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Abstract

A general method for calculation of the full isometry group of a Riemannian solvmanifold is presented.

Using it we determine the full isometry group of the non-symmetric quaternionic Kähler solvmanifolds M: $\mathcal{T}$-, $\mathcal{W}$- and $\mathcal{V}$-spaces.

As an application we prove that the isometry group acts transitively on the twistor space and on the $SO(3)$-principal ("3-Sasakian") bundle of M and that the manifold M does not admit quotients of finite volume.

As other application, we give a simple description of the quaternionic Kähler solvmanifolds in terms of a certain spinorial module S of the group $Spin(3, 3+k)$. The Lie bracket is defined by means of the unique embedding of the vector module $V = \mathbb{R}^{3,3+k}$ into $\wedge^2 S$. We describe also the group of isometries which preserves the principal Kähler submanifold $\mathcal{U} \subset M$.

Introduction

We recall that a quaternionic Kähler manifold is a Riemannian manifold M^{4n} with holonomy group Hol contained in $Sp(1)Sp(n)$. These are Einstein ma-

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nifolds if $n > 1$. The only known examples of quaternionic Kähler manifolds of positive scalar curvature are symmetric and in one-to-one correspondence with the simple compact Lie algebras (Wolf spaces). On the other hand, many examples of non-symmetric quaternionic Kähler manifolds of negative scalar curvature are known. The first such examples were constructed in [A 2], where the classification of quaternionic Kähler manifolds admitting a transitive splittable solvable group of isometries was given. One additional series of such manifolds, missing in the classification [A 2] was discovered by the physicists B. de Wit and A. Van Proeyen in the context of supergravity models and the original algebraic classification was completed in [C]. The final result is that there exist three such series of non-symmetric homogeneous quaternionic Kähler manifolds depending on integer parameters: $\mathcal{T}(p)$, $\mathcal{W}(p,q)$ and $\mathcal{V}(l,k)$ $(k \not\equiv 0\ (4))$, $\mathcal{V}(p,q;k)$ $(k \equiv 0\ (4))$. We will call them quaternionic Kähler solvmanifolds.

Quaternionic Kähler solvmanifolds and the associated "principal Kähler submanifolds" are studied by physicists (s. [Ce], [C-F-G], [dW-VP 0-2] and [dW-V-VP]) in the context of $N = 2$ supergravity. They describe the coupling of n vector multiplets to $N = 2$ supergravity in dimension $d = 4$ in terms of a meromorphic function $F(X)$ on $\mathbb{C}^{n+1}$ which is homogeneous of degree 2. They associate with F a Kähler metric g on (an open subset of) the projective space $P\mathbb{C}^{n+1}$ with the Kähler potential

$$K = \ln\left[\sum_{i,j=0}^{n}(\partial_i\partial_j F + \overline{\partial_i\partial_j F})\frac{X^i}{X^0}\frac{X^j}{X^0}\right].$$

The corresponding metric is called *special Kähler*. Using dimensional reduction from $d = 4$ to $d = 3$, to every special Kähler metric g they associate a metric $\tilde{g}$, which appear to be quaternionic Kähler and is also called *special*. The metric $\tilde{g}$ is explicitely expressed in terms of F, it is homogeneous if g is homogeneous. If $\tilde{g}$ is the metric of a quaternionic Kähler solvmanifold, then g is the metric of the essential part $\mathcal{U}_0$ of the principal Kähler submanifold $\mathcal{U} = \mathcal{F}_0 \times \mathcal{U}_0$ (associated with the principal Kählerian subalgebra $\mathfrak{u} = \mathfrak{f}_0 + \mathfrak{u}_0$). In [dW-VP 0-2] and [dW-V-VP] B. de Wit, F. Vanderseypen and A. Van Proeyen studied the isometries of special geometries and in particular described the full isometry algebra of quaternionic Kähler solvmanifolds in terms of symmetries of the Lagrangian associated to F.

In the present paper we determine the full isometry algebra and the full isometry group of the quaternionic Kähler solvmanifolds using the purely algebraic approach of Lie group theory. It seems that our results are consistent with that of B. de Wit, F. Vanderseypen and A. Van Proeyen.

Now we sketch the structure of the paper. In section 1 a general method for calculation of the full isometry Lie algebra and the full isometry Lie group of Riemannian solvmanifolds is presented. For simplicity we assume that the simply transitive solvable Lie group satisfies some additional conditions which are fulfilled for the quaternionic Kähler solvmanifolds (also for homogeneous bounded domains and for homogeneous spaces of non-positive curvature).

Let (M, g) be a Riemannian manifold with a simply transitive group of isometries $\mathcal{L}$. We can equip the Lie algebra $\mathfrak{l}$ of $\mathcal{L}$ with the Euclidean scalar product $\langle , \rangle$ induced by g. If M is simply connected it can be reconstructed from the metric Lie algebra $(\mathfrak{l}, \langle , \rangle)$.

Under some conditions we prove that the description of the full isometry Lie algebra $\mathfrak{g}(M)$ reduces to the description of so-called suitable SR-decompositions $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$ of $(\mathfrak{l}, \langle , \rangle)$, s. Thm. 1.1. An SR-decomposition is a semi-direct orthogonal decomposition into an ideal $(\mathfrak{l}_r, \langle , \rangle)$ and the Iwasawa algebra $(\mathfrak{l}_s, \langle , \rangle)$ of some semi-simple Lie algebra $\mathfrak{s}$ with scalar product associated to a symmetric Riemannian metric on the corresponding manifold $\mathcal{L}_s$. It is suitable if the adjoint representation $ad : \mathfrak{l}_s \to \mathfrak{Der}(\mathfrak{l}_r)$ is extended to a representation $\rho : \mathfrak{s} \to \mathfrak{Der}(\mathfrak{l}_r)$ with some properties.

If $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$ is a suitable SR-decomposition with maximal dimension of $\mathfrak{l}_s$, then $\mathfrak{g}(M) = (\mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r)) \oplus \mathfrak{l}_r$, where $\mathfrak{d}_0(\mathfrak{l}_r)$ denotes the Lie algebra of skew-symmetric derivations of $\mathfrak{l}_r$ which commute with $\rho(\mathfrak{s})$.

Starting from the description of the full isometry algebra, we can describe the full isometry group in terms of outer automorphisms of the semisimple Lie algebra $\mathfrak{s}$ (s. Thm. 1.5, Cor. 1.8 and Cor. 1.9).

In section 2 we apply this algorithm to the quaternionic Kähler solvmanifolds and determine the full isometry algebra of the $\mathcal{T}$-, $\mathcal{W}$- and $\mathcal{V}$-spaces (s. Thm. 2.5, Thm. 2.11 and Thm 2.18). In all the cases the maximal semisimple Lie subalgebra of non-compact type $\mathfrak{s} \subset \mathfrak{g}(M)$ has the form $\mathfrak{s} = \mathfrak{so}(3, 3+k)$.

In section 3 we determine the automorphism group of the Lie algebra $\mathfrak{so}(3, 3+k)$ (s. Prop. 3.1) and use this to determine the full (not necessarily connected) isometry group (s. Thm. 3.6, Thm. 3.7 and Thm. 3.9).

Our results have the following consequences:

1. The isometry group of the quaternionic Kählerian solvmanifold M acts transitively on the twistor space and on the $SO(3)$-principal bundle of M (s. Thm. 3.10).

2. A quaternionic Kähler solvmanifold M admits a (smooth) quotient M/Γ of finite volume if and only if M is symmetric (s. Cor. 1.10 and Cor. 1.11); here Γ is a discrete group of isometries.

Theorems 3.6, 3.7 and 3.9 give a new simple description of the quaternionic Kähler solvmanifolds in terms of certain spinorial representation of $Spin(3, 3+k)$.

