

**A Heterotic Sigma Model
with Novel Target Geometry**

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**A HETEROTIC SIGMA MODEL
WITH NOVEL TARGET GEOMETRY**

by

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Abstract

We construct a $(1, 2)$ heterotic sigma model whose target space geometry consists of a transitive Lie algebroid with complex structure on a Kaehler manifold. We show that, under certain geometrical and topological conditions, there are two distinguished topological half-twists of the heterotic sigma model leading to A and B type half-topological models. Each of these models is characterized by the usual topological BRST operator, stemming from the heterotic $(0, 2)$ supersymmetry, and a second BRST operator anticommuting with the former, originating from the $(1, 0)$ supersymmetry. These BRST operators combined in a certain way provide each half-topological model with two inequivalent BRST structures and, correspondingly, two distinct perturbative chiral algebras and chiral rings. The latter are studied in detail and characterized geometrically in terms of Lie algebroid cohomology in the quasiclassical limit.

Keywords: quantum field theory in curved spacetime; geometry, differential geometry and topology. PACS: 04.62.+v 02.40.-k

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1 Introduction

The superstring theories which, in the low energy limit, yield effective four-dimensional field theories with space-time supersymmetry are described by $(0, 2)$ superconformal field theories whose right-moving $U(1)$ charges satisfy suitable integrality conditions. (See [1]– [3] for background and extensive referencing.) The subject of $(0, 2)$ superconformal field theories is therefore of a considerable physical interest. Nevertheless, the amount of research work dedicated to these models has been very limited in comparison to the $(2, 2)$ models, which constitute a special subclass. The reason of this, of course, is that the smaller amount of world-sheet supersymmetry makes them harder to study.

Sigma models are nonlinear interacting quantum field theories and, so, are rather difficult to deal with. In general, they can be described only perturbatively in the large target space volume limit. However, as originally showed by Witten in ref. [4], important aspects of supersymmetric sigma models are captured by means of topological twist. A topologically twisted supersymmetric sigma model is characterized by the existence of a nilpotent BRST charge and the associated operator BRST cohomology. On a flat world-sheet, the untwisted parent model and the twisted daughter model are equivalent. The power of cohomological methods makes then it possible to study rather effectively the sector of the former corresponding to the operator BRST cohomology of the latter. On a non flat world-sheet, the two models are no longer equivalent, but the twisted model still exists and can be fruitfully studied.

As found by Witten in ref. [5], the $(2, 2)$ supersymmetric sigma model on a Kaehler manifold can be topologically twisted in two inequivalent ways. The operator BRST cohomology of the twisted sigma models is known as chiral ring. It consists of operator classes constant on the world-sheet and is endowed with a natural multiplicative structure. The twisted models are therefore topological

field theories. The A model chiral ring is isomorphic to the target manifold quantum cohomology, a deformation of de Rham cohomology related to Gromov–Witten invariants and Floer homology, and depends only on the target’s Kaehler structure. The B model chiral ring is isomorphic to the sheaf cohomology of the exterior algebra of the target manifold holomorphic tangent bundle and depends only on the target’s complex structure. For Calabi–Yau manifolds, A and B models are further related by mirror symmetry [6].

The $(0, 2)$ supersymmetric sigma model on a Kaehler manifold can be topologically twisted in just one way. The operator BRST cohomology of the twisted sigma model is known as chiral algebra. It consists of operator classes varying holomorphically on the world–sheet and is endowed with a natural product structure of the operator product expansion type. The twisted model, therefore, is not a topological field theory but a holomorphic field theory.

As remarked again by Witten in ref. [7], the chiral algebra of the twisted $(0, 2)$ sigma model is akin to the chiral ring of the twisted $(2, 2)$ sigma models, but only to a point. Like the chiral ring, it is independent from the target manifold’s metric, but, unlike the chiral ring, it receives complicated perturbative and world–sheet instanton corrections, because the model’s lack of left–moving supersymmetry. In spite of these difficulties, the chiral algebra remains an object of central importance in the theory. The interest in the subject has been revived in recent years, after Witten’s discovery that the perturbative chiral algebra can be formulated as the cohomology of a sheaf of chiral differential operators [8, 9]. Almost simultaneously, Kapustin [10] reached independently the same conclusion and, subsequently, Tan [11]– [14] generalized the analysis to heterotic $(0, 2)$ supersymmetric sigma models with left–moving fermions.

The perturbative chiral algebra contains a sector formed by classes with non singular operator product expansion, the $(0, 2)$ chiral ring, which shares many

properties with and is the true counterpart of the $(2, 2)$ chiral ring, as suggested by its name [15, 16, 17]. The $(0, 2)$ chiral ring is of considerable interest of its own and lends itself to a nice geometrical and topological characterization as the $(2, 2)$ one.

In this paper, we construct a heterotic $(1, 2)$ supersymmetric sigma model whose target space geometry consists of a transitive Lie algebroid E with complex structure on a Kaehler manifold M . There are several reasons why this is an interesting endeavour, as we now explain

Lie algebroids were first studied by Pradines in the early sixties as a vector bundle generalization of Lie algebras [18]. Lie algebroids provide a very general and flexible framework for studying a wide range of geometrical structures [19]. Recently, they proved to be essential in the formulation of generalized complex geometry [20, 21], which has attracted much attention in the string theory community for its applications to type II flux compactifications [22] and in the construction of topological sigma models with generalized Kaehler target geometry [23]–[25]. The study of sigma models with a Lie algebroid target geometry constitutes therefore a useful theoretical laboratory capable of providing important clues about the eventual construction of supersymmetric sigma models describing the propagation of strings in more general flux backgrounds.

Applications of Lie algebroids to sigma models are plentiful [27]. However, virtually all of them are concerned with smooth Lie algebroids and the sigma models they yield require gauge fixing and, so, are not immediately suitable for quantization. Our experience with sigma models with higher world-sheet supersymmetry suggests instead that, if Lie algebroids are to have any role, it is the holomorphic rather than the smooth ones which should be involved. In our construction, in fact, holomorphic Lie algebroids enter in an essential way.

As it turns out, the heterotic Lie algebroid sigma model worked out by us

enjoys a $(0, 2)$ world-sheet supersymmetry, as all heterotic sigma models, and an extra $(1, 0)$ supersymmetry originating from the Lie algebroid's geometry. Heterotic sigma models with $(1, 2)$ supersymmetry were studied long ago by Hull and Witten in ref. [26]. However, the origin of the left-moving supersymmetry in that case was completely different. The extra supersymmetry, we hope, should constrain the model enough to make the study of important structures as the chiral algebra and the chiral ring more manageable without rendering the model too 'ad hoc' to be interesting.

There are two distinguished topological half-twists of the heterotic Lie algebroid sigma model, which lead to an A and a B type half-topological sigma model. Each half-topological model is characterized by the usual topological BRST operator, stemming from the heterotic $(0, 2)$ supersymmetry, and a second BRST operator anticommuting with the former, originating from the $(1, 0)$ supersymmetry. These BRST operators combined in a certain way provide each half-topological model with two inequivalent BRST structures and, correspondingly, also two distinct perturbative chiral algebras and chiral rings. Albeit we do not yet fully understand its ultimate origin and implications, we consider this to be one of the most salient features of our sigma models.

The first BRST structure answers to the usual one of other half-topological heterotic sigma models. The second BRST structure is instead completely novel. Though the two structures share a number of properties, the second one is, in the appropriate sense explained in the body of the paper, "more topological" than the first. However, in general, the second structure's chiral algebra seemingly does not lend itself to a chiral differential operator sheafy description the same way the first structure's does.

The perturbative chiral rings of the two half-topological Lie algebroid sigma models can be characterized geometrically in the quasiclassical limit, as expected.

In particular, for the second BRST structure, they are isomorphic to certain Lie algebroid cohomologies. In this way, the problem of computation of genus 0 perturbative chiral ring correlators can be phrased in Lie algebroid theoretic terms.

The plan of the paper is as follows. In sect. 2, the theory of smooth Lie algebroids is reviewed with particular emphasis on the transitive case. In sect. 3, the theory of Lie algebroid complex and holomorphic structures is expounded, providing the necessary geometric toolkit for the construction of Lie algebroid sigma models. In sect. 4, the heterotic Lie algebroid sigma model is constructed, its supersymmetry is unveiled and its basic quantum aspects described. In sect. 5, which is the core part of the paper, it is shown that there are two distinguished twists of the Lie algebroid heterotic sigma model previously constructed, the half-topological Lie algebroid sigma models, each of which is characterized by two inequivalent BRST structures. Various topics such as dependence on the target space geometry, perturbative chiral algebras and rings and anomalies are analyzed in detail. In sect. 6, the chiral rings of the half-topological Lie algebroid sigma models are studied in the quasiclassical limit and described in terms of the target Lie algebroid's cohomology. In sect. 7, we summarize our results and indicate future lines of research.

2 Lie algebroids

2.1 Lie algebroids

Lie algebroids are vector bundles which have the same structural properties as the tangent bundle of a manifold. On the other hand, Lie algebroids are anchored bundles and, so, their definition requires a prior independent definition of the tangent bundle. The latter, therefore, is not merely a particular example of Lie algebroid. See ref. [19] for an exhaustive treatment of the subject.

A *real Lie algebroid* is a smooth real vector bundle E over a manifold M equipped with a smooth bundle map $\rho_E : E \mapsto T_M$, called the *anchor*, and an \mathbb{R} -bilinear bracket $[\cdot, \cdot]_E : \Gamma(E) \times \Gamma(E) \mapsto \Gamma(E)$ with the following properties.

1) $[\cdot, \cdot]_E$ is a Lie bracket so that $\Gamma(E)$ is a Lie algebra:

$$[s, t]_E + [t, s]_E = 0, \quad (2.1)$$

$$[s, [t, u]_E]_E + [t, [u, s]_E]_E + [u, [s, t]_E]_E = 0, \quad (2.2)$$

for $s, t, u \in \Gamma(E)$.

2) ρ_E defines a Lie algebra homomorphism of $\Gamma(E)$ into $\Gamma(T_M)$:

$$\rho_E[s, t]_E = [\rho_E s, \rho_E t]_{T_M}, \quad (2.3)$$

for $s, t \in \Gamma(E)$, where $[\cdot, \cdot]_{T_M}$ is the usual Lie bracket of vector fields of M .

3) The generalized Leibniz rule holds:

$$[s, ft]_E = f[s, t]_E + (\rho_E s f)t, \quad (2.4)$$

for $f \in C^\infty(M)$ and $s, t \in \Gamma(E)$.

The prototype Lie algebroid over M is the tangent bundle T_M : the anchor ρ_{T_M} is the identity 1_{T_M} and the bracket is the usual Lie bracket $[\cdot, \cdot]_{T_M}$. Lie algebroids generalize Lie algebras: a Lie algebra can be viewed as a Lie algebroid over the singleton manifold $M = \text{pt}$.

A (*base preserving*) *morphism* of two Lie algebroids E, E' over M is a vector bundle morphism $\varphi : E \mapsto E'$ such that

$$\rho_{E'}\varphi = \rho_E, \quad (2.5)$$

$$\varphi[s, t]_E = [\varphi s, \varphi t]_{E'}, \quad (2.6)$$

with $s, t \in \Gamma(E)$.

If L, E are two Lie algebroids over M and L is a subbundle of E , then L is a *Lie subalgebroid* of E , if the natural injection $\iota : L \mapsto E$ is a Lie algebroid morphism.

2.2 Transitive Lie algebroids

In this paper, we are interested mainly in transitive Lie algebroids. A Lie algebroid E is said *transitive*, if its anchor ρ_E is surjective. When E is transitive, $\ker \rho_E$ is a subbundle of E , called the *adjoint bundle* of E , and $\ker \rho_E$ inherits from E a structure of Lie algebroid rendering it a Lie subalgebroid of E . Since the anchor $\rho_{\ker \rho_E}$ of $\ker \rho_E$ vanishes, $\ker \rho_E$ is a Lie algebra bundle, that is vector bundle whose typical fiber is a fixed Lie algebra \mathfrak{g} .

If E is transitive, $\Gamma(\ker \rho_E)$ is a Lie ideal of $\Gamma(E)$. The quotient bundle $\tilde{E} = E / \ker \rho_E$ inherits then from E a structure of Lie algebroid. The anchor $\rho_{\tilde{E}}$ of \tilde{E} defines a Lie algebroid isomorphism of \tilde{E} onto T_M .

If E is transitive, then there exists a canonical exact sequence of Lie algebroids

$$0 \longrightarrow \ker \rho_E \xrightarrow{\iota_E} E \xrightarrow{\rho_E} T_M \longrightarrow 0 \quad (2.7)$$

called *Atiyah sequence*. A *splitting*¹ of E is a bundle map $\sigma : T_M \mapsto E$ such that

$$\rho_E \sigma = 1_{T_M}. \quad (2.8)$$

¹ Splittings are often called connections. We shall reserve the term connection for a related notion (see below)

Unlike ρ_E , σ does not induce in general a morphism of the Lie algebras $\Gamma(T_M)$ and $\Gamma(E)$. The failure to do so is measured by the *curvature* of σ , the field $G_{E\sigma} \in \Gamma(\wedge^2 T_M^* \otimes \ker \rho_E)$ defined by

$$G_{E\sigma}(x, y) = [\sigma x, \sigma y]_E - \sigma[x, y]_{T_M}, \quad (2.9)$$

with $x, y \in \Gamma(T_M)$. The splitting σ is *flat*, if $G_{E\sigma} = 0$.

2.3 Lie algebroid connections

Let E be a Lie algebroid. An E Lie algebroid connection on a vector bundle V is a bilinear map $D : \Gamma(E) \times \Gamma(V) \mapsto \Gamma(V)$ satisfying

$$D_{fs}v = fD_s v, \quad (2.10)$$

$$D_s(fv) = fD_s v + (\rho_E s f)v, \quad (2.11)$$

for any $f \in C^\infty(M)$, $s \in \Gamma(E)$, $v \in \Gamma(V)$. The curvature of the connection D is the field $R_D \in \Gamma(\wedge^2 E^* \otimes \text{End}(V))$ defined by

$$R_D(s, t)v = D_s D_t v - D_t D_s v - D_{[s, t]_E} v, \quad (2.12)$$

where $s, t \in \Gamma(E)$, $v \in \Gamma(V)$. D is said *flat*, if $R_D = 0$. In that case, D is called a *representation* of the Lie algebroid E in the vector bundle V .

When $V = E$, it is also possible to define the torsion of D , which is the field $T_D \in \Gamma(\wedge^2 E^* \otimes E)$ defined by

$$T_D(s, t)u = D_s t - D_t s - [s, t]_E, \quad (2.13)$$

where $s, t \in \Gamma(E)$. D is said *torsionless*, if $T_D = 0$.

A T_M Lie algebroid connection on V is an ordinary connection on V . In that case, the definition of curvature and, when $V = T_M$, of torsion given above reproduce the usual ones. The notion of Lie algebroid connection is however more

general. For instance, while for a generic $s \in \Gamma(E)$, D_s is a 1st order differential operator on $\Gamma(V)$, when s is valued in $\ker \rho_E$, D_s is simply a field of $\Gamma(\text{End}(E))$.

With any E Lie algebroid connection D on E , there is associated canonically an E Lie algebroid connection on any of the vector bundles which can be constructed from E by dualizing and tensoring in the usual way. We shall denote these connections with the same symbol D . In particular, the associated connection on $\text{End}(E)$ is defined by the relation $(D_s A)t = D_s(At) - AD_s t$ with $t \in \Gamma(E)$, for $s \in \Gamma(E)$, $A \in \Gamma(\text{End}(E))$.

Let E be a transitive Lie algebroid, V be a vector bundle and D be an E connection on V . If σ is a splitting of E , then

$$(D_\sigma)_x v = D_{\sigma x} v, \quad (2.14)$$

with $x \in \Gamma(T_M)$, $v \in \Gamma(V)$, is an ordinary connection on V . The curvatures of D and D_σ are related as

$$R_{D_\sigma}(x, y)v = R_D(\sigma x, \sigma y)v + D_{G_{E\sigma}(x, y)}v, \quad (2.15)$$

for with $x, y \in \Gamma(T_M)$, $v \in \Gamma(V)$.

For a transitive Lie algebroid E , the expression

$$D_{E s}^{\text{ad}} z = [s, z]_E, \quad (2.16)$$

with $s \in \Gamma(E)$, $z \in \Gamma(\ker \rho_E)$, defines a canonical E connection on $\ker \rho_E$. D_E^{ad} is flat. Conversely, for a generic splitting σ of E , $D_{E\sigma}^{\text{ad}}$ is not flat, being

$$R_{D_{E\sigma}^{\text{ad}}}(x, y)z = [G_{E\sigma}(x, y), z]_{\ker \rho_E}, \quad (2.17)$$

for $x, y \in \Gamma(T_M)$, $z \in \Gamma(\ker \rho_E)$. $D_{E\sigma}^{\text{ad}}$ is flat if $G_{E\sigma}$ is valued in the center of $\ker \rho_E$, the subbundle of $\ker \rho_E$ corresponding to the center $Z\mathfrak{g}$ of the typical fiber \mathfrak{g} of $\ker \rho_E$.

2.4 Complex Lie algebroids

For a real vector bundle V , let $V^c = V \otimes \mathbb{C}$ be the complexification of V . A *complex Lie algebroid* is a smooth complex vector bundle W over a manifold M equipped with a smooth bundle map $\rho_W : W \mapsto T_M^c$ and a \mathbb{C} -bilinear bracket $[\cdot, \cdot]_W : \Gamma(W) \times \Gamma(W) \mapsto \Gamma(W)$ such that (2.1)–(2.4) hold with T_M replaced by T_M^c throughout. If E is a real Lie algebroid, the complexification E^c of E has an obvious induced structure of complex Lie algebroid. All the constructions illustrated above extend without change to complex Lie algebroids.

3 Complex structures and holomorphic Lie algebroids

3.1 Holomorphic Lie algebroids

Lie algebroid theory can be formulated also in the holomorphic setting [28]–[30]. In this case, however, one must take into account the fact that, in general, a complex manifold \mathcal{M} admits an infinite dimensional algebra of holomorphic functions and, similarly, a holomorphic vector bundle \mathcal{V} over \mathcal{M} admits an infinite dimensional space of holomorphic sections only locally on \mathcal{M} . For this reason, one is forced to work locally on each open set U of \mathcal{M} . To each relation of the smooth theory, there then correspond infinitely many formally analogous relations of the holomorphic theory, one for each open set U and these relations are compatible with inclusions of open sets $U \subset V$. Formally, as is well-known, this can be done employing sheaf theory. We denote by $\mathcal{O}_{\mathcal{M}}$ the sheaf of holomorphic functions of \mathcal{M} and by $\mathcal{O}_{\mathcal{V}}$ the sheaf of holomorphic sections of \mathcal{V} : for any open set U of \mathcal{M} , $\mathcal{O}_{\mathcal{M}}(U)$ is the algebra of holomorphic functions on U and $\mathcal{O}_{\mathcal{V}}(U)$ is the space of holomorphic sections of \mathcal{V} on U .

A *holomorphic Lie algebroid* is a holomorphic vector bundle \mathcal{E} over a complex manifold \mathcal{M} equipped with a holomorphic bundle map $\rho_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{T}_{\mathcal{M}}$ and a bilinear bracket $[\cdot, \cdot]_{\mathcal{E}} : \mathcal{O}_{\mathcal{E}} \times \mathcal{O}_{\mathcal{E}} \mapsto \mathcal{O}_{\mathcal{E}}$ commuting with restrictions such that (2.1)–(2.4) are satisfied at the sheaf level (that is with $\Gamma(E)$, $\Gamma(T_M)$ and $C^\infty(M)$ replaced throughout by $\mathcal{O}_{\mathcal{E}}(U)$, $\mathcal{O}_{\mathcal{T}_{\mathcal{M}}}(U)$ and $\mathcal{O}_{\mathcal{M}}(U)$, respectively, for all open sets U of \mathcal{M} .)

One can define the notions of transitive holomorphic Lie algebroid and holomorphic splitting and those of holomorphic algebroid connections as in the smooth theory in obvious fashion.

3.2 Lie algebroid complex structures

Let V be a real vector bundle over a manifold M . An *almost complex structure*

of V is a field $K \in \Gamma(\text{End}(V))$ satisfying

$$K^2 = -1_V. \quad (3.1)$$

We denote by K^c the extension of K to the complexification V^c of V and by V^+, \bar{V}^+ the $\pm i$ eigenbundles of K^c ². When $V = T_M$, we recover the customary notion of almost complex structure.

Let E be a real Lie algebroid over M and let J be an almost complex structure of E . J is said a *complex structure*, if it satisfies the integrability condition

$$N_{EJ} = 0, \quad (3.2)$$

where $N_{EJ} \in \Gamma(\wedge^2 E^* \otimes E)$ is the Nijenhuis field of J defined by

$$N_{EJ}(s, t) = [Js, Jt]_E - [s, t]_E - J([Js, t]_E + [s, Jt]_E), \quad (3.3)$$

with $s, t \in \Gamma(E)$. The integrability condition (3.2) is equivalent to the closedness of $\Gamma(E^+)$ under the Lie bracket $[\cdot, \cdot]_{E^c}$. When $E = T_M$, we recover the customary notion of complex structure.

Let E be a real Lie algebroid over M . A *Lie algebroid complex structure* of E is a pair (J, J_M) , where J and J_M are complex structures of E and T_M , respectively, such that the anchor ρ_E of E is complex linear,

$$\rho_E J = J_M \rho_E. \quad (3.4)$$

A *Lie algebroid with complex structure* is a real Lie algebroid E endowed with a Lie algebroid complex structure (J_E, J_{EM}) .

A *Lie algebroid with complex base* is a complex vector bundle W over a manifold M together with a complex structure J_{WM} of T_M , a smooth bundle map $\rho_W : W \mapsto T_M^+$ and a \mathbb{C} -bilinear bracket $[\cdot, \cdot]_W : \Gamma(W) \times \Gamma(W) \mapsto \Gamma(W)$ such that (2.1)–(2.4)

² Usually, the notation $V^{1,0}, V^{0,1}$ is used for these eigenbundles. We employ the notation V^+, \bar{V}^+ for convenience.

hold with E and T_M replaced by W and T_M^+ , respectively. T_M^+ itself is a complex Lie algebroid with complex base.

If E is a Lie algebroid with complex structure, then E^+ has an obvious induced structure of Lie algebroid with complex base. \overline{E}^+ also does. A lot more happens however.

The complex structure J_{EM} ensures the existence of a holomorphic coordinate atlas on M and, so, renders M a complex manifold, henceforth denoted by \mathcal{M} . The complex vector bundle T_M^+ has a canonical holomorphic structure and is thus a holomorphic vector bundle \mathcal{T}_M over \mathcal{M} , the holomorphic tangent bundle of \mathcal{M} . Further, the Lie bracket $[\cdot, \cdot]_{T_M^+}$ restricts to the Lie bracket $[\cdot, \cdot]_{\mathcal{T}_M}$ on local holomorphic sections of \mathcal{T}_M . In this way, \mathcal{T}_M gets endowed with a canonical structure of holomorphic Lie algebroid. The natural questions arise about whether the complex structure J_E similarly endows the vector bundle E^+ with a canonical holomorphic structure making it a holomorphic vector bundle \mathcal{E} over \mathcal{M} and whether \mathcal{E} is also canonically a holomorphic Lie algebroid. Things however are not as simple as for T_M , as the following analysis shows.

