal{S}$. It was also shown to be a deformation of the canonical symplectic structure for a relativistic particle, i.e. $\{x_\mu, p_\nu\} = \eta_{\mu\nu}$, $\eta = \text{diag}\,(-1,1,1,1)$. In Sec. 3 (and in the appendix), we shall write γ in terms of space-time coordinates x and the momenta $\tilde{p}$. This gives an expression analogous to the canonical orbital angular momentum for a relativistic particle. Only here γ is given in terms of an infinite series in x and $\tilde{p}$ (and λ), which reduces to the usual expression for the orbital angular momentum when $\lambda \to 0$. We show that the deformed Pauli-Lubanski vector is zero for this realization (for any value of λ), and hence we conclude that we have a description of a particle with zero spin. We also remark that if the classical Hamiltonian for the system is taken to be the momentum squared (i.e., det $\tilde{p}$) times a Lagrange multiplier, then the resulting dynamics is identical to that of a free massless particle (for any value of λ). We thus arrive at a one parameter family of canonically inequivalent descriptions for the photon. Upon quantization, the resulting states are expected to transform under the action of the quantum Lorentz group.

In Sec. 4, we show how the algebra (2-5) can be realized when spin is present. Unlike in the canonical theory, we find that the spin associated with a particle must have nonvanishing Poisson brackets with both the space-time coordinates and the momenta. This is a consequence of the result that $\mathcal{O}$, unlike $\mathcal{S}$, is not a Poisson-Lie group. When the mass shell constraint is taken for the Hamiltonian, the classical spin is found to have a trivial dynamics, i.e. there is no spin precession, and this is just as in the canonical theory.[10]

In Sec. 5, we give a preliminary discussion of the quantization of the Poisson bracket algebra (2-5), while concluding remarks are made in Sec. 6.

2 The Deformed Poincaré Algebra

Here we will examine the two distinct Poisson manifolds $\mathcal{S}$ and $\mathcal{O}$, associated respectively with the space of symmetries and the space of classical observables. As stated in the introduction, we shall identify the former with the six-dimensional Lorentz group [or more precisely, its covering group $SL(2,C)$], having Poisson brackets $\{\ ,\ \}_\mathcal{S}$ corresponding to that of a Poisson-Lie group [9],[11]. $\mathcal{O}$ will be assumed to be $\mathbf{R}^4 \times SL(2,C)$ with Poisson brackets $\{\ ,\ \}_\mathcal{O}$ defining a one-parameter deformation of the Poincaré algebra. The deformed algebra is essentially given by eqs. (2-5).

We first review the Poisson structure on $\mathcal{S}$. [9],[11] For simplicity of notation we shall drop the subscripts on the Poisson brackets.

2.1 Symmetries

Let g be a 2×2 complex unimodular matrix which we use to parametrize $\mathcal{S}$. For $\mathcal{S}$ to be a Poisson-Lie group its Poisson brackets must be compatible with left and right group multiplication. Such brackets are:

$$\{\underset{1}{g}, \underset{2}{g}\} = [\, r\ ,\ \underset{1}{g}\,\underset{2}{g}\,]\ , \tag{7}$$

where we again utilize tensor product notation, with $\underset{1}{g} = g \otimes 1\!\!1$, $\underset{2}{g} = 1\!\!1 \otimes g$ and the $r-$ matrix defined in eq. (6). Since the latter is proportional to λ, the group elements have zero Poisson brackets in the limit $\lambda \to 0$, and this once again corresponds to the canonical limit. The Jacobi identity holds due to the $r-$matrix satisfying the (modified) classical Yang-Baxter equation. (The Leibniz identity for the Poisson brackets is assumed here and throughout this article.) It can also be checked from eq. (7) that $\det(g)$ has zero Poisson brackets with all components of g and hence we may consistently set $\det(g)=1$.

The transformations of the observables will involve g as well as its hermitean conjugate. Therefore in addition to eq. (7) we will also need to know the Poisson brackets of $g^\dagger$ or $\bar{g}=(g^\dagger)^{-1}$. For this we demand that the Poisson structure for g and $g^\dagger$ is consistent with complex conjugation, antisymmetry and the Jacobi identity. All three of these conditions are met for the following set of relations [11]:

$$\{\underset{1}{g}, \underset{2}{\bar{g}}\} = [\, r \,,\ \underset{1}{g}\underset{2}{\bar{g}}\,] \,, \tag{8}$$

$$\{\underset{1}{\bar{g}}, \underset{2}{g}\} = [\, r^\dagger \,,\ \underset{1}{\bar{g}}\underset{2}{g}\,] \,, \tag{9}$$

$$\{\underset{1}{\bar{g}}, \underset{2}{\bar{g}}\} = [\, r \,,\ \underset{1}{\bar{g}}\underset{2}{\bar{g}}\,] \,. \tag{10}$$

As indicated earlier, the Poisson brackets (7) and (8-10) coincide with the classical limit of the $SL_q(2,C)$ commutation relations given in refs.[7], [9].

We note that eqs. (7) and (10) can be rewritten with r replaced by $r^\dagger$. This is because $r-r^\dagger$ serves as an adjoint invariant for $SL(2,C)$. More specifically, using the matrix representation (6) for r we have the identity

$$r - r^\dagger = i\lambda(2\Pi - 1\!\!1) \,, \tag{11}$$

where $1\!\!1$ is the unit operator (now acting on the entire tensor product space) and Π is the permutation operator, i.e. Π switches the two vector spaces. Thus for example, $\underset{1}{g}\,\Pi = \Pi\,\underset{2}{g}$ and $\underset{2}{g}\,\Pi = \Pi\,\underset{1}{g}$.

2.2 Observables

Here we discuss the Poisson structure on $\mathcal{O}$ given in eqs. (2-5) expressed in terms of the ten observables contained in $\tilde{p}$ and γ. The former transforms as a Lorentz vector, i.e. it corresponds to the $(\frac{1}{2},\frac{1}{2})$ representation of the Lorentz group, and we associate it with momenta, while the latter transforms as the $(1,0)$ and $(0,1)$ representations and we associate it with angular momenta. We shall here show that the Poisson structure (2-5) is a deformation of the standard Poincaré algebra and that it is preserved under the Lie-Poisson action of the Lorentz group.

We first discuss the Poisson structure (2) for the momenta $\tilde{p}$. Actually, this structure was already given in ref. [11]. There we wrote $\tilde{p}$ as a 2×2 hermitean matrix

$$\tilde{p} = \begin{pmatrix} -p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_0 - p_3 \end{pmatrix} \;, \tag{12}$$

p_μ being the space-time components. Under a Lorentz transformation

$$\tilde{p} \rightarrow \tilde{p}' = \bar{g}\tilde{p}g^{-1} \;. \tag{13}$$

The Poisson structure for $\tilde{p}$ is required to be preserved under these transformations, where we assume the Poisson brackets (7) and (8-10) for g and $\bar{g}$. (As stated earlier, the symmetries g and $\bar{g}$ are assumed to have zero Poisson brackets with all observables.) The Poisson brackets are also required to be skewsymmetric, invariant under hermitean conjugation and satisfy the Jacobi identity.

A solution to all of the above requirements is eq. (2). It is easy to check that eq. (2) is preserved under Lorentz transformations,

$$\begin{aligned} \{\underset{1}{\tilde{p}}, \underset{2}{\tilde{p}}\} \rightarrow \{\underset{1}{\tilde{p}'}, \underset{2}{\tilde{p}'}\} \quad &= \{\underset{1}{\bar{g}}\,\underset{1}{\tilde{p}}\,\underset{1}{g}^{-1}, \underset{2}{\bar{g}}\,\underset{2}{\tilde{p}}\,\underset{2}{g}^{-1}\} \\ &= r\,\underset{1}{\tilde{p}'}\,\underset{2}{\tilde{p}'} + \underset{1}{\tilde{p}'}\,\underset{2}{\tilde{p}'}r^\dagger - \underset{2}{\tilde{p}'}r^\dagger\,\underset{1}{\tilde{p}'} - \underset{1}{\tilde{p}'}r\,\underset{2}{\tilde{p}'} \;. \end{aligned} \tag{14}$$

Since the $r-$matrix is proportional to λ, all of the brackets are zero in the limit $\lambda \rightarrow 0$ and we recover the canonical result. The skewsymmetry of the bracket and invariance under hermitean conjugation is also easily checked.