For example, we can describe the quaternionic Kähler manifold $\mathcal{T}(p)$, $p \geq 1$, as follows. Let $(V = \mathbb{R}^4, \omega_0)$ be Euclidean 4-space with standard symplectic form ω_0. Then $\wedge^2 V$ as $Sp(V)$-module can be decomposed as

$$\wedge^2 V = \wedge_0^2 V + \mathbb{R}\omega^0 ,$$

where $\omega^0 = \omega_0^{-1} \in \wedge^2 V$. We define a Lie algebra structure on the vector space

$$\mathfrak{t}_r = V \otimes \mathbb{R}^p + \wedge_0^2 V + \mathbb{R}\omega^0$$

by

$$ad_{\omega^0}|V \otimes \mathbb{R}^p = \frac{1}{2} Id, \quad ad_{\omega^0}| \wedge_0^2 V = Id\,, \quad [\wedge_0^2 V, V \otimes \mathbb{R}^p + \wedge_0^2 V] = 0\,;$$

$$[v \otimes x, w \otimes y] = (v \wedge w)_0 \langle x, y\rangle\,, \quad v, w \in V, \quad x, y \in \mathbb{R}^p\,,$$

where the subindex 0 denotes the natural projection $\wedge^2 V \to \wedge_0^2 V$ and $\langle , \rangle$ is the standard scalar product on $\mathbb{R}^p$. We denote by $\langle , \rangle$ the scalar product on $\mathfrak{t}_r$ induced by the given scalar products on V and $\mathbb{R}^p$. By definition the decomposition of $\mathfrak{t}_r$ is orthogonal,

$$\langle v \wedge w, v \wedge w\rangle = \langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2 \quad \text{and} \quad \langle \omega^0, \omega^0\rangle = 2\,.$$

The symplectic group $Sp(V) = Sp(4, \mathbb{R})$ acts on the Lie algebra $\mathfrak{t}_r$ as group of automorphisms. Its maximal compact subgroup $U(2)$ preserves the scalar product on $\mathfrak{t}_r$. Consider the linear group $Sp(V) \cdot \mathcal{T}_r \subset Aut(\mathfrak{t}_r)$ with Lie algebra $\mathfrak{sp}(V) \oplus \mathfrak{t}_r \subset \mathfrak{Der}(\mathfrak{t}_r)$. Then

$$\mathcal{T}(p) \cong \frac{Sp(V) \cdot \mathcal{T}_r}{U(2)}\,.$$

Let $\mathfrak{sp}(V) = \mathfrak{u}(2) + \mathfrak{m}$ be a Cartan decomposition and $\langle , \rangle_{\mathfrak{m}}$ be a certain $\mathfrak{u}(2)$-invariant scalar product on $\mathfrak{m}$. Then the quaternionic Kähler metric on $\mathcal{T}(p) = (Sp(V) \cdot \mathcal{T}_r)/U(2)$ is induced by the scalar product $\langle , \rangle_{\mathfrak{m}} + \langle , \rangle$ on $\mathfrak{m} + \mathfrak{t}_r \cong T_{U(2)}\mathcal{T}(p)$.

Remark that in this case we may identify $Sp(4, \mathbb{R}) = Spin_0(3, 2)$; then V is the semi-spinor module of $Spin(3, 2)$ and $\wedge_0^2 V = \mathbb{R}^{3,2}$ the vector representation of $Spin(3, 2)$. A similar description is given for the other two series.

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Contents

1 Isometry Groups of Riemannian Solvmanifolds

1.1 Admissible Lie Algebras

Definition 1.1 *A* **metric Lie algebra** *is a Lie algebra* $\mathfrak{l}$ *together with a Euclidean scalar product* $\langle , \rangle$. *We say that a metric Lie algebra is* **irreducible** *if it cannot be decomposed into a direct orthogonal sum of two metric Lie algebras. A solvable metric Lie algebra* $(\mathfrak{l}, \langle , \rangle)$ *is said to be* **admissible** *if*

(i) it is an orthogonal sum of an Abelian Cartan subalgebra $\mathfrak{a}$ *and of its derived Lie algebra* $\mathfrak{n} = [\mathfrak{l}, \mathfrak{l}]$ *:*

$$\mathfrak{l} = \mathfrak{a} + \mathfrak{n} \tag{1}$$

(ii) the operators $ad_A|\mathfrak{n}$, $A \in \mathfrak{a} - \{0\}$, *are non-zero and symmetric and*

(iii) there exists an element $A^0 \in \mathfrak{a}$ such that $ad_{A^0}|\mathfrak{n} > 0$.

We denote by $\mathcal{R} \subset \mathfrak{a}^*$ the set of roots of $\mathfrak{l}$ with respect to the Cartan subalgebra $\mathfrak{a}$. We have the following orthogonal root space decomposition

$$\mathfrak{l} = \mathfrak{a} + \sum_{\alpha \in \mathcal{R}} \mathfrak{n}_\alpha, \quad \mathfrak{n}_\alpha = \{X \in \mathfrak{n} | \quad ad_A X = \alpha(A) X\}.$$

Remark 1: $\mathfrak{l}$ has trivial center: $\mathfrak{zent}(\mathfrak{l}) = 0$. The nilradical of $\mathfrak{l}$ is $\mathfrak{n}$.

Example 1: Let $\mathfrak{s}$ be a semisimple Lie algebra of non-compact type and $\mathfrak{s} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ its Iwasawa decomposition, where $\mathfrak{k}$ is a maximal compact subalgebra, that is the Lie algebra of a maximal compact subgroup of the adjoint group, $\mathfrak{n}$ is a nilpotent subalgebra and $\mathfrak{a}$ is a maximal ad-diagonalizable subalgebra normalizing $\mathfrak{n}$. Then $\mathfrak{i}(\mathfrak{s}) = \mathfrak{a} + \mathfrak{n}$ equipped with the scalar product $\langle , \rangle_B$ induced on $\mathfrak{i}(\mathfrak{s}) \cong \mathfrak{s}/\mathfrak{k}$ by the Cartan-Killing-form B is admissible. The metric Lie algebra $(\mathfrak{i}(\mathfrak{s}), \langle , \rangle_B)$ is unique up to isomorphism (of metric Lie algebras). More generally we will consider Euclidean scalar products $\langle , \rangle$ on $\mathfrak{i}(\mathfrak{s})$ which are obtained from $\langle , \rangle_B$ via scaling along the irreducible summands of $(\mathfrak{i}(\mathfrak{s}), \langle , \rangle_B)$. We say that the admissible Lie algebra $(\mathfrak{i}(\mathfrak{s}), \langle , \rangle)$ is a **symmetric Iwasawa algebra** associated with $\mathfrak{s}$. We remark that the scalar products $\langle , \rangle$ described above are precisely the Euclidean scalar products such that $(\mathfrak{i}(\mathfrak{s}), \langle , \rangle)$ is symmetric in the sense of Def. 1.2. In this case $\mathcal{R} \cup -\mathcal{R}$ is a root system, the system of restricted roots of the real Lie algebra $\mathfrak{s}$.

Example 2: Let $\mathfrak{l} = \mathfrak{l}_1 \oplus\!\!\!\!+ \mathfrak{l}_2$ be an orthogonal semidirect sum of admissible Lie algebras $\mathfrak{l}_i = \mathfrak{a}_i + \mathfrak{n}_i$ $(i = 1, 2)$, i.e. $[\mathfrak{l}_1, \mathfrak{l}_2] \subset \mathfrak{l}_1$. A necessary and sufficient condition for $\mathfrak{l}$ to be admissible is that the operators ad_A, $A \in \mathfrak{a}_1 + \mathfrak{a}_2$, be symmetric.