3.3 Lie algebroid holomorphic structures

Let E be a Lie algebroid over M with complex structure. The $\overline{\partial}$ operator of E^+ is the differential operator $\overline{\partial}_{E^+}$ on $\Gamma(E^+)$ defined by

$$\overline{\partial}_{E^+} t = \Pi_E[\overline{s}, t]_{E^c}, \quad (3.5)$$

with $s, t \in \Gamma(E^+)$, where $\Pi_E = (1_E - iJ_E)/2 \in \Gamma(\text{End}(E^c))$ is the projector field on E^+ . $\overline{\partial}_{E^+}$ is a \overline{E}^+ Lie algebroid connection on E^+ , that is

$$\overline{\partial}_{E^+} (f\overline{s})t = f\overline{\partial}_{E^+} t, \quad (3.6)$$

$$\overline{\partial}_{E^+} (ft) = f\overline{\partial}_{E^+} t + (\overline{\rho}_{E^+} \overline{s} f)t, \quad (3.7)$$

for any $f \in C_c^\infty(M)$, $s, t \in \Gamma(E^+)$ (see eqs. (2.10), (2.11)). The integrability of the complex structure J_E entails that the curvature $R_{\bar{\partial}_{E^+}}$ of $\bar{\partial}_{E^+}$ vanishes,

$$R_{\bar{\partial}_{E^+}}(\bar{s}, \bar{t})u = \bar{\partial}_{E^+\bar{s}}\bar{\partial}_{E^+\bar{t}}u - \bar{\partial}_{E^+\bar{t}}\bar{\partial}_{E^+\bar{s}}u - \bar{\partial}_{E^+[\bar{s}, \bar{t}]_{E^+}}u = 0, \quad (3.8)$$

for $s, t, u \in \Gamma(E^+)$.

The above construction can be repeated in the case where $E = T_M$. As is readily checked, $\bar{\partial}_{T_M^+}$ is the usual Cauchy–Riemann operator on $\Gamma(T_M^+)$. We shall see that $\bar{\partial}_{E^+}$ is a natural generalization of $\bar{\partial}_{T_M^+}$, as suggested by the notation.

The operator $\bar{\partial}_{E^+}$ intertwines naturally with the anchor and the Lie bracket of the Lie algebroid E^+ . Indeed, one has

$$\bar{\partial}_{T_M^+\bar{\rho}_{E^+\bar{s}}}(\rho_{E^+}t) = \rho_{E^+}\bar{\partial}_{E^+\bar{s}}t, \quad (3.9)$$

for $s, t \in \Gamma(E^+)$ and, further,

$$\bar{\partial}_{E^+\bar{s}}[u, v]_{E^+} = [\bar{\partial}_{E^+\bar{s}}u, v]_{E^+} - [\bar{\partial}_{E^+\bar{s}}v, u]_{E^+} + \bar{\partial}_{E^+v\bar{s}}u - \bar{\partial}_{E^+u\bar{s}}v, \quad (3.10)$$

for $s, u, v \in \Gamma(E^+)$, where $u_{\bar{s}} = \overline{\bar{\partial}_{E^+u}s}$.

Let us call a local section t of E^+ \mathcal{E} -holomorphic if

$$\bar{\partial}_{E^+\bar{s}}t = 0, \quad s \in \Gamma(E^+). \quad (3.11)$$

In general, this equation has only local solutions. We denote by $\mathcal{O}_{\mathcal{E}}$ the sheaf of solutions of (3.11). Using (3.7), one verifies that, if $f \in \mathcal{O}_{\mathcal{M}}(U)$ and $s \in \mathcal{O}_{\mathcal{E}}(U)$, then $fs \in \mathcal{O}_{\mathcal{E}}(U)$ as well. Thus, $\mathcal{O}_{\mathcal{E}}$ is a sheaf of $\mathcal{O}_{\mathcal{M}}$ modules.

Does the operator $\bar{\partial}_{E^+\bar{s}}$ defines a holomorphic structure on E^+ making it a holomorphic vector bundle \mathcal{E} over \mathcal{M} in analogy to T_M^+ ? In that case, is \mathcal{E} naturally also a holomorphic Lie algebroid again in analogy to T_M^+ ?

In order $\bar{\partial}_{E^+\bar{s}}$ to define a holomorphic structure on E^+ , it is necessary that, locally, there exist \mathcal{E} -holomorphic frames of E^+ , that is that the sheaf $\mathcal{O}_{\mathcal{E}}$ is

locally free. This is not the case in general. Let us assume however that it is. Then, E^+ becomes a holomorphic vector bundle \mathcal{E} .

Eq. (3.9) is not sufficient to conclude that the anchor ρ_{E^+} is holomorphic, since in general not every field $x \in \Gamma(T_M^+)$ is of the form $x = \rho_{E^+} s$ for some field $s \in \Gamma(E^+)$. (3.9) is anyway compatible with the holomorphy of ρ_{E^+} . So, we can assume ρ_{E^+} does have this property.

Eq. (3.10) implies that, if U is an open subset of M and $u, v \in \mathcal{O}_{\mathcal{E}}(U)$, $[u, v]_{E^+} \in \mathcal{O}_{\mathcal{E}}(U)$ as well. Thus, $\mathcal{O}_{\mathcal{E}}$ is closed under the Le brackets $[\cdot, \cdot]_{E^+}$.

In summary, in order E^+ to become a holomorphic vector bundle \mathcal{E} over \mathcal{M} , we have to assume that the sheaf $\mathcal{O}_{\mathcal{E}}$ is locally free. Further, in order \mathcal{E} to be a holomorphic Lie algebroid, we have to assume further that ρ_{E^+} is holomorphic. When the Lie algebroid E is transitive, we are able to make definitely stronger statements.

3.4 The transitive case

Let E be a transitive Lie algebroid and let J be an almost complex structure of E . Suppose that $\ker \rho_E$ is invariant under J . Then, in addition to the primary Nijenhuis field N_{EJ} defined in (3.3), we can define a secondary Nijenhuis field $\tilde{N}_{EJ} \in \Gamma(E^* \otimes \ker \rho_E^* \otimes \ker \rho_E)$ by

$$\tilde{N}_{EJ}(s, t) = [Js, Jt]_E + [s, t]_E - J([Js, t]_E - [s, Jt]_E), \quad (3.12)$$

with $s \in \Gamma(E)$, $t \in \Gamma(\ker \rho_E)$. The vanishing of \tilde{N}_{EJ} is equivalent to the property that $[\bar{s}, t]_{E^c} \in \Gamma(\ker \rho_{E^+})$ for $s \in \Gamma(E^+)$, $t \in \Gamma(\ker \rho_{E^+})$, as is easily verified.

Let E be a transitive Lie algebroid with complex structure³. It is readily seen that the Lie algebroid with complex base E^+ is then also transitive.

³ We note that, as $\rho_E N_{EJ_E}(s, t) = N_{T_M J_{EM}}(\rho_E s, \rho_E t)$ for $s, t \in \Gamma(E)$ by (3.3) and ρ_E is surjective, the integrability of J_E automatically implies that of J_{EM} .

As, by (3.4), $\ker \rho_E$ is invariant under J_E , the secondary Nijenhuis field \tilde{N}_{EJ_E} is defined. It can then be shown that the vanishing of \tilde{N}_{EJ_E} ,

$$\tilde{N}_{EJ_E} = 0, \quad (3.13)$$

is a necessary and sufficient condition for the sheaf $\mathcal{O}_{\mathcal{E}}$ to be locally free. Here is a sketch of the proof. $\mathcal{O}_{\mathcal{E}}$ is locally free if and only if E^+ admits locally an \mathcal{E} -holomorphic frame. Let $\{e_i\}$ be a local frame of E^+ . There is an \mathcal{E} -holomorphic frame of E^+ if and only if we can find a local invertible matrix function T^i_j such that $\bar{\partial}_{E+\bar{s}}(T^j_i e_j) = 0$ for $s \in \Gamma(E^+)$. Now, the anchor $\rho_{E^+} : E^+ \mapsto T_M^+$ is surjective. There is thus a bundle map $\sigma^+ : T_M^+ \mapsto E^+$ such that $\rho_{E^+} \sigma^+ = 1_{E^+}$. The above equation then splits into two equations: *i*) $\bar{\partial}_{E+\bar{s}}(T^j_i e_j) = 0$ for $s \in \Gamma(\ker \rho_{E^+})$ and *ii*) $\bar{\partial}_{E+\bar{\sigma}+\bar{x}}(T^j_i e_j) = 0$ for $x \in \Gamma(T_M^+)$. Eq. *i* reduces to $\Pi_E[\bar{s}, e_i]_{E^c} = 0$ and thus to (3.13). Eq. *ii* is a genuine differential equation for T^i_j . It has a solution provided the appropriate integrability condition is satisfied. The condition indeed is as a consequence of the Jacobi identity obeyed by the Lie bracket $[\cdot, \cdot]_{E^c}$.

Since E^+ is transitive, (3.9) is sufficient to conclude that the anchor ρ_{E^+} is holomorphic, as by the surjectivity of ρ_{E^+} , every field $x \in \Gamma(T_M^+)$ is of the form $x = \rho_{E^+} s$ for some field $s \in \Gamma(E^+)$.

In summary, if E is a transitive Lie algebroid with complex structure satisfying (3.13), then $\mathcal{O}_{\mathcal{E}}$ is a locally free sheaf and ρ_{E^+} is holomorphic. E^+ has thus a canonical holomorphic structure rendering it a transitive holomorphic Lie algebroid \mathcal{E} over \mathcal{M} . Since $\mathcal{O}_{\mathcal{E}}$ is the sheaf of sections of \mathcal{E} and is defined by eq. (3.11), we can interpret $\bar{\partial}_{E^+}$ as the *Cauchy–Riemann operator* on $\Gamma(E^+)$.

3.5 Induced structures on $\ker \rho_E$ and \tilde{E}

Suppose that E is a transitive Lie algebroid with complex structure satisfying (3.13), as above.

Since $\ker \rho_E$ is invariant under J_E , J_E defines almost complex structures on the vector bundles $\ker \rho_E$ and $\tilde{E} = E/\ker \rho_E \simeq T_M$. In this way, both $\ker \rho_E$ and \tilde{E} are Lie algebroids with complex structure. $\ker \rho_E$ is not transitive, as its anchor vanishes identically. \tilde{E} , conversely, is transitive, as its anchor is a vector bundle isomorphism. Further, \tilde{E} satisfies (3.13) trivially.

Therefore, the above theory does not apply to $\ker \rho_E$ while it does to \tilde{E} . Nevertheless, the holomorphic structure of E^+ induces naturally holomorphic structures on $(\ker \rho_E)^+ \simeq \ker \rho_{E^+}$ and $(\tilde{E})^+ \simeq \tilde{E}^+$ and, in the latter case, the holomorphic structure is precisely the one associated with the structure of transitive Lie algebroid with complex structure of \tilde{E} . In fact, as the anchor $\rho_{\mathcal{E}}$ of \mathcal{E} is holomorphic, $\ker \rho_{\mathcal{E}}$ is a holomorphic Lie subalgebroid of \mathcal{E} and, so, a holomorphic Lie algebroid. Consequently, $\tilde{\mathcal{E}} = \mathcal{E}/\ker \rho_{\mathcal{E}} \simeq \mathcal{T}_M$ is a holomorphic Lie algebroid.

3.6 Standard complex connections

Let E be a transitive Lie algebroid and let σ be a splitting of E (cf. subsect. 2.2). We define a field $\omega \in \Gamma(\text{End}(E))$ by

$$\omega = 1_E - \sigma\rho_E. \quad (3.14)$$

As $\omega^2 = \omega$, ω is a projector field. In fact, ω projects on $\ker \rho_E$.

Let E be a transitive Lie algebroid and let (J, J_M) be a Lie algebroid complex structure of E . A splitting σ of E is said *complex* with respect to (J, J_M) if

$$\sigma J_M = J\sigma. \quad (3.15)$$

Compare with condition (3.4).

Let E be a transitive Lie algebroid with complex structure satisfying the condition (3.13). Let σ be a complex splitting of E and let D_M be an ordinary connection on M such that the complex structure J_{EM} is parallel, so that

$$D_{Mx}J_{EM} = 0, \quad (3.16)$$

with $x \in \Gamma(T_M)$. We now set

$$D_s t = \sigma D_{M\rho_E s}(\rho_E t) + [s, \omega t]_E + H_{E\sigma}(\rho_E s, \rho_E t), \quad (3.17)$$

with $s, t \in \Gamma(T_M)$, where the field $H_{E\sigma} \in \Gamma(T_M^{*\otimes 2} \otimes \ker \rho_E)$ is defined by

$$\begin{aligned} H_{E\sigma}(x, y) = \frac{1}{4} & \left(G_{E\sigma}(J_{EM}x, J_{EM}y) + G_{E\sigma}(x, y) \right. \\ & \left. + J_E(G_{E\sigma}(J_{EM}x, y) - G_{E\sigma}(x, J_{EM}y)) \right), \end{aligned} \quad (3.18)$$

with $x, y \in \Gamma(T_M)$, $G_{E\sigma}$ being the curvature of the splitting σ (cf. eq. (2.9)). Directly from the definition (3.17), we verify that D is an E Lie algebroid connection on E (cf. eqs. (2.10), (2.11)). Note that, when $E = T_M$, $D = D_M$. We call connections of this form *standard*. They will play an important role in the formulation of the Lie algebroid heterotic sigma model. The last term in the right hand side of (3.17) may be dropped without compromising D being a connection. The reason why it is added is to make D naturally related to the operator $\bar{\partial}_{E^+}$, as shown momentarily.

The complex structure J_E of E is parallel with respect to D ,

$$D_s J_E = 0, \quad (3.19)$$

with $s \in \Gamma(E)$. Another distinguished property of D is the following. We define an ordinary connection on the vector bundle $\text{Hom}(E, T_M)$, denoted also by D_M , by the relation $(D_{Mx}\phi)t = D_{Mx}(\phi t) - \phi D_{\sigma x}t$ with $t \in \Gamma(E)$, for $x \in \Gamma(T_M)$, $\phi \in \Gamma(\text{Hom}(E, T_M))$. We further view the anchor of E as a field $\rho_E \in \Gamma(\text{Hom}(E, T_M))$. Then, as is easily verified, ρ_E is parallel with respect to D_M ,

$$D_{Mx}\rho_E = 0, \quad (3.20)$$

with $x \in \Gamma(T_M)$.

The torsion T_D of D is given by the formula

$$\begin{aligned}
T_D(s, t) &= \sigma T_{D_M}(\rho_E s, \rho_E t) + [s, \omega t]_E - [t, \omega s]_E \\
&\quad - \omega[s, t]_E + H_{E\sigma}(\rho_E s, \rho_E t) - H_{E\sigma}(\rho_E t, \rho_E s),
\end{aligned} \tag{3.21}$$

with $s, t \in \Gamma(E)$. We note that, when $T_{D_M} = 0$, T_D is generally non vanishing, but $\ker \rho_E$ valued.

The curvature R_D of D is given by the formula

$$\begin{aligned}
R_D(s, t)u &= \sigma R_{D_M}(\rho_E s, \rho_E t)\rho_E u - H_{E\sigma}(\rho_E[s, t]_E, \rho_E u) \\
&\quad + [s, H_{E\sigma}(\rho_E t, \rho_E u)]_E - [t, H_{E\sigma}(\rho_E s, \rho_E u)]_E \\
&\quad + H_{E\sigma}(\rho_E s, D_{M\rho_E t}(\rho_E u)) - H_{E\sigma}(\rho_E t, D_{M\rho_E s}(\rho_E u)),
\end{aligned} \tag{3.22}$$

with $s, t, u \in \Gamma(E)$.

For $s \in \Gamma(E)$ and $t \in \Gamma(\ker \rho_E)$, we have

$$D_s t = [s, t]_E. \tag{3.23}$$

So, upon restriction to $\Gamma(\ker \rho_E)$, D reduces to the canonical E connection D_E^{ad} of the adjoint bundle $\ker \rho_E$ (cf. eq. (2.17)).

Since, for $s \in \Gamma(E)$, $t \in \Gamma(\ker \rho_E)$, we have $D_s t \in \Gamma(\ker \rho_E)$, D induces also an E Lie algebroid connection on \tilde{E} , where $\tilde{E} = E/\ker \rho_E$ (cf. subsect. 2.2),

$$D_s t = \sigma D_{M\rho_E s}(\rho_E t) \quad \text{mod } \ker \rho_E, \tag{3.24}$$

with $s \in \Gamma(E)$, $t \in \Gamma(\tilde{E})$. As the right hand side of (3.24) vanishes for $s \in \Gamma(\ker \rho_E)$, D induces an \tilde{E} Lie algebroid connection on \tilde{E} . Recalling that $\tilde{E} \simeq T_M$ and that $\rho_E : \tilde{E} \mapsto T_M$ is a Lie algebroid isomorphism with inverse $\sigma : T_M \mapsto \tilde{E}$, we realize that this connection is essentially the connection D_M entering in the definition (3.17) of D .

The E Lie algebroid connection D on E extends by complexification to an E^c Lie algebroid connection D^c on E^c . All the properties of D found above generalize to D^c without change.

By (3.19), $\Gamma(E^+)$ is invariant under D^c . In this way, by restriction, we obtain an \overline{E}^+ Lie algebroid connection $\overline{\mathcal{D}}^+$ on E^+ (cf. eqs. (3.6), (3.7)). Explicitly,

$$\overline{\mathcal{D}}^+_{\bar{s}}t = D^c_{\bar{s}}t, \quad (3.25)$$

where $s, t \in \Gamma(E^+)$. The remarkable property of this $\overline{\mathcal{D}}^+$ is that, if $\overline{\mathcal{D}}_{M^+} = \overline{\partial}_{T_{M^+}}$, the Cauchy–Riemann operator of T_{M^+} , then, likewise,

$$\overline{\mathcal{D}}^+ = \overline{\partial}_{E^+}, \quad (3.26)$$

the Cauchy–Riemann operator of E^+ defining its canonical holomorphic structure (cf. eq. (3.5)). We remark that, much as D_M is not uniquely fixed by the requirement that $\overline{\mathcal{D}}_{M^+} = \overline{\partial}_{T_{M^+}}$, so D is not the only connection enjoying the important property (3.26).

3.7 Local expressions

Let E be a transitive Lie algebroid with complex structure satisfying (3.13). Then, as shown earlier, the base M of E is automatically a complex manifold \mathcal{M} and E^+ has a canonical holomorphic structure making it a holomorphic Lie algebroid \mathcal{E} over \mathcal{M} . Consequently, over any sufficiently small open set U of M , there exists a holomorphic frame $\{\partial_a\}$ of $\mathcal{T}_{\mathcal{M}}$ associated with each set of holomorphic coordinates $\{z^a\}$ of \mathcal{M} and a holomorphic frame $\{e_i\}$ of \mathcal{E} .

Since the anchor $\rho_{\mathcal{E}}$ of \mathcal{E} is holomorphic, we have

$$\rho_{\mathcal{E}}e_i = \rho^a_i \partial_a, \quad (3.27)$$

where the anchor structure functions ρ^a_i are holomorphic

$$\overline{\partial}_{\bar{a}}\rho^b_k = 0. \quad (3.28)$$

As $\mathcal{O}_{\mathcal{E}}(U)$ is closed under the Lie bracket $[\cdot, \cdot]_{\mathcal{E}}$, we have

$$[e_j, e_i]_{\mathcal{E}} = f^k_{ij}e_k, \quad (3.29)$$

where the bracket structure functions f^k_{ij} are holomorphic

$$\bar{\partial}_{\bar{a}} f^k_{ij} = 0. \quad (3.30)$$

The holomorphy of the anchor $\rho_{\mathcal{E}}$ allows us to choose adapted holomorphic frames. Here and in the following, we use the Latin letters u, v, w, x, y as holomorphic $\ker \rho_{\mathcal{E}}$ indices and the Latin letters p, q, r, s, t as holomorphic $\mathcal{E}/\ker \rho_{\mathcal{E}}$ indices. For an adapted holomorphic frame, we have $\{e_i\} = \{e_u\} \cup \{e_p\}$, where $\{e_u\}$ is a holomorphic frame of $\ker \rho_{\mathcal{E}}$,

$$\rho_{\mathcal{E}} e_u = 0. \quad (3.31)$$

(3.31) implies the important relation

$$\rho^a_u = 0. \quad (3.32)$$

Since $[s, t]_{\mathcal{E}} \in \mathcal{O}_{\ker \rho_{\mathcal{E}}}(U)$ if either $s \in \mathcal{O}_{\ker \rho_{\mathcal{E}}}(U)$ or $t \in \mathcal{O}_{\ker \rho_{\mathcal{E}}}(U)$, we have

$$f^p_{uv} = 0, \quad f^p_{uq} = 0. \quad (3.33)$$

Let us now view $\mathcal{O}_{\mathcal{E}}(U)$ as a subspace of $\Gamma(E^+|_U)$. From (3.5), it is readily verified that $[\bar{s}, t]_{E^c} = 0$ whenever $s, t \in \mathcal{O}_{\mathcal{E}}(U)$. From (3.29) and this observation, we have then

$$[e_j, e_j]_{E^c} = f^k_{ij} e_k, \quad [\bar{e}_i, e_j]_{E^c} = 0 \quad \text{and c. c.} \quad (3.34)$$

Therefore, only f^k_{ij} and c. c. are non vanishing.

From (2.1)–(2.4), the structure functions ρ^a_i, f^k_{ij} satisfy

$$f^i_{jk} + f^i_{kj} = 0, \quad (3.35)$$

$$f^i_{jm} f^m_{kl} + f^i_{km} f^m_{lj} + f^i_{lm} f^m_{jk} \quad (3.36)$$

$$+ \rho^a_j \partial_a f^i_{kl} + \rho^a_k \partial_k f^i_{lj} + \rho^a_l \partial_a f^i_{jk} = 0,$$

$$\rho^b_i \partial_b \rho^a_j - \rho^b_j \partial_b \rho^a_i - f^k_{ij} \rho^a_k = 0. \quad (3.37)$$

A host of more explicit relations can be obtained for an adapted frame by splitting the frame index i as (u, p) and using (3.32), (3.33).

In our treatment, the complex Lie algebroid E is always endowed with a complex splitting σ (cf. eq. (3.15)). By (3.15), we have

$$\sigma\partial_a = \sigma^i{}_a e_i \quad \text{and c. c.} \quad (3.38)$$

σ is characterized by its curvature $G_{E\sigma}$ (cf. eq. (2.9)). The components $G_{E\sigma}$ are defined by the relations

$$G_{E\sigma}{}^c(\partial_a, \partial_b) = G^i{}_{ab} e_i, \quad G_{E\sigma}{}^c(\bar{\partial}_{\bar{a}}, \partial_b) = G^i{}_{\bar{a}b} e_i + G^{\bar{i}}{}_{\bar{a}b} e_{\bar{i}} \quad \text{and c. c.} \quad (3.39)$$

In the right hand side of the first of these relations, a term of the form $G^{\bar{i}}{}_{ab} e_{\bar{i}}$ is absent, as this automatically vanishes by the complex nature of σ . Applying the definition (2.9), by a straightforward calculation, we find

$$G^i{}_{ab} = \partial_a \sigma^i{}_b - \partial_b \sigma^i{}_a + f^i{}_{jk} \sigma^j{}_a \sigma^k{}_b, \quad G^i{}_{\bar{a}b} = \bar{\partial}_{\bar{a}} \sigma^i{}_b \quad \text{and c. c.} \quad (3.40)$$

Note that, since $G_{E\sigma}$ is $\ker \rho_E$ valued, we have

$$G^p{}_{ab} = 0, \quad G^p{}_{\bar{a}b} = 0 \quad \text{and c. c.}, \quad (3.41)$$

as may be readily checked. Note also that the vanishing of the components $G^u{}_{\bar{a}b}$ is equivalent to the holomorphy of σ .