In terms of the space-time components of $\tilde{p}$, eq. (2) can be written as

$$\begin{aligned} \{p_i, p_j\} &= 2\lambda\epsilon_{ijk}p_k(p_0 + p_3) \;, \\ \{p_i, p_0\} &= 0 \;, \\ & \qquad\qquad i,j,k = 1,2,3 \;. \end{aligned} \tag{15}$$

Thus the time component p_0 is in the center of the algebra. Also in the center is the magnitude of spatial components $\sqrt{p_ip_i}$ and consequently the invariant mass-squared, i.e. $p^\mu p_\mu = \det(\tilde{p})$. We expect that analogous central elements appear in the quantum theory, indicating that simultaneous measurements of the "energy", magnitude of the "momentum" and one spatial component of p_μ are possible.[4]

We next take up the Poisson structure of the angular momenta which we denote by j. j can be represented by a 2×2 complex traceless matrix. Actually, we find it more convenient however to deal with the exponentiation of j which we denote by $\gamma = e^{i\lambda j}$. Like g, γ is unimodular, i.e. $\det(\gamma) = 1$, and hence it is an $SL(2,C)$ matrix. Our space of classical observables is thus $\mathbf{R}^4 \times SL(2,C)$.

Under Lorentz transformations

$$\gamma \rightarrow \gamma' = g\gamma g^{-1} \;. \tag{16}$$

The Poisson structure for γ is required to be covariant under this transformation. It is also required to be antisymmetric, consistent with the constraint $\det(\gamma)=1$ and satisfy the Jacobi identity. A solution is eq. (3). Under a Lorentz transformation,

$$\begin{aligned}\{\underset{1}{\gamma},\underset{2}{\gamma}\}\rightarrow\{\underset{1}{\gamma}',\underset{2}{\gamma}'\} \quad &= \{\underset{1}{g}\,\underset{1}{\gamma}\,\underset{1}{g}^{-1},\underset{2}{g}\,\underset{2}{\gamma}\,\underset{2}{g}^{-1}\} \\ &= r^\dagger\underset{1}{\gamma}'\underset{2}{\gamma}'+\underset{1}{\gamma}'\underset{2}{\gamma}'r-\underset{2}{\gamma}'r\underset{1}{\gamma}'-\underset{1}{\gamma}'r^\dagger\underset{2}{\gamma}'\,. \end{aligned} \tag{17}$$

and hence eq. (3) is preserved. ¿From eq. (3) it can also be checked from that $\det(\gamma)$ has zero Poisson brackets with all components of γ and hence we may consistently set $\det(\gamma)=1$.

In addition to eq. (3), we need to specify the Poisson brackets of $\gamma^\dagger$ or $\bar\gamma=(\gamma^\dagger)^{-1}$. For this we again demand that it be preserved under Lorentz transformations and that it is consistent with complex conjugation, antisymmetry and the Jacobi identity. All of these conditions are met for eq. (4) along with the following relations:

$$\{\underset{1}{\bar\gamma},\underset{2}{\bar\gamma}\} \;=\; r\,\underset{1}{\bar\gamma}\,\underset{2}{\bar\gamma}+\underset{1}{\bar\gamma}\,\underset{2}{\bar\gamma}\,r^\dagger-\underset{2}{\bar\gamma}\,r^\dagger\,\underset{1}{\bar\gamma}-\underset{1}{\bar\gamma}\,r\,\underset{2}{\bar\gamma}\;, \tag{18}$$

$$\{\underset{1}{\bar\gamma},\underset{2}{\gamma}\} \;=\; r^\dagger\,\underset{1}{\bar\gamma}\,\underset{2}{\gamma}+\underset{1}{\bar\gamma}\,\underset{2}{\gamma}\,r^\dagger-\underset{2}{\gamma}\,r^\dagger\,\underset{1}{\bar\gamma}-\underset{1}{\bar\gamma}\,r^\dagger\,\underset{2}{\gamma}\;. \tag{19}$$

The remaining Poisson brackets of the observables are between $\tilde p$ and the group variables γ and $\bar\gamma$. For them we find eq. (5) along with

$$\{\underset{1}{\tilde p},\underset{2}{\bar\gamma}\}=r\,\underset{1}{\tilde p}\,\underset{2}{\bar\gamma}+\underset{1}{\tilde p}\,\underset{2}{\bar\gamma}\,r-\underset{2}{\bar\gamma}\,r^\dagger\,\underset{1}{\tilde p}-\underset{1}{\tilde p}\,r\,\underset{2}{\bar\gamma}\;. \tag{20}$$

Using eqs. (4) and (20), we have checked that $\{\det(\gamma),\tilde p\}=\{\det(\gamma),\bar\gamma\}=0$ and therefore that these Poisson brackets are consistent with the condition of unimodularity.

The Poisson structure of all ten observables is given by eqs. (2-5) and (19-20). [Actually, we only need to specify eqs. (2-5) as the remaining relations are obtained by hermitean conjugation.] We have used algebraic manipulation packages to verify the Jacobi identity for $\tilde p$, γ and $\bar\gamma$.

We next show that the algebra generated by $\tilde p$, γ and $\bar\gamma$ is a deformation of the standard Poincaré algebra, the latter being recovered in the limit $\lambda\rightarrow 0$. For this we substitute $\gamma=e^{i\lambda j}$ and $\bar\gamma=e^{i\lambda j^\dagger}$ into the Poisson bracket relations and expand around $\lambda=0$, keeping only the lowest order contributions. As stated earlier, eq. (2) gives

$$\{\underset{1}{\tilde p},\underset{2}{\tilde p}\}\rightarrow 0\;. \tag{21}$$

The lowest order contributions to eqs. (3) and (4) are quadratic in λ, yielding

$$\{\underset{1}{j},\underset{2}{j}\}\rightarrow 2\Pi\,(\underset{2}{j}-\underset{1}{j})\;,\qquad \{\underset{1}{j},\underset{2}{j}^\dagger\}\rightarrow 0\;, \tag{22}$$

where we used eq. (11). Lastly, from eq. (5) we get

$$\{ \underset{1}{\tilde{p}} , \underset{2}{j} \} \to \underset{1}{\tilde{p}} \; (2\Pi - 1\!\mathrm{l}) \; . \tag{23}$$

Eqs. (21-23) define the Poincaré algebra. It can be expressed in a more familiar, i.e.

$$\begin{aligned} \{p_\mu, p_\nu\} &= 0 \;, &(24)\\ \{j_{\mu\nu}, j_{\rho\sigma}\} &= \eta_{\mu\rho} j_{\nu\sigma} + \eta_{\nu\sigma} j_{\mu\rho} + \eta_{\mu\sigma} j_{\rho\nu} + \eta_{\nu\rho} j_{\sigma\mu} \;, &(25)\\ \{p_\mu, j_{\nu\rho}\} &= \eta_{\mu\rho} p_\nu - \eta_{\mu\nu} p_\rho \;, &(26)\\ \eta &= \mathrm{diag}(-1,1,1,1) \;, &(27) \end{aligned}$$

upon applying the matrix representation [cf. eq. (12)] for $\tilde{p}$, along with the following representation for the 2×2 complex traceless matrix j:

$$j = \begin{pmatrix} -ij_{12} + j_{30} & -ij_{23} - ij_{20} - j_{31} + j_{10} \\ -ij_{23} + ij_{20} + j_{31} + j_{10} & ij_{12} - j_{30} \end{pmatrix} \tag{28}$$

2.3 Casimirs

Like the Poincaré algebra, the algebra generated by $\tilde{p}$, γ and $\bar{\gamma}$ has two central elements, which we will associate with "mass" and "spin".

With regard to the mass, this classical Casimir function is identical in form to that of the Poincaré algebra. (This however isn't the case at the quantum level [4]. To define the latter one normally introduces a deformed determinant.) That is, $p^\mu p_\mu$ =det$(\tilde{p})$ is the Casimir function. ¿From eqs. (2), (5) and (20), we have that

$$\{\det\,(\tilde{p}), \tilde{p}\} = \{\det\,(\tilde{p}), \gamma\} = \{\det\,(\tilde{p}), \bar{\gamma}\} = 0 \; . \tag{29}$$

and therefore that it is in the center of the algebra.