1.2 Holonomy Conditions

The notion of **covariant derivative** is defined for metric Lie algebras by means of the following **Koszul formula** $(X, Y, Z \in \mathfrak{l})$:

$$2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle - \langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle .$$

Definition 1.2 *Let $\mathfrak{l}$ be a metric Lie algebra. The* **Kostant algebra** $\mathfrak{kos}$ *is the Lie subalgebra of $\mathfrak{so}(\mathfrak{l})$ generated by the endomorphisms ∇_X $(X \in \mathfrak{l})$.*

The **holonomy algebra** $\mathfrak{hol}$ *is the ideal of $\mathfrak{kos}$ generated by the expressions of the type*

$$[\nabla_{X_1}, \ldots [\nabla_{X_k}, R(X_{k+1}, X_{k+2})] \ldots] , \quad X_1, X_2, \ldots X_{k+2} \in \mathfrak{l}, \quad k = 0, 1, 2, \ldots .$$

$\mathfrak{l}$ is said to be **symmetric** *if the curvature tensor is annihilated by $\mathfrak{hol}$.*

Definition 1.3 *Let V be a Euclidean vector space. A* **complex structure** *on V is a skew-symmetric endomorphism J satisfying $J^2 = -Id$.*

A **quaternionic structure** *on V is a Lie subalgebra $\mathfrak{q}$ of $\mathfrak{so}(V)$ generated by two anticommuting complex structures J_1 and J_2.*

Definition 1.4 *A metric Lie algebra $\mathfrak{l}$ with complex structure J is said to be* **Kählerian**, *if*

$$[\mathfrak{hol}, J] = 0\,.$$

A metric Lie algebra $\mathfrak{l}$ of dimension > 4 (resp. $= 4$) with quaternionic structure $\mathfrak{q}$ is said to be **quaternionic Kählerian**, *if*

$$[\mathfrak{hol}, \mathfrak{q}] \subset \mathfrak{q}$$

(resp. if $\mathfrak{q}$ annihilates the curvature tensor).

Example 3: The basic examples of admissible Kählerian Lie algebras are the following.

A **key algebra** $\mathfrak{f} = span\{G, H\}$ **with root** $\mu \geq 0$ is defined in terms of the orthonormal basis $G = JH, H$ by the formula $[H, G] = \mu G$. Up to scaling $\mathfrak{f} = \mathfrak{i}(\mathfrak{su}(1,1))$.

Given $\mathfrak{f}$ and a Euclidean vector space $\mathfrak{x}$ with complex structure, $\mathfrak{e} = \mathfrak{f} + \mathfrak{x}$ carries a canonical Euclidean scalar product $\langle , \rangle$ and complex structure J. The structure of an **elementary Kählerian Lie algebra** with key subalgebra $\mathfrak{f}$ is defined on $\mathfrak{e}$ by the formulas

$$ad_H|\mathfrak{x} = \frac{\mu}{2} Id, \quad ad_G|\mathfrak{x} = 0 \quad \text{and} \quad [X, Y] = \mu\langle JX, Y\rangle G \quad \text{for} \quad X, Y \in \mathfrak{x}\,.$$

The metric Lie algebra $\mathfrak{e}$ is determined up to isomorphism by $n = \dim_{\mathbb{C}} \mathfrak{x}$ and μ. If we wish to specify these parameters, we shall write $\mathfrak{e} = \mathfrak{e}(n+1, \mu)$. Up to scaling $\mathfrak{e}(n+1, \mu) = \mathfrak{i}(\mathfrak{su}(1, n+1))$.

Elementary Kählerian Lie algebras can be used as building blocks for the construction of more complicated admissible Kählerian Lie algebras via semidirect sums (v. Example 4).

Example 4: Let $\mathfrak{l} = \mathfrak{l}_1 \oplus_\varphi \mathfrak{l}_2$ be an admissible orthogonal semidirect sum of admissible Kählerian Lie algebras $(\mathfrak{l}_i, \langle , \rangle_i, J_i)$, $i = 1, 2$ (v. Example 2). The admissible Lie algebra $\mathfrak{l}$ is Kählerian with complex structure $J = J_1 + J_2$ if and only if

1. $\varphi : \mathfrak{l}_2 \to \mathfrak{der}(\mathfrak{l}_1)$ preserves the Kähler form $\rho_1 = \langle J_1 \cdot, \cdot \rangle$ of $\mathfrak{l}_1$, i.e. $\rho_1(\varphi(X)Y, Z) + \rho_1(Y, \varphi(X)Z) = 0$ for all $X \in \mathfrak{l}_2$, $Y, Z \in \mathfrak{l}_1$ and

2. $\varphi(J_2X)^{sym} = J_1\varphi(X)^{sym}$ for all $X \in \mathfrak{l}_2$, where $\varphi(X)^{sym}$ denotes the symmetric part of $\varphi(X)$.

Such symplectic representations were studied for elementary Kählerian Lie algebras by S.G. Gindikin, I.I. Pyateckiĭ-Shapiro and E.B. Vinberg (v. e.g. [G-PS-V] and [PS]).

According to I.I. Pyatetskiĭ-Shapiro [PS] every homogeneous bounded domain in $\mathbb{C}^n$ with Bergmann-metric admits a simply transitive group of isometries such that the corresponding metric Lie algebra $\mathfrak{l}$ can be decomposed as $\mathfrak{l} = \mathfrak{e}_1 \bar{\oplus} \mathfrak{e}_2 \bar{\oplus} \cdots \bar{\oplus} \mathfrak{e}_k$, where the $\mathfrak{e}_i$ are elementary and the semidirect sum is of the type presented in the example.

Example 5: The classification of admissible quaternionic Kählerian Lie algebras follows from the classification of quaternionic Kählerian solvmanifolds which is due to the first author [A 2] and was completed by B. de Wit, A. Van Proeyen [dW-VP 1] and the second author [C]. Besides the (non-Abelian) symmetric examples, which are precisely $\mathfrak{i}(\mathfrak{so}(p+4,4))$ $(p \geq -1)$, $\mathfrak{i}(\mathfrak{su}(m,2))$, $\mathfrak{i}(\mathfrak{sp}(m,1))$ $(m \geq 1)$, $\mathfrak{i}(\mathfrak{g}_2^{(2)})$, $\mathfrak{i}(\mathfrak{f}_4^{(4)})$, $\mathfrak{i}(\mathfrak{e}_6^{(2)})$, $\mathfrak{i}(\mathfrak{e}_7^{(-5)})$ and $\mathfrak{i}(\mathfrak{e}_8^{(-24)})$, this classification yields also infinitely many non-symmetric admissible quaternionic Kählerian Lie algebras. The classification reduces essentially to the classification of admissible Kählerian Lie algebras $(\mathfrak{u}, \langle,\rangle, J)$ of the type $\mathfrak{u} = \mathfrak{f}_0 \bar{\oplus} \mathfrak{u}_0$ $(\mathfrak{f}_0 = span\{G_0, H_0\} = \mathfrak{e}(1,1)$, $\mathfrak{u}_0 = \mathfrak{e}(n_1+1, \mu_1) \bar{\oplus} \mathfrak{e}(n_2+1,\mu_2) \bar{\oplus} \cdots \bar{\oplus} \mathfrak{e}(n_k+1,\mu_k))$ which admit a Q-representation. We recall that a Q-representation is a representation $T : \mathfrak{u} \to End(\tilde{\mathfrak{u}})$ which satisfies a list of technical conditions (v. [A 2] Def. 5.3). $\tilde{\mathfrak{u}}$ is isometric to $\mathfrak{u}$ via an isometry $\tilde{}: \mathfrak{u} \to \tilde{\mathfrak{u}}$ and carries a natural symplectic structure $\hat{\rho} = \langle \hat{J}\cdot, \cdot\rangle$, which is given by the complex structure $\hat{J}$: $\hat{J}\tilde{U}_0 = -\widetilde{JU}_0$, $\hat{J}\tilde{F}_0 = \widetilde{JF}_0$, $U_0 \in \mathfrak{u}_0$, $F_0 \in \mathfrak{f}_0$. $T|\mathfrak{u}_0$ is symplectic with respect to $\hat{\rho}$.

Given a Q-representation, $\mathfrak{l} = \mathfrak{u} \bar{+} \tilde{\mathfrak{u}}$ carries a canonical structure of admissible quaternionic Kählerian Lie algebra such that

(i) $\mathfrak{u}$ is a subalgebra of $\mathfrak{l}$ (called principal Kählerian subalgebra);

(ii) $ad_U|\tilde{\mathfrak{u}} = T_U$ for all $U \in \mathfrak{u}$ and

(iii) $[\tilde{U}, \tilde{V}] = \hat{\rho}(\tilde{U}, \tilde{V})G_0$ for all $U, V \in \mathfrak{u}$.

The quaternionic structure $\mathfrak{q} = span\{J_\alpha | \alpha = 1,2,3\}$ is defined as follows:

$$J_1U := JU\,, \quad J_1\tilde{U} := -\widetilde{JU}\,, \quad J_2U := \tilde{U}, \quad J_2\tilde{U} := -U\,, \quad J_3 = J_1J_2\,,$$

for $U \in \mathfrak{u}$. The main examples of Q-representations will be presented in section 2.