In view of its application to the construction of the Lie algebroid sigma model, we provide the local expression of the standard connection D (cf. eq. (3.17)). We assume that the connection D_M is such that $\bar{\mathcal{D}}_M^+ = \bar{\partial}_{T_M^+}$, so that (3.26) holds. (3.26) and the holomorphy condition $\bar{\partial}_{E^+} e_i = 0$ entail that

$$D^c{}_{\bar{e}_i} e_j = 0 \quad \text{and c. c.} \quad (3.42)$$

The coefficients of the connection are defined by the relation

$$D^c{}_{e_i} e_j = A^k{}_{ij} e_k \quad \text{and c. c.}, \quad (3.43)$$

which follows from the invariance of $\Gamma(E^+)$ under D^c . The coefficients A^k_{ij} can be computed directly using (3.17),

$$A^k_{ij} = \sigma^k_c \rho^a_i \rho^b_j A_M^c{}_{ab} - \rho^a_i \rho^b_j \partial_a \sigma^k_b + f^k_{il} \omega^l_j, \quad (3.44)$$

where the components of ω are defined by $\omega e_i = \omega^j_i e_j$ with $\omega^p_i = 0$, in accordance with (3.14). In particular, we have

$$A^v_{iu} = f^v_{iu}, \quad (3.45)$$

as implied by the property (3.23), and

$$A^p_{ij} = 0 \quad \text{for } (i, j) \neq (q, r), \quad (3.46)$$

as follows from (3.24).

The local expressions of the torsion and the curvature of D^c can also be obtained. We shall not exhibit them however, since they will not be needed in the following.

4 The heterotic Lie algebroid sigma model

4.1 The heterotic Lie algebroid sigma model: classical theory

As every sigma model, the Lie algebroid heterotic sigma model is characterized classically by the following data: *a*) the target space geometry, *b*) the world-sheet geometry, *c*) the field content, *d*) the action and *e*) its symmetries.

The target space geometry consists of a transitive Lie algebroid E with complex structure over a manifold M satisfying (3.13) (cf. subsects. 2.2, 3.2). E is further furnished with a complex splitting σ . Finally, M is endowed with a metric g which is Kaehler with respect its complex structure J_{EM} .

The world-sheet is a Riemann surface Σ equipped with a spin structure $K_\Sigma^{1/2}$, a tensor square root of the canonical line bundle K_Σ of Σ .

The fields of the model are an embedding field $x \in \text{Map}(\Sigma, M)$, two fermion fields $\chi_{\bar{\theta}} \in \Gamma(\overline{K}_\Sigma^{1/2} \otimes x^* \Pi \tilde{E}^+)$, $\bar{\chi}_{\bar{\theta}} \in \Gamma(\overline{K}_\Sigma^{1/2} \otimes x^* \Pi \overline{\tilde{E}^+})$, two more fermion fields $\lambda_\theta \in \Gamma(K_\Sigma^{1/2} \otimes x^* \Pi \ker \rho_{E^+})$, $\lambda^*_\theta \in \Gamma(K_\Sigma^{1/2} \otimes x^* \Pi \ker \rho_{E^+}^*)$ and two boson fields $l_{\theta\bar{\theta}} \in \Gamma(K_\Sigma^{1/2} \otimes \overline{K}_\Sigma^{1/2} \otimes x^* \ker \rho_{E^+})$, $l^*_{\theta\bar{\theta}} \in \Gamma(K_\Sigma^{1/2} \otimes \overline{K}_\Sigma^{1/2} \otimes x^* \ker \rho_{E^+}^*)$ ⁴. Here, \tilde{E}^+ is the quotient vector bundle $E^+/\ker \rho_{E^+}$ and $\ker \rho_{E^+}^*$ is the dual vector bundle of $\ker \rho_{E^+}$. Π is the parity reversal operator which associates with any vector bundle V its odd counterpart ΠV . x^*V denotes the pull-back of a vector bundle V over M by the map x .

The action of the Lie algebroid heterotic sigma model is

$$S = \int_\Sigma d^2z \left[\frac{1}{2} g_{\bar{a}b}(x) (\bar{\partial}_{\bar{z}} x^{\bar{a}} \partial_z x^b + \partial_z x^{\bar{a}} \bar{\partial}_{\bar{z}} x^b) + i g_{\bar{a}b} \rho^{\bar{a}}_{\bar{p}} \rho^b_q(x) \bar{\chi}^{\bar{p}}_{\bar{\theta}} D_z \chi^q_{\bar{\theta}} \right. \\ \left. + i \lambda^*_{u\theta} \bar{D}_{\bar{z}} \lambda^u_\theta + f^u_{wv} G^w_{\bar{a}b} \rho^{\bar{a}}_{\bar{p}} \rho^b_q(x) \lambda^*_{u\theta} \lambda^v_\theta \bar{\chi}^{\bar{p}}_{\bar{\theta}} \chi^q_{\bar{\theta}} - l^*_{u\theta\bar{\theta}} l^u_{\theta\bar{\theta}} \right]. \quad (4.1)$$

Here, $G^u_{\bar{a}b}$ is the (1, 1) component of the curvature $G_{E\sigma}$ of σ (cf. eq. (3.40)). D_z , $\bar{D}_{\bar{z}}$ are the components of the connection x^*D_σ of x^*E yielded via pull-back by

⁴ In the string theory literature, the spinor indices $\theta, \bar{\theta}$ are generally denoted by $-, +$

x of the connection D_σ of E associated with a standard E Lie algebroid connection D of E and the splitting σ as in eq. (2.14). D , in turn, is constructed using σ and the customary Levi–Civita connection ∇_M of T_M according to (3.17). The connection x^*D_σ extends to x^*E^c by complexification and then to x^*E^+ by restriction, on account of (3.19). Since x^*D_σ leaves $x^*\ker\rho_{E^+}$ invariant, by (3.20), x^*D_σ induces a connection on both $x^*\ker\rho_{E^+}$ and $x^*\tilde{E}^+$. Explicitly, using (3.44), we have

$$D_z\phi^p = \partial_z\phi^p + (\Gamma^a{}_{cb}\sigma^p{}_a\rho^b{}_q - \partial_c\sigma^p{}_b\rho^b{}_q)(x)\partial_zx^c\phi^q, \quad \text{and c. c.}, \quad (4.2a)$$

$$\overline{D}_z\psi^u = \overline{\partial}_z\psi^u + f^u{}_{iv}\sigma^i{}_a(x)\overline{\partial}_zx^a\psi^v, \quad (4.2b)$$

$$\overline{D}_z\psi^*_u = \overline{\partial}_z\psi^*_u - f^v{}_{iu}\sigma^i{}_a(x)\overline{\partial}_zx^a\psi^*_v. \quad (4.2c)$$

Here and in the following, the abbreviation “c. c.” denotes target space complex conjugation only and is thus inert on the world–sheet.

The action S of the heterotic sigma model has a high amount of symmetry, as we now show.

The $(0, 2)$ supersymmetry field variations are given by

$$\delta_\alpha x^a = i\alpha^{\bar{\theta}}\rho^a{}_p(x)\chi^p_{\bar{\theta}} \quad \text{and c. c.}, \quad (4.3a)$$

$$\delta_\alpha\chi^p_{\bar{\theta}} = -\frac{i}{2}\alpha^{\bar{\theta}}f^p{}_{qr}(x)\chi^q_{\bar{\theta}}\chi^r_{\bar{\theta}} - \overline{\alpha}^{\bar{\theta}}\sigma^p{}_a(x)\overline{\partial}_zx^a \quad \text{and c. c.}, \quad (4.3b)$$

$$\delta_\alpha\lambda^u_\theta = i\alpha^{\bar{\theta}}(-f^u{}_{iv}\sigma^i{}_a\rho^a{}_p(x)\chi^p_{\bar{\theta}}\lambda^v_\theta + l^u_{\theta\bar{\theta}}), \quad (4.3c)$$

$$\delta_\alpha l^u_{\theta\bar{\theta}} = -i\alpha^{\bar{\theta}}f^u{}_{iv}\sigma^i{}_a\rho^a{}_p(x)\chi^p_{\bar{\theta}}l^v_{\theta\bar{\theta}} \quad (4.3d)$$

$$+ i\overline{\alpha}^{\bar{\theta}}(i\overline{D}_z\lambda^u_\theta + f^u{}_{wv}G^w{}_{\bar{a}b}\rho^{\bar{a}}{}_{\bar{p}}\rho^b{}_q(x)\overline{\chi}^{\bar{p}}_{\bar{\theta}}\chi^q_{\bar{\theta}}\lambda^v_\theta),$$

$$\delta_\alpha\lambda^*_{u\theta} = i\alpha^{\bar{\theta}}f^v{}_{iu}\sigma^i{}_a\rho^a{}_p(x)\chi^p_{\bar{\theta}}\lambda^*_{v\theta} + i\overline{\alpha}^{\bar{\theta}}l^*_{u\theta\bar{\theta}}, \quad (4.3e)$$

$$\delta_\alpha l^*_{u\theta\bar{\theta}} = i\alpha^{\bar{\theta}}(f^v{}_{iu}\sigma^i{}_a\rho^a{}_p(x)\chi^p_{\bar{\theta}}l^*_{v\theta\bar{\theta}} \quad (4.3f)$$

$$+ i\overline{D}_z\lambda^*_{u\theta} - f^v{}_{wu}G^w{}_{\bar{a}b}\rho^{\bar{a}}{}_{\bar{p}}\rho^b{}_q(x)\overline{\chi}^{\bar{p}}_{\bar{\theta}}\chi^q_{\bar{\theta}}\lambda^*_{v\theta}),$$

where the spinors $\alpha^{\bar{\theta}}, \overline{\alpha}^{\bar{\theta}} \in \Gamma(\Pi\overline{K}_\Sigma^{-1/2})$ satisfy the antiholomorphy condition

$$\partial_z \alpha^{\bar{\theta}} = \partial_z \bar{\alpha}^{\theta} = 0 \quad (4.4)$$

and complex conjugation interchanges $\alpha^{\bar{\theta}}$, $\bar{\alpha}^{\theta}$ ⁵.

The sigma model action S enjoys $(0, 2)$ supersymmetry, so that

$$\delta_\alpha S = 0, \quad (4.5)$$

provided the splitting σ satisfies the condition

$$G^u{}_{ab} = 0, \quad (4.6)$$

where $G^u{}_{ab}$ is $(2, 0)$ component of the curvature $G_{E\sigma^c}$ of σ (cf. eq. (3.40)). As a matter of fact, the weaker condition $f^v{}_{wu} G^w{}_{ab} = 0$ would be sufficient for $(0, 2)$ supersymmetry. However, in the analysis of the cohomological properties of the topological twisted version of the model studied in the next section, it will emerge that the stronger condition (4.6) is required. (4.6) will thus be assumed right away.

The $(1, 0)$ supersymmetry field variations are given by

$$s_\xi x^a = 0, \quad \text{and c. c.}, \quad (4.7a)$$

$$s_\xi \chi^{\bar{\theta}} = 0, \quad \text{and c. c.}, \quad (4.7b)$$

$$s_\xi \lambda^u{}_\theta = -\frac{i}{2} \xi^\theta f^u{}_{vw}(x) \lambda^v{}_\theta \lambda^w{}_\theta, \quad (4.7c)$$

$$s_\xi l^u{}_{\theta\bar{\theta}} = -i \xi^\theta f^u{}_{vw}(x) \lambda^v{}_\theta l^w{}_{\theta\bar{\theta}}, \quad (4.7d)$$

$$s_\xi \lambda^*{}_{u\theta} = i \xi^\theta f^w{}_{vu}(x) \lambda^v{}_\theta \lambda^*{}_{w\theta}, \quad (4.7e)$$

$$s_\xi l^*{}_{u\theta\bar{\theta}} = i \xi^\theta f^w{}_{vu}(x) \lambda^v{}_\theta l^*{}_{w\theta\bar{\theta}}. \quad (4.7f)$$

where the spinor $\xi^\theta \in \Gamma(\Pi K_\Sigma^{-1/2})$ satisfies the holomorphy condition

$$\bar{\partial}_{\bar{z}} \xi^\theta = 0. \quad (4.8)$$

The sigma model action S enjoys also $(1, 0)$ supersymmetry,

⁵ In our conventions, if ξ, η are complex fermion fields, $\overline{\xi\eta} = \bar{\eta}\bar{\xi} = -\bar{\xi}\bar{\eta}$.

$$s_\xi S = 0, \tag{4.9}$$

no further restriction on the target space geometry being required. While the $(0, 2)$ supersymmetry is a common property of all heterotic sigma models, this $(1, 0)$ supersymmetry is a special feature of our Lie algebroid sigma model. The model has therefore a total $(1, 2)$ supersymmetry. The $(0, 2)$ and $(1, 0)$ parts of this play however quite different roles in the topological twisted version of the model, as we shall see.

As originally noticed in ref. [5], on a compact Riemann surface Σ , the space of holomorphic sections of the holomorphic line bundles $K_\Sigma^{p/2}$ with $p \in \mathbb{Z}$ is only finite dimensional. For a world-sheet of this type, conditions (4.4), (4.8) have only the trivial solutions $\alpha^{\bar{\theta}} = \bar{\alpha}^{\bar{\theta}} = 0$, $\xi^\theta = 0$, and, so, the $(0, 2)$ and $(1, 0)$ supersymmetries of the sigma model are empty (except in genus 1). To have a full infinite dimensional space of solutions of (4.4), (4.8), it is necessary that Σ has punctures. Customarily, one assumes that Σ is the flat punctured complex plane $\mathbb{C} \setminus \{0\}$. In the next section, we shall see that the Lie algebroid heterotic sigma model, in a topologically twisted form, can be formulated on a world-sheet that is a general compact Riemann surface Σ .

On a flat world-sheet, in which the supersymmetry parameters $\alpha^{\bar{\theta}}$, $\bar{\alpha}^{\bar{\theta}}$ and ξ^θ can be taken constant, the $(1, 2)$ supersymmetry algebra

$$[\delta_\alpha, \delta_\beta] = i(\bar{\alpha}^{\bar{\theta}}\beta^{\bar{\theta}} - \bar{\beta}^{\bar{\theta}}\alpha^{\bar{\theta}})\bar{\partial}_{\bar{z}}, \tag{4.10a}$$

$$[s_\xi, s_\eta] = 0, \tag{4.10b}$$

$$[\delta_\alpha, s_\xi] \simeq 0 \tag{4.10c}$$

is verified, provided the $(0, 2)$ supersymmetry condition (4.6) is satisfied. Here and in the following \simeq denotes equality holding *on-shell*. We remark that (4.6) is required only by the validity of (4.10a).

The sigma model enjoys also an antiholomorphic R and a holomorphic flavour

symmetry, under which the sigma model fields $x, \chi_{\bar{\theta}}, \bar{\chi}_{\bar{\theta}}, \lambda_{\theta}, \lambda^*_{\theta}, l_{\theta\bar{\theta}}, l^*_{\theta\bar{\theta}}$ have R /flavour charges $(0, 0), (-1, 0), (1, 0), (0, 1), (0, -1), (-1, 1), (1, -1)$, respectively. The overall normalization of the two charges is conventional. Different conventions are used in [11]–[14], [17].

The supersymmetry field variations (4.3), (4.7) respect the R /flavour charges of the sigma model fields provided that the supersymmetry parameters $\alpha^{\bar{\theta}}, \bar{\alpha}^{\bar{\theta}}, \xi^{\theta}$ are assigned R /flavour charges $(1, 0), (-1, 0), (0, -1)$ respectively. It is natural to assume this and we shall do so henceforth.

4.2 The Lie algebroid heterotic sigma model: quantum aspects

The Lie algebroid heterotic sigma model is a particular heterotic sigma model and, so, the usual consistency requirements of the quantum field theory associated with the latter apply.

In order the sigma model quantum functional integral to be meaningfully defined, the fermion determinants must combine to yield a function on the space of all embeddings x . As is well-known, to this end, the associated determinant line bundle \mathcal{L} must be trivial. In the heterotic sigma model, this is achieved if the complex vector bundles $T_M^+, \ker \rho_{E^+}$ satisfy the condition

$$\text{ch}_2(T_M^+) = \text{ch}_2(\ker \rho_{E^+}), \quad (4.11)$$

where $\text{ch}_2(W)$ denotes the degree 4 part of the Chern character of a complex vector bundle W . Above, the isomorphism $\tilde{E}^+ \simeq T_M^+$ has been taken into account.

The R and flavour symmetry are generally anomalous. The anomalies manifest themselves as vacuum background R and flavour charges $\Delta q_R, \Delta q_L$. Δq_R is the difference of the numbers of the $\bar{\chi}_{\bar{\theta}}$, and the $\chi_{\bar{\theta}}$ zero-modes. Similarly, Δq_L is the difference of the numbers of the λ_{θ} and the λ^*_{θ} zero-modes. $\Delta q_R, \Delta q_L$ are

so given by the index formulae

$$\Delta q_R = \int_{\Sigma} x^* c_1(T_M^+), \quad (4.12)$$

$$\Delta q_L = \int_{\Sigma} x^* c_1(\ker \rho_{E^+}). \quad (4.13)$$

Quantum sigma model correlators can be non zero only if the vacuum charges are soaked up by those of the inserted operators.

Further topological restrictions will arise in the topologically twisted version of the sigma model studied in the next section. In that case, the fact that the target manifold M and the gauge vector bundle $\ker \rho_E$ are encompassed in the Lie algebroid E makes the consistency requirements somewhat more restrictive and, so, also more predictive.

5 The half-topological Lie algebroid sigma models

5.1 Topological twisting: classical theory

As is well-known, a supersymmetric field theory on a flat space-time does not remain supersymmetric on a more general space-time with non trivial topology. The reason for this is that supercharges are constant spinors and constant spinors exist only on very few space-times. A systematic way of constructing supersymmetric field theories on general space-times is by topological twist of supersymmetric field theories on flat ones. The twist is implemented by modifying the space-time covariance of the fields depending on their R and flavour symmetry properties in such a way that one or more supercharges become scalar and thus can be constant on a general background. Twisting is not always possible or may be possible in more than one way. See ref. [5] for a readable exposition of twisting in sigma models

We shall now consider the topological twist of the Lie algebroid heterotic sigma model constructed in sect. 4. A sigma model field is either the embedding field $x \in \text{Map}(\Sigma, M)$ or a section $\phi \in \Gamma(\zeta)$ of some vector bundle ζ over Σ . A twist prescription leaves x unchanged and replaces any other field $\phi \in \Gamma(\zeta)$ having R /flavour charges (q_R, q_L) with a twisted counterpart $\phi_{\text{tw}} \in \Gamma(\overline{K}_\Sigma^{\eta_R q_R/2} \otimes K_\Sigma^{\eta_L q_L/2} \otimes \zeta)$, where η_R, η_L are integers independent from ϕ whose values define the type of the twist.

The topological twist of the fields implies a simultaneous twist of the spinor supersymmetry parameters $\alpha^{\bar{\theta}}, \bar{\alpha}^{\bar{\theta}} \in \Gamma(\overline{K}_\Sigma^{-1/2})$, $\xi^\theta \in \Gamma(K_\Sigma^{-1/2})$ of the $(0, 2)$, $(1, 0)$ supersymmetry field variations (4.3), (4.7) so that to ensure the world-sheet covariance of the twisted form of these latter. In fact, inspection reveals that the twist of $\alpha^{\bar{\theta}}, \bar{\alpha}^{\bar{\theta}}, \xi^\theta$ must be carried out using the same prescription as that of the fields, where $\alpha^{\bar{\theta}}, \bar{\alpha}^{\bar{\theta}}, \xi^\theta$ have the R /flavour charges $(1, 0)$, $(-1, 0)$, $(0, -1)$ found in sect. 4, respectively.

To ensure that one of the two $(0, 2)$ supercharges becomes a world-sheet scalar upon twisting, as required, we must twist either $\bar{\alpha}^{\bar{\theta}}$ into $\bar{\alpha} \in \Gamma(\Pi 1_\Sigma)$ or $\alpha^{\bar{\theta}}$ into $\alpha \in \Gamma(\Pi 1_\Sigma)$. This restricts the choice of η_R to $\eta_R = \mp 1$, as is immediately seen. Similarly, to ensure that the $(1, 0)$ supercharge becomes a world-sheet scalar upon twisting, ξ^θ must be twisted into $\xi \in \Gamma(\Pi 1_\Sigma)$. This fixes η_L to $\eta_L = -1$. The topological supersymmetry is the residual twisted supersymmetry associated with either $\bar{\alpha}$ or α and ξ .

We are thus left with the twists defined by $(\eta_R, \eta_L) = (\mp 1, -1)$. They are called *type A* and *B* twists and they lead to the *type A* and *B half-topological Lie algebroid sigma models*, respectively, which are studied in this section. We work first at the classical level and postpone the analysis of their quantum consistency to the second part of this section.

5.2 The type A Lie algebroid sigma model

The type A Lie algebroid sigma model arises from the implementation of the type A twist in the heterotic Lie algebroid sigma model (cf. subsect. 4.1).

The field content of the A model is obtained from that of the heterotic model by turning each field of the latter into its twisted counterpart. Proceeding in this way, the embedding field $x \in \text{Map}(\Sigma, M)$ is left unchanged, the fermion fields $\chi_{\bar{\theta}} \in \Gamma(\bar{K}_\Sigma^{1/2} \otimes x^* \Pi \tilde{E}^+)$, $\bar{\chi}_{\bar{\theta}} \in \Gamma(\bar{K}_\Sigma^{1/2} \otimes x^* \Pi \bar{\tilde{E}}^+)$ are replaced by fermion fields $\chi_{\bar{z}} \in \Gamma(\bar{K}_\Sigma \otimes x^* \Pi \tilde{E}^+)$, $\bar{\chi} \in \Gamma(x^* \Pi \bar{\tilde{E}}^+)$, the fermion fields $\lambda_\theta \in \Gamma(K_\Sigma^{1/2} \otimes x^* \Pi \ker \rho_{E^+})$, $\lambda^*_\theta \in \Gamma(K_\Sigma^{1/2} \otimes x^* \Pi \ker \rho_{E^+*})$ turn into fermion fields $\lambda \in \Gamma(x^* \Pi \ker \rho_{E^+})$, $\lambda^*_z \in \Gamma(K_\Sigma \otimes x^* \Pi \ker \rho_{E^+*})$ and the boson fields $l_{\theta\bar{\theta}} \in \Gamma(K_\Sigma^{1/2} \otimes \bar{K}_\Sigma^{1/2} \otimes x^* \ker \rho_{E^+})$, $l^*_{\theta\bar{\theta}} \in \Gamma(K_\Sigma^{1/2} \otimes \bar{K}_\Sigma^{1/2} \otimes x^* \ker \rho_{E^+*})$ yield boson fields $l_{\bar{z}} \in \Gamma(\bar{K}_\Sigma \otimes x^* \ker \rho_{E^+})$, $l^*_z \in \Gamma(K_\Sigma \otimes x^* \ker \rho_{E^+*})$.