With regard to the spin, the second Casimir function can be defined as the square of a new vector $\tilde{w}$ which we now define:

$$\tilde{w} = \frac{1}{2\lambda} \, (\bar{\gamma}^{-1} \tilde{p} \gamma - \tilde{p}) \; . \tag{30}$$

$\tilde{w}$ is a 2×2 hermitean matrix, so we can write:

$$\tilde{w} = \begin{pmatrix} -w_0 + w_3 & w_1 - iw_2 \\ w_1 + iw_2 & -w_0 - w_3 \end{pmatrix} \; . \tag{31}$$

It is also a deformation of the standard Pauli-Lubanski vector. To see this, we substitute $\gamma = e^{i\lambda j}$ and $\bar{\gamma} = e^{i\lambda j^\dagger}$ in eq. (30) and expand around $\lambda = 0$, yielding

$$\tilde{w} = \frac{i}{2}(\tilde{p} j - j^\dagger \tilde{p}) + \mathcal{O}(\lambda) \;, \tag{32}$$

the zeroth order term in λ being the Pauli-Lubanski vector. Under Lorentz transformations, $\tilde{w}$ transforms as $\tilde{p}$ does, i.e. $\tilde{w} \to \tilde{w}' = \bar{g}\tilde{w}g^{-1}$, and in addition, we find that its Poisson brackets with the observables γ and $\bar{\gamma}$ are identical in form to those of $\tilde{p}$ with γ and $\bar{\gamma}$, i.e.

$$\{ \underset{1}{\tilde{w}} , \underset{2}{\gamma} \} = r^\dagger \underset{1}{\tilde{w}} \underset{2}{\gamma} + \underset{1}{\tilde{w}} \underset{2}{\gamma} r - \underset{2}{\gamma} r^\dagger \underset{1}{\tilde{w}} - \underset{1}{\tilde{w}} r^\dagger \underset{2}{\gamma} , \tag{33}$$

$$\{ \underset{1}{\tilde{w}} , \underset{2}{\bar{\gamma}} \} = r \underset{1}{\tilde{w}} \underset{2}{\bar{\gamma}} + \underset{1}{\tilde{w}} \underset{2}{\bar{\gamma}} r - \underset{2}{\bar{\gamma}} r^\dagger \underset{1}{\tilde{w}} - \underset{1}{\tilde{w}} r \underset{2}{\bar{\gamma}} . \tag{34}$$

The Poisson brackets $\tilde{w}$ with $\tilde{p}$ are given by

$$\{ \underset{1}{\tilde{w}} , \underset{2}{\tilde{p}} \} = r \underset{1}{\tilde{w}} \underset{2}{\tilde{p}} + \underset{1}{\tilde{w}} \underset{2}{\tilde{p}} r^\dagger - \underset{2}{\tilde{p}} r^\dagger \underset{1}{\tilde{w}} - \underset{1}{\tilde{w}} r \underset{2}{\tilde{p}} , \tag{35}$$

which in terms of space-time components, eq. (35) can be expressed as follows:

$$\begin{aligned} \{w_i, p_j\} &= 2\lambda\Big(\epsilon_{ij\ell}(p_0 + p_3) - \delta_{j3}\epsilon_{ik\ell}p_k\Big)w_\ell \ , \\ \{w_i, p_0\} &= 2\lambda\epsilon_{ijk}p_j w_k \qquad i,j,k = 1,2,3 \ , \\ \{w_0, p_\mu\} &= 0 \ . \end{aligned} \tag{36}$$

We then find that, in analogy to eq. (29),

$$\{\det\,(\tilde{w}), \tilde{p}\} = \{\det\,(\tilde{w}), \gamma\} = \{\det\,(\tilde{w}), \bar{\gamma}\} = 0 \ , \tag{37}$$

and hence that $w_\mu w^\mu = \det(\tilde{w})$ is a classical Casimir function.

We expect that there are no additional independent Casimir functions and therefore that the symplectic leaves in $\mathcal{O}$ are eight-dimensional, just as is the case with the Poincaré algebra. In the two sections which follow, we shall show how to parametrize the symplectic leaves with variables which one can naturally associate with position, momenta and spin.

For completeness we compute the Poisson brackets for $\tilde{w}$ with itself. From eqs. (33-35), we find

$$\{ \underset{1}{\tilde{w}} , \underset{2}{\tilde{w}} \} = r \underset{1}{\tilde{w}} \underset{2}{\tilde{w}} + \underset{1}{\tilde{w}} \underset{2}{\tilde{w}} r^\dagger - \underset{2}{\tilde{w}} r^\dagger \underset{1}{\tilde{w}} - \underset{1}{\tilde{w}} r \underset{2}{\tilde{w}} - i\Pi(\underset{1}{\tilde{w}} \underset{2}{\tilde{p}} - \underset{2}{\tilde{w}} \underset{1}{\tilde{p}}) , \tag{38}$$

or in terms of the space-time components of $\tilde{w}$,

$$\begin{aligned} \{w_i, w_j\} &= \epsilon_{ijk}\Big(p_0 w_k - w_0 p_k + 2\lambda(p_0 + p_3)p_k\Big) \ , \\ \{w_0, w_i\} &= \epsilon_{ijk}p_j w_k \qquad i,j,k = 1,2,3 \ . \end{aligned} \tag{39}$$

¿From eqs. (36) and (39), we deduce that the set of commuting operators in the quantum theory can be enlarged to those associated with

$$p_0, \ p_3, \ w_0 \text{ and } w_i p_i \ ,$$

in addition to the two Casimirs $p_\mu p^\mu$ and $w_\mu w^\mu$.

[We note that in ref. [4], a set of commuting operators for the spinless particle was found for a similar system. The set contained operators associated with p_0, p_3 and the third component of angular momentum. All of these variables were shown to have a discrete quantum spectrum for the case of a particle with nonzero mass. (From ref. [11], we surmise that such a particle is not free, but instead has a nontrivial interaction with the space-time.) With regard to the variables p_0 and p_3, we expect that a similar spectrum will occur for us. We do not know what the third component of angular momentum corresponds to in our formalism, nor do we know if w_0 and $w_i p_i$ can be included in the set of commuting operators of ref. [4].]

3 Spin Zero Realization

Here we discuss a realization of the deformed Poincaré algebra (2-5) in terms of space-time coordinates x_μ and the momenta p_μ. The realization, is based on the system described in [11], where we deformed the canonical symplectic structure for a relativistic particle. As we shall see, this realization has the Casimir $\det(\tilde{w})$ equal to zero, and we therefore can associate it with the description of a (deformed) spinless relativistic particle. Actually, here we get the even stronger constraint that $\tilde{w} = 0$, or from (30),

$$\tilde{p}\gamma = \bar{\gamma}\tilde{p} \; . \tag{40}$$

This is analogous to what is obtained in the *canonical theory* of spinless particles, where all of the components of the Pauli-Lubanski vector vanish.