1.3 Riemannian Homogeneous Spaces associated with an admissible Lie Algebra

Let an admissible Lie algebra $(\mathfrak{l}, \langle , \rangle)$ be given and let $\mathcal{L}$ be the corresponding simply connected Lie group. We denote by g the left-invariant Riemannian metric on $\mathcal{L}$ defined by $\langle , \rangle$. We say that $(\mathcal{L}, g)$ is the Riemannian homogeneous space associated with $(\mathfrak{l}, \langle , \rangle)$. Then the definitions of section 1.2 may be reformulated in terms of the homogeneous Riemannian space $(\mathcal{L}, g)$ as follows: The Koszul formula gives the covariant derivative of a left-invariant vector field Y^* on $\mathcal{L}$ corresponding to $Y \in \mathfrak{l}$ in the point $e \in \mathcal{L}$ under the identification $T_e\mathcal{L} = \mathfrak{l}$. The holonomy algebra $\mathfrak{hol}$ is the Lie algebra of the holonomy group in the point e of the manifold $(\mathcal{L}, g)$. Symmetric (resp. Kähler, resp. quaternionic Kähler) metric Lie algebras correspond exactly to symmetric (resp. Kähler, resp. quaternionic Kähler) manifolds $(\mathcal{L}, g)$. Moreover the Riemannian manifold $(\mathcal{L}, g)$ is irreducible if and only if the admissible Lie algebra $(\mathfrak{l}, \langle , \rangle)$ is irreducible (v. [A 1]). Remark that for arbitrary metric Lie algebras this is not true.

We will describe an algorithm to determine the full isometry group $I(\mathcal{L}, g)$ and the full isometry algebra $Lie\, I(\mathcal{L}, g)$ of the homogeneous Riemannian manifold $(\mathcal{L}, g)$.

1.4 Full Isometry Group and Algebra

Definition 1.5 *An* **SR-decomposition** *of an admissible Lie algebra* $\mathfrak{l}$ *is an orthogonal semidirect sum* $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$*, where*

(i) $\mathfrak{l}_s = \mathfrak{i}(\mathfrak{s})$ *is a symmetric Iwasawa algebra of a semisimple Lie algebra* $\mathfrak{s}$ *of non-compact type (i.e. without compact ideals* $\neq 0$*).*

(ii) $\mathfrak{l}_r$ *is some admissible ideal or trivial.*

It was proved by S.I. Araki [Ar] that the Lie algebra $\mathfrak{i}(\mathfrak{s})$ uniquely determines $\mathfrak{s}$. Note that if $\mathfrak{l} = \mathfrak{a} + \mathfrak{n}$, $\mathfrak{l}_s = \mathfrak{a}_s + \mathfrak{n}_s$ and $\mathfrak{l}_r = \mathfrak{a}_r + \mathfrak{n}_r$ are the orthogonal decompositions (1) then $\mathfrak{a} = \mathfrak{a}_s + \mathfrak{a}_r$, $\mathfrak{n} = \mathfrak{n}_s + \mathfrak{n}_r$, $[\mathfrak{a}_r, \mathfrak{l}_s] = 0$, $\mathfrak{n}_r = \sum_{\alpha \in \mathcal{R}_r} \mathfrak{n}_\alpha$ and $\mathfrak{n}_s = \sum_{\alpha \in \mathcal{R}_s} \mathfrak{n}_\alpha$, where $\mathcal{R}_r = \{\alpha \in \mathcal{R} |\ \alpha|\mathfrak{a}_r \neq 0\}$, $\mathcal{R}_s = \{\alpha \in \mathcal{R} |\ \alpha|\mathfrak{a}_r = 0\}$ and the set of roots $\mathcal{R}$ has the decomposition $\mathcal{R} = \mathcal{R}_r \cup \mathcal{R}_s$.

We denote by $\mathfrak{der}(\mathfrak{l})$ respectively $\mathfrak{d}(\mathfrak{l})$ the Lie algebra of derivations respectively skew-symmetric derivations of $\mathfrak{l}$. Remark that the semidirect sum $\mathfrak{d}(\mathfrak{l}) +\!\!\!\!\supset \mathfrak{l}$ is an isometry Lie algebra on $(\mathcal{L}, g)$, this means that it is the Lie algebra of some Lie group of isometries of the Riemannian manifold $(\mathcal{L}, g)$ (v. [K-N 1] Ch. VI Thm. 3.6. and Thm. 7.2.). But in general it is not the full isometry algebra, i.e. the Lie algebra of the full isometry group. For an SR-decomposition $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$ we denote by $\mathfrak{d}_0(\mathfrak{l}_r)$ the Lie algebra of derivations of $\mathfrak{l}_r$ which can be extended to derivations of $\mathfrak{l}$ vanishing on $\mathfrak{l}_s$.

Definition 1.6 *An SR-decomposition $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$, $\mathfrak{l}_s = \mathfrak{i}(\mathfrak{s})$ of an irreducible admissible Lie algebra $\mathfrak{l}$ is called* **suitable**, *if the representation $ad : \mathfrak{l}_s \to \mathfrak{Der}(\mathfrak{l}_r)$ is extended to a representation $\rho : \mathfrak{s} \to \mathfrak{Der}(\mathfrak{l}_r)$ such that $\mathfrak{k}^\rho := \rho(\mathfrak{s}) \cap \mathfrak{d}(\mathfrak{l}_r)$ is a maximal compact subalgebra in $\rho(\mathfrak{s})$. A suitable SR-decomposition is called* **maximal** *if* $\dim \mathfrak{l}_s$ *is maximal.*

Remark 2: From the irreducibility it follows that ρ is faithful.

The following theorem gives an algorithm which allows to compute the full isometry algebra of the homogeneous Riemannian manifold $(\mathcal{L}, g)$ associated with an irreducible admissible Lie algebra $(\mathfrak{l}, \langle,\rangle)$.

Theorem 1.1 *Let $(\mathfrak{l}, \langle,\rangle)$ be an irreducible admissible Lie algebra. Any suitable SR-decomposition $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$, $\mathfrak{l}_s = \mathfrak{i}(\mathfrak{s})$ (with representation ρ) defines an isometry algebra $\mathfrak{g} \supset \mathfrak{l}$ of the corresponding homogeneous Riemannian manifold $(\mathcal{L}, g)$, which is described as follows:*

$$\mathfrak{g} = \mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r) + \mathfrak{l}_r \,, \tag{2}$$

$\mathfrak{l}_r$ is an ideal with the natural action of the Lie algebra $\mathfrak{d}_0(\mathfrak{l}_r)$ and the action of $\mathfrak{s}$ defined by the representation $\rho : \mathfrak{s} \to \mathfrak{Der}(\mathfrak{l}_r)$ and $[\mathfrak{s}, \mathfrak{d}_0(\mathfrak{l}_r)] = 0$. Moreover we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{l}\,, \quad \mathfrak{s} = \mathfrak{k}_s + \mathfrak{l}_s \quad \text{and} \quad \mathfrak{k} \cap \mathfrak{l} = 0\,,$$

where $\mathfrak{k} = \mathfrak{k}_s + \mathfrak{d}_0(\mathfrak{l}_r)$ is a maximal compact subalgebra of $\mathfrak{g}$ and $\mathfrak{k}_s = \rho^{-1}(\mathfrak{k}^\rho)$ is a maximal compact subalgebra of $\mathfrak{s}$.

A maximal suitable SR-decomposition is unique and defines the full isometry algebra $\mathfrak{g}$.

Proof. Let $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$ be a suitable SR-decomposition and $\mathfrak{g}$ the associated Lie algebra described in the theorem. It is immediate that $\mathfrak{g} = \mathfrak{k} + \mathfrak{l}$ with stated properties.

To prove that $\mathfrak{g}$ is an isometry Lie algebra on $(\mathcal{L}, g)$ it is sufficient to construct a connected Lie group $\mathcal{G}$ with $Lie\,\mathcal{G} = \mathfrak{g}$ such that the connected subgroup $\mathcal{K}$ with $Lie\,\mathcal{K} = \mathfrak{k}$ is compact and hence closed. Then we may identify $\mathcal{L} \cong \mathcal{G}/\mathcal{K}$. For this we need Lemma 1.2.