The action of the type A sigma model is obtained from that of the heterotic model (cf. eq. (4.1)) by substituting each field of the latter with its twisted

counterpart. Proceeding in this way, we obtain

$$S_A = \int_{\Sigma} d^2z \left[\frac{1}{2} g_{\bar{a}b}(x) (\bar{\partial}_{\bar{z}} x^{\bar{a}} \partial_z x^b + \partial_z x^{\bar{a}} \bar{\partial}_{\bar{z}} x^b) + i g_{\bar{a}b} \rho^{\bar{a}}_{\bar{p}} \rho^b_q(x) \bar{\chi}^{\bar{p}} D_z \chi^q_{\bar{z}} \right. \\ \left. + i \lambda^*_{uz} \bar{D}_{\bar{z}} \lambda^u + f^u_{uv} G^w_{\bar{a}b} \rho^{\bar{a}}_{\bar{p}} \rho^b_q(x) \lambda^*_{uz} \lambda^v \bar{\chi}^{\bar{p}} \chi^q_{\bar{z}} - l^*_{uz} l^u_{\bar{z}} \right]. \quad (5.1)$$

The $(0, 2)$ supersymmetry field variations of the A model are obtained from those of the heterotic model (cf. eq. (4.3)) by the simultaneous twist of the sigma model fields and the supersymmetry parameters. Under type A twist, the supersymmetry parameters $\alpha^{\bar{\theta}}, \bar{\alpha}^{\bar{\theta}} \in \Gamma(\bar{K}_{\Sigma}^{-1/2})$ turn into $\alpha^{\bar{z}} \in \Gamma(\Pi \bar{K}_{\Sigma}^{-1})$, $\bar{\alpha} \in \Gamma(\Pi 1_{\Sigma})$, respectively. The primary type A topological field variations are the truncated $(0, 2)$ supersymmetry field variations yielded by setting $\alpha^{\bar{z}} = 0$. Proceeding as indicated and formally dividing by $\bar{\alpha}$, these read

$$\delta_A x^a = 0, \quad \delta_A x^{\bar{a}} = i \rho^{\bar{a}}_{\bar{p}}(x) \bar{\chi}^{\bar{p}}, \quad (5.2a)$$

$$\delta_A \chi^p_{\bar{z}} = -\sigma^p_a(x) \bar{\partial}_{\bar{z}} x^a, \quad \delta_A \bar{\chi}^{\bar{p}} = -\frac{i}{2} f^{\bar{p}}_{\bar{q}r}(x) \bar{\chi}^{\bar{q}} \bar{\chi}^{\bar{r}}, \quad (5.2b)$$

$$\delta_A \lambda^u = 0, \quad (5.2c)$$

$$\delta_A l^u_{\bar{z}} = -\bar{D}_{\bar{z}} \lambda^u + i f^u_{uv} G^w_{\bar{a}b} \rho^{\bar{a}}_{\bar{p}} \rho^b_q(x) \bar{\chi}^{\bar{p}} \chi^q_{\bar{z}} \lambda^v, \quad (5.2d)$$

$$\delta_A \lambda^*_{uz} = i l^*_{uz}, \quad (5.2e)$$

$$\delta_A l^*_{uz} = 0. \quad (5.2f)$$

The $(0, 2)$ supersymmetry of the heterotic model action, eq. (4.5), implies that the A model action S_A is invariant under the primary topological symmetry,

$$\delta_A S_A = 0. \quad (5.3)$$

In analogous fashion, the $(1, 0)$ supersymmetry field variations of the A model are obtained from those of the heterotic model (cf. eq. (4.7)) by the simultaneous twist of the sigma model fields and the supersymmetry parameters. Under type A twist, the $(1, 0)$ supersymmetry parameter $\xi^{\theta} \in \Gamma(K_{\Sigma}^{-1/2})$ turns into $\xi \in \Gamma(\Pi 1_{\Sigma})$.

The secondary type A topological field variations result in this way. After formally dividing by ξ , they read

$$s_A x^a = 0, \quad s_A x^{\bar{a}} = 0, \quad (5.4a)$$

$$s_A \chi^p_{\bar{z}} = 0, \quad s_A \bar{\chi}^{\bar{p}} = 0, \quad (5.4b)$$

$$s_A \lambda^u = -\frac{i}{2} f^u_{vw}(x) \lambda^v \lambda^w, \quad (5.4c)$$

$$s_A l^u_{\bar{z}} = -i f^u_{vw}(x) \lambda^v l^w_{\bar{z}}, \quad (5.4d)$$

$$s_A \lambda^*_{uz} = i f^w_{vu}(x) \lambda^v \lambda^*_{wz}, \quad (5.4e)$$

$$s_A l^*_{uz} = i f^w_{vu}(x) \lambda^v l^*_{wz}. \quad (5.4f)$$

The $(1, 0)$ supersymmetry of the heterotic model action, eq. (4.9), implies that the A model action S_A enjoys also the secondary type A topological symmetry,

$$s_A S_A = 0. \quad (5.5)$$

As the heterotic sigma model it comes from, the type A sigma model enjoys an R and a flavour symmetry, in which each twisted field has the same R and flavour charge as its untwisted parent. The R /flavour charges of the twisted fields $x, \chi_{\bar{z}}, \bar{\chi}, \lambda, \lambda^*_z, l_{\bar{z}}, l^*_z$ are thus $(0, 0), (-1, 0), (1, 0), (0, 1), (0, -1), (-1, 1), (1, -1)$, respectively.

5.3 The type B Lie algebroid sigma model

The type B Lie algebroid sigma model arises from the implementation of the type B twist in the heterotic Lie algebroid sigma model.

The field content of the B model is obtained from that of the heterotic model by turning each field of the latter into its twisted counterpart. Proceeding in this way, the embedding field $x \in \text{Map}(\Sigma, M)$ is left unchanged, the fermion fields $\chi_{\bar{\theta}} \in \Gamma(\bar{K}_\Sigma^{1/2} \otimes x^* \Pi \tilde{E}^+)$, $\bar{\chi}_{\bar{\theta}} \in \Gamma(\bar{K}_\Sigma^{1/2} \otimes x^* \Pi \tilde{E}^+)$ turn into fermion fields $\chi \in$

$\Gamma(x^*\Pi\tilde{E}^+)$, $\bar{\chi}_{\bar{z}} \in \Gamma(\bar{K}_\Sigma \otimes x^*\Pi\tilde{E}^+)$, the fermion fields $\lambda_\theta \in \Gamma(K_\Sigma^{1/2} \otimes x^*\Pi \ker \rho_{E^+})$, $\lambda^*_\theta \in \Gamma(K_\Sigma^{1/2} \otimes x^*\Pi \ker \rho_{E^+}^*)$ give rise to fermion fields $\lambda \in \Gamma(x^*\Pi \ker \rho_{E^+})$, $\lambda^*_z \in \Gamma(K_\Sigma \otimes x^*\Pi \ker \rho_{E^+}^*)$ and the boson fields $l_{\theta\bar{\theta}} \in \Gamma(K_\Sigma^{1/2} \otimes \bar{K}_\Sigma^{1/2} \otimes x^* \ker \rho_{E^+})$, $l^*_{\theta\bar{\theta}} \in \Gamma(K_\Sigma^{1/2} \otimes \bar{K}_\Sigma^{1/2} \otimes x^* \ker \rho_{E^+}^*)$ yield boson fields $l \in \Gamma(x^* \ker \rho_{E^+})$, $l^*_{z\bar{z}} \in \Gamma(K_\Sigma \otimes \bar{K}_\Sigma \otimes x^* \ker \rho_{E^+}^*)$.

As for the A model, the action of the type B sigma model is obtained from that of the heterotic model (cf. eq. (4.1)) by substituting each field of the latter with its twisted counterpart. The result is

$$S_B = \int_\Sigma d^2z \left[\frac{1}{2} g_{\bar{a}b}(x) (\bar{\partial}_{\bar{z}} x^{\bar{a}} \partial_z x^b + \partial_z x^{\bar{a}} \bar{\partial}_{\bar{z}} x^b) + i g_{\bar{a}b} \rho^{\bar{a}}_{\bar{p}} \rho^b_q(x) \bar{\chi}^{\bar{p}}_{\bar{z}} D_z \chi^q \right. \\ \left. + i \lambda^*_{uz} \bar{D}_{\bar{z}} \lambda^u + f^u_{uv} G^w_{\bar{a}b} \rho^{\bar{a}}_{\bar{p}} \rho^b_q(x) \lambda^*_{uz} \lambda^v \bar{\chi}^{\bar{p}}_{\bar{z}} \chi^q - l^*_{uz\bar{z}} l^u \right]. \quad (5.6)$$

Again as for the A model, the $(0, 2)$ supersymmetry field variations of the B model are obtained from those of the heterotic model (cf. eq. (4.3)) by the simultaneous twist of the sigma model fields and the supersymmetry parameters. Under B twist, the $(0, 2)$ supersymmetry parameters $\alpha^{\bar{\theta}}, \bar{\alpha}^{\bar{\theta}} \in \Gamma(\bar{K}_\Sigma^{-1/2})$ become $\alpha \in \Gamma(\Pi 1_\Sigma)$, $\bar{\alpha}^{\bar{z}} \in \Gamma(\Pi \bar{K}_\Sigma^{-1})$, respectively. The primary type B topological field variations are the truncated $(0, 2)$ supersymmetry field variations yielded by setting $\bar{\alpha}^{\bar{z}} = 0$. Proceeding as said and formally eliminating α , these read

$$\delta_B x^a = i \rho^a_p(x) \chi^p, \quad \delta_B x^{\bar{a}} = 0, \quad (5.7a)$$

$$\delta_B \chi^p = -\frac{i}{2} f^p_{qr}(x) \chi^q \chi^r, \quad \delta_B \bar{\chi}^{\bar{p}}_{\bar{z}} = -\sigma^{\bar{p}}_{\bar{a}}(x) \bar{\partial}_{\bar{z}} x^{\bar{a}}, \quad (5.7b)$$

$$\delta_B \lambda^u = -i f^u_{iv} \sigma^i_a \rho^a_p(x) \chi^p \lambda^v + i l^u, \quad (5.7c)$$

$$\delta_B l^u = -i f^u_{iv} \sigma^i_a \rho^a_p(x) \chi^p l^v, \quad (5.7d)$$

$$\delta_B \lambda^*_{uz} = i f^v_{iu} \sigma^i_a \rho^a_p(x) \chi^p \lambda^*_{vz}, \quad (5.7e)$$

$$\delta_B l^*_{uz\bar{z}} = i f^v_{iu} \sigma^i_a \rho^a_p(x) \chi^p l^*_{vz\bar{z}} \quad (5.7f)$$

$$- \bar{D}_{\bar{z}} \lambda^*_{uz} - i f^v_{wu} G^w_{\bar{a}b} \rho^{\bar{a}}_{\bar{p}} \rho^b_q(x) \bar{\chi}^{\bar{p}}_{\bar{z}} \chi^q \lambda^*_{vz}.$$

The $(0, 2)$ supersymmetry of the heterotic model action, eq. (4.5), implies that the B model action S_B is invariant under the primary topological symmetry,

$$\delta_B S_B = 0. \quad (5.8)$$

In analogous fashion, just as in the A model, the $(1, 0)$ supersymmetry field variations of the B model are obtained from those of the heterotic model (cf. eq. (4.7)) by the simultaneous twist of the sigma model fields and the supersymmetry parameters. Under type B twist, the $(1, 0)$ supersymmetry parameter $\xi^\theta \in \Gamma(K_\Sigma^{-1/2})$ turns again into $\xi \in \Gamma(\Pi 1_\Sigma)$. The secondary type B topological field variations result in this way. After formally dividing by ξ , they read

$$s_B x^a = 0, \quad s_B x^{\bar{a}} = 0, \quad (5.9a)$$

$$s_B \chi^p = 0, \quad s_B \bar{\chi}^{\bar{p}} = 0, \quad (5.9b)$$

$$s_B \lambda^u = -\frac{i}{2} f^u{}_{vw}(x) \lambda^v \lambda^w, \quad (5.9c)$$

$$s_B l^u = -i f^u{}_{vw}(x) \lambda^v l^w, \quad (5.9d)$$

$$s_B \lambda^*{}_{uz} = i f^w{}_{vu}(x) \lambda^v \lambda^*{}_{wz}, \quad (5.9e)$$

$$s_B l^*{}_{uz\bar{z}} = i f^w{}_{vu}(x) \lambda^v l^*{}_{wz\bar{z}}. \quad (5.9f)$$

The $(1, 0)$ supersymmetry of the heterotic model action, eq. (4.9), implies that the B model action S_B enjoys also the secondary type B topological symmetry,

$$s_B S_B = 0. \quad (5.10)$$

As the heterotic sigma model, the type B sigma model enjoys an R and a flavour symmetry, in which each twisted field has the same R and flavour charge as its untwisted parent. However, in order to comply later with established cohomological conventions, it is convenient to redefine the sign of the R charge. The R /flavour charges of the twisted fields $x, \chi, \bar{\chi}_{\bar{z}}, \lambda, \lambda^*{}_z, l, l^*{}_{z\bar{z}}$ are therefore

$(0, 0), (1, 0), (-1, 0), (0, 1), (0, -1), (1, 1), (-1, -1)$, respectively.

5.4 Half-topological nature of the twisted sigma models

The twisted sigma models illustrated above have a number of features, which make them candidate topological field theories. In the unified treatment presented below, the label t stands for the types A, B .

The algebra of the twisted sigma model fields has a double grading, where the bidegree of each elementary field is given by its R /flavour charges.

The primary and the secondary topological field variation operators δ_t, s_t are odd, as they flip the statistics of the fields, and in fact of bidegree $(1, 0)$ and $(0, 1)$, respectively. The $(1, 2)$ supersymmetry algebra (4.10) entails that

$$\delta_t^2 = 0, \tag{5.11a}$$

$$s_t^2 = 0, \tag{5.11b}$$

$$\delta_t s_t + s_t \delta_t \simeq 0, \tag{5.11c}$$

provided the $(0, 2)$ supersymmetry condition (4.6) is satisfied. (Recall that \simeq denotes equality on-shell.) (4.6) is required only by the validity of (5.11a).

The nilpotency of δ_t and s_t shows that they are BRST operators. In spite of this and the fact that they both originate by topological twist of the heterotic sigma model, they play quite different roles in the twisted sigma models. As we shall now show, δ_t is a true topological field variation operator, while s_t is akin to a gauge theoretic Slavnov operator. Indeed, s_t represents a non trivial symmetry of the half-topological sigma models, which, being intimately related to the adjoint bundle $\ker \rho_E$ of E , we shall call *adjoint symmetry*.

The existence of a nilpotent BRST operator and the associated BRST cohomology is a universal feature of all topological sigma models, but it is not by itself sufficient to render a sigma model genuinely topological. Other require-

ments must be satisfied: a) the sigma model action should be BRST-exact up to a topological term and b) the stress tensor should be BRST-exact. Let us check whether our twisted sigma models fulfil these taking δ_t as BRST operator.

The first requirement is fulfilled. Indeed, S_t has the following structure

$$S_t = \delta_t \Psi_t + \eta_t I_{\text{top}}, \quad (5.12)$$

where Ψ_t is the gauge fermion

$$\Psi_A = \int_{\Sigma} d^2 z \left(-g_{\bar{a}b} \rho^b{}_q(x) \chi^q{}_{\bar{z}} \partial_z x^{\bar{a}} + i \lambda^*{}_{uz} l^u{}_{\bar{z}} \right), \quad (5.13)$$

$$\Psi_B = \int_{\Sigma} d^2 z \left(-g_{\bar{a}b} \rho^{\bar{a}}{}_{\bar{p}}(x) \bar{\chi}^{\bar{p}}{}_{\bar{z}} \partial_z x^b + i l^*{}_{uz\bar{z}} \lambda^u \right), \quad (5.14)$$

$\eta_A = -1, \eta_B = 1$ and I_{top} is the topological action,

$$I_{\text{top}} = \int_{\Sigma} d^2 z \frac{1}{2} g_{\bar{a}b}(x) (\partial_z x^{\bar{a}} \bar{\partial}_{\bar{z}} x^b - \bar{\partial}_{\bar{z}} x^{\bar{a}} \partial_z x^b) = \int_{\Sigma} x^* \omega, \quad (5.15)$$

$\omega = \frac{i}{2} g_{\bar{a}b} dz^{\bar{a}} \wedge dz^b$ being the Kaehler form of M .

The second requirement, conversely, is not fulfilled. The components of the stress tensor are in fact found to be

$$T_{tzz} = -g_{\bar{a}b}(x) \partial_z x^{\bar{a}} \partial_z x^b - i \lambda^*{}_{uz} D_z \lambda^u, \quad (5.16a)$$

$$T_{t\bar{z}\bar{z}} = -g_{\bar{a}b}(x) \bar{\partial}_{\bar{z}} x^{\bar{a}} \bar{\partial}_{\bar{z}} x^b + T'_{t\bar{z}\bar{z}}, \quad (5.16b)$$

$$T_{t\bar{z}z} = 0, \quad (5.16c)$$

where $T'_{t\bar{z}\bar{z}}$ is given by the expressions

$$T'_{A\bar{z}\bar{z}} = i g_{\bar{a}b} \rho^{\bar{a}}{}_{\bar{p}} \rho^b{}_q(x) \bar{D}_{\bar{z}} \bar{\chi}^{\bar{p}} \chi^q{}_{\bar{z}}, \quad (5.17)$$

$$T'_{B\bar{z}\bar{z}} = -i g_{\bar{a}b} \rho^{\bar{a}}{}_{\bar{p}} \rho^b{}_q(x) \bar{\chi}^{\bar{p}}{}_{\bar{z}} \bar{D}_{\bar{z}} \chi^q, \quad (5.18)$$

in the two twisted models. The stress tensor component T_{zz} is δ_t -closed

$$\delta_t T_{zz} \simeq 0, \quad (5.19)$$

but it is not δ_t -exact. The stress tensor component $T_{\bar{z}\bar{z}}$, instead, is δ_t -exact,

$$T_{t\bar{z}\bar{z}} = \delta_t G_{t\bar{z}\bar{z}}, \quad (5.20)$$

where $G_{t\bar{z}\bar{z}}$ is given by the expressions

$$G_{A\bar{z}\bar{z}} = g_{\bar{a}\bar{b}} \rho_{\bar{q}}^{\bar{b}}(x) \chi_{\bar{z}}^{\bar{q}} \bar{\partial}_{\bar{z}} x^{\bar{a}}, \quad (5.21)$$

$$G_{B\bar{z}\bar{z}} = g_{\bar{a}\bar{b}} \rho_{\bar{p}}^{\bar{a}}(x) \bar{\chi}_{\bar{z}}^{\bar{p}} \bar{\partial}_{\bar{z}} x^{\bar{b}}. \quad (5.22)$$

We find in this way that our twisted sigma models are not topological field theories. Since only one of the two non trivial components of the stress tensor is BRST-exact, they are *half-topological* sigma models [17].

What is the role of the BRST operator s_t in all this? Any attempt to formulate the twisted sigma models as topological field theories taking s_t as BRST operator necessarily fails. Since s_t acts trivially on the embedding and the \tilde{E}^+ , \bar{E}^+ valued fields, there is no way of generating the corresponding terms of the action by letting s_t act on a suitable gauge fermion. Nevertheless, δ_t and s_t combined yield a distinct BRST operator with properties completely analogous to those of δ_t above, as we now show.

For any $u \in \mathfrak{g}$, we can construct a BRST operator

$$\delta_t(u) = \delta_t + u s_t. \quad (5.23)$$

This is indeed nilpotent on-shell on account of (5.11),

$$\delta_t(u)^2 \simeq 0. \quad (5.24)$$

One may be tempted to extend the domain of definition of $\delta_t(u)$ to $u \in \mathfrak{g}$ by setting $\delta'_t(u') = u' \delta_t + s_t$. However, as just noticed, the primary and secondary BRST operators play an asymmetrical role in the sigma models, the first one only being the counterpart of the usual BRST operator of other twisted heterotic sigma models. So, it seems unlikely that $\delta'_t(u')$ would have any significance.

With the BRST operator $\delta_t(u)$, there is associated a BRST cohomology depending a priori on u . Let us see how. Let f_t be the flavour charge. As δ_t and s_t have flavour charges 0 and 1, respectively, we have $[f_t, \delta_t] = 0$ and $[f_t, s_t] = s_t$. So, for $\zeta \in \mathbb{C}$, we have

$$\exp(\zeta f_t) \delta_t(u) \exp(-\zeta f_t) = \delta_t(u \exp(\zeta)). \quad (5.25)$$

It follows that the $\delta_t(u)$ -cohomology is independent from $u \in \mathbb{C} \setminus \{0\}$ up to isomorphism. Thus, we really have to consider only the BRST charges $\delta_t(0)$ and, say, $\delta_t(1)$. Note that $\delta_t(0) = \delta_t$ is the primary BRST charge while $\delta_t(1) = \delta_t + s_t$ is the total BRST charge.

Using (5.13), (5.14) and (5.4), (5.9), we find that

$$s_t \Psi_t \simeq 0. \quad (5.26)$$

Further, we have

$$s_t T_{tzz} = 0 \quad (5.27)$$

and

$$s_t G_{t\bar{z}\bar{z}} = 0, \quad (5.28)$$

as is straightforwardly verified using once more (5.4), (5.9). By these relations, it is possible to extend the construction carried out in the first half of this subsection, replacing δ_t by $\delta_t + s_t$ throughout. In this way, (5.12) becomes

$$S_t \simeq (\delta_t + s_t) \Psi_t + \eta_t I_{\text{top}}. \quad (5.29)$$

Likewise, (5.19) and (5.20) take the form

$$(\delta_t + s_t) T_{zz} \simeq 0 \quad (5.30)$$

and

$$T_{t\bar{z}\bar{z}} = (\delta_t + s_t) G_{t\bar{z}\bar{z}}. \quad (5.31)$$

So, T_{zz} and $T_{t\bar{z}\bar{z}}$ are also $\delta_t + s_t$ -closed and $\delta_t + s_t$ -exact, respectively.

The half-topological sigma models have therefore two distinct BRST cohomological structures with analogous formal properties, one associated with δ_t , the other associated with $\delta_t + s_t$. They will emerge over and over again at various points below and will constitute the main theme of our analysis. This feature of the sigma models raises a number of questions. Do both these BRST structures have a counterpart in the quantum theory? If so, which properties do they have and how do they relate to each other? In the rest of this section, we shall try to answer these queries.

5.5 Dependence on target space geometry

The target space geometry of the half-topological sigma models is characterized by several fields: the Lie bracket $[\cdot, \cdot]_E$, the anchor ρ_E , the Lie algebroid complex structure (J_E, J_{EM}) , the complex splitting σ and the Kaehler metric g . The cohomological nature of the half-topological sigma models suggests that, when some of these fields are varied, the action varies by a BRST-exact term.

Consider first a variation h of the metric g , the other target space fields remaining fixed. Since the variation must preserve the Hermiticity and the Kaehler-ness of the metric g , the $(2, 0)$, $(0, 2)$ components of h vanish and the $(1, 1)$ component satisfies the equation

$$\partial_{\bar{a}} h_{\bar{b}c} - \partial_{\bar{b}} h_{\bar{a}c} = 0 \quad \text{and c. c.} \quad (5.32)$$

Inspecting the expression of the primary field variation operator δ_t (cf. eqs. (5.2), (5.7)), we observe that it does not depend on the metric. From (5.12)–(5.15), we find then that the resulting variation of the action S_t is given by

$$\Delta_h S_t = \delta_t \Delta_h \Psi_t + \eta_t \Delta_h I_{\text{top}}, \quad (5.33)$$

where $\Delta_h \Psi_t$ is the variation of the gauge fermion Ψ_t ,

$$\Delta_h \Psi_A = \int_{\Sigma} d^2 z \left(- h_{\bar{a}b} \rho^b{}_q(x) \chi^q{}_{\bar{z}} \partial_z x^{\bar{a}} \right), \quad (5.34)$$

$$\Delta_h \Psi_B = \int_{\Sigma} d^2 z \left(- h_{\bar{a}b} \rho^{\bar{a}}{}_{\bar{p}}(x) \bar{\chi}^{\bar{p}}{}_{\bar{z}} \partial_z x^b \right) \quad (5.35)$$

and $\Delta_h I_{\text{top}}$ is the variation of the topological term I_{top} ,

$$\Delta_h I_{\text{top}} = \int_{\Sigma} d^2 z \frac{1}{2} h_{\bar{a}b}(x) (\partial_z x^{\bar{a}} \bar{\partial}_{\bar{z}} x^b - \bar{\partial}_{\bar{z}} x^{\bar{a}} \partial_z x^b) = \int_{\Sigma} x^* \Delta_h \omega, \quad (5.36)$$

with $\Delta_h \omega = \frac{i}{2} h_{\bar{a}b} dz^{\bar{a}} \wedge dz^b$. Thus, $\Delta_h S_t$ is δ_t -exact only if $\Delta_h I_{\text{top}} = 0$. This happens when either the (1,1) form $\Delta_h \omega$ is exact or the embedding field x is homotopically trivial.