The momentum matrix $\tilde{p}$ was introduced previously in eq. (12). With regard to the space-time coordinates x_μ, we find it convenient to define the 2×2 hermitean matrix

$$x = \begin{pmatrix} -x_0 - x_3 & -x_1 + ix_2 \\ -x_1 - ix_2 & -x_0 + x_3 \end{pmatrix} \; . \tag{41}$$

In contrast to $\tilde{p}$, x transforms according to

$$x \rightarrow x' = g x \bar{g}^{-1} \; . \tag{42}$$

As stated previously, the Poisson structure for x and $\tilde{p}$ is required to be a deformation of the canonical Poisson brackets for a relativistic particle. In addition it should be preserved under the Lie-Poisson action of the Lorentz group, satisfy the Jacobi identity, and be hermitean. The following Poisson brackets are consistent with all of the above conditions:

$$\{ \underset{1}{x} , \underset{2}{x} \} \;=\; r \underset{1}{x} \underset{2}{x} + \underset{1}{x} \underset{2}{x} r^\dagger - \underset{2}{x} r \underset{1}{x} - \underset{1}{x} r^\dagger \underset{2}{x} \; , \tag{43}$$

$$\{ \underset{1}{x} , \underset{2}{\tilde{p}} \} \;=\; r \underset{1}{x} \underset{2}{\tilde{p}} + \underset{1}{x} \underset{2}{\tilde{p}} r^\dagger - \underset{2}{\tilde{p}} r \underset{1}{x} - \underset{1}{x} r^\dagger \underset{2}{\tilde{p}} - \Pi(\underset{1}{f}^\dagger + \underset{2}{f}) \; , \tag{44}$$

along with eq. (2). f is a 2×2 complex matrix. For eq. (44) to be preserved under Lorentz transformations, it must transform like γ, i.e.

$$f \rightarrow g f g^{-1} \; . \tag{45}$$

f must be a function of λ. This is since in order to recover the canonical relations, we need that f tends to the unit matrix $1\!\!1$ when $\lambda \to 0$, while it cannot be $1\!\!1$ for all λ because some work shows that it would not then satisfy the Jacobi identity. In this regard, the issue of the Jacobi identity was only partially addressed in [11]. Here we find (with the aid of algebraic manipulation packages) that the Jacobi identity involving the position and momentum variables holds provided that f satisfies the following Poisson brackets with x and $\tilde{p}$:

$$\{ \underset{1}{\tilde{p}} , \underset{2}{f} \} \;=\; r^\dagger \underset{1}{\tilde{p}}\, \underset{2}{f} + \underset{1}{\tilde{p}}\, \underset{2}{f}\, r - \underset{2}{f}\, r^\dagger \underset{1}{\tilde{p}} - \underset{1}{\tilde{p}}\, r^\dagger \underset{2}{f} + i\lambda \underset{1}{\tilde{p}}\, \underset{2}{f} \;, \tag{46}$$

$$\{ \underset{1}{x} , \underset{2}{f} \} \;=\; r^\dagger \underset{1}{x}\, \underset{2}{f} + \underset{1}{x}\, \underset{2}{f}\, r^\dagger - \underset{2}{f}\, r\, \underset{1}{x} - \underset{1}{x}\, r^\dagger \underset{2}{f} - i\lambda \underset{1}{x}\, \underset{2}{f} \;. \tag{47}$$

In the above, it appears that we have enlarged the phase space spanned by x and $\tilde{p}$ to also include the variables f. However it is not necessary to regard f as independent variables. Rather, it is possible to express f in terms of x and $\tilde{p}$, while still being consistent with Poisson brackets (46) and (47), as well as with the canonical limit $f \to 1\!\!1$. This was done in ref. [11], where we wrote f according to:

$$f = \exp\{i\lambda J\} \;, \quad \text{where} \quad \sin \lambda J = \lambda x \tilde{p} \;. \tag{48}$$

By this we meant the following Taylor expansion:

$$\begin{aligned} f \;&=\; 1\!\!1 + i\lambda x\tilde{p} - \frac{1}{2}(\lambda x\tilde{p})^2 - \frac{1}{8}(\lambda x\tilde{p})^4 \; ... \\ &\qquad ... \; - \frac{1\cdot 3\cdot 5\cdot \; ... \; \cdot (2n-3)}{2\cdot 4\cdot 6\cdot \; ... \; \cdot 2n}(\lambda x\tilde{p})^{2n} \; ... \end{aligned} \tag{49}$$

It is easily seen that this expression transforms as in eq. (45) and that it tends to the unit matrix $1\!\!1$ when $\lambda \to 0$. We show that it agrees with eq. (46) in the appendix. In a similar manner, it can be shown to agree with eq. (47). Also in a similar manner, we can use eq. (48) to obtain the brackets for f with itself and with $f^\dagger$. We find:

$$\{ \underset{1}{f} , \underset{2}{f} \} \;=\; r^\dagger \underset{1}{f}\, \underset{2}{f} + \underset{1}{f}\, \underset{2}{f}\, r - \underset{2}{f}\, r\, \underset{1}{f} - \underset{1}{f}\, r^\dagger \underset{2}{f} \;, \tag{50}$$

$$\{ \underset{1}{f} , \underset{2}{f^\dagger} \} \;=\; r\, \underset{1}{f}\, \underset{2}{f^\dagger} + \underset{1}{f}\, \underset{2}{f^\dagger}\, r - \underset{2}{f^\dagger}\, r\, \underset{1}{f} - \underset{1}{f}\, r\, \underset{2}{f^\dagger} \;. \tag{51}$$

We have checked that these relations are consistent with the Jacobi identity. We note that the former Poisson bracket is identical in form to the Poisson brackets (3) for γ, the only difference between them being that while γ appearing in eq. (3) is unimodular, f is not. In this regard, we note that we are not allowed to set $\det(f)$ equal to one, because although $\det(f)$ has zero brackets with f and $f^\dagger$ [which follows from eqs. (50) and (51)], it does not have zero brackets with the coordinates or the momenta. Instead from eq. (46), we find

$$\{x, \det\,(f)\} \;=\; -2i\lambda x \,\det\,(f) \;,$$

$$\{\tilde{p}, \det(f)\} = 2i\lambda\tilde{p}\det(f)\,. \tag{52}$$

Alternatively, we can easily define a unimodular matrix from f by simply dividing by $\sqrt{\det(f)}$. This is in fact how we make the identification with γ. Thus we set

$$\gamma = \frac{f}{\sqrt{\det(f)}}\,. \tag{53}$$

It then follows that if we again write $\gamma = e^{i\lambda j}$ and use eq. (48), then j corresponds to the traceless part of J,

$$j = J - \frac{1}{2}\mathbb{1}\,\mathrm{Tr}\,J\,. \tag{54}$$

Upon keeping terms linear in λ in the expression (49) for we see that j reduces to the usual expression for orbital angular momentum in the canonical limit $\lambda \to 0$.

Using eq. (53) we can recover the Poisson brackets (3-5) starting from the brackets for f [specifically, (46), (50) and (51)]. Since we also have eq. (2), we obtain a realization of the entire deformed Poincaré algebra. Using eqs. (47) and (52), we can further obtain the brackets between the space-time coordinates x and γ:

$$\{\underset{1}{x}, \underset{2}{\gamma}\} = r^\dagger \underset{1}{x}\,\underset{2}{\gamma} + \underset{1}{x}\,\underset{2}{\gamma}\, r^\dagger - \underset{2}{\gamma}\, r\, \underset{1}{x} - \underset{1}{x}\, r^\dagger \underset{2}{\gamma}\,. \tag{55}$$

It now only remains to show that the condition (40) is satisfied implying that the deformed Pauli-Lubanski vector is zero . This follows using results from the appendix, specifically eqs. (91) and (93), which lead to

$$\tilde{p}e^{i\lambda J} = e^{i\lambda J^\dagger}\tilde{p}\,, \tag{56}$$

If we now divide both sides of this equation by $\sqrt{\det(f)}$, we get eq. (40) and hence the description of a (deformed) spinless particle. We therefore associate the expression for γ in terms of x and $\tilde{p}$ with the 'orbital angular momentum'.

In [11], we made some remarks concerning the dynamics for such a system. There we showed that if the standard mass shell constraint is chosen for the Hamiltonian function, i.e.

$$H = \alpha(\det(\tilde{p}) - m^2)\,, \tag{57}$$

then it, along with the Poisson brackets (2), (43) and (44), yields a nontrivial interaction of the particle with the space-time when λ and m are different from zero. (α denotes a Lagrange multiplier). In [11] we solved for the particle trajectory and found that it originates and terminates at singularities. The particle has a lifetime equal to

$$\left|\frac{\mathrm{Tr}(\gamma p)}{\lambda\det(\tilde{p})}\right|, \tag{58}$$

where $p = \sigma_2\tilde{p}^T\sigma_2$, T denoting transpose and σ_2 being the second Pauli matrix. Thus when λ and m are different from zero, it appears that the dynamics corresponds to a kind of 'virtual' particle which is off its 'physical' mass shell. (By 'physical' mass shell we mean that the square of the *canonical* momentum - not p_μ - is equal to m^2.)