Lemma 1.2 *The adjoint representation $\rho : \mathfrak{g} \to \mathfrak{Der}(\mathfrak{l}_r)$ is faithful.*

Proof. Suppose that $\mathfrak{h} = \ker \rho \neq 0$. By remarks 1 and 2 $\mathfrak{h} \cap \mathfrak{l}_r = \mathfrak{zent}(\mathfrak{l}_r) = 0$ and $\mathfrak{h} \cap (\mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r)) = 0$. Hence, the projection $\mathfrak{b}$ of $\mathfrak{h}$ onto $\mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r)$ is a non-trivial ideal of $\mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r)$. Since any semisimple Lie algebra has a non-trivial compact subalgebra and $\mathfrak{d}_0(\mathfrak{l}_r)$ is compact , there exists $X \in \mathfrak{b}$ such that $\rho(X) \neq 0$ has purely imaginary eigenvalues. Choose $Y \in \mathfrak{l}_r$ such that $\rho(X+Y) = 0$. Then $\rho(Y) \neq 0$ has purely imaginary eigenvalues. This is impossible because $\mathfrak{l}_r$ is admissible. □

We define $\mathcal{G}$ as the group of automorphisms of the Lie algebra $\mathfrak{l}_r$ generated by the linear Lie algebra $\rho(\mathfrak{g}) \subset \mathfrak{der}(\mathfrak{l}_r)$. By Lemma 1.2 the adjoint representation $\rho : \mathfrak{g} \to \mathfrak{der}(\mathfrak{l}_r)$ is faithful, hence $Lie\, \mathcal{G} = \mathfrak{g}$.

Denote by $\mathcal{S}$, $A^0(\mathfrak{l}_r)$ and $\mathcal{L}_r$ the connected linear Lie groups which correspond to the summands of $\mathfrak{g} = (\mathfrak{s} \oplus \mathfrak{d}_0(\mathfrak{l}_r)) \,+\!\!\!\!\!\supset \mathfrak{l}_r$. They are closed in $GL(\mathfrak{l}_r)$ because $\mathfrak{s}$ is semisimple and hence algebraic, $\mathfrak{d}_0(\mathfrak{l}_r)$ is algebraic by definition and $\mathcal{L}_r$ is splittable solvable with trivial center. This implies that $\mathcal{G} = \mathcal{S} A^0(\mathfrak{l}_r) \mathcal{L}_r$ is a closed linear group. Denote by $\mathcal{K}$ the subgroup of $\mathcal{G}$ with $Lie\, \mathcal{K} = \rho(\mathfrak{k}) \cong \mathfrak{k}_s + \mathfrak{d}_0(\mathfrak{l}_r)$. Since $\rho(\mathfrak{k})$ consists of skew-symmetric operators the closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ in $GL(\mathfrak{l}_r)$ (and hence in $\mathcal{G}$) is compact. On the other hand $\mathcal{K} = \mathcal{K}_s A^0(\mathfrak{l}_r)$ is closed in $\mathcal{G}$ since the second factor is compact and $\mathcal{K}_s$ (the Lie group generated by $\mathfrak{k}_s$) is a closed subgroup of $\mathcal{S}$ and hence of $\mathcal{G}$. This shows that $\mathcal{K}$ is a maximal compact subgroup of $\mathcal{G}$. This proves that every suitable SR-decomposition $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$ gives rise to an isometry algebra $\mathfrak{g} \supset \mathfrak{l}$ of $(\mathcal{L}, g)$.

To prove the last statement let $(\mathcal{L}, g)$ be the Riemannian homogeneous space associated with the irreducible admissible Lie algebra $(\mathfrak{l}, \langle , \rangle)$. Denote by $\mathfrak{g}$ the full isometry algebra of $(\mathcal{L}, g)$ and by $\mathfrak{k}$ the isotropy subalgebra of the point $e \in \mathcal{L}$ in $\mathcal{G}$; it is a maximal compact subalgebra of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{l}$ and $\mathfrak{k} \cap \mathfrak{l} = 0$, because $\mathcal{L}$ acts simply transitively. Now we make use of two lemmas.

Lemma 1.3 *There exists a decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{l} = \mathfrak{s} + \mathfrak{s}_c + \mathfrak{r}$, where $\mathfrak{r}$ is the radical and $\mathfrak{s} + \mathfrak{s}_c$ is the splitting of a maximal semisimple subalgebra into a direct sum of a non-compact and a maximal compact ideal, such that*

a) $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$, $\mathfrak{l}_s = \mathfrak{i}(\mathfrak{s}) = \mathfrak{l} \cap \mathfrak{s}$, $\mathfrak{l}_r = \mathfrak{l} \cap \mathfrak{r}$, *is a suitable SR-decomposition. As following Def. 1.5 the corresponding decompositions (1) are denoted by* $\mathfrak{l} = \mathfrak{a} + \mathfrak{n}$, $\mathfrak{l}_s = \mathfrak{a}_s + \mathfrak{n}_s$ *and* $\mathfrak{l}_r = \mathfrak{a}_r + \mathfrak{n}_r$.

b) $\mathfrak{k} = \mathfrak{k}_s + \mathfrak{s}_c + \mathfrak{k}_r$, *where* $\mathfrak{k}_s = \mathfrak{k} \cap \mathfrak{s}$ *and* $\mathfrak{k}_r = \mathfrak{k} \cap \mathfrak{r}$ *are maximal compact subalgebras of* $\mathfrak{s}$ *and* $\mathfrak{r}$ *respectively.*

c) $\mathfrak{s} = \mathfrak{k}_s + \mathfrak{l}_s$, $\mathfrak{r} = \mathfrak{k}_r + \mathfrak{l}_r$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{l} = (\mathfrak{k}_s + \mathfrak{s}_c + \mathfrak{k}_r) + \mathfrak{l}_s + \mathfrak{l}_r$;

d) $[\mathfrak{k}_r, \mathfrak{s} + \mathfrak{s}_c + \mathfrak{k}_r] = [\mathfrak{s}, \mathfrak{s}_c] = 0$*;*

e) $[\mathfrak{k} + \mathfrak{s}, \mathfrak{a}_r] = 0$ *and* $\mathfrak{n}_r$ *is the nilradical of* $\mathfrak{g}$*.*

Proof. The proof can be found in the literature, s. [Wolt] 2.1 Satz, cf. [A 1] and [A-W 2]. In [Wolt] non-positive curvature is assumed. This ensures the existence of a simply transitive solvable group of isometries which gives rise to a Lie algebra of "non-positive curvature type". In our paper the existence of such a group is assumed from the very beginning, since admissible Lie algebras are of non-positive curvature type. □

Lemma 1.4 *Under the notation of the preceding lemma:*

1) The adjoint representation $\rho : \mathfrak{g} \to \mathfrak{Der}(\mathfrak{l}_r)$ *is faithful.*

2) $\rho(\mathfrak{s}_c + \mathfrak{k}_r) = \mathfrak{d}_0(\mathfrak{l}_r)$*, where* $\mathfrak{d}_0(\mathfrak{l}_r)$ *is defined after Def. 1.5.*

Proof. The proof of 1) is as the proof of Lemma 1.2. To prove 2) we remark that

$$\rho(\mathfrak{s}_c + \mathfrak{k}_r) \subset \mathfrak{d}_0(\mathfrak{l}_r)\,.$$

Indeed, $ad(\mathfrak{s}_c + \mathfrak{k}_r)$ is a Lie algebra of derivations of $\mathfrak{g}$ which acts trivially on $\mathfrak{s}$ by Lemma1.3 d), in particular it acts trivially on $\mathfrak{l}_s$ and hence it acts on $\mathfrak{l}$ and $ad(\mathfrak{s}_c + \mathfrak{k}_r)|\mathfrak{l}_r = \rho(\mathfrak{s}_c + \mathfrak{k}_r) \subset \mathfrak{d}_0(\mathfrak{l}_r)$.