Consider next a variation τ of the splitting σ , the other target space fields remaining fixed. Since the variation must preserve the basic relations (2.8), (3.15), only the components $\tau^u{}_a$ of τ are non vanishing. Further, the (0,2) supersymmetry condition (4.6) requires that

$$\partial_a \tau^u{}_b - \partial_b \tau^u{}_a + f^u{}_{vi} \tau^v{}_a \sigma^i{}_b + f^u{}_{iv} \sigma^i{}_a \tau^v{}_b = 0, \quad (5.37)$$

as follows easily from (3.40). Inspecting the expression of the primary field variation operator δ_t (cf. eqs. (5.2), (5.7)), we observe that it does depend on the splitting σ . Therefore, we cannot expect the resulting variation of the action S_t to be given by an expression analogous to (5.33). However, using directly the explicit expression of S_t (cf. eqs. (5.1), (5.6)) and recalling that of the secondary field variation operator s_t (cf. eqs. (5.4), (5.9)), we find that

$$\Delta_{\tau} S_t = s_t \Delta_{\tau} \Upsilon_t, \quad (5.38)$$

where $\Delta_{\tau} \Upsilon_t$ is given by

$$\Delta_{\tau} \Upsilon_A = \int_{\Sigma} d^2 z \lambda^*{}_{uz} \left(\tau^u{}_a(x) \bar{\partial}_{\bar{z}} x^a - i \partial_{\bar{a}} \tau^u{}_b \rho^{\bar{a}}{}_{\bar{p}} \rho^b{}_q(x) \bar{\chi}^{\bar{p}} \chi^q{}_{\bar{z}} \right), \quad (5.39)$$

$$\Delta_{\tau} \Upsilon_B = \int_{\Sigma} d^2 z \lambda^*{}_{uz} \left(\tau^u{}_a(x) \bar{\partial}_{\bar{z}} x^a - i \partial_{\bar{a}} \tau^u{}_b \rho^{\bar{a}}{}_{\bar{p}} \rho^b{}_q(x) \bar{\chi}^{\bar{p}}{}_{\bar{z}} \chi^q \right). \quad (5.40)$$

We discover in this way that, though $\Delta_\tau S_t$ is not δ_t -exact, it is s_t -exact.

All this may look a little bit puzzling. To shed light on this matter, we begin with noticing that, for a variation h of the metric, one has

$$s_t \Delta_h \Psi_t = 0. \quad (5.41)$$

Therefore, (5.33) could be cast as

$$\Delta_h S_t = (\delta_t + s_t) \Delta_h \Psi_t + \eta_t \Delta_h I_{\text{top}}. \quad (5.42)$$

Next, for a variation τ of the splitting, one has

$$\delta_t \Delta_\tau \Upsilon_t \simeq 0, \quad (5.43)$$

in virtue of (5.37). Thus, (5.33) could be cast as

$$\Delta_\tau S_t \simeq (\delta_t + s_t) \Delta_\tau \Upsilon_t. \quad (5.44)$$

We reach therefore the following conclusions.

Under a variation of the Kaehler metric g , the variation of the action S_t minus the topological term I_{top} is both δ_t - and $\delta_t + s_t$ -exact. Conversely, under a variation of the splitting σ compatible with the $(0, 2)$ supersymmetry condition (4.6), the variation of the action S_t is only $\delta_t + s_t$ -exact.

As far as we can see, the variation of the action S_t resulting from the variation of the Lie bracket $[\cdot, \cdot]_E$, the anchor ρ_E and the Lie algebroid complex structure (J_E, J_{EM}) is neither δ_t - nor $\delta_t + s_t$ -exact. S_t depends non trivially on those target space fields. We shall come back to this point in sect. 7.

5.6 Perturbative renormalization and conformal invariance

In sigma model perturbation theory, at 1-loop, the classical action S_t of the half-topological sigma models undergoes a renormalization of the form

$$\Delta S_t = \delta_t \Delta \Psi_t, \quad (5.45)$$

where $\Delta\Psi_t$ is the renormalization suffered by the gauge fermion Ψ_t ,

$$\Delta\Psi_A = \int_{\Sigma} d^2z \left(-\kappa_{A1} R_{\bar{a}b} \rho^b{}_q(x) \chi^q{}_{\bar{z}} \partial_z x^{\bar{a}} + i\kappa_{A2} f^u{}_{wv} g^{\bar{a}b} G^w{}_{\bar{a}b}(x) \lambda^*{}_{uz} l^v{}_{\bar{z}} \right), \quad (5.46)$$

$$\Delta\Psi_B = \int_{\Sigma} d^2z \left(-\kappa_{B1} R_{\bar{a}b} \rho^{\bar{a}}{}_{\bar{p}}(x) \bar{\chi}^{\bar{p}}{}_{\bar{z}} \partial_z x^b + i\kappa_{B2} f^u{}_{wv} g^{\bar{a}b} G^w{}_{\bar{a}b}(x) l^*{}_{uz} \lambda^v \right), \quad (5.47)$$

$R_{\bar{a}b}$ being the Ricci tensor of the metric $g_{\bar{a}b}$. The κ_{ti} are certain cut-off dependent constants. Therefore, ΔS_t is δ_t -exact.

The renormalization $\Delta\Psi_t$ of the gauge fermion is s_t -closed,

$$s_t \Delta\Psi_t \simeq 0, \quad (5.48)$$

as is straightforward to verify using (5.4), (5.9). Therefore, the renormalization ΔS_t could be also expressed as

$$\Delta S_t \simeq (\delta_t + s_t) \Delta\Psi_t. \quad (5.49)$$

Consequently, ΔS_t is $\delta_t + s_t$ -exact as well.

Because of (5.16c), the half-topological sigma models are conformally invariant at the classical level. The cut-off dependence of the κ_{ti} indicates however that conformal invariance is generally broken at the quantum level. Conformal invariance is preserved quantum mechanically only if $R_{\bar{a}b} = 0$, i. e. when $g_{\bar{a}b}$ is Calabi-Yau, and the splitting $\sigma^i{}_a$ satisfies the appropriate version of the Uhlenbeck-Yau equation, namely

$$f^u{}_{wv} g^{\bar{a}b} G^w{}_{\bar{a}b} = 0 \quad (5.50)$$

(cf. eq. (3.40)), as usual. Even if these restrictions on the target space geometry do not hold, the renormalization of the classical action remains δ_t - as well as $\delta_t + s_t$ -exact. We thus expect conformal invariance to continue to hold at the level of BRST cohomology in the appropriate sense to be defined.

The above conclusion is based on perturbation theory. Its validity is therefore restricted to the perturbative regime where the effects of world-sheet instantons

are neglected. In fact, conformal invariance can be broken by world-sheet instantons non perturbatively [7]. The following analysis assumes conformal invariance and thus it holds only perturbatively.

5.7 BRST charges

Having clarified the role of the two BRST cohomological structures of the half-topological sigma models at the classical and perturbative quantum level, let us now consider their quantum operator realization.

In the quantum theory, the field variation operator δ_t becomes a BRST charge Q_t ⁶ characterized by the property that, for any local operator \mathcal{O}

$$[Q_t, \mathcal{O}] = \delta_t \mathcal{O}, \quad (5.51)$$

in the quasiclassical limit, $[\cdot, \cdot]$ denoting here and below a graded commutator. Similarly, the field variation operator s_t becomes a BRST charge Λ_t such that

$$[\Lambda_t, \mathcal{O}] = s_t \mathcal{O} \quad (5.52)$$

again quasiclassically.

As δ_t is nilpotent (cf. eq.(5.11a)), so should be the associated charge Q_t

$$Q_t^2 = 0. \quad (5.53)$$

Likewise, the nilpotence of s_t (cf. eq.(5.11b)) leads to the nilpotence of Λ_t

$$\Lambda_t^2 = 0. \quad (5.54)$$

As δ_t and s_t anticommute (cf. eq.(5.11c)), so should their associated charges,

$$[Q_t, \Lambda_t] = 0. \quad (5.55)$$

⁶ In the string theory literature, Q_A, Q_B are generally denoted by \bar{Q}_+, Q_+ , respectively.

By (5.51), (5.52), the BRST charges corresponding to the two BRST field variation operators δ_t , $\delta_t + s_t$ of the sigma models at the quantum level are Q_t , $Q_t + \Lambda_t$. By (5.53)–(5.55), they are both nilpotent as required.

5.8 Operator BRST cohomology and chiral algebra

At the quantum level, the half-topological sigma models are characterized by their operator BRST cohomology. Since, in the present case, we have two BRST structures whose treatment is totally analogous, it is convenient to employ a uniform notation. We denote the two topological field variation operators δ_t , $\delta_t + s_t$ as $\delta^{(1)}_t$, $\delta^{(2)}_t$ and the associated BRST charges by Q_t , $Q_t + \Lambda_t$ as $Q^{(1)}_t$, $Q^{(2)}_t$, respectively.

The basic property of the charge $Q^{(\nu)}_t$ is its nilpotence,

$$Q^{(\nu)}_t{}^2 = 0. \quad (5.56)$$

Because of it, for any local operator \mathcal{O} , we have

$$[Q^{(\nu)}_t, [Q^{(\nu)}_t, \mathcal{O}]] = 0. \quad (5.57)$$

Consequently, with $Q^{(\nu)}_t$, there is associated an operator BRST cohomology. A local operator \mathcal{O} is $Q^{(\nu)}_t$ -closed if

$$[Q^{(\nu)}_t, \mathcal{O}] = 0. \quad (5.58)$$

A $Q^{(\nu)}_t$ -closed local operator \mathcal{O} is $Q^{(\nu)}_t$ -exact if

$$\mathcal{O} = [Q^{(\nu)}_t, \mathcal{X}]. \quad (5.59)$$

for some local operator \mathcal{X} . The *operator BRST cohomology* is just the operator $Q^{(\nu)}_t$ -cohomology, that is the quotient of the algebra of $Q^{(\nu)}_t$ -closed local operators by the ideal of $Q^{(\nu)}_t$ -exact ones.

The *topological operators* \mathcal{O} are precisely the $Q^{(\nu)}_t$ -closed ones. The quantum correlators of topological operators depend only on their $Q^{(\nu)}_t$ -cohomology classes, as $\langle [Q^{(\nu)}_t, \mathcal{X}] \rangle = 0$ for any operator \mathcal{X} . Thus, inside correlators, we have

$$\mathcal{O} \approx \mathcal{O} + [Q^{(\nu)}_t, \mathcal{X}]. \quad (5.60)$$

We have seen above that the classical half-topological sigma models are conformally invariant; $T_{z\bar{z}}$ vanishes (cf. eq. (5.16c)). Further, T_{zz} is $\delta^{(\nu)}_t$ -closed (cf. eqs. (5.19), (5.30)) and $T_{\bar{z}\bar{z}}$ is $\delta^{(\nu)}_t$ -exact (cf. eqs. (5.20), (5.31)). These properties do not automatically extend to the quantum sigma models.

The $\delta^{(\nu)}_t$ -exactness of the renormalization of the classical action (cf. eqs. (5.45), (5.49)) indicates that conformal invariance holds in the quantum theory at the level of $Q^{(\nu)}_t$ -cohomology. This statement needs to be qualified [7, 10, 17].

In a quantum conformal field theory with central charge c , it is more natural to employ, in place of the covariant stress tensor T used so far, the conformal stress tensor T^c . The components of the two are related as

$$T^c_{zz} = T_{zz} - \frac{c}{12\pi} R_{zz}, \quad (5.61a)$$

$$T^c_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}} - \frac{c}{12\pi} R_{\bar{z}\bar{z}}, \quad (5.61b)$$

$$T^c_{z\bar{z}} = T_{z\bar{z}} - \frac{c}{12\pi} R_{z\bar{z}}, \quad (5.61c)$$

where R_{zz} , $R_{\bar{z}\bar{z}}$ and $R_{z\bar{z}}$ are the components of the projective connection and the Ricci tensor of the world-sheet metric $h_{z\bar{z}}$ ⁷. In what follows, we assume that $h_{z\bar{z}}$ has constant Ricci scalar⁸. Conformal invariance implies that $T^c_{z\bar{z}} = 0$ and that

⁷ We recall that $R_{zz} = \partial_z \Gamma^z_{zz} - \frac{1}{2}(\Gamma^z_{zz})^2$, $R_{\bar{z}\bar{z}} = \bar{\partial}_{\bar{z}} \Gamma^{\bar{z}}_{\bar{z}\bar{z}} - \frac{1}{2}(\Gamma^{\bar{z}}_{\bar{z}\bar{z}})^2$ and $R_{z\bar{z}} = -\bar{\partial}_{\bar{z}} \Gamma^z_{z\bar{z}} = -\partial_z \Gamma^{\bar{z}}_{\bar{z}z}$ where $\Gamma^z_{zz} = \partial_z \ln h_{z\bar{z}}$, $\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \bar{\partial}_{\bar{z}} \ln h_{z\bar{z}}$ are the components of the Levi-Civita connection. $R_{z\bar{z}}$ can be expressed as $R_{z\bar{z}} = \frac{1}{2} R h_{z\bar{z}}$, where R is the Ricci scalar.

⁸ According to the uniformization theorem, this is always possible. Upon doing so, we have $\bar{\partial}_{\bar{z}} R_{zz} = 0$, $\partial_z R_{\bar{z}\bar{z}} = 0$.

$$\bar{\partial}_{\bar{z}} T^c_{zz} = 0, \partial_z T^c_{\bar{z}\bar{z}} = 0.$$

In the half-topological sigma models, which are conformally invariant only at the level of $Q^{(\nu)}_t$ -cohomology, weaker conditions hold. The sigma models are still characterized by their central charge c_t , as any ordinary conformal field theory. However, $T^c_{t\bar{z}\bar{z}}$ vanishes only up to a $Q^{(\nu)}_t$ -exact term. So, we have

$$T^c_{t\bar{z}\bar{z}} = [Q^{(\nu)}_t, G^{(\nu)}_{t\bar{z}\bar{z}}], \quad (5.62)$$

for some operator $G^{(\nu)}_{t\bar{z}\bar{z}}$. Similarly, T^c_{tzz} is holomorphic and $T^c_{t\bar{z}\bar{z}}$ is antiholomorphic only up to a $Q^{(\nu)}_t$ -exact term. We find indeed the relations

$$\bar{\partial}_{\bar{z}} T^c_{tzz} = -[Q^{(\nu)}_t, \nabla_z G^{(\nu)}_{t\bar{z}\bar{z}}], \quad \partial_z T^c_{t\bar{z}\bar{z}} = -[Q^{(\nu)}_t, \bar{\nabla}_{\bar{z}} G^{(\nu)}_{t\bar{z}\bar{z}}], \quad (5.63)$$

which follow from (5.62) and the covariant stress energy conservation equations, $\bar{\nabla}_{\bar{z}} T_{tzz} + \nabla_z T_{t\bar{z}\bar{z}} = 0$, $\nabla_z T_{t\bar{z}\bar{z}} + \bar{\nabla}_{\bar{z}} T_{tzz} = 0$.

In the classical theory, the component $T_{t\bar{z}\bar{z}}$ of the stress tensor is $\delta^{(\nu)}_t$ -exact (cf. eqs. (5.20), (5.31)). In the quantum theory, an analogous property holds. The component $T^c_{t\bar{z}\bar{z}}$ of the stress tensor is $Q^{(\nu)}_t$ -exact up to a central term. More precisely, we have

$$T^c_{t\bar{z}\bar{z}} + \frac{c_t}{12\pi} R_{\bar{z}\bar{z}} = [Q^{(\nu)}_t, G^{(\nu)}_{t\bar{z}\bar{z}}], \quad (5.64)$$

for some operator $G^{(\nu)}_{t\bar{z}\bar{z}}$ ⁹. The validity of (5.64) can be argued as follows. We first note that the left hand side of (5.64) is just the component $T_{t\bar{z}\bar{z}}$ of the covariant stress tensor. We then recall that an infinitesimal deformation of the complex structure of the world-sheet Σ , corresponding to the infinitesimal Beltrami differentials $(\mu^z_{\bar{z}}, \bar{\mu}^{\bar{z}}_z)$, results in an insertion of the form $\int_{\Sigma} d^2z (\mu^z_{\bar{z}} T_{tzz} + \bar{\mu}^{\bar{z}}_z T_{t\bar{z}\bar{z}})$ inside correlators. Now, in the quantum theory, the action undergoes

⁹ The central term $\frac{c_t}{12\pi} R_{\bar{z}\bar{z}}$ in the left hand side ensures that the latter transforms as a quadratic differential under a change of world-sheet coordinates as the left hand side does.

a 1-loop renormalization ΔS_t of the form (5.45), (5.49), which keeps it $\delta^{(\nu)}_t$ -exact up to a topological term. Under the deformation, the Beltrami differential component $\bar{\mu}^{\bar{z}}_z$ enters only in the variation of $\Delta\Psi_t$ but not in that of $\delta^{(\nu)}_t$, as is easy to realize inspecting the expressions of $\delta^{(\nu)}_t$ (cf. eqs. (5.2), (5.7), (5.4), (5.9)) and of $\Delta\Psi_t$ (cf. eqs. (5.46), (5.47)). $\bar{\mu}^{\bar{z}}_z$ therefore couples to a $\delta^{(\nu)}_t$ -exact object indicating that (5.64) must hold quantum mechanically.

In the classical theory, the component T_{tzz} of the stress tensor is $\delta^{(\nu)}_t$ -closed (cf. eqs. (5.19), (5.30)). However, we cannot expect this property to extend at the quantum level as for $T_{t\bar{z}\bar{z}}$. T^c_{tzz} is not $Q^{(\nu)}_t$ -closed in general. This can be seen by an argument similar to the one expounded in the previous paragraph. Under a deformation of the world-sheet complex structure, the Beltrami differential component $\mu^z_{\bar{z}}$ enters in the variation of $\delta^{(\nu)}_t$ in the 1-loop renormalization ΔS_t of the action and, so, it does not necessarily couple to a $\delta^{(\nu)}_t$ -closed object. If, however, this does nevertheless happens, then T^c_{tzz} will be $Q^{(\nu)}_t$ -closed. This is the case in particular when ΔS_t vanishes, i. e. when $g_{\bar{a}b}$ is Calabi-Yau and the splitting σ satisfies the Uhlenbeck-Yau equation (5.50) and the half-topological sigma models are strictly conformal quantum mechanically. Weaker conditions on the target space geometry may however suffice.

Suppose now that the world-sheet Σ is the standard flat punctured complex plane $\setminus \{0\}$. Since, by (5.63), T^c_{tzz} , $T^c_{t\bar{z}\bar{z}}$ are (anti)holomorphic up to a $Q^{(\nu)}_t$ -exact term, we can expand $T^c_{t\bar{z}\bar{z}}$, T^c_{tzz} in Laurent series in the usual way, $T^c_{tzz} = \sum_n L_{tn} z^{-n-2}$, $T^c_{t\bar{z}\bar{z}} = \sum_n \bar{L}_{tn} \bar{z}^{-n-2}$. The Laurent modes L_{tn} , \bar{L}_{tn} however are defined only up to a $Q^{(\nu)}_t$ -exact ambiguity.

As $T^c_{t\bar{z}\bar{z}}$ is $Q^{(\nu)}_t$ -exact up to a central term, by (5.64), the modes \bar{L}_{tn} also are. Their adjoint action thus descends on the operator $Q^{(\nu)}_t$ -cohomology and is in fact trivial¹⁰. Conversely, as T^c_{tzz} is not $Q^{(\nu)}_t$ -closed in general, the modes L_{tn}

¹⁰ If T is a $Q^{(\nu)}_t$ -closed operator, then the adjoint action of T is defined on the operator

also are not. Consequently, their adjoint action is not defined on the operator $Q^{(\nu)}_t$ -cohomology. However, translation invariance guarantees that ∂_z commutes with $Q^{(\nu)}_t$. As $L_{t-1} = \partial_z$, we have $[Q^{(\nu)}_t, L_{t-1}] = 0$. Moreover, as $Q^{(\nu)}_t$ is a scalar, it commutes with the spin operator S_t . Since $S_t = L_{t0} - \bar{L}_{t0}$ and \bar{L}_{t0} is $Q^{(\nu)}_t$ -exact, we have $[Q^{(\nu)}_t, L_{t0}] = 0$. Hence, the adjoint action of L_{t-1} and L_{t0} still descends on the operator $Q^{(\nu)}_t$ -cohomology [17]. When T^c_{tzz} happens to be $Q^{(\nu)}_t$ -closed, all the modes L_{tn} also are and their adjoint action is thus defined on the operator $Q^{(\nu)}_t$ -cohomology.

As observed in [7], these properties have important consequences. If \mathcal{O} is any $Q^{(\nu)}_t$ -closed operator, then $[\bar{L}_{t0}, \mathcal{O}]$ and $\bar{\partial}_z \mathcal{O} = [\bar{L}_{t-1}, \mathcal{O}]$ are $Q^{(\nu)}_t$ -exact. Therefore, the operator $Q^{(\nu)}_t$ -cohomology consists of antiholomorphic scaling dimension 0 holomorphic operator classes. So, if \mathcal{O}_α is a basis of $Q^{(\nu)}_t$ -closed operators modulo $Q^{(\nu)}_t$ -exact ones representing the $Q^{(\nu)}_t$ -cohomology, the coefficients $c_{\alpha\beta\gamma}(z)$ entering in the operator product expansion

$$\mathcal{O}_\alpha(z)\mathcal{O}_\beta(0) \approx \sum_\gamma c_{\alpha\beta\gamma}(z)\mathcal{O}_\gamma(0) \quad (5.65)$$

are in fact all meromorphic. The holomorphic operator $Q^{(\nu)}_t$ -cohomology, equipped with the operator product expansion structure (5.65), constitutes the *chiral algebra* $\mathcal{A}^{(\nu)}_t$ of the half-topological sigma model.

Since L_{t-1} and L_{t0} act on the operator $Q^{(\nu)}_t$ -cohomology, $\mathcal{A}^{(\nu)}_t$ carries a natural action of ∂_z and is further graded according to holomorphic scaling dimension.

The spectrum of holomorphic scaling dimensions h of $\mathcal{A}^{(\nu)}_t$ consists of non negative integers. This can be shown as follows. At the classical level, operator

$Q^{(\nu)}_t$ -cohomology. Indeed, as $[Q^{(\nu)}_t, T] = 0$, $[Q^{(\nu)}_t, [T, \mathcal{O}]] = 0$ for any operator \mathcal{O} such that $[Q^{(\nu)}_t, \mathcal{O}] = 0$ and $[T, \mathcal{O}] = \pm[Q^{(\nu)}_t, [T, \mathcal{X}]]$ whenever $\mathcal{O} = [Q^{(\nu)}_t, \mathcal{X}]$. Further, if T is a $Q^{(\nu)}_t$ -exact operator, then the adjoint action of T on the operator $Q^{(\nu)}_t$ -cohomology is trivial. Indeed, $T = [Q^{(\nu)}_t, V]$ for some operator V and, so, $[T, \mathcal{O}] = [Q^{(\nu)}_t, [V, \mathcal{O}]]$ for any operator \mathcal{O} such that $[Q^{(\nu)}_t, \mathcal{O}] = 0$.

cohomology classes of $\mathcal{A}^{(\nu)}_t$ with non vanishing h can be constructed using exclusively the field λ^*_{uz} and the ∂_z derivatives of the fields x^a , $x^{\bar{a}}$ and λ^*_{uz} . (The ∂_z derivatives of the fields χ^p , $\bar{\chi}^{\bar{p}}$ as well as the fields l^u , l^*_u can be eliminated using the field equations.) The resulting classes have non negative integer h . Their spin is therefore $s = h$, since $s = h - \bar{h}$, where \bar{h} is the antiholomorphic scaling dimension of the classes, which always vanishes. At the quantum level, there are consequently no perturbative quantum corrections of the values of h , since the spin $s = h$ must remain integer.