We note that the expression (58) is singular for the case of a massless particle, i.e. $m = 0$, indicating that such a particle has an infinite lifetime. Actually, we can show that the Hamiltonian (57) with $m = 0$ [along with the Poisson brackets (2), (43) and (44)] describes a free photon (or any massless particle) for arbitrary values of λ. For this we compute the Hamilton equations of motion:

$$\dot{x} = \frac{d}{d\tau}x = \alpha\{x, \det(\tilde{p})\} = -\alpha(fp + pf^\dagger) , \tag{59}$$

$$\dot{\tilde{p}} = \frac{d}{d\tau}\tilde{p} = \alpha\{\tilde{p}, \det(\tilde{p})\} = 0 . \tag{60}$$

It then follows that for $m = 0$,

$$\dot{x}\tilde{p} = -\alpha(fp + pf^\dagger)\tilde{p} = 0 , \tag{61}$$

where we have used the mass shell constraint $\tilde{p}p = p\tilde{p} = \det(\tilde{p})\mathbb{1} = 0$, in addition to the series expansion eq. (49) for f. The traceless part of eq. (61) is equivalent to $\dot{x}_\mu p_\nu - \dot{x}_\nu p_\mu = 0$ and hence

$$\dot{x}_\mu = \kappa p_\nu . \tag{62}$$

The trace of eq. (61) implies that $\dot{x}^\mu p_\mu = 0$, and thus gives no constraint on the proportionality constant κ. We therefore arrive at a *free* light-like trajectory. Furthermore, since λ is arbitrary, we get an entire family of canonically inequivalent Hamiltonian descriptions of a photon trajectory. Upon quantization, the resulting states are expected to transform covariantly under the action of the quantum Lorentz group.

Actually, to truly describe a photon, we should introduce a spin and check that its equation of motion is the usual one. That is, there should be no classical spin precession. Spin will be introduced in the next section. There we indeed find that there is no precession of the classical spin (even for the case $m \neq 0$).

4 Inclusion of Spin

In the canonical theory, spin is introduced as an additional term in the angular momentum. This term is defined to have zero Poisson brackets with coordinates and momenta, and it is the only term in the angular momentum which contributes to the Pauli-Lubanski vector. We shall look for an analogous prescription for including spin in our deformed Poincaré algebra. It should reduce to the canonical prescription in the limit $\lambda \to 0$. In this regard, there is of course no unique prescription for including spin in the deformed Poincaré algebra. In what follows, our choice shall be to multiply the $SL(2,C)$ matrix γ obtained in the previous section on the right by another $SL(2,C)$ matrix γ_s. The former is to be regarded as the orbital angular momentum, while γ_s plays the role of 'spin'. Thus

$$\gamma \to \gamma\gamma_s = \frac{f\gamma_s}{\sqrt{\det(f)}} . \tag{63}$$

To get back the canonical prescription, i.e. $j \to j + s$, when $\lambda \to 0$, we can take $\gamma_s = e^{i\lambda s}$, s being a traceless complex matrix. Furthermore, using eq. (63) we get that the 'spin' γ_s, and not the 'orbital angular momentum' γ [subject to eq. (40)], contributes to the deformed

Pauli-Lubanski vector eq. (30), analogous to what happens in the canonical theory. [This would not have been the case if instead of eq. (63), we had multiplied γ on the left by γ_s.] We next show that *unlike* in the canonical theory, the spin has nonzero Poisson brackets with momentum and position, and that this is a consequence of the fact that the space spanned by the matrices γ does not form a Poisson-Lie group (unlike the space spanned by matrices g).

Because $\mathcal{S}$ is a Poisson-Lie group the Poisson structure for the $SL(2,C)$ matrices g is preserved under left or right group multiplication, i.e. the Poisson brackets (7) are compatible with the group product[12]-[15]. As we show below, the analogous statement does not, however, apply to the Poisson structure for the $SL(2,C)$ matrices γ. That is, the Poisson brackets (3) are not compatible with group multiplication and hence the space spanned by γ is not a Poisson-Lie group.

To see that the Poisson structure for symmetries is preserved under group multiplication, one defines a variable $g' \in SL(2,C)$ which satisfies the same relations as g,

$$\{\underset{1}{g}', \underset{2}{g}'\} = [\, r \,, \;\underset{1}{g}'\underset{2}{g}'\,] \,, \tag{64}$$

in addition to $\{\underset{1}{g}', \underset{2}{g}\} = 0$. Then under right multiplication $g \to gg'$, we get

$$\begin{aligned}\{\underset{1}{g}, \underset{2}{g}\} \to \{\underset{1}{g}\underset{1}{g}', \underset{2}{g}\underset{2}{g}'\} &= [\, r \,, \;\underset{1}{g}\underset{2}{g}\,]\underset{1}{g}'\underset{2}{g}' + \underset{1}{g}\underset{2}{g}[\, r \,, \;\underset{1}{g}'\underset{2}{g}'\,] \\ &= [\, r \,, \;(\underset{1}{g}\underset{1}{g}')(\underset{2}{g}\underset{2}{g}')\,] \,,\end{aligned} \tag{65}$$

and hence that the Poisson structure given by (7) is preserved.

Let us now try the same thing for the observables γ. We define $\gamma_s \in SL(2,C)$ to satisfy the same relations as γ,

$$\{\underset{1}{\gamma}_s, \underset{2}{\gamma}_s\} = r^\dagger \underset{1}{\gamma}_s\underset{2}{\gamma}_s + \underset{1}{\gamma}_s\underset{2}{\gamma}_s r - \underset{2}{\gamma}_s r \underset{1}{\gamma}_s - \underset{1}{\gamma}_s r^\dagger \underset{2}{\gamma}_s \,, \tag{66}$$

in addition to $\{\underset{1}{\gamma}_s, \underset{2}{\gamma}\} = 0$. But it is easily checked that this Poisson structure is not preserved under right (or left) multiplication $\gamma \to \gamma\gamma_s$, and therefore that $f\gamma_s/\sqrt{\det(f)}$ does not give a realization of the relations (3).

To proceed further we shall drop the assumption that the product space $\{\gamma\} \times \{\gamma_s\}$ has a product Poisson structure, i.e. we drop the assumption that the spin has zero Poisson brackets with the orbital angular momentum γ (and hence also with the coordinates and momenta) in the deformed theory, i.e. $\{\underset{1}{\gamma}_s, \underset{2}{\gamma}\} \neq 0$. Instead we take

$$\{\underset{1}{\gamma}_s, \underset{2}{\gamma}\} = r^\dagger \underset{1}{\gamma}_s \underset{2}{\gamma} + \underset{1}{\gamma}_s \underset{2}{\gamma} r^\dagger - \underset{2}{\gamma} r^\dagger \underset{1}{\gamma}_s - \underset{1}{\gamma}_s r^\dagger \underset{2}{\gamma} \;, \tag{67}$$

or equivalently,

$$\{\underset{1}{\gamma}, \underset{2}{\gamma}_s\} = r \underset{1}{\gamma} \underset{2}{\gamma}_s + \underset{1}{\gamma} \underset{2}{\gamma}_s r - \underset{2}{\gamma}_s r \underset{1}{\gamma} - \underset{1}{\gamma} r \underset{2}{\gamma}_s \;. \tag{68}$$

It then can be checked that the product $\gamma\gamma_s$ carries the same Poisson structure as γ [i.e. eq. (3)],

$$\{\underset{1}{\gamma}\underset{1}{\gamma}_s, \underset{2}{\gamma}\underset{2}{\gamma}_s\} = r^\dagger(\underset{1}{\gamma}\underset{1}{\gamma}_s)(\underset{2}{\gamma}\underset{2}{\gamma}_s)+(\underset{1}{\gamma}\underset{1}{\gamma}_s)(\underset{2}{\gamma}\underset{2}{\gamma}_s)r-(\underset{2}{\gamma}\underset{2}{\gamma}_s)r(\underset{1}{\gamma}\underset{1}{\gamma}_s)-(\underset{1}{\gamma}\underset{1}{\gamma}_s)r^\dagger(\underset{2}{\gamma}\underset{2}{\gamma}_s) \;. \tag{69}$$

Thus, in this sense we can preserve the Poisson structure for the observables. We note that the Poisson bracket relations for γ and γ_s are identical in form to those of γ and $\gamma^\dagger$. Then since the Jacobi identity holds for the latter variables, it must also hold for γ and γ_s. In addition, we have that $\{\det(\gamma), \gamma_s\} = \{\det(\gamma_s), \gamma\} = 0$ and therefore the Poisson brackets (67) are consistent with the unimodularity of both γ and γ_s. We note that eq. (67) is also consistent with the canonical theory, because if we write $\gamma = e^{i\lambda j}$ and $\gamma_s = e^{i\lambda s}$ then to lowest order in λ, we get that s has zero Poisson brackets with j.