Now it is sufficient to check that

$$\rho(\mathfrak{k}_s) \cap \mathfrak{d}_0(\mathfrak{l}_r) = 0\,.$$

Let $X \in \mathfrak{k}_s$ be such that $\rho(X) \in \mathfrak{d}_0(\mathfrak{l}_r)$, i.e. $\rho(X)$ can be extended to $D \in \mathfrak{d}(\mathfrak{l})$ with $D|\mathfrak{l}_s = 0$. Since $\mathfrak{g}$ is the full isometry algebra, such a derivation defines an element $Y \in \mathfrak{k}$ such that $ad_Y|\mathfrak{l} = D$. The difference $X - Y$ is in $\ker \rho = 0$ (v. 1)). Hence $ad_X \mathfrak{l} \subset \mathfrak{l}$ and $ad_X|\mathfrak{l}_s = 0$. This implies that $X = 0$, because the isotropy representation of the infinitesimal symmetric space $\mathfrak{s}/\mathfrak{k}_s$ is faithful. □

¿From Lemma 1.3 and 1.4 it follows that the full isometry algebra of $(\mathcal{L}, g)$ admits a decomposition (2), where $\mathfrak{s}$ is the non-compact part of some Levi subalgebra $\mathfrak{s} + \mathfrak{s}_c$ and $\mathfrak{d}_0(\mathfrak{l}_r) = \mathfrak{s}_c + \mathfrak{k}_r$, where $\mathfrak{k}_r$ is some maximal compact subalgebra of the radical $\mathfrak{r} = \mathfrak{k}_r + \mathfrak{l}_r$. Moreover, all the properties stated in the theorem are fulfilled. Hence we proved that the full isometry algebra is associated with some suitable SR-decomposition $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$. Let $\mathfrak{l} = \mathfrak{l}'_s + \mathfrak{l}'_r$ be an other suitable SR-decomposition and $\mathfrak{g}' = \mathfrak{s}' + \mathfrak{d}_0(\mathfrak{l}'_r) + \mathfrak{l}'_r$ the associated isometry algebra. We may embed $\mathfrak{g}' \subset \mathfrak{g}$ such that $\mathfrak{s}' \subset \mathfrak{s}$ due to the conjugacy of maximal semisimple subalgebras. Then $\mathfrak{l}'_s = \mathfrak{s}' \cap \mathfrak{l} \subset \mathfrak{l}_s = \mathfrak{s} \cap \mathfrak{l}$ and

$\dim \mathfrak{l}'_s \leq \dim \mathfrak{l}_s$. If $\dim \mathfrak{l}'_s = \dim \mathfrak{l}_s$ then $\mathfrak{l}'_s = \mathfrak{l}_s$ and $\mathfrak{l}'_r = \mathfrak{l}_r$ as orthogonal complement of $\mathfrak{l}_s$ in $\mathfrak{l}$. This finishes the proof of Theorem 1.1. □

Remark 3: Note that a subalgebra of a semisimple Lie algebra $\mathfrak{s}$ which generates a compact subgroup of $Ad\,\mathcal{S}$ may generate a non-compact subgroup of a Lie group $\mathcal{S}$ with $Lie\,\mathcal{S} = \mathfrak{s}$.

Let $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$ be the maximal suitable SR-decomposition of an irreducible admissible Lie algebra $(\mathfrak{l}, \langle , \rangle)$. We identify the full isometry algebra $\mathfrak{g}(\mathfrak{l}) = \mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r) + \mathfrak{l}_r$, provided by Theorem 1.1, with the linear algebra $\rho(\mathfrak{g}(\mathfrak{l})) \subset \mathfrak{Der}(\mathfrak{l}_r)$, using the faithful representation ρ and denote by $\mathcal{G}(\mathfrak{l}) \subset Aut(\mathfrak{l}_r)$ the corresponding linear group. It is the connected component of the unity of the full isometry group $I(\mathcal{L}, g)$

Now we describe $I(\mathcal{L}, g)$. Denote by $\mathcal{K}(\mathfrak{l})$ the group of all orthogonal automorphisms of the metric Lie algebra $\mathfrak{l}_r$ which normalize the subalgebra $\mathfrak{s} \subset \mathfrak{Der}(\mathfrak{l}_r)$. Note that $\mathcal{K}(\mathfrak{l})$ normalizes $\mathfrak{g}(\mathfrak{l})$.

Theorem 1.5 *Let $(\mathfrak{l}, \langle , \rangle)$ be an irreducible admissible Lie algebra and $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$ its maximal suitable SR-decomposition. Assume that the maximal semisimple subalgebra $\mathfrak{s}$ of non-compact type of the full isometry algebra $\mathfrak{g}(\mathfrak{l}) = \mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r) + \mathfrak{l}_r$ is simple. Then the (full) isometry group is*

$$I(\mathcal{L}, g) = \mathcal{K}(\mathfrak{l}) \cdot \mathcal{G}(\mathfrak{l}) \subset Aut(\mathfrak{l}_r)\,.$$

Moreover $\mathcal{K}(\mathfrak{l})$ is identified with the stabilizer $\mathcal{K}$ of the point $e \in \mathcal{L}$ and $\mathcal{K}(\mathfrak{l}) \cap \mathcal{G}(\mathfrak{l})$ is the connected component of unity in $\mathcal{K}(\mathfrak{l})$.

Remark 4: The theorem remains true also when $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k$ is not simple under the condition that for any two isomorphic ideals $\mathfrak{s}_i$, $\mathfrak{s}_j$ the corresponding Iwasawa ideals $\mathfrak{i}(\mathfrak{s}_i)$, $\mathfrak{i}(\mathfrak{s}_j)$ of $\mathfrak{l}_s$ are isomorphic as metric Lie algebras.

Proof. Denote by $\mathcal{K}$ the stabilizer of the point $e \in \mathcal{L}$ in $I(\mathcal{L}, g)$.

Lemma 1.6 *1) The adjoint representation of $\mathcal{K}$ on $\mathfrak{g}(\mathfrak{l})$ preserves the decomposition $\mathfrak{g}(\mathfrak{l}) = \mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r) + \mathfrak{l}_r$.*
2) The adjoint representation of $\mathcal{K}$ on $\mathfrak{l}_r$ is faithful.

Proof. 1) The stabilizer $\mathcal{K}_0$ of the point e in $\mathcal{G}(\mathfrak{l})$ is a (connected) maximal compact subgroup of $\mathcal{G}(\mathfrak{l})$. It may be written as

$$\mathcal{K}_0 = \mathcal{K}_s \cdot A^0(\mathfrak{l}_r)\,,$$

where $\mathcal{K}_s$ and $A^0(\mathfrak{l}_r)$ are the subgroups of $\mathcal{G}(\mathfrak{l})$ corresponding to $\mathfrak{k}_s$ (maximal compact subalgebra of $\mathfrak{s}$) and $\mathfrak{d}_0(\mathfrak{l}_r)$. It is clear that $\mathcal{K}_0$ preserves the decomposition $\mathfrak{g}(\mathfrak{l}) = \mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r) + \mathfrak{l}_r$. Any $k \in \mathcal{K}$ preserves $\mathfrak{k} = Lie\,\mathcal{K}$ and transforms the polar decomposition $\mathfrak{g}(\mathfrak{l}) = \mathfrak{k} + \mathfrak{l}$ into a polar decomposition $\mathfrak{g}(\mathfrak{l}) = \mathfrak{k} + Ad_k\mathfrak{l}$. Since any two polar decompostions are conjugated (s. [A 1]), there exists $k_0 \in \mathcal{K}_0$ such that Ad_{kk_0} preserves $\mathfrak{k}$ and $\mathfrak{l}$. Then it preserves also the decompositions

$$\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r, \quad \mathfrak{g}(\mathfrak{l}) = (\mathfrak{k}_s + \mathfrak{l}_s) + \mathfrak{d}_0(\mathfrak{l}_r) + \mathfrak{l}_r\,.$$

Hence, Ad_k preserves the decomposition $\mathfrak{g}(\mathfrak{l}) = \mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r) + \mathfrak{l}_r$. This proves 1).