For dimensional reasons, the coefficients $c_{\alpha\beta}^\gamma(z)$ entering in the operator product expansion (5.65) have the form

$$c_{\alpha\beta}^\gamma(z) = \frac{f_{\alpha\beta}^\gamma}{z^{h_\alpha+h_\beta-h_\gamma}}, \quad (5.66)$$

where $f_{\alpha\beta}^\gamma$ is a constants and h_ι is the scaling dimension of \mathcal{O}_ι .

Since $h_\iota \geq 0$, the operator classes \mathcal{O}_α with $h_\alpha = 0$ have a non singular operator product expansion. When inserted in the same point they therefore form a ring $\mathcal{R}^{(\nu)}_t$, the *chiral ring* of the half-topological model sigma model, the heterotic counterpart of the chiral ring of (2, 2) theories. Correlators of chiral ring classes are holomorphic and hence constant when the world-sheet Σ is compact.

$\mathcal{A}^{(\nu)}_t$ is graded also by R and flavour charge. In the operator product expansion (5.65), the R and flavour charges of the two sides must match. Then, in (5.66), $f_{\alpha\beta}^\gamma = 0$ unless $q_{R\alpha} + q_{R\beta} - q_{R\gamma} = q_{L\alpha} + q_{L\beta} - q_{L\gamma} = 0$, where $q_{R\iota}$, $q_{L\iota}$ are the R /flavour charges of \mathcal{O}_ι .

5.9 Topological twist and anomalies

At the quantum level, the half-topological sigma models worked out above must satisfy the same consistency requirement as the parent heterotic sigma model, namely that the fermion determinants combine to yield a function on the space of all embeddings x or, equivalently, that the associated determinant

line bundle \mathcal{L} be trivial (cf. subsect. 4.2). In the half-topological sigma models, this is achieved if the complex vector bundles T_M^+ , $\ker \rho_{E^+}$ satisfy the condition (4.11) relating their Chern characters and a further condition, viz

$$\frac{1}{2}c_1(T_\Sigma^+)((2\bar{s} - 1)c_1(T_M^+) - (2s - 1)c_1(\ker \rho_{E^+})) = 0 \quad (5.67)$$

¹¹, where \bar{s} and $1 - s$ are the spins of the fermion fields χ and λ (that is, from the world-sheet point of view, χ and λ are sections of $\overline{K}_\Sigma^{\bar{s}}$ and K_Σ^{1-s} , respectively). We note that (5.67) is automatically verified in the untwisted case, where $s, \bar{s} = 1/2$, but not so in general in the twisted ones, where $s, \bar{s} \neq 1/2$. Now, $s = 1, \bar{s} = 1$, for the A twist, and $s = 1, \bar{s} = 0$, for the B twist. Then, (5.67) becomes

$$-\frac{1}{2}c_1(T_\Sigma^+)(\eta_t c_1(T_M^+) + c_1(\ker \rho_{E^+})) = 0, \quad (5.68)$$

where $\eta_A = -1, \eta_B = 1$, as before.

In order (5.68) to be satisfied, it suffices that either $c_1(T_\Sigma^+) = 0$ or $\eta_t c_1(T_M^+) + c_1(\ker \rho_{E^+}) = 0$. Assume that the latter condition obtains. Then, on account of (4.11), the 1st and 2nd Chern classes of the vector bundles $T_M^+, \ker \rho_{E^+}$ satisfy

$$-\eta_t c_1(T_M^+) = c_1(\ker \rho_{E^+}), \quad (5.69)$$

$$c_2(T_M^+) = c_2(\ker \rho_{E^+}). \quad (5.70)$$

Let us see how these topological restrictions affect the Lie algebroid E^+ .

Since the Lie algebroid E^+ is transitive, it fits into the Atiyah sequence

$$0 \longrightarrow \ker \rho_{E^+} \xrightarrow{\iota_{E^+}} E^+ \xrightarrow{\rho_{E^+}} T_M^+ \longrightarrow 0 \quad (5.71)$$

(cf. eq. (2.7)), which implies the relation $c(E^+) = c(\ker \rho_{E^+})c(T_M^+)$ among total Chern classes. Using (5.69), (5.70), we find

¹¹ Here, $T_\Sigma^+, T_M^+, \ker \rho_{E^+}$ are shorthands for the box product vector bundles over $\Sigma \times M$ $T_\Sigma^+ \boxtimes 1_M, 1_\Sigma \boxtimes T_M^+, 1_\Sigma \boxtimes \ker \rho_{E^+}$, respectively.

$$c_1(E^+) = (1 - \eta_t)c_1(T_M^+), \quad (5.72)$$

$$c_2(E^+) = 2c_2(T_M^+) - \eta_t c_1(T_M^+)^2. \quad (5.73)$$

By (5.72), the cohomology class $c_1(E^+)$ is even for the A model and vanishes for the B model. If $E = T_M$, then necessarily $c_i(E^+) = 0$, $i = 1, 2$. In that case, further, M is Calabi–Yau.

At the de Rham cohomology level, condition (5.69) is required by the consistency of the twist of the heterotic sigma model. As we recalled at the beginning of this section, twisting amounts to altering the covariance of the sigma model fields depending on their R and flavour charges. In the sigma model, twisting is implemented by coupling the world-sheet spin connection to the appropriate linear combination J_{tw} of the R and a flavour currents J_R, J_L . With s and \bar{s} defined as earlier, we have

$$J_{\text{tw}} = (2\bar{s} - 1)J_R + (1 - 2s)J_L. \quad (5.74)$$

In our case, J_{tw} turns out to be

$$J_t = -\frac{1}{2}(\eta_t J_R + J_L). \quad (5.75)$$

To ensure that the quantum twisted fields have the same covariance as their classical counterparts, the current J_t must be non-anomalous, i. e. the corresponding vacuum background charge must vanish,

$$\Delta q_t = -\frac{1}{2}(\eta_t \Delta q_R + \Delta q_L) = 0. \quad (5.76)$$

Using (4.12), (4.13), (5.76) can be cast compactly as

$$-\frac{1}{2} \int_{\Sigma} (\eta_t x^* c_1(T_M^+) + x^* c_1(\ker \rho_{E^+})) = 0. \quad (5.77)$$

(5.69) is a sufficient condition for (5.77) to be satisfied.

While we require that the twisted general covariance is non-anomalous, the

global R and flavour symmetries still could, and generally will, suffer an anomaly after twisting. The vacuum background R and flavour charges, with s and \bar{s} defined as above, are given by

$$\Delta q_{R\bar{s}} = d(1 - 2\bar{s})(\ell - 1) + \int_{\Sigma} x^* c_1(T_M^+), \quad (5.78)$$

$$\Delta q_{Ls} = r(1 - 2s)(\ell - 1) + \int_{\Sigma} x^* c_1(\ker \rho_{E^+}), \quad (5.79)$$

where ℓ is the genus of Σ and $d = \text{rank } T_M^+$, $r = \text{rank } \ker \rho_{E^+}$. In the untwisted case, where $s, \bar{s} = 1/2$, we recover (4.12), (4.13). In the twisted cases, taking into account that we have conventionally redefined the sign of the R charge in the B model, we find the expressions

$$\Delta q_{Rt} = -d(\ell - 1) - \eta_t \Delta q_R, \quad (5.80)$$

$$\Delta q_{Lt} = -r(\ell - 1) + \Delta q_L, \quad (5.81)$$

where $\Delta q_R, \Delta q_L$ given by (4.12), (4.13). Again, quantum sigma model correlators can be non zero only if these vacuum charges are soaked up by those of the inserted operators.

5.10 [Algebra–geometric conditions](#)

There are certain algebra–geometric conditions which must be added to the topological anomaly cancellation conditions (4.11) and (5.68) of the half–topological sigma models. They ensure the existence of a natural trace on the relevant chiral ring which, in turn, is required for the consistent computation of correlators of chiral ring operator classes. We refer the reader to ref. [16] (see also ref. [17]) for a detailed discussion of this point. Here, we shall limit ourselves to outline the argument.

Let us consider for definiteness the string tree level $\ell = 0$. In the perturbative regime, assumed in our analysis, world–sheet instantons do not contribute and

the sigma model path integral localizes at constant embeddings $x \in \text{Map}(\Sigma, M)$. The moduli space of constant embeddings is just the target space M . Thus, the quantum correlator $\langle \mathcal{O} \rangle$ of any chiral ring element \mathcal{O} should be expressible as an integral over M of a suitable top form $\omega_{\mathcal{O}}$ depending on \mathcal{O} .

$$\langle \mathcal{O} \rangle = \int_M \omega_{\mathcal{O}}. \quad (5.82)$$

To ensure the convergence of the integral, we require M to be compact. Clearly, \mathcal{O} determines only the de Rham cohomology class $[\omega_{\mathcal{O}}]_{\text{dR}}$ of $\omega_{\mathcal{O}}$.

From (5.80), (5.81), in the situation considered, the vacuum background R and flavor charges are $\Delta q_{Rt} = d$ and $\Delta q_{Lt} = r$. Therefore, the correlator $\langle \mathcal{O} \rangle$ can be non vanishing only if the R and flavour charges of \mathcal{O} have the values d and r , respectively. Correspondingly, the class $[\omega_{\mathcal{O}}]_{\text{dR}}$ is trivial unless this condition is met.

The algebro-geometric conditions mentioned at beginning of this subsection are sufficient conditions for the existence of the appropriate correspondence $\mathcal{O} \mapsto [\omega_{\mathcal{O}}]_{\text{dR}}$. They take the form of isomorphism relations between certain (anti)holomorphic line bundles (denoted below by \cong).

For the type A sigma model, there is just one condition, viz

$$\wedge^r \ker \rho_{\mathcal{E}}^* \cong \wedge^d \mathcal{T}_{\mathcal{M}}^*. \quad (5.83)$$

For the type B sigma model, the statement of the conditions is slightly more involved. To begin with, we recall that the target space geometry involves a splitting σ of E satisfying (3.15) and whose curvature $G_{E\sigma}{}^c$ has vanishing $(2, 0)$ component (cf. eq. (4.6)). Let $\mathcal{D}_{E\sigma}^{\text{ad}}$ be the ordinary $(1, 0)$ connection of $\ker \rho_{E^+}$ obtained from the connection $D_{E\sigma}^{\text{ad}}$ by complexification and restriction to $\Gamma(\ker \rho_{E^+})$ (cf. subsect. 2.3). By relation (2.17), being $G_{E\sigma}{}^{2,0} = 0$, $\mathcal{D}_{E\sigma}^{\text{ad}}$ has vanishing $(2, 0)$ curvature. $\mathcal{D}_{E\sigma}^{\text{ad}}$ therefore defines an antiholomorphic structure on the complex vector bundle $\ker \rho_{E^+}$ making it an antiholomorphic vector bundle $\mathcal{K}_{E^+\sigma}$ over \mathcal{M} .

The first condition then reads

$$\wedge^r \mathcal{K}_{E+\sigma}^* \cong 1_{\mathcal{M}}. \quad (5.84)$$

The second condition requires that \mathcal{M} is a Calabi–Yau manifold,

$$\wedge^d \mathcal{T}_{\mathcal{M}}^* \cong 1_{\mathcal{M}}. \quad (5.85)$$

For the type A sigma model, (5.83) implies that

$$c_1(T_M^+) = c_1(\ker \rho_{E+}). \quad (5.86)$$

(5.86) is just (5.69). For the type B sigma model, (5.84), (5.85) together lead to

$$c_1(T_M^+) = c_1(\ker \rho_{E+}) = 0. \quad (5.87)$$

(5.87) is compatible with but more restrictive than (5.69).

For the second BRST structure of the half-twisted sigma models, there is a further condition: the typical fiber \mathfrak{g}^+ of the Lie algebra bundle $\ker \rho_{E+}$ must be *unimodular*, that is $\text{tr ad } X = 0$ for any $X \in \mathfrak{g}^+$. Explicitly, this can be phrased as a simple condition on the structure functions $f^u{}_{vw}$, namely

$$f^v{}_{uv} = 0. \quad (5.88)$$

For the B model, the unimodularity condition has another important consequence: unlike $\mathcal{K}_{E+\sigma}$, the antiholomorphic line bundle $\wedge^r \mathcal{K}_{E+\sigma}^*$ is actually independent from the splitting σ ¹². Hence, so is also condition (5.84).

The justification of the restrictions listed above will be provided in sect. 6 below, once the structure of the sigma model chiral rings will have been unveiled.

¹² The $(1,0)$ connection induced by $\mathcal{D}_{E\sigma}^{\text{ad}}$ on $\wedge^r \ker_{E+}$ is given locally by $\partial_a - f^v{}_{iv} \sigma^i{}_a$. Under a variation τ of σ at fixed ρ_E (cf. subsect. 5.5), the variation of the induced connection is $-f^v{}_{uv} \tau^u{}_a$, which vanishes if (5.88) holds. It follows that the antiholomorphic structure of $\mathcal{K}_{E+\sigma}$ is indeed independent from σ .

6 Lie algebroid cohomology and chiral ring

6.1 Lie algebroid cohomology

In this section, we shall unveil the relation between the cohomology of certain Lie algebroids with complex base belonging to the target space geometry of the heterotic Lie algebroid sigma model and the BRST cohomology defining the chiral rings of the half-twisted sigma models in the perturbative quasiclassical regime. This will provide an elegant geometric interpretation of those rings.

A Lie algebroid E over M is characterized by its cohomology $H_{LA}^*(E)$. This is defined as follows.

Let $E[1]$ be the parity shifted form of E . In each trivializing neighborhood, $E[1]$ is coordinatized by base coordinates x^A of degree 0 and fiber coordinates ξ^I of degree 1. The algebra of functions of $E[1]$, $\text{Fun}(E[1])$, is then graded according to the polynomial degree in the ξ^I .

There exists a canonical degree 1 vector field Q_E on $E[1]$ given locally by

$$Q_E = \rho^A_I \xi^I \partial_{x^A} - \frac{1}{2} f^I_{JK} \xi^J \xi^K \partial_{\xi^I}, \quad (6.1)$$

where ρ^A_I, f^I_{JK} are the anchor and bracket structure functions of E with respect to a local frame $\{e_I\}$ ¹³. From (2.1)–(2.3), it follows that Q_E is homological,

$$Q_E^2 = 0. \quad (6.2)$$

The Lie algebroid cohomology $H_{LA}^*(E)$ is then the cohomology of the complex $(\text{Fun}(E[1]), Q_E)$. When $E = T_M$, it reduces to the customary de Rham cohomology $H_{d_M}^*(M)$ of M , as is immediate to see. So, it can be considered a natural generalization of the latter.

The Lie algebroid cohomology $H_{LA}^*(W)$ of a complex Lie algebroid W is de-

¹³ The structure functions are defined as usual by the relations $\rho_E e_I = \rho^A_I \partial_A$ and $[e_I, e_J]_E = f^k_{IJ} e_K$ and obey relations formally identical to (3.35)–(3.37).

fined in the same fashion. If E is a Lie algebroid, then $H_{LA}^*(E^c)$ is the complexification of $H_{LA}^*(E)$.

6.2 Complex and holomorphic Lie algebroid cohomology

Let W be a Lie algebroid with complex base M (cf. subsect. 3.2). W is characterized by a Lie algebroid cohomology $H_{LA}^*(W)$ defined in an analogous fashion. One considers the parity shifted vector bundle $W[1]$ and coordinatizes it by degree 0 base coordinates $z^a, z^{\bar{a}}$ and degree 1 fiber coordinates ξ^i . $\text{Fun}(W[1])$ is then graded according to the polynomial degree in the ξ^i . Next, one constructs a canonical degree 1 holomorphic vector field on $W[1]$, viz

$$Q_W = \rho^a{}_i \xi^i \partial_{z^a} - \frac{1}{2} f^i{}_{jk} \xi^j \xi^k \partial_{\xi^i}, \quad (6.3)$$

with the anchor and bracket structure functions $\rho^a{}_i, f^i{}_{jk}$ taken with respect to a local frame $\{e_i\}$ of W . Then, due to (2.1)–(2.3), Q_W is homological,

$$Q_W^2 = 0. \quad (6.4)$$

$H_{LA}^*(W)$ is the cohomology of the complex $(\text{Fun}(W[1]), Q_W)$. When $W = T_M^+$, we obtain the holomorphic Dolbeault cohomology of M , $H_{\partial_M}^{*,0}(M)$.

Let \mathcal{E} be a holomorphic Lie algebroid on a complex manifold \mathcal{M} (cf. subsect. 3.1). Then, with \mathcal{E} , there is associated a holomorphic Lie algebroid cohomology $H_{LA}^*(\mathcal{E})$ as follows. Instead of a complex, we have a sheaf of complexes $(\text{Fun}(\mathcal{E}[1]), \mathcal{Q}_{\mathcal{E}})$, where $\text{Fun}(\mathcal{E}[1])$ is the sheaf such that, for any open set U of M , $\text{Fun}(\mathcal{E}[1])(U)$ is the algebra of holomorphic functions on $\mathcal{E}|_U[1]$ and $\mathcal{Q}_{\mathcal{E}}$ is defined according to (6.3) using a holomorphic frame $\{e_i\}$. The Lie algebroid cohomology $H_{LA}^*(\mathcal{E})$ is the sheaf associated with the presheaf $H'_{LA}^*(\mathcal{E})$ such that, for any open set U of M , $H'_{LA}^*(\mathcal{E})(U)$ is the cohomology of $(\text{Fun}(\mathcal{E}[1])(U), \mathcal{Q}_{\mathcal{E}})$.

Suppose that E is a transitive Lie algebroid with complex structure satisfying (3.13), the case mainly treated in this paper. Then, E^+ is in particular a Lie

algebroid with complex base and, as such, it is equipped with its cohomology $H_{LA}^*(E^+)$.

Using (6.3), we can extend Q_{E^+} to a homological vector field on $E^c[1]$. However, while Q_{E^+} does not depend on the framing $\{e_i\}$ of E^+ used, the resulting extension does. There is nevertheless a canonical choice of the extension. Since E^+ has a canonical holomorphic structure making it a holomorphic Lie algebroid \mathcal{E} , we can use the associated holomorphic framing $\{e_i\}$. We denote the resulting extension by Q_E ¹⁴. The homological vector field Q_{E^c} of E^c splits as

$$Q_{E^c} = Q_E + \overline{Q}_E, \quad (6.5)$$

as is easy to check. The homological vector field $Q_{\mathcal{E}}$ of \mathcal{E} is simply the restriction of Q_E to $\text{Fun}(\mathcal{E}[1])$.

From now on, we assume that E is a transitive Lie algebroid with complex structure satisfying (3.13). Further, we pick a splitting σ of E satisfying condition (3.15) and whose curvature $G_{E\sigma}{}^c$ has vanishing $(2, 0)$ component (cf. eq. (4.6)).

6.3 The Lie algebroid \hat{E}^+ and its cohomology

Now, we are going to construct a distinguished Lie subalgebroid \hat{E}^+ of E^c , which is a Lie algebroid with complex base and whose cohomology turns out to be intimately related to the BRST cohomology defining the type A half-topological sigma model chiral ring.

Consider the complex vector bundle $\hat{E}^+{}_{\sigma} = \overline{T}_M^+ \oplus \ker \rho_{E^+}$. We endow $\hat{E}^+{}_{\sigma}$

¹⁴ A change of the smooth frame $\{e_i\}$ induces a change of fiber coordinates ξ^i , in which the new fiber coordinates depend holomorphically on the old ones but generally non holomorphically on the base coordinates. The derivatives ∂_a in the two frames differ so by terms proportional to $\partial_{\bar{i}}$. Therefore, the right hand side of (6.3) is covariant only up to terms of the latter form. These are inert on $\text{Fun}(E^+[1])$ but not on $\text{Fun}(E^c[1])$. If, however, we restrict to holomorphic frames, the $\partial_{\bar{i}}$ terms vanish and the right hand side of (6.3) is covariant.

with the anchor defined by

$$\rho_{\widehat{E}^+_\sigma}(\bar{x} \oplus u) = \bar{x}, \quad (6.6)$$

with $x \in \Gamma(T_M^+)$, $u \in \Gamma(\ker \rho_{E^+})$, and with the Lie bracket defined by

$$[\bar{x} \oplus u, \bar{y} \oplus v]_{\widehat{E}^+_\sigma} = \overline{[x, y]_{T_M^+}} \oplus ([\sigma^c \bar{x}, v]_{E^c} - [\sigma^c \bar{y}, u]_{E^c} + [u, v]_{E^c}), \quad (6.7)$$

with $x, y \in \Gamma(T_M^+)$, $u, v \in \Gamma(\ker \rho_{E^+})$. Condition (3.13) ensures that the right hand side of (6.7) belongs to $\Gamma(\widehat{E}^+_\sigma)$. Condition (4.6) is required by the fulfillment of the Jacobi identity. \widehat{E}^+_σ acquires in this way a structure of Lie algebroid with complex base, as is readily verified.

The bundle map $\widehat{\omega}_\sigma : \widehat{E}^+_\sigma \rightarrow E^c$ defined by $\widehat{\omega}_\sigma(\bar{x} \oplus u) = \sigma^c \bar{x} + u$ for $x \in \Gamma(T_M^+)$, $u \in \Gamma(\ker \rho_{E^+})$ is a monomorphism of complex Lie algebroids (cf. subsect. 2.1). Let \widehat{E}^+ be the image of $\widehat{\omega}_\sigma$ in E^c . Then, \widehat{E}^+ is a complex Lie subalgebroid of E^c and a Lie algebroid with complex base. Further, $\widehat{\omega}_\sigma : \widehat{E}^+_\sigma \rightarrow \widehat{E}^+$ is an isomorphism of Lie algebroids with complex base. From now on, we identify \widehat{E}^+_σ and \widehat{E}^+ leaving the isomorphism $\widehat{\omega}_\sigma$ understood.

Let us compute the homological vector field $Q_{\widehat{E}^+}$. As E^+ has a canonical holomorphic structure, there is a holomorphic framing $\{e_i\}$. Then, $\{\partial_{\bar{a}}\} \cup \{e_u\}$ constitutes a distinguished framing of \widehat{E}^+ . Let $\zeta^{\bar{a}}$, c^u be the corresponding fiber coordinates of $\widehat{E}^+[1]$. Then

$$Q_{\widehat{E}^+} = \zeta^{\bar{a}} \partial_{z\bar{a}} - \frac{1}{2} f^u{}_{vw} c^v c^w \partial_{cu}. \quad (6.8)$$

It is straightforward to check that $Q_{\widehat{E}^+}$ is nilpotent

$$Q_{\widehat{E}^+}^2 = 0, \quad (6.9)$$

as required.