As stated before, in the canonical theory the spin has zero Poisson brackets with the momenta. Here, however, if the Poisson bracket (5) is to be preserved under eq. (63), we need that $\{\underset{1}{\gamma}_s, \underset{2}{\tilde{p}}\} \neq 0$. Specifically, we need

$$\{\underset{1}{\tilde{p}}, \underset{2}{\gamma}_s\} = r^\dagger \underset{1}{\tilde{p}} \underset{2}{\gamma}_s + \underset{1}{\tilde{p}} \underset{2}{\gamma}_s r - \underset{2}{\gamma}_s r^\dagger \underset{1}{\tilde{p}} - \underset{1}{\tilde{p}} r \underset{2}{\gamma}_s \;. \tag{70}$$

For then

$$\{\underset{1}{\tilde{p}}, \underset{2}{\gamma}\underset{2}{\gamma}_s\} = r^\dagger \underset{1}{\tilde{p}}(\underset{2}{\gamma}\underset{2}{\gamma}_s) + \underset{1}{\tilde{p}}(\underset{2}{\gamma}\underset{2}{\gamma}_s)r - (\underset{2}{\gamma}\underset{2}{\gamma}_s)r^\dagger \underset{1}{\tilde{p}} - \underset{1}{\tilde{p}} r^\dagger(\underset{2}{\gamma}\underset{2}{\gamma}_s) \;, \tag{71}$$

and the relation (5) is preserved. We have checked that eq. (70) is consistent with the Jacobi identity for γ, $\tilde{p}$ and γ_s. We also verified that $\{\det(\gamma_s), \tilde{p}\} = 0$ and that $\tilde{p}$ has zero brackets with s in the limit $\lambda \to 0$.

In the above [cf. eq. (70)], we found that the spin γ_s does not have zero Poisson brackets with the momenta. It also doesn't have zero Poisson brackets with the position. That $\{\underset{1}{\gamma}_s, \underset{2}{x}\} \neq 0$ is easily seen because if it were not so we would not then be able to recover the correct brackets (67) for γ_s with the orbital angular momenta γ [given as a function of $x\tilde{p}$ in eqs. (49) and (53)]. What works instead is

$$\{\underset{1}{x}, \underset{2}{\gamma_s}\} = r\underset{1}{x}\,\underset{2}{\gamma_s} + \underset{1}{x}\,\underset{2}{\gamma_s} r^\dagger - \underset{2}{\gamma_s} r \underset{1}{x} - \underset{1}{x} r^\dagger \underset{2}{\gamma_s} \,. \tag{72}$$

¿From it and eq. (70) we find that

$$\{(\underset{1}{x}\,\underset{1}{\tilde{p}})^n, \underset{2}{\gamma_s}\} = r(\underset{1}{x}\,\underset{1}{\tilde{p}})^n \underset{2}{\gamma_s} + (\underset{1}{x}\,\underset{1}{\tilde{p}})^n \underset{2}{\gamma_s} r - \underset{2}{\gamma_s} r(\underset{1}{x}\,\underset{1}{\tilde{p}})^n - (\underset{1}{x}\,\underset{1}{\tilde{p}})^n r \underset{2}{\gamma_s} \,. \tag{73}$$

Hence the Poisson brackets between γ_s and *any* polynomial function of $x\tilde{p}$ has precisely the same form as the Poisson brackets between γ_s and γ. Thus

$$\{\underset{1}{f}, \underset{2}{\gamma_s}\} = r\underset{1}{f}\,\underset{2}{\gamma_s} + \underset{1}{f}\,\underset{2}{\gamma_s} r - \underset{2}{\gamma_s} r \underset{1}{f} - \underset{1}{f} r \underset{2}{\gamma_s} \,. \tag{74}$$

From previous arguments we then also know that $\{\det(f), \gamma_s\} = 0$. The Poisson brackets (67) between γ and γ_s are recovered by dividing both sides of eq. (74) by $\sqrt{\det(f)}$. ¿From eq. (72) it also follows that $\{\underset{1}{x}, \underset{2}{s}\} \to 0$ when $\lambda \to 0$ and hence we recover the usual canonical limit.

To summarize, we have obtained a realization of the deformed Poincare algebra defined by eqs. (2-5) with $\det(\tilde{w}) \neq 0$. (Actually for this we also need to have the Poisson brackets between γ_s and $\bar{\gamma}$. We shall assume that a consistent set of such brackets exist.) Thus unlike the previous section, we now have a particle with spin. We get back the canonical description of a spinning particle when $\lambda \to 0$. Under Lorentz transformations, γ_s must transform as does γ, i.e. $\gamma_s \to g\gamma_s g^{-1}$. It is easy to check that all Poisson brackets with γ_s are preserved under such transformations and once again that the Lorentz group induces a Lie-Poisson action on the space of observables. (Here we assume as usual that the classical observables have zero Poisson brackets with the classical symmetries.)

Finally, we remark about the spin dynamics. For this purpose it is of interest to compute the Poisson bracket of γ_s with $\det(\tilde{p})$. From eq. (70), we find that this Poisson bracket vanishes. This means that if the Hamiltonian function for the system is once again chosen to be the usual mass shell constraint (57), then the classical spin has a trivial dynamics (i.e., there is no precession). This is just as in the canonical formulation of a classical spinning particle.[10] In other words, the particle interaction which is present due to the highly nontrivial Poisson structure does not affect the spin.

5 Towards Quantization

Here we make some preliminary remarks concerning quantization. We plan to address this issue more fully in a subsequent publication.

There exists a standard quantization scheme (deformation quantization) which can be applied for the symmetries which takes $\mathcal{S}$ to a Hopf algebra, specifically, $SL_q(2,C)$.[9] We remark on this first. We then comment on a possible quantization of the classical observables. The system which results appears to be different from $q-$Poincaré algebras discussed previously in the literature. [3-7]

With regard to the symmetries, one standardly replaces g by an $SL_q(2,C)$ matrix which we denote by T and the Poisson brackets (7) by the corresponding quantum commutation relations. The matrix elements in T are constrained by the condition that its 'deformed' determinant is equal to one, and this is the analogue of the unimodularity condition on g. The commutation relations are given in terms of a quantum R matrix, satisfying the usual quantum Yang-Baxter equations, and can be written according to

$$\underset{12}{R}\,\underset{1}{T}\,\underset{2}{T} = \underset{2}{T}\,\underset{1}{T}\,\underset{12}{R}\ . \tag{75}$$

This algebra can presumably be realized on the space $\mathcal{S}$ of classical symmetries with the use of a star product. In order to recover the correct classical limit one only needs that $\underset{12}{R} \rightarrow 1\!\!1 - i\hbar r + \mathcal{O}(\hbar^2)$ when $\hbar \rightarrow 0$. In addition to eq. (75), one needs the quantum analogues of the Poisson brackets (8) and (10). For this we introduce another $SL_q(2,C)$ matrix $\bar{T}$ which in analogy to the classical observable $\bar{g}$ is defined by $\bar{T} = T^{\dagger^{-1}}$. Then along with eq. (75), we write

$$\underset{12}{R}\,\underset{1}{T}\,\underset{2}{\bar{T}} = \underset{2}{\bar{T}}\,\underset{1}{T}\,\underset{12}{R}\ , \tag{76}$$

$$\underset{12}{R}\,\underset{1}{\bar{T}}\,\underset{2}{\bar{T}} = \underset{2}{\bar{T}}\,\underset{1}{\bar{T}}\,\underset{12}{R}\ , \tag{77}$$

which reduces to eqs. (8) and (10) when $\hbar \rightarrow 0$. By switching vector space indices 1 and 2, we see that we can replace $\underset{12}{R}$ in eqs. (75) and (77) by $\underset{21}{R}^{-1}$. Thus $\underset{12}{R}\,\underset{21}{R}$ must commute with $\underset{2}{T}\,\underset{1}{T}$ and $\underset{2}{\bar{T}}\,\underset{1}{\bar{T}}$. This is analogous to the statement that $r - r^\dagger$ is an adjoint invariant in the classical theory.