2) Let $Ad_k|\mathfrak{l}_r = Id$ for some $k \in \mathcal{K}$. Then for any $A \in \mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r)$, $X \in \mathfrak{l}_r$ we have

$$Ad_k[A, X] = [Ad_kA, X] = [A, X]$$

therefore $[A - Ad_kA, X] = 0$. This implies $Ad_kA = A$ and $Ad_k|\mathfrak{g}(\mathfrak{l}) = Id$, since the representation of $\mathfrak{s} + \mathfrak{d}_0(\mathfrak{l}_r)$ on $\mathfrak{l}_r$ is faithful. ¿From the faithfulness of the isotropy representation it follows now that $k = id$. □

The lemma shows that $\mathcal{K} \subset \mathcal{K}(\mathfrak{l})$.

Lemma 1.7 $\mathcal{K}_0 = \mathcal{K}(\mathfrak{l})_0$

Proof. The group $Ad\mathcal{K}(\mathfrak{l})_0|\mathfrak{s}$ is a compact connected group of automorphisms of a semisimple Lie algebra, hence it consists of inner automorphisms. On the other hand it contains $Ad\mathcal{K}_s$, where $\mathcal{K}_s$ is maximal compact in $\mathcal{S}$. This implies $Ad\mathcal{K}(\mathfrak{l})_0|\mathfrak{s} = Ad\mathcal{K}_s$. Consider a one-parameter subgroup $\varphi(t) = \exp tX$ in $\mathcal{K}(\mathfrak{l})_0$. Modulo $\mathcal{K}_s \subset \mathcal{K}_0 = \mathcal{K}_s \cdot A^0(\mathfrak{l}_r)$ we may assume that $\varphi(t)$ commutes with $\mathfrak{s}$. This implies that $X \in \mathfrak{d}_0(\mathfrak{l}_r)$ (s. Def. 1.5) and hence $\varphi(t) \in A^0(\mathfrak{l}_r) \subset \mathcal{K}_0$. □

As before, due to the conjugacy of polar decompositions, any $\varphi \in \mathcal{K}(\mathfrak{l})$ after multiplication by some element from $\mathcal{K}_0 = \mathcal{K}(\mathfrak{l})_0$ preserves $\mathfrak{l}$. Then it preserves $\mathfrak{l}_r = \mathfrak{l} \cap \mathfrak{r}$, where $\mathfrak{r}$ is the radical of $\mathfrak{g}(\mathfrak{l})$ (s. Lemma 1.3), and $\mathfrak{l}_s = \mathfrak{l} \cap \mathfrak{s}$.

Now we prove that φ preserves the scalar product on $\mathfrak{l}$. By definition φ is orthogonal on $\mathfrak{l}_r$. Identify $\mathfrak{l}_s \cong \mathfrak{s}/\mathfrak{k}_s$ and note that $\langle,\rangle$ on $\mathfrak{l}_s$ is the unique (up to scaling) $\mathfrak{k}_s$-invariant scalar product on $\mathfrak{l}_s$, since $\mathfrak{s}$ is simple. This scalar product is invariant under φ, because φ together with $\mathcal{K}_0$ generates a compact Lie group with Lie algebra $\mathfrak{k} = \mathfrak{k}_s + \mathfrak{d}_0(\mathfrak{l}_r)$ which preserves $\mathfrak{k}_s = \mathfrak{k} \cap \mathfrak{s}$. So φ is an orthogonal automorphism of $(\mathfrak{l}, \langle,\rangle)$. By [K-N 1] Ch. 6, Th. 3.6 and Thm. 7.2, it is identified with some element in the stabilizer $\mathcal{K}$. Hence we proved that $\mathcal{K}(\mathfrak{l}) = \mathcal{K}$.□

Corollary 1.8 *If all automorphisms of* $\mathfrak{s}$ *are inner, i.e.* $Aut(\mathfrak{s}) = Int(\mathfrak{s})$, *then*

$$I(\mathcal{L}, g) = \mathcal{G}^s = \mathcal{S} \cdot \mathcal{A}^s(\mathfrak{l}_r) \cdot \mathcal{L}_r\,,$$

where $\mathcal{A}^s(\mathfrak{l}_r)$ is the group of orthogonal automorphisms of $\mathfrak{l}_r$ which commute with $\mathfrak{s}$.

Corollary 1.9 *Assume that $Aut(\mathfrak{s}) = Int(\mathfrak{s}) \cup \varphi Int(\mathfrak{s})$ and that φ preserves the maximal compact subalgebra $\mathfrak{k}_s \subset \mathfrak{s}$. If φ can be extended to an automorphism $\tilde{\varphi} \in Aut(\mathfrak{s} +\!\!\!\!\supset \mathfrak{l}_r)$ which preserves the decomposition $\mathfrak{s} + \mathfrak{l}_r$ and is orthogonal on $\mathfrak{l}_r$ then*

$$I(\mathcal{L}, g) = \mathcal{G}^s \cup \tilde{\varphi}\mathcal{G}^s .$$

In the opposite case $I(\mathcal{L}, g) = \mathcal{G}^s$.

Remark 5: It is no restriction of generality to assume that $\varphi\mathfrak{k}_s = \mathfrak{k}_s$, since any two maximal compact subalgebras of $\mathfrak{s}$ are conjugated by an inner automorphism.

Remark 6: A similar statement holds if $Aut(\mathfrak{s})$ has more than two connected components.

Proof. It is sufficient to prove that if φ cannot be extended, then $\mathcal{K} = \mathcal{K}(\mathfrak{l}) = \mathcal{K}_s \cdot A^s(\mathfrak{l}_r)$. Let $k \in \mathcal{K}(\mathfrak{l})$. Then k preserves $\mathfrak{s}$, $\mathfrak{k} = Lie\,\mathcal{K}$ and $\mathfrak{k}_s = \mathfrak{s} \cap \mathfrak{k}$. ¿From the asumption we know that $Ad_k|\mathfrak{s} = \varphi Ad_a$ or Ad_a, where a is an element of the normalizer $N_{\mathcal{S}}(\mathfrak{k}_s) = N_{\mathcal{S}}(\mathcal{K}_s)$. Now we prove that $N_{\mathcal{S}}(\mathcal{K}_s) = \mathcal{K}_s$. Without restriction we may assume that the symmetric space $\mathcal{S}/\mathcal{K}_s$ is irreducible. Suppose that $a \in N_{\mathcal{S}}(\mathcal{K}_s) - \mathcal{K}_s$. This implies that $a\mathcal{K}_s \in \mathcal{S}/\mathcal{K}_s$ is a fixpoint of $\mathcal{K}_s$ and hence $\mathcal{K}_s$ fixes the unique geodesic through the points $\mathcal{K}_s$ and $a\mathcal{K}_s$, which contradicts the irreducibility of $\mathcal{S}/\mathcal{K}_s$. Hence, we proved that $a \in \mathcal{K}_s$. In the first case we can write $\varphi = Ad_{ka^{-1}}|\mathfrak{s}$. This shows that φ can be extended to an automorphism $Ad_{ka^{-1}}$ with the stated properties, which contradicts the assumption. Hence we have $Ad_k|\mathfrak{s} = Ad_a|\mathfrak{s}$ and therefore $b = ka^{-1} \in A^s(\mathfrak{l}_r)$. This shows $k = ba \in A^s(\mathfrak{l}_r) \cdot \mathcal{K}_s$ and $\mathcal{K} = A^s(\mathfrak{l}_r) \cdot \mathcal{K}_s$. $\square$

Corollary 1.10 *Let $(\mathcal{L}, g)$ be the Riemannian homogeneous space associated to an admissible Lie algebra $(\mathfrak{l}, \langle,\rangle)$. Then the following conditions are equivalent.*

1) The full isometry group $I(\mathcal{L}, g)$ is unimodular.

2) $(\mathcal{L}, g)$ is symmetric.

3) $(\mathcal{L}, g)$ has a compact quotient (as Riemannian manifold).

4) $(\mathcal{L}, g)$ has a quotient of finite volume.