Next, we observe that $Q_{\widehat{E}^+}$ decomposes as

$$Q_{\widehat{E}^+} = \widetilde{Q}_{\widehat{E}^+} + \Lambda_{\widehat{E}^+}, \quad (6.10)$$

where $\tilde{Q}_{\hat{E}^+}$, $\Lambda_{\hat{E}^+}$ are given by

$$\tilde{Q}_{\hat{E}^+} = \zeta^{\bar{a}} \partial_{z\bar{a}}, \quad (6.11)$$

$$\Lambda_{\hat{E}^+} = -\frac{1}{2} f^u{}_{vw} c^v c^w \partial_{cu}. \quad (6.12)$$

$\tilde{Q}_{\hat{E}^+}$, $\Lambda_{\hat{E}^+}$ are nilpotent and anticommute

$$\tilde{Q}_{\hat{E}^+}^2 = 0, \quad (6.13)$$

$$\Lambda_{\hat{E}^+}^2 = 0, \quad (6.14)$$

$$\tilde{Q}_{\hat{E}^+} \Lambda_{\hat{E}^+} + \Lambda_{\hat{E}^+} \tilde{Q}_{\hat{E}^+} = 0. \quad (6.15)$$

By (6.13), $(\text{Fun}(\hat{E}^+[1]), \tilde{Q}_{\hat{E}^+})$ is a complex. The algebra $\text{Fun}(\hat{E}^+[1])$ is isomorphic to the algebra of $\wedge^* \ker_{\mathcal{E}}^*$ -valued $(0, *)$ forms $\Omega^{0,*}(M, \wedge^* \ker_{\mathcal{E}}^*)$. Under the isomorphism, $\tilde{Q}_{\hat{E}^+}$ is identified with the standard Dolbeault operator $\bar{\partial}_{\wedge^* \ker_{\mathcal{E}}^*}$. The cohomology of $(\text{Fun}(\hat{E}^+[1]), \tilde{Q}_{\hat{E}^+})$ is thus isomorphic to the Dolbeault cohomology $H_{\bar{\partial}}^*(\wedge^* \ker_{\mathcal{E}}^*)$, that is the sheaf cohomology $H^*(\mathcal{O}_{\wedge^* \ker_{\mathcal{E}}^*})$.

We note that $\text{Fun}(\hat{E}^+[1])$ is bigraded according to the polynomial degree in $\zeta^{\bar{a}}$, c^u and that, by (6.13)–(6.15), $(\text{Fun}(\hat{E}^+[1]), \tilde{Q}_{\hat{E}^+}, \Lambda_{\hat{E}^+})$ is a double complex. By (6.10), the total cohomology of this latter is the cohomology of the complex $(\text{Fun}(\hat{E}^+[1]), Q_{\hat{E}^+})$, that is the Lie algebroid cohomology $H_{LA}^*(\hat{E}^+)$. This can therefore be computed, at least in principle, using standard spectral sequence methods ¹⁵.

¹⁵ $\ker \rho_{E^+}$ is a Lie algebroid with vanishing anchor. It has a canonical holomorphic structure induced by that of E^+ , with which there are associated a holomorphic framing $\{e_u\}$ and fiber coordinates c^u of $\ker \rho_{E^+}[1]$. The homological vector field $Q_{\ker \rho_{E^+}}$ is given by $Q_{\ker \rho_{E^+}} = -\frac{1}{2} f^w{}_{uv} c^u c^v \partial_{cw}$. $Q_{\ker \rho_{E^+}}$ is therefore the fiberwise Chevalley–Eilenberg differential of the Lie algebra bundle $\ker \rho_{E^+}$ and, so, $H_{LA}^*(\ker \rho_{E^+})$ is the space of sections of a vector bundle whose typical fiber the Chevalley–Eilenberg cohomology $H_{CE}^*(\mathfrak{g}^+)$ of the typical fiber \mathfrak{g}^+ of $\ker \rho_{E^+}$. As is apparent, $Q_{\ker \rho_{E^+}}$ is formally identical to the homological vector field Λ_E . This furnishes an interpretation of the latter.

6.4 Lie algebroid cohomology of \widehat{E}^+ and A model chiral ring

Now, we are ready to unveil the relation between the Lie algebroid cohomology of \widehat{E}^+ and the A model chiral ring.

As explained in subsect. 5.8, the chiral ring $\mathcal{R}^{(\nu)}_A$ of the ν -th BRST structure of the type A half-topological sigma model consists of operator $Q^{(\nu)}_A$ -cohomology classes of vanishing scaling dimensions h, \bar{h} . Each class is represented by a scalar local operator \mathcal{O} ¹⁶ such that $[Q^{(\nu)}_A, \mathcal{O}] = 0$ defined modulo an operator of the form $[Q^{(\nu)}_A, \mathcal{X}]$ with \mathcal{X} an arbitrary scalar local operator¹⁷. Let \mathbb{F}_A denote the algebra of the A model scalar local operators. Then, $(\mathbb{F}_A, [Q^{(\nu)}_A, \cdot])$ is a complex and the chiral ring $\mathcal{R}^{(\nu)}_A$ is its cohomology.

The operators $\mathcal{F} \in \mathbb{F}_A$ are all of the form

$$\mathcal{F} = \sum_{m,n} \frac{1}{m!n!} \phi_{\bar{p}_1 \dots \bar{p}_m; u_1 \dots u_n}(x) \bar{\chi}^{\bar{p}_1} \dots \bar{\chi}^{\bar{p}_m} \lambda^{u_1} \dots \lambda^{u_n}. \quad (6.16)$$

The fields $\chi^p_{\bar{z}}, \lambda^*_{uz}$ as well as all the $\partial_z, \bar{\partial}_{\bar{z}}$ derivatives of fields cannot contribute to \mathcal{F} because of its scalar nature.

In the quasiclassical limit, there is a canonical isomorphism $\varsigma_A : \mathbb{F}_A \mapsto \text{Fun}(\widehat{E}^+[1])$ defined as follows. Redefine the fiber coordinates of $\widehat{E}^+[1]$ as

$$\gamma^{\bar{p}} = \sigma^{\bar{p}}_{\bar{a}} \zeta^{\bar{a}}. \quad (6.17)$$

Then, for any operator $\mathcal{F} \in \mathbb{F}_A$ of the form (6.16), $\varsigma_A(\mathcal{F}) \in \text{Fun}(\widehat{E}^+[1])$ reads

$$\varsigma_A(\mathcal{F}) = \sum_{m,n} \frac{1}{m!n!} \phi_{\bar{p}_1 \dots \bar{p}_m; u_1 \dots u_n} \gamma^{\bar{p}_1} \dots \gamma^{\bar{p}_m} c^{u_1} \dots c^{u_n}. \quad (6.18)$$

Under the isomorphism ς_A , we have

$$\varsigma_A([Q^{(\nu)}_A, \mathcal{F}]) = Q^{(\nu)}_{\widehat{E}^+} \varsigma_A(\mathcal{F}), \quad (6.19)$$

¹⁶ Here and below, the adjective “scalar” means “world-sheet scalar”.

¹⁷ Since the BRST charge $Q^{(\nu)}_A$ is itself a scalar, it cannot change the world-sheet covariance of the local operators \mathcal{F} which it acts upon. Therefore, \mathcal{X} must be also be a scalar.

where $Q^{(\nu)}_{\hat{E}^+}$ is a certain homological vector field on $\hat{E}^+[1]$,

$$Q^{(\nu)}_{\hat{E}^+}{}^2 = 0. \quad (6.20)$$

Relation (6.19) states that ς_A is a chain map of the complexes $(\mathbf{F}_A, [Q^{(\nu)}_A, \cdot])$, $(\text{Fun}(\hat{E}^+[1]), Q^{(\nu)}_{\hat{E}^+})$. Therefore, in the quasiclassical limit, the chiral ring $\mathcal{R}^{(\nu)}_A$ is isomorphic to the cohomology of $(\text{Fun}(\hat{E}^+[1]), Q^{(\nu)}_{\hat{E}^+})$. Next, we are going to compute $Q^{(\nu)}_{\hat{E}^+}$ in such regime, having in mind that $Q^{(\nu)}_{\hat{E}^+}$ may receive complicated perturbative quantum corrections.

Using (4.6), we find that

$$\tilde{Q}_{\hat{E}^+} = \rho^{\bar{a}}_{\bar{p}} \gamma^{\bar{p}} \partial_{z^{\bar{a}}} - \frac{1}{2} f^{\bar{p}}_{\bar{q}\bar{r}} \gamma^{\bar{q}} \gamma^{\bar{r}} \partial_{\gamma^{\bar{p}}}, \quad (6.21)$$

while $\Lambda_{\hat{E}^+}$ is still given by expression (6.12). From (6.17), (6.21), we find easily

$$\tilde{Q}_{\hat{E}^+} z^a = 0, \quad \tilde{Q}_{\hat{E}^+} z^{\bar{a}} = \rho^{\bar{a}}_{\bar{p}} \gamma^{\bar{p}} \quad (6.22a)$$

$$\tilde{Q}_{\hat{E}^+} \gamma^{\bar{p}} = -\frac{1}{2} f^{\bar{p}}_{\bar{q}\bar{r}} \gamma^{\bar{q}} \gamma^{\bar{r}}, \quad (6.22b)$$

$$\tilde{Q}_{\hat{E}^+} c^u = 0, \quad (6.22c)$$

From (6.12), we obtain

$$\Lambda_{\hat{E}^+} z^a = 0, \quad \Lambda_{\hat{E}^+} z^{\bar{a}} = 0, \quad (6.23a)$$

$$\Lambda_{\hat{E}^+} \gamma^{\bar{p}} = 0, \quad (6.23b)$$

$$\Lambda_{\hat{E}^+} c^u = -\frac{1}{2} f^u_{vw} c^v c^w. \quad (6.23c)$$

Comparing (5.2) and (6.22), we find that the action of the primary topological field variation operator δ_A on \mathbf{F}_A is such that

$$\varsigma_A(\delta_A \mathcal{F}) = \tilde{Q}_{\hat{E}^+} \varsigma_A(\mathcal{F}), \quad (6.24)$$

for any operator $\mathcal{F} \in \mathbf{F}_A$. Likewise, comparing (5.4) and (6.23), we find that the action of the secondary topological field variation operator s_A is such that

$$\varsigma_A(s_A \mathcal{F}) = \Lambda_{\hat{E}^+} \varsigma_A(\mathcal{F}). \quad (6.25)$$

Consider the first BRST structure of the A model. In the quasiclassical limit, $Q^{(1)}_A = Q_A$ acts as δ_A (cf. subsect. 5.7, eq. (5.51)). From (6.19), (6.24), in the same regime, we have therefore

$$Q^{(1)}_{\hat{E}^+} = \tilde{Q}_{\hat{E}^+}. \quad (6.26)$$

We conclude that, *at the classical level, the chiral ring $\mathcal{R}^{(1)}_A$ is isomorphic to the cohomology of $(\text{Fun}(\hat{E}^+[1]), \tilde{Q}_{\hat{E}^+})$ and, hence, to the Dolbeault cohomology $H_{\bar{\partial}}^*(\wedge^* \ker \mathcal{E}^*)$ or the sheaf cohomology $H^*(\mathcal{O}_{\wedge^* \ker \mathcal{E}^*})$.* The first BRST structure is the one normally envisaged in twisted heterotic sigma models. Here, we have recovered a well-known result about the heterotic chiral ring [16]–[17].

Next, consider the second BRST structure. In the quasiclassical limit, $Q^{(2)}_A = Q_A + \Lambda_A$ acts as $\delta_A + s_A$ (cf. subsect. 5.7, eqs. (5.51), (5.52)). From (6.19), (6.24), (6.25), in the same regime, we have then

$$Q^{(2)}_{\hat{E}^+} = \tilde{Q}_{\hat{E}^+} + \Lambda_{\hat{E}^+} = Q_{\hat{E}^+}, \quad (6.27)$$

where (6.10) has been used. We conclude that, *at the classical level, the chiral ring $\mathcal{R}^{(2)}_A$ is isomorphic to the cohomology of $(\text{Fun}(\hat{E}^+[1]), Q_{\hat{E}^+})$, that is the Lie algebroid cohomology $H_{LA}^*(\hat{E}^+)$.*

Next, we consider the problem of computing correlators of A model chiral ring classes at genus $\ell = 0$ in the quasiclassical limit. This matter was discussed preliminarily in subsect.5.10, which the reader is referred to. Recall that the quantum correlator $\langle \mathcal{O} \rangle$ of any chiral ring element $\mathcal{O} \in \mathcal{R}^{(\nu)}_A$ should be expressible as an integral over M of a suitable top degree de Rham cohomology class $[\omega_{\mathcal{O}}]_{\text{dR}}$ depending on \mathcal{O} . As we have found above, with \mathcal{O} there corresponds a cohomology class of the complex $(\text{Fun}(\hat{E}^+[1]), Q^{(\nu)}_{\hat{E}^+})$. This indicates that there must exist a map $\hat{\varphi} : \text{Fun}(\hat{E}^+[1]) \mapsto \text{Fun}(T_M[1])$ such that, for any chiral ring element $\mathcal{O} \in \mathcal{R}^{(\nu)}_A$, one has

$$\langle \mathcal{O} \rangle = \int_{T_M[1]} \widehat{\varphi} \omega, \quad (6.28)$$

where $\omega \in \text{Fun}(\widehat{E}^+[1])$ with $Q^{(\nu)}_{\widehat{E}^+} \omega = 0$ is a representative of \mathcal{O} . In (6.28), $\widehat{\varphi} \omega$ is to be thought of as a non homogeneous form on M , whose top degree part is integrated on M . The map $\widehat{\varphi}$ must have the following properties. First,

$$\widehat{\varphi} Q^{(\nu)}_{\widehat{E}^+} \omega = (-1)^{d-r} d \widehat{\varphi} \omega, \quad (6.29)$$

for $\omega \in \text{Fun}(\widehat{E}^+[1])$, where $d = \zeta^a \partial_a + \zeta^{\bar{a}} \partial_{\bar{a}}$ is the de Rham differential in supergeometric form. In virtue of (6.29), the right hand side of (6.28) depends only on the $Q^{(\nu)}_{\widehat{E}^+}$ -cohomology class of ω and, so, is defined on $\mathcal{R}^{(\nu)}_A$. Second,

$$\int_{T_M[1]} \widehat{\varphi} \omega = 0, \quad (6.30)$$

whenever $\omega \in \text{Fun}(\widehat{E}^+[1])$ does not have bidegree (d, r) . This ensures that the selection rule on the chiral ring correlators found earlier in subsect. 5.10 is satisfied.

Let (5.83) hold. Then, the holomorphic line bundle $\wedge^d \mathcal{T}_{\mathcal{M}}^* \otimes \wedge^r \ker \rho_{\mathcal{E}}$ is holomorphically trivial and, so, it has a nowhere vanishing holomorphic section v . v induces an algebra morphism $\widehat{\varphi}_v : \text{Fun}(\widehat{E}^+[1]) \mapsto \text{Fun}(T_M[1])$ defined by

$$\widehat{\varphi}_v \omega = \frac{1}{d!r!} v_{a_1 \dots a_d} \zeta^{u_1 \dots u_r} \zeta^{a_1} \dots \zeta^{a_d} \partial_{c u_1} \dots \partial_{c u_r} \omega, \quad (6.31)$$

with $\omega \in \text{Fun}(\widehat{E}^+[1])$. By the holomorphy of v , $\widehat{\varphi}_v$ fulfils (6.29), for the first BRST structure. It does so also for the second BRST structure, provided that the unimodularity condition (5.88) holds. Moreover, $\widehat{\varphi}_v$ fulfils (6.30) for either structures. In this way, we have justified the algebro-geometric conditions (5.83) and (5.88) for the A model introduced in subsect. 5.10.

6.5 The Lie algebroid E^+ and its cohomology

We are now going to show that the Lie algebroid with complex base E^+ has

a structure analogous to that of the Lie algebroid \widehat{E}^+ constructed above and that its cohomology is intimately related to the BRST cohomology defining the type B half-topological sigma model chiral ring.

Consider the complex vector bundle $E^+_\sigma = T_M^+ \oplus \ker \rho_{E^+}$. We endow E^+_σ with the anchor defined by

$$\rho_{E^+_\sigma}(x \oplus u) = x \quad (6.32)$$

with $x \in \Gamma(T_M^+)$, $u \in \Gamma(\ker \rho_{E^+})$, and with the Lie bracket defined by

$$[x \oplus u, y \oplus v]_{E^+_\sigma} = [x, y]_{T_M^+} \oplus ([\sigma^c x, v]_{E^c} - [\sigma^c y, u]_{E^c} + [u, v]_{E^c}) \quad (6.33)$$

with $x, y \in \Gamma(T_M^+)$, $u, v \in \Gamma(\ker \rho_{E^+})$. Again, condition (4.6) is required by the fulfillment of the Jacobi identity. E^+_σ acquires in this way a structure of Lie algebroid with complex base, as is immediately verified.

The bundle map $\varpi_\sigma : E^+_\sigma \rightarrow E^c$ defined by $\varpi_\sigma(x \oplus u) = \sigma^c x + u$ for $x \in \Gamma(T_M^+)$, $u \in \Gamma(\ker \rho_{E^+})$ is a monomorphism of complex Lie algebroids. The image of ϖ_σ in E^c is nothing but the Lie algebroid E^+ , as follows from relation (3.14). $\varpi_\sigma : E^+_\sigma \rightarrow E^+$ is then an isomorphism of Lie algebroids with complex base. Again, for simplicity, we identify E^+_σ and E^+ leaving the isomorphism ϖ_σ understood.

Let us compute the homological vector field Q_{E^+} . As E^+ has a canonical holomorphic structure, there is a holomorphic framing $\{e_i\}$. By the above findings, $\{\partial_a\} \cup \{e_u\}$ constitutes another distinguished framing of E^+ . Let ζ^a, c^u be the corresponding fiber coordinates of $E^+[1]$. Then,

$$Q_{E^+} = \zeta^a (\partial_{za} - f^u{}_{iv} \sigma^i{}_a c^v \partial_{cu}) - \frac{1}{2} f^u{}_{vw} c^v c^w \partial_{cu}. \quad (6.34)$$

It is straightforward to check that Q_{E^+} is nilpotent

$$Q_{E^+}^2 = 0, \quad (6.35)$$

as required ¹⁸.

Next, we observe that Q_{E^+} decomposes as

$$Q_{E^+} = \tilde{Q}_{E^+} + \Lambda_{E^+}, \quad (6.36)$$

where \tilde{Q}_{E^+} , Λ_{E^+} are given by

$$\tilde{Q}_{E^+} = \zeta^a (\partial_{z_a} - f^u{}_{iv} \sigma^i{}_a c^v \partial_{c_u}), \quad (6.37)$$

$$\Lambda_{E^+} = -\frac{1}{2} f^u{}_{vw} c^v c^w \partial_{c_u}. \quad (6.38)$$

\tilde{Q}_{E^+} , Λ_{E^+} are nilpotent and anticommute

$$\tilde{Q}_{E^+}^2 = 0, \quad (6.39)$$

$$\Lambda_{E^+}^2 = 0, \quad (6.40)$$

$$\tilde{Q}_{E^+} \Lambda_{E^+} + \Lambda_{E^+} \tilde{Q}_{E^+} = 0. \quad (6.41)$$

We note that (6.39) depends crucially on the validity of (4.6).

By (6.39), $(\text{Fun}(E^+[1]), \tilde{Q}_{E^+})$ is a complex. The algebra $\text{Fun}(E^+[1])$ is isomorphic to the algebra of $\wedge^* \mathcal{K}_{E^+\sigma}$ -valued $(*, 0)$ forms $\Omega^{*,0}(M, \wedge^* \mathcal{K}_{E^+\sigma})$, where the antiholomorphic vector bundle $\mathcal{K}_{E^+\sigma}$ was defined in subsect. 5.10. Under the isomorphism, \tilde{Q}_{E^+} is identified with the $(1, 0)$ connection $\mathcal{D}_{E^+\sigma}^{\text{ad}}$ with vanishing $(2, 0)$ curvature that defines the antiholomorphic structure of $\mathcal{K}_{E^+\sigma}$ and, so, with the holomorphic Dolbeault operator $\partial_{\wedge^* \mathcal{K}_{E^+\sigma}}$. The cohomology of $(\text{Fun}(\hat{E}^+[1]), \tilde{Q}_{\hat{E}^+})$ is thus isomorphic to the holomorphic Dolbeault cohomology $H_{\partial}^*(\wedge^* \mathcal{K}_{E^+\sigma})$, that is the sheaf cohomology $H^*(\overline{\mathcal{O}}_{\wedge^* \mathcal{K}_{E^+\sigma}})$. This cohomology depends in general on the splitting σ , since $\mathcal{K}_{E^+\sigma}$ does.

¹⁸ Q_{E^+} extends to a homological vector field on $E^c[1]$. Since the framing $\{\partial_a\} \cup \{e_u\}$ used in the construction of Q_{E^+} is not holomorphic, the extension differs from the canonical extension \mathcal{Q}_E defined above eq. (6.5), which employs the holomorphic framing $\{e_i\}$ (cf. footnote 14). Indeed, one has $\mathcal{Q}_E = Q_{E^+} - G^{\bar{u}}{}_{\bar{a}\bar{b}} \zeta^{\bar{a}} \zeta^{\bar{b}} \partial_{c_{\bar{u}}}$. So, \mathcal{Q}_E , Q_{E^+} have the same restriction on $\text{Fun}(E^+[1])$ but generally differ on $\text{Fun}(E^c[1])$.

We note that $\text{Fun}(E^+[1])$ is bigraded according to the polynomial degree in ζ^a , c^u and that, by (6.39)–(6.41), $(\text{Fun}(E^+[1]), \tilde{Q}_{E^+}, \Lambda_{E^+})$ is a double complex. By (6.36), the total cohomology of this double complex is the cohomology of the complex $(\text{Fun}(E^+[1]), Q_{E^+})$, that is the Lie algebroid cohomology $H_{LA}^*(E^+)$. Again, the latter can be computed using spectral sequence methods. It is also manifestly independent from the splitting σ .

6.6 Lie algebroid cohomology of E^+ and B model chiral ring

Now, we are going to uncover the relation between the Lie algebroid cohomology of E^+ and the B model chiral ring. In outline, the analysis follows the same lines as that done for the A model.

From subsect. 5.8, the chiral ring $\mathcal{R}^{(\nu)}_B$ of the ν -th BRST structure of the type B half-topological sigma model consists of operator $Q^{(\nu)}_B$ -cohomology classes of vanishing scaling dimensions h, \bar{h} . Each class is represented by a scalar local operator \mathcal{O} such that $[Q^{(\nu)}_B, \mathcal{O}] = 0$ defined modulo an operator of the form $[Q^{(\nu)}_B, \mathcal{X}]$ with \mathcal{X} any scalar local operator. Let \mathbb{F}_B denote the algebra of the B model scalar local operators. Then, $(\mathbb{F}_B, [Q^{(\nu)}_B, \cdot])$ is a complex and the chiral ring $\mathcal{R}^{(\nu)}_B$ is its cohomology.

The operators $\mathcal{F} \in \mathbb{F}_B$ are all of the form

$$\mathcal{F} = \sum_{m,n} \frac{1}{m!n!} \phi_{p_1 \dots p_m; u_1 \dots u_n}(x) \chi^{p_1} \dots \chi^{p_m} \lambda^{u_1} \dots \lambda^{u_n}, \quad (6.42)$$

the fields $\bar{\chi}^{\bar{p}}_{\bar{z}}$, λ^*_{uz} as well as all the $\partial_z, \bar{\partial}_{\bar{z}}$ derivatives of fields not contributing because of the scalar nature of \mathcal{F} .