Concerning the quantum observables, we shall require that its algebra is preserved under the action of the quantum symmetries, in analogy to what happens in the classical theory. We also want that this algebra is consistent with the classical Poisson bracket algebra in the limit $\hbar \rightarrow 0$. A quantum algebra for the momenta was already given in ref. [7] which is consistent with these properties, so we will adopt it here. There one replaces the classical variable $\tilde{p}$ by a 2×2 matrix P whose elements are operator-valued. The Poisson brackets (2) are replaced by what were refered to as reflection equations,

$$\underset{12}{R}\,\underset{1}{P}\,\underset{12}{R}^{-1}\underset{2}{P} = \underset{2}{P}\,\underset{21}{R}^{-1}\underset{1}{P}\,\underset{21}{R}\ . \tag{78}$$

With this choice, one can easily obtain the correct classical limit. For this one notes that using the matrix expression (6) for r, one gets that $\underset{21}{R} \rightarrow 1\!\mathrm{l} + i\hbar r^\dagger + \mathcal{O}(\hbar^2)$ when $\hbar \rightarrow 0$. Furthermore, as desired, the commutation relations (78) are preserved under $SL_q(2,C)$ transformations. Here in analogy to eq. (13), one assumes that P transforms as a vector under the quantum Lorentz group, i.e.

$$P \rightarrow P' = \bar{T} P T^{-1} \,, \tag{79}$$

and that the matrix elements of T and $\bar{T}$ commute with those of P. Then using the relations (75-77), one gets that the left hand side of (78) transforms according to

$$\begin{aligned}
\underset{12}{R}\,\underset{1}{P}\,\underset{12}{R}^{-1}\underset{2}{P} \rightarrow \underset{12}{R}\,\underset{1}{P}'\,\underset{12}{R}^{-1}\underset{2}{P}' &= \underset{2}{\bar{T}}\,\underset{1}{\bar{T}}\,\underset{12}{R}\,\underset{1}{P}\,\underset{12}{R}^{-1}\underset{2}{P}\,\underset{1}{T}^{-1}\underset{2}{T}^{-1} \\
&= \underset{2}{\bar{T}}\,\underset{1}{\bar{T}}\,\underset{2}{P}\,\underset{21}{R}^{-1}\underset{1}{P}\,\underset{21}{R}\,\underset{1}{T}^{-1}\underset{2}{T}^{-1} \\
&= \underset{2}{P}'\,\underset{21}{R}^{-1}\underset{1}{P}'\,\underset{21}{R} \,,
\end{aligned} \tag{80}$$

and hence that eq. (78) is preserved. Here we may assume that the quantum matrix P is hermitean, analogous to the fact that the classical matrix $\tilde{p}$ is hermitean. It is easy to check that this is consistent with the transformation property (79). It is also consistent with the commutation relation (78) provided that we have the following condition on the quantum $R-$matrix.

$$\underset{12}{R}^{\dagger} = \underset{21}{R} \,. \tag{81}$$

It remains to specify the quantum analogues of Poisson brackets (3-5). For this we associate the classical observables γ and $\bar{\gamma}$ with operator-valued 2×2 matrices Γ and $\bar{\Gamma}$, which in analogy to (16) transform as

$$\Gamma \rightarrow \Gamma' = T\Gamma T^{-1} \,, \quad \bar{\Gamma} \rightarrow \bar{\Gamma}' = \bar{T}\bar{\Gamma}\bar{T}^{-1} \,. \tag{82}$$

under the action of $SL_q(2,C)$. As with t, we can assume that its matrix elements in T are constrained by the condition that its 'deformed' determinant is equal to one, in analogy to the unimodularity condition on γ. We propose that the Γ's satisfy the following commutation relations amongst themselves and with P:

$$\underset{21}{R}^{-1}\underset{1}{\Gamma}\,\underset{21}{R}\,\underset{2}{\Gamma} = \underset{2}{\Gamma}\,\underset{12}{R}\,\underset{1}{\Gamma}\,\underset{12}{R}^{-1} \,, \tag{83}$$

$$\underset{12}{R}\,\underset{1}{\Gamma}\,\underset{12}{R}^{-1}\underset{2}{\bar{\Gamma}} = \underset{2}{\bar{\Gamma}}\,\underset{12}{R}\,\underset{1}{\Gamma}\,\underset{12}{R}^{-1} \,, \tag{84}$$

$$\underset{21}{R}^{-1}\underset{1}{P}\,\underset{21}{R}\,\underset{2}{\Gamma} = \underset{2}{\Gamma}\,\underset{21}{R}^{-1}\underset{1}{P}\,\underset{12}{R}^{-1} \,. \tag{85}$$

It can be checked that from these relations one recovers the correct quadratic algebra, i.e. eqs. (3-5), as $\hbar \to 0$. Also, using eqs. (75-77) and the assumption that the matrix elements of T and $\bar{T}$ commute with those of Γ and $\bar{\Gamma}$, it can be checked that the commutation relations (83- 85) are preserved under $SL_q(2,C)$ transformations. The procedure is analogous to that used in eq. (80).

If we define $\bar{\Gamma} = \Gamma^{\dagger^{-1}}$ in analogy to what was done in the classical theory, then by taking the hermitean conjugates of eqs. (83-85) we get the quantum analogues of the Poisson brackets (18-20). Using (81), we get

$$\underset{12}{R}\,\underset{1}{\bar{\Gamma}}\,\underset{12}{R^{-1}}\,\underset{2}{\bar{\Gamma}} = \underset{2}{\bar{\Gamma}}\,\underset{21}{R^{-1}}\,\underset{1}{\bar{\Gamma}}\,\underset{21}{R}\ , \tag{86}$$

$$\underset{21}{R^{-1}}\,\underset{1}{\bar{\Gamma}}\,\underset{21}{R}\,\underset{2}{\Gamma} = \underset{2}{\Gamma}\,\underset{21}{R^{-1}}\,\underset{1}{\bar{\Gamma}}\,\underset{21}{R}\ , \tag{87}$$

$$\underset{12}{R}\,\underset{1}{P}\,\underset{12}{R^{-1}}\,\underset{2}{\bar{\Gamma}} = \underset{2}{\bar{\Gamma}}\,\underset{21}{R^{-1}}\,\underset{1}{P}\,\underset{12}{R^{-1}}\ , \tag{88}$$

which correspond to eqs. (18-20) when $\lambda \to 0$.

Although the set of symmetry operators $\{T\}$ defines a Hopf algebra, the same does not seem to be the case for the set of quantum operators $\{\Gamma\}$. In this regard we do not know how to define a coproduct for the latter. This is not too surprising since the set of classical variables $\{\gamma\}$ did not define a Poisson-Lie group. For this reason it appears that our quantum algebra defined in eqs. (78) and (83-85) differs from those given previously.[3-7] Eqs. (78) and (83-85) define a set the quadratic commutation relations between the quantum mechanical observables P,Γ and $\bar{\Gamma}$, which nevertheless are preserved under the action of a Hopf algebra. Whether or not it is necessary to impose higher order relations remains to be checked. ¿From the observables it is possible to construct the quantum mechanical deformed Pauli-Lubanski vector, and also Casimir operators corresponding to mass and spin. It should then be possible to look for eigenvectors of these operators along with the quantum analogues of p_0, p_3, w_0 and $w_i p_i$. We can also hope to obtain realizations of the quantum algebra for the cases of a spinless and spinning relativistic particle in a manner similar to what was done in Secs. 3 and 4.

6 Concluding Remarks

Here we outline additional future avenues of research.