Analogously to the A model, in the quasiclassical limit, there is a canonical isomorphism $\varsigma_B : \mathbb{F}_B \mapsto \text{Fun}(E^+[1])$. Define new fiber coordinates of $E^+[1]$ by

$$\gamma^p = \sigma^p_a \zeta^a. \quad (6.43)$$

Then, for any operator $\mathcal{F} \in \mathbb{F}_B$ of the form (6.42), $\varsigma_B(\mathcal{F}) \in \text{Fun}(E^+[1])$ is

$$\varsigma_B(\mathcal{F}) = \sum_{m,n} \frac{1}{m!n!} \phi_{p_1 \dots p_m; u_1 \dots u_n} \gamma^{p_1} \dots \gamma^{p_m} c^{u_1} \dots c^{u_n}. \quad (6.44)$$

Again, analogously to the A model, the isomorphism ς_B has the property that

$$\varsigma_B([Q^{(\nu)}_B, \mathcal{F}]) = Q^{(\nu)}_{E^+} \varsigma_B(\mathcal{F}), \quad (6.45)$$

where $Q^{(\nu)}_{E^+}$ is a certain homological vector field on $E^+[1]$,

$$Q^{(\nu)}_{E^+}{}^2 = 0. \quad (6.46)$$

By (6.45), ς_B is a chain map of the complexes $(\mathbf{F}_B, [Q^{(\nu)}_B, \cdot])$, $(\text{Fun}(E^+[1]), Q^{(\nu)}_{E^+})$ and, so, the quasiclassical chiral ring $\mathcal{R}^{(\nu)}_B$ is isomorphic to the cohomology of $(\text{Fun}(E^+[1]), Q^{(\nu)}_{E^+})$. We are now going to compute $Q^{(\nu)}_{E^+}$ in that regime.

Using (4.6), we find that

$$\tilde{Q}_{E^+} = \rho^a_p \gamma^p (\partial_{z^a} - f^v_{iu} \sigma^i_a c^u \partial_{c^v}) - \frac{1}{2} f^r_{pq} \gamma^p \gamma^q \partial_{\gamma^r}. \quad (6.47)$$

while Λ_{E^+} is still given by expression (6.38). From (6.43), (6.47), we find easily

$$\tilde{Q}_{E^+} z^a = \rho_p \gamma^p, \quad \tilde{Q}_{E^+} z^{\bar{a}} = 0, \quad (6.48a)$$

$$\tilde{Q}_{E^+} \gamma^p = -\frac{1}{2} f^p_{qr} \gamma^q \gamma^r, \quad (6.48b)$$

$$\tilde{Q}_{E^+} c^u = -f^u_{iv} \sigma^i_a \rho^a_p \gamma^p c^v \quad (6.48c)$$

while, from (6.38), we obtain

$$\Lambda_{E^+} z^a = 0, \quad \Lambda_{E^+} z^{\bar{a}} = 0, \quad (6.49a)$$

$$\Lambda_{E^+} \gamma^p = 0, \quad (6.49b)$$

$$\Lambda_{E^+} c^u = -\frac{1}{2} f^u_{vw} c^v c^w. \quad (6.49c)$$

By comparing (5.7) with (6.48) and (5.9) with (6.49), we find that the actions of the topological field variation operators δ_B, s_B on \mathbf{F}_B satisfy

$$\varsigma_B(\delta_B \mathcal{F}) = \tilde{Q}_{E^+} \varsigma_B(\mathcal{F}), \quad (6.50)$$

and

$$\varsigma_B(s_B \mathcal{F}) = \Lambda_{E^+} \varsigma_B(\mathcal{F}). \quad (6.51)$$

for any operator $\mathcal{F} \in \mathbf{F}_B$, extending the A model findings.

Consider the first BRST structure of the B model. In the quasiclassical limit, $Q^{(1)}_B = Q_B$ acts as δ_B (cf. subsect. 5.7, eq. (5.51)). From (6.45), (6.50), in the same regime, we have therefore

$$Q^{(1)}_{E^+} = \tilde{Q}_{E^+}. \quad (6.52)$$

It follows that, *at the classical level, the chiral ring $\mathcal{R}^{(1)}_B$ is isomorphic to the cohomology of $(\text{Fun}(\widehat{E}^+[1]), \tilde{Q}_{E^+})$ and, hence, to the holomorphic Dolbeault cohomology $H_{\partial}^*(\wedge^* \mathcal{K}_{E^+\sigma^*})$ or the sheaf cohomology $H^*(\overline{\mathcal{O}}_{\wedge^* \mathcal{K}_{E^+\sigma^*}})$.*

Next, consider the second BRST structure. In the quasiclassical limit, $Q^{(2)}_B = Q_B + \Lambda_B$ acts as $\delta_B + s_B$ (cf. subsect. 5.7, eqs. (5.51), (5.52)). From (6.45), (6.50), (6.51), in the same regime, we have then

$$Q^{(2)}_{E^+} = \tilde{Q}_{E^+} + \Lambda_{E^+} = Q_{E^+}, \quad (6.53)$$

where (6.36) has been used. Consequently, *at the classical level, the chiral ring $\mathcal{R}^{(2)}_B$ is isomorphic to the cohomology of $(\text{Fun}(E^+[1]), Q_{E^+})$, that is the Lie algebroid cohomology $H_{LA}^*(E^+)$.*

The problem of computing correlators of B model chiral ring classes at genus $\ell = 0$ in the quasiclassical limit can be treated along the same lines as the A model. As we have found above, with any chiral ring class $\mathcal{O} \in \mathcal{R}^{(\nu)}_B$ there corresponds a cohomology class of the complex $(\text{Fun}(E^+[1]), Q^{(\nu)}_{E^+})$. Thus, there must exist a map $\varphi : \text{Fun}(E^+[1]) \mapsto \text{Fun}(T_M[1])$ such that, for $\mathcal{O} \in \mathcal{R}^{(\nu)}_B$,

$$\langle \mathcal{O} \rangle = \int_{T_M[1]} \varphi \omega, \quad (6.54)$$

where $\omega \in \text{Fun}(E^+[1])$ with $Q^{(\nu)}_{E^+} \omega = 0$ is a representative of \mathcal{O} . The mapping

φ must have the properties that

$$\varphi Q^{(\nu)}_{E^+\omega} = (-1)^{d-r} d\varphi\omega, \quad (6.55)$$

for $\omega \in \text{Fun}(E^+[1])$, to make the right hand side of (6.54) depend only on the $Q^{(\nu)}_{E^+}$ -cohomology class of ω and, thus, defined on $\mathcal{R}^{(\nu)}_B$, and that

$$\int_{T_M[1]} \varphi\omega = 0, \quad (6.56)$$

whenever $\omega \in \text{Fun}(E^+[1])$ does not have bidegree (d, r) , to ensure that the selection rule on the chiral ring correlators of subsect. 5.10 is satisfied.

Assume that (5.84), (5.85) hold. As the antiholomorphic line bundle $\wedge^r \mathcal{K}_{E^+\sigma^*}$ is antiholomorphically trivial, it admits a nowhere vanishing antiholomorphic section v . Similarly, as the holomorphic line bundle $\wedge^d \mathcal{T}_{\mathcal{M}^*}$, is holomorphically trivial, it possesses a nowhere vanishing holomorphic sections Ω . v, Ω induce an algebra morphism $\varphi_{v\Omega} : \text{Fun}(E^+[1]) \mapsto \text{Fun}(T_M[1])$ defined by

$$\varphi_{v\Omega}\omega = \frac{1}{d!r!} \bar{\Omega}_{\bar{a}_1 \dots \bar{a}_d} v^{u_1 \dots u_r} \zeta^{\bar{a}_1} \dots \zeta^{\bar{a}_d} \partial_{cu_1} \dots \partial_{cu_r} \omega, \quad (6.57)$$

with $\omega \in \text{Fun}(E^+[1])$. By the antiholomorphy of v and the holomorphy of Ω $\varphi_{v\Omega}$ fulfils (6.55), for the first BRST structure. It does so also for the second BRST structure, provided, again, that the unimodularity condition (5.88) holds. Moreover, $\varphi_{v\Omega}$ fulfils (6.56) for either structures. In this way, as for the A model, we have justified the algebro-geometric conditions (5.84), (5.85) and (5.88) for the B model introduced in subsect. 5.10.

6.7 Lie algebroid chiral algebras

In this final subsection, we make some remarks concerning the chiral algebras of the half-topological sigma models.

Consider first the chiral algebra $\mathcal{A}^{(\nu)}_A$ of the A model. Since the local operators and the BRST charge can be described locally along M , it is possible

to consider operators that are well-defined not throughout M , but only on local neighborhoods of M [7]. $Q^{(\nu)}_A$ -cohomology classes of operators defined in a neighborhood have operator product expansions involving classes of operators defined in the same neighborhood and they can be restricted to smaller neighborhoods and patched on unions of neighborhoods in a natural fashion. So, what we actually have is a sheaf of chiral algebras $\mathcal{C}^{(\nu)}_A$.

An operator class \mathcal{O} of $\mathcal{A}^{(\nu)}_A$ is characterized by its scaling dimensions $(h, 0)$ and R /flavour charges (q_R, q_L) . A local operator \mathcal{F} representing \mathcal{O} must then have the same properties. \mathcal{F} must so be a function of the fields $x^a, x^{\bar{a}}, \lambda^u, \lambda^*_{uz}$ and their ∂_z derivatives and the field $\bar{\chi}^{\bar{p}}$ with h z indices, q_R factors $\bar{\chi}^{\bar{p}}$ and q_L λ^u minus λ^*_{uz} possibly differentiated factors. The fields $\chi^p_{\bar{z}}$ and $\bar{\partial}_{\bar{z}}$ derivatives of fields cannot appear in the expression of \mathcal{F} because its antiholomorphic scaling dimension vanishes. The ∂_z derivatives of $\bar{\chi}^{\bar{p}}$ and the fields $l^u_{\bar{z}}, l^*_{uz}$ are not included as they can be eliminated using the field equations. It can be shown that the operators \mathcal{F} of this form are in one-to-one correspondence with the elements of $\Omega^{0, q_R}(\mathcal{W}_{h, q_L})$, where \mathcal{W}_{h, q_L} is a certain complicated holomorphic vector bundle depending on h, q_L constructed with $\mathcal{T}_M, \ker \varepsilon$ and their duals.

Consider now the first BRST structure of the A model. In the quasiclassical limit, the BRST charge $Q^{(1)}_A$ acts on the local operators \mathcal{F} of fixed scaling dimensions $(h, 0)$ and flavour charge q_L but varying R charge q_R as the Dolbeault operator $\bar{\partial}_{\mathcal{W}_{h, q_L}}$ in many though not all cases, by reasons analogous to those found in the treatment of chiral ring $\mathcal{R}^{(1)}_A$ ¹⁹. For the sake of the argument, let us leave aside this technical complication and pretend that $Q^{(1)}_A$ acts classically as $\bar{\partial}_{\mathcal{W}_{h, q_L}}$. Since $\bar{\partial}_{\mathcal{W}_{h, q_L}}$ obeys the $\bar{\partial}$ -Poincaré lemma, the cohomology of $\bar{\partial}_{\mathcal{W}_{h, q_L}}$ is trivial on any local neighborhood of M . The operator $Q^{(1)}_A$ -cohomology is therefore locally trivial classically on the local operators \mathcal{F} of scaling dimensions $(h, 0)$ and

¹⁹ In the case where $E = T_M$ considered by Witten in [7], this is actually always true.

flavour charge q_L . Since h and q_L are arbitrary, the operator $Q^{(1)}_A$ -cohomology is locally trivial classically in general.

It is believed that perturbative quantum corrections cannot create cohomology classes [7]. Taking this for granted, it follows that the operator $Q^{(1)}_A$ -cohomology is locally trivial also quantum mechanically in the perturbative regime we assume. By an argument totally analogous to that used to prove the Dolbeault-Čech isomorphism in algebraic geometry, one then shows that the operator $Q^{(1)}_A$ -cohomology, that is the chiral algebra $\mathcal{A}^{(1)}_A$, is isomorphic to the Čech cohomology of the sheaf $\mathcal{A}^{(1)}_A{}^0$ of R charge 0 $Q^{(1)}_A$ -closed operators. In [7, 10], Witten and Kapustin showed independently that, for $E = T_M$ with M Calabi-Yau, the Čech cohomology of $\mathcal{A}^{(1)}_A{}^0$ could be related to that of the sheaf of chiral differential operators introduced in [8], [9]. This result was later generalized by Tan in [11]–[14] to heterotic sigma models with general gauge bundle. It extends also to our type A sigma model for the first BRST structure.

Does the above generalize to the type A sigma model for the second BRST structure? To be sure, not in a straightforward fashion. The argument just outlined for the first structure uses in an essential way the fact that, in the quasiclassical limit, $Q^{(1)}_A$ reduces to the appropriate Dolbeault operator $\bar{\partial}$ and that, by the $\bar{\partial}$ -Poincaré lemma, the $\bar{\partial}$ -cohomology is locally trivial. For the second structure, one expects that $Q^{(2)}_A$, again quasiclassically, reduces instead to some generalization of the Lie algebroid differential $Q_{\hat{E}^+}$ studied in subsect. 6.3 by arguments analogous to those employed in the study of the chiral ring $\mathcal{R}^{(2)}_A$ in subsect. 6.4. The point here is that there is no analog of the Poincaré lemma for $Q_{\hat{E}^+}$. Thus, the operator $Q^{(2)}_A$ -cohomology is not locally trivial in general and $\mathcal{A}^{(2)}_A$ does not have a Čech description the same way $\mathcal{A}^{(1)}_A$ does.

The above analysis can be repeated along the same lines for the B model with completely similar conclusions.

7 Conclusions and outlook

7.1 Summary of results

In this paper, we have constructed a heterotic sigma model whose target space geometry consists of a transitive Lie algebroid E with complex structure on a Kaehler manifold M satisfying certain natural geometric conditions, the heterotic Lie algebroid sigma model. We have then found that the model possesses a primary $(0, 2)$ supersymmetry, which it shares with the other heterotic sigma models [3], and a novel secondary $(1, 0)$ supersymmetry ensuing from the adjoint bundle $\ker \rho_E$, the kernel of the anchor ρ_E of E .

This opens new possibilities for topological twisting. The two twist prescriptions, considered in this paper, lead to half-topological sigma models of type A and B , as already studied in the literature. These sigma models, however, because of the higher amount of supersymmetry of the parent untwisted model stemming from the Lie algebroid target space geometry, are characterized not only by the usual primary topological BRST operator but also by a secondary Slavnov-like BRST operator anticommuting with the former. As a consequence, the sigma models possess two inequivalent BRST structures, which share many, but not all, properties.

Quantum mechanically, in the perturbative regime in which there are no world-sheet instanton contributions, the two half-topological sigma models constructed by us are conformally invariant at the level of BRST cohomology and are characterized by a holomorphic chiral algebra \mathcal{A} and a chiral ring \mathcal{R} with respect to both BRST structures, on a par with the half-topological sigma models considered in earlier studies [15]–[17]. The difference between the structures manifests itself with regard to the dependence of the chiral algebra correlators on the target space geometric data. The classical action of our sigma models contains also a Kaehler metric g of M and a splitting σ of E . The chiral alge-

bra correlators are independent from g for both BRST structures while they are independent from σ for the second BRST structure only. The second structure is therefore “more topological” than the first and, from a Lie algebroid theoretic point of view, also more natural. Indeed, as we have seen, the second structure’s chiral ring is directly related to the target space Lie algebroid cohomology in a way the first structure’s one is not.

Our conclusions rest on a considerable amount of educated guessing and are therefore to a certain extent conjectural. We do not know whether the two BRST structures coexist in the same quantum field theory or, else, whether their pertain to two different quantum field theories sharing the same quasiclassical limit. Happily, our conclusions apparently do not hinge on the answer to this questions. In any case, as is well-known, the inclusion of non perturbative world-sheet instanton effects can radically alter the perturbative picture that emerges from our analysis.

7.2 Open problems

There are several issues which we have not touched and, related to these, problems still open.

Lie algebroid sigma models with a regular but non transitive target.

A Lie algebroid E is regular if its anchor ρ_E has locally constant rank [19]. Every transitive Lie algebroid is regular, but not viceversa. The natural question arises whether it would be possible to generalize our analysis and produce a heterotic Lie algebroid sigma model with a regular non transitive target. Since our construction relies in an essential way on the use of splittings, which exist only for transitive Lie algebroids, the task seems hopeless at first glance. However, as the base M of a regular Lie algebroid E admits a foliation such that the restriction of E to each leaf of M is transitive, it is conceivable that a Lie algebroid sigma

model with regular target may be built as a family of Lie algebroid sigma models with transitive target parametrized by the foliation of M and connected in some way.

Lie algebroid sigma models with $(2, 2)$ supersymmetry.

It would be desirable to have a Lie algebroid sigma model with $(2, 2)$ supersymmetry. For a target space geometry consisting of a transitive Lie algebroid E with complex structure over a Kaehler base M , of the kind considered in this paper, in the same way as the $(0, 2)$ heterotic sigma model arises by coupling fermionic degrees of freedom in $\ker \rho_{E^+}$ to the basic $(0, 2)$ sigma model over M in a way compatible with $(0, 2)$ supersymmetry, the $(2, 2)$ sigma model should arise by coupling fermionic degrees of freedom in $\ker \rho_{E^c}$ to the basic $(2, 2)$ sigma model over M in a way compatible with $(2, 2)$ supersymmetry. The closure of the $(2, 2)$ supersymmetry algebra apparently requires the splitting σ to be flat. While every transitive Lie algebroid admits a flat splitting locally, it generally does not globally [19]. So, it is likely that such $(2, 2)$ Lie algebroid sigma model, assuming that it does exist, will turn out to be only a very mild generalization of the usual one. Alternatively, one may try to define the $(2, 2)$ sigma model locally in target space and then try to glue the resulting field theories in a globally meaningful way, though it is difficult to fathom how this could be done.

Description of the chiral algebra.

In this paper, we concentrated on the chiral rings of the two half-topological Lie algebroid sigma models for each BRST structure and said comparatively little about their ambient chiral algebras (cf. subsect. 6.7). The point is that the condition $[Q^{(\nu)}_t, \mathcal{O}] = 0$ is not very constraining for a local operator \mathcal{O} of positive holomorphic scaling dimension. This is what renders chiral algebras very complicated objects which resist any attempts at a simple description such as that of chiral rings. Needless to say, much work remains to be done on this aspect of

the theory.

Deformation of the target Lie algebroid structure.

In general, infinitesimal deformations of the target space geometry of a sigma model result in the insertion of integrated vertex operators in correlators. For the sigma models studied in this paper, the target geometry consists of a transitive Lie algebroid E with complex structure over a manifold M satisfying (3.13) (cf. subsects. 2.2, 3.2). Its deformation involves that of *a*) the Lie bracket structure $[\cdot, \cdot]_E$, *b*) the anchor structure ρ_E , *c*) the fiber complex structure J_E and *d*) the base complex structure J_{EM} subject to various constraints. Further, the deformations reducible to symmetry transformations must be modded out. This analysis requires a considerable amount of extra work and is left for future work.

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References

- [1] J. Distler and B. R. Greene, “Aspects of (2,0) String Compactifications”, Nucl. Phys. B **304** (1988) 1.
- [2] E. Witten, “Phases of $N = 2$ theories in two dimensions”, Nucl. Phys. B **403** (1993) 159 [arXiv:hep-th/9301042].
- [3] J. Distler, “Notes on (0,2) superconformal field theories”, arXiv:hep-th/9502012.
- [4] E. Witten, “Topological sigma models”, Commun. Math. Phys. **118** (1988) 411.
- [5] E. Witten, “Mirror manifolds and topological field theory”, in “Essays on mirror manifolds”, ed. S. T. Yau, International Press, Hong Kong, (1992) 120, [arXiv:hep-th/9112056].
- [6] K. Hori *et al.*, “Mirror symmetry”, *Providence, USA: AMS (2003) 929 p.*
- [7] E. Witten, “Two-dimensional models with (0,2) supersymmetry: Perturbative aspects”, arXiv:hep-th/0504078.
- [8] F. Malikov, V. Schechtman and A. Vaintrob, “Chiral de Rham complex”, Commun. Math. Phys. **204** (1999) 439 [arXiv:math/9803041].
- [9] V. Gorbounov, F. Malikov and V. Schechtman, “Gerbes of chiral differential operator”, Math. Res. Lett. **7** (2000) 55 [arXiv:math.AG/9906117].
- [10] A. Kapustin, “Chiral de Rham complex and the half-twisted sigma-model”, arXiv:hep-th/0504074.

- [11] M. C. Tan, “Two–dimensional twisted sigma models and the theory of chiral differential operators”, *Adv. Theor. Math. Phys.* **10** (2006) 759 [arXiv:hep-th/0604179].
- [12] M. C. Tan, “The half-twisted orbifold sigma model and the chiral de Rham complex”, *Adv. Theor. Math. Phys.* **12** (2008) 547 [arXiv:hep-th/0607199].
- [13] M. C. Tan, “Equivariant cohomology of the Cchiral de Rham complex and the half—twisted gauged sigma model”, *Adv. Theor. Math. Phys.* **13** (2009) 897 [arXiv:hep-th/0612164].
- [14] M. C. Tan, “Two–dimensional twisted sigma models, the mirror chiral de Rham complex, and twisted generalised mirror symmetry”, *JHEP* **0707** (2007) 013 [arXiv:0705.0790 [hep-th]].
- [15] A. Adams, A. Basu and S. Sethi, “(0,2) duality”, *Adv. Theor. Math. Phys.* **7** (2004) 865 [arXiv:hep-th/0309226].
- [16] E. Sharpe, “Notes on correlation functions in (0,2) theories”, arXiv:hep-th/0502064.
- [17] A. Adams, J. Distler and M. Ernebjerg, “Topological heterotic rings”, *Adv. Theor. Math. Phys.* **10** (2006) 657 [arXiv:hep-th/0506263].
- [18] J. Pradines, “Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux”, *C. R. Acad. Sci. Paris* **264 A** (1967) 245.
- [19] K. C. H. Mackenzie, “General theory of Lie groupoids and Lie algebroids”, London Mathematical Society Lecture Note Series, vol. 213, Cambridge University Press, Cambridge, UK, 2005.

- [20] N. Hitchin, “Generalized Calabi-Yau manifolds”, *Q. J. Math.* **54** (2003), no. 3, 281 [arXiv:math.DG/0209099].
- [21] M. Gualtieri, “Generalized complex geometry”, Oxford University Ph. D. Thesis, United Kingdom (2003), arXiv:math.DG/0401221.
- [22] M. Grana, “Flux compactifications in string theory: A comprehensive review”, *Phys. Rept.* **423** (2006) 91 [arXiv:hep-th/0509003].
- [23] A. Kapustin and Y. Li, “Topological sigma-models with H-flux and twisted generalized complex manifolds”, arXiv:hep-th/0407249.
- [24] R. Zucchini, “A topological sigma model of biKähler geometry”, *JHEP* **0601** (2006) 041 [arXiv:hep-th/0511144].
- [25] R. Zucchini, “BiHermitian supersymmetric quantum mechanics”, *Class. Quant. Grav.* **24** (2007) 2073 [arXiv:hep-th/0611308].
- [26] C. M. Hull and E. Witten, “Supersymmetric sigma models and the heterotic string”, *Phys. Lett. B* **160** (1985) 398.
- [27] A. Kotov and T. Strobl, “Generalizing geometry - algebroids and sigma models”, arXiv:1004.0632 [hep-th].
- [28] S. Evens, J.-H. Lu and A. Weinstein, “Transverse measures, the modular class, and a cohomology pairing for Lie algebroids”, *Quart. J. Math. Oxford Ser. (2)* **50** no. 200 (1999) 417 [arXiv:dg-ga/9610008].
- [29] C. Laurent-Gengoux, M. Stiennon and P. Xu, “Holomorphic Poisson manifolds and holomorphic Lie algebroids”, *Int. Math. Res. Not. Art. ID* **088** (2008) 46, [arXiv:0707.4253 [math.DG]].
- [30] U. Bruzzo and V. Rubtsov, “Cohomology of skew-holomorphic Lie algebroids”, arXiv:1003.1823 [math.CV].