We have obtained a deformation of the Poincaré algebra which is covariant with respect to the Lie-Poisson action of the Lorentz group. It is of interest to know whether or not this algebra can also be made to be covariant under the Lie-Poisson action of the translation group, and hence under the action of the full Poincaré group. With regard to Lorentz transformations alone, it is also of interest to know whether or not the angular momenta γ can somehow play the role of generators of the transformation, as in the canonical theory, and also whether or not the momenta $\tilde{p}$ can somehow play the role of generators of translations. One thing which is clear is that infinitesimal Lorentz transformations are not obtained (as in the canonical theory) by simply taking Poisson brackets with γ. Similarly, translations are not obtained by simply taking Poisson brackets with $\tilde{p}$. [An analogous problem was solved in [16] upon studying the system of a isotropic rigid rotator. That system was invariant under the Lie-Poisson action of the chiral symmetry group. There we were able to find the generators of the chiral symmetry, and they took values in a group which was dual to the symmetry group. Similar novel features are anticipated here.]

In Secs. 3 and 4, realizations for the deformed Poincaré algebra were found which were associated with a single relativistic particle. It is then of interest to know how one constructs representations for two or more particles. This is not straightforward because as we found in Sec. 4, the Poisson structure for the observables γ is not compatible with the product for the Lorentz group, i.e. the space spanned by $\{\gamma\}$ did not define a Poisson-Lie group. Also, it can be checked that the Poisson structure for the observables $\tilde{p}$ is not compatible with addition of the momenta. Just as we found in the case of a single particle that the spin does not commute with the orbital angular momentum, we can conclude the angular momenta for different particles cannot commute, and we also suspect that the momenta of different particles does not commute.

Acknowledgements: We are grateful to G. Marmo for helpful discussions. We were supported in part by the Department of Energy, USA, under contract number DE-FG05-84ER40141.

Appendix

Here we show that the Poisson brackets (46) for $\tilde{p}$ with f can be deduced using the realization for f in terms of x and $\tilde{p}$ given in eq. (48), along with the Poisson brackets (2) and (44).

We start by computing the brackets for $\tilde{p}$ with $\sin\lambda J = \lambda x\tilde{p}$. From eqs. (2) and (44) we get

$$\begin{aligned}\{\underset{1}{\tilde{p}},\sin(\lambda\underset{2}{J})\} &= r^\dagger\underset{1}{\tilde{p}}\,\sin(\lambda\underset{2}{J}) + \underset{1}{\tilde{p}}\,\sin(\lambda\underset{2}{J})r^\dagger - \sin(\lambda\underset{2}{J})r^\dagger\underset{1}{\tilde{p}} - \underset{1}{\tilde{p}}\,r^\dagger\sin(\lambda\underset{2}{J}) \\ &\quad + \lambda(\underset{1}{f}^\dagger + \underset{2}{f})\underset{1}{\tilde{p}}\,\Pi\ .\end{aligned} \tag{89}$$

To determine the brackets of $\tilde{p}$ with f, we also need $\{\underset{1}{\tilde{p}},\cos(\lambda\underset{2}{J})\}$. We can deduce it by knowing the brackets for $\tilde{p}$ with $\cos^2\lambda J = 1-\sin^2\lambda J$, which are easily obtained from eq. (89),

$$\{\underset{1}{\tilde{p}},\cos^2(\lambda\underset{2}{J})\} = r^\dagger\underset{1}{\tilde{p}}\,\cos^2(\lambda\underset{2}{J}) + \underset{1}{\tilde{p}}\,\cos^2(\lambda\underset{2}{J})r^\dagger - \cos^2(\lambda\underset{2}{J})r^\dagger\underset{1}{\tilde{p}} - \underset{1}{\tilde{p}}\,r^\dagger\cos^2(\lambda\underset{2}{J})$$

$$-\lambda\Big(\sin(\lambda \underset{2}{J})(\underset{1}{f}^{\dagger}+\underset{2}{f})+(\underset{1}{f}^{\dagger}+\underset{2}{f})\sin(\lambda \underset{1}{J}^{\dagger})\Big)\underset{1}{\tilde{p}}\,\Pi\ , \tag{90}$$

where we have used

$$\tilde{p}\ \sin(\lambda J)=\sin(\lambda J^{\dagger})\ \tilde{p}\ . \tag{91}$$

Then a solution is

$$\{\underset{1}{\tilde{p}},\cos(\lambda \underset{2}{J})\} \;=\; r^{\dagger}\underset{1}{\tilde{p}}\ \cos(\lambda \underset{2}{J})+\underset{1}{\tilde{p}}\ \cos(\lambda \underset{2}{J})r^{\dagger}-\cos(\lambda \underset{2}{J})r^{\dagger}\underset{1}{\tilde{p}}\ -\ \underset{1}{\tilde{p}}\,r^{\dagger}\cos(\lambda \underset{2}{J})$$

$$-\,i\lambda(\underset{1}{f}^{\dagger}-\underset{2}{f})\underset{1}{\tilde{p}}\,\Pi\ . \tag{92}$$

To check that eq. (90) follows from eq. (92), we can apply the identities

$$\tilde{p}\ \cos(\lambda J)=\cos(\lambda J^{\dagger})\ \tilde{p}\ , \tag{93}$$

and

$$\sin(\lambda \underset{2}{J})(\underset{1}{f}^{\dagger}+\underset{2}{f})+(\underset{1}{f}^{\dagger}+\underset{2}{f})\sin(\lambda \underset{1}{J}^{\dagger})=i\cos(\lambda \underset{2}{J})(\underset{1}{f}^{\dagger}-\underset{2}{f})+i(\underset{1}{f}^{\dagger}-\underset{2}{f})\cos(\lambda \underset{1}{J}^{\dagger})\ . \tag{94}$$

The former identity follows after writing J as an infinite series in $x\tilde{p}$, while the latter follows after writing $f=\cos(\lambda J)+i\sin(\lambda J)$. Finally, from eqs. (89) and (92), and using a third identity, i.e. eq. (11), we then obtain the desired result eq. (46).

References

[1] P. Podles and S. Woronowicz, Commun. Math. Phys. **130**, 381 (1990).

[2] U. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura, Z. Phys. **C 48**, 159 (1990).

[3] W. B. Schmidke, J. Wess and B. Zumino, Z. Phys. **C 52**, 471 (1991).

[4] O. Ogievetsky, W. B. Schmidke, J. Wess and B. Zumino, Commun. Math. Phys. **150**, 495 (1992); M.Pillin, W. B. Schmidke and J. Wess, Nucl. Phys. **B403**, 223 (1993).

[5] X.-C. Song, Z. Phys. **C 55**, 417 (1992); M. Pillin, Max Planck preprint MPI-Ph/93-61; U. Meyer, DAMTP 94-10.

[6] M.Pillin, W. B. Schmidke and J. Wess, Nucl. Phys. **B403**, 223 (1993).

[7] J. A. de Azcárraga, P. P. Kulish and F. Rodenas, Lett. Math. Phys. **32** 173 (1994); Valencia preprint FTUV 94-21; FTUV 94-64.

[8] Zakrzewski, S. *"Poisson Poincaré groups"* Warsaw preprint.

[9] See for example, L. Takhtajan in *Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory*, M-L. Ge and B-H. Zhao (eds.) (World Scientific, 1990).

[10] See for example, A. P. Balachandran, G. Marmo, B. Skagerstam and A. Stern, *Gauge Symmetries and Fibre Bundles, Applications to Particle Dynamics*, Lecture Notes in Physics **188** (Springer-Verlag, 1983).

[11] A. Simoni, A. Stern and I. Yakushin, University of Alabama preprint UAHEP 951.

[12] V.G. Drinfel'd, Sov. Math. Doklady **27** (1983) 68; in Proc. Int. Congr. Math. (Berkekey), vol. 1 Academic Press, New York, 1986.

[13] M.A. Semenov-Tian-Shansky, Publ. RIMS, Kyoto University 21, no. 6 (1985) 1237; Theor. Math. Phys. **93** (1992) 312 (in Russian).

[14] J.-H. Lu and A. Weinstein, J. Diff. Geom. **31**, 501 (1991).

[15] A. Yu. Alekseev and A. Z. Malkin, Paris preprint PAR- LPTHE 93-08; A. Yu. Alekseev and I. T. Todorov, Vienna preprint.

[16] G. Marmo, A. Simoni and A. Stern, Int. J. of Mod. Phys. **A10** 99 (1995); A. Stern and I. Yakushin, to appear in Mod. Phys. Lett **A**.