Proof. It is well known that 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 1). We have to show 1) $\Rightarrow$ 2). We may assume that $\mathfrak{l}$ is irreducible. Let $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$ be a maximal suitable SR-decomposition, in particular $\mathfrak{l}_r$ is admissible. According to Theorem 1.1 $\mathfrak{l}_r$ is an ideal in the full isometry algebra $\mathfrak{g} = (\mathfrak{s} \oplus \mathfrak{d}_0(\mathfrak{l}_r)) \dot{+} \mathfrak{l}_r$. Therefore $\mathfrak{l}_r$ is unimodular and admissible, hence trivial. This implies that $\mathfrak{g} = \mathfrak{s}$ is semisimple and $\mathfrak{l} = \mathfrak{i}(\mathfrak{s})$ is its symmetric Iwasawa algebra. □

Corollary 1.11 *The quaternionic Kählerian solvmanifolds classified in [A 2] do admit quotients of finite volume if and only if they are symmetric.*

Proof. Since the corresponding solvable metric Lie algebras are admisible (v. [C] Prop. II.29) the result follows from the preceding corollary. □

2 Full Isometry Algebra of Quaternionic Kählerian Solvmanifolds

In this section we will determine the full isometry algebra of quaternionic Kählerian solvmanifolds $(\mathcal{L}, g)$ using the algorithm described in section 1. The non-symmetric examples are naturally grouped into three families, which will be treated parallelly: $\mathcal{T}$-, $\mathcal{W}$- and $\mathcal{V}$-spaces. These families contain symmetric examples as well; they correspond to special values of the integer parameters and will be excluded in our discussion.

We will begin by recalling the definition of each of the three families for convenience of the reader. Since the corresponding metric Lie algebras $\mathfrak{l} = \mathfrak{a} + \mathfrak{n}$ are admissible (v. [C]) and irreducible (v. [Be] 14.45 Thm.) we can apply Thm. 1.1. The first step is to list all possible decompositions $\mathfrak{n} = \mathfrak{n}_1 \bar{\dot{+}} \mathfrak{n}_2$ of $\mathfrak{n} = [\mathfrak{l}, \mathfrak{l}]$ into sums of root-spaces of $\mathfrak{a}$. The main step is to prove the existence of a suitable SR-decomposition $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$ such that $\mathfrak{n}_s = \mathfrak{n}_1$ and $\mathfrak{n}_r = \mathfrak{n}_2$, where $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$ is a decomposition from the list such that $\mathfrak{n}_1 \neq \mathfrak{n}$ is maximal. The last property implies that $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_r$ is a *maximal* suitable SR-decomposition. Thanks to Thm. 1.1 this provides us with an explicit description of the principal part $\mathfrak{s} \dot{+} \mathfrak{l}_r$ ($\mathfrak{s} = \mathfrak{i}(\mathfrak{l}_s)$) of the full isometry algebra $\mathfrak{g} = (\mathfrak{s} \oplus \mathfrak{d}_0(\mathfrak{l}_r)) \dot{+} \mathfrak{l}_r$. The remaining step is the straightforward computation of $\mathfrak{d}_0(\mathfrak{l}_r)$.

At this point the following observation is pertinent. Denote by $(\mathcal{L}, g)$ the quaternionic Kählerian solvmanifold associated with a non-symmetric admissible quaternionic Kählerian Lie algebra $(\mathfrak{l}, \langle,\rangle, \mathfrak{q})$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{l}$ be the full isometry algebra of $(\mathcal{L}, g)$. Consider the linear isotropy algebra $\bar{\mathfrak{k}} \subset \mathfrak{so}(\mathfrak{l})$, i.e. the image of the isotropy algebra $\mathfrak{k}$ under the isotropy representation on $\mathfrak{g}/\mathfrak{k} \cong \mathfrak{l}$. Since $\mathfrak{l}$ is quaternionic Kählerian and non-symmetric, we know (s.

[Be] 10.92 Cor.) that $\mathfrak{hol} = \mathfrak{n}(\mathfrak{q}) = \mathfrak{q} + \mathfrak{z}(\mathfrak{q})$ ($\cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$), where $\mathfrak{n}(\mathfrak{q})$ resp. $\mathfrak{z}(\mathfrak{q})$ denotes the normalizer resp. the centralizer of the quaternionic structure $\mathfrak{q}$ in $\mathfrak{so}(\mathfrak{l})$. Therefore

$$\bar{\mathfrak{k}} \subset \mathfrak{n}(\mathfrak{hol}) = \mathfrak{n}(\mathfrak{n}(\mathfrak{q})) = \mathfrak{n}(\mathfrak{q})\,.$$

We shall see that the projection of $\bar{\mathfrak{k}}$ on $\mathfrak{q}$ is always surjective. This implies that the twistor space and the canonical SO_3-principal bundle over the quaternionic Kählerian solvmanifold are *homogeneous* under the isometry group of the base.

The following notion will be used frequently.

Definition 2.1 *Let φ be a representation of a key algebra $\mathfrak{f} = span\{G = JH, H\}$ with root μ on a Euclidean vector space $(\mathfrak{x}, \langle,\rangle)$ with complex structure J_1. We say that $\mathfrak{x} = \mathfrak{x}_+ \bar{+} \mathfrak{x}_-$, $\mathfrak{x}_+ = J_1\mathfrak{x}_-$, is a* **weight decomposition** *if*

$$\varphi(H)|\mathfrak{x}_\pm = \pm\frac{\mu}{2} Id\,, \quad \varphi(G)|\mathfrak{x}_+ = 0 \quad \textit{and} \quad \varphi(G)|\mathfrak{x}_- = -\mu J_1\,.$$

Note that φ is symplectic with respect to $\rho_1 = \langle J_1\cdot, \cdot\rangle$ and that $\varphi(JF)^{sym} = J_1\varphi(F)^{sym}$ for all $F \in \mathfrak{f}$ (cf. Example 4).

2.1 $\mathcal{T}$-Spaces

The family $\mathfrak{t} = \mathfrak{t}(p)$, $p = 0, 1, 2 \ldots$, of admissible quaternionic Kählerian Lie algebras is defined by a family of admissible Kählerian Lie algebras $\mathfrak{u} = \mathfrak{u}(p)$ and their unique Q-representations T as explained in Example 5. $\mathfrak{u} = \mathfrak{f}_0 \bar{\oplus} \mathfrak{u}_0$ can be described as follows:

$$\mathfrak{u}_0 = \mathfrak{e}_1 \bar{\oplus} \mathfrak{f}_2\,, \quad \mathfrak{e}_1 = \mathfrak{f}_1 + \mathfrak{x}_1 = \mathfrak{e}(p+1, 1)\,, \quad \mathfrak{f}_2 = \mathfrak{e}(1, \frac{1}{\sqrt{2}})\,;$$

the key algebras $\mathfrak{f}_i = span\{G_i, H_i\}$, $i = 0, 1, 2$, commute and $ad_{\mathfrak{f}_2}|\mathfrak{x}_1$ has a weight decomposition $\mathfrak{x}_1 = \mathfrak{x}_+ + \mathfrak{x}_-$. $\mathfrak{u}$ has a unique Q-representation $T : \mathfrak{u} \to End(\tilde{\mathfrak{u}})$. With respect to the orthonormal basis

$$\tilde{P}_+ := \frac{1}{2}\left(\tilde{H}_0 + \tilde{H}_1 + \sqrt{2}\tilde{H}_2\right)\,, \quad \tilde{Q}_+ := \hat{J}\tilde{P}_- = \frac{1}{2}\left(\tilde{G}_0 - \tilde{G}_1 + \sqrt{2}\tilde{G}_2\right)\,,$$

$$\tilde{P}_0 := \frac{1}{\sqrt{2}}\left(\tilde{G}_0 + \tilde{G}_1\right)\,, \quad \tilde{Q}_0 := \hat{J}\tilde{P}_0 = \frac{1}{\sqrt{2}}\left(-\tilde{H}_0 + \tilde{H}_1\right)\,,$$

$$\tilde{P}_- := \frac{1}{2}\left(\tilde{H}_0 + \tilde{H}_1 - \sqrt{2}\tilde{H}_2\right)\,, \quad \tilde{Q}_- := \hat{J}\tilde{P}_+ = \frac{1}{2}\left(\tilde{G}_0 - \tilde{G}_1 - \sqrt{2}\tilde{G}_2\right)$$

of $\tilde{\mathfrak{f}}_0 + \tilde{\mathfrak{f}}_1 + \tilde{\mathfrak{f}}_2$ it is given by ($X_\pm \in \mathfrak{x}_\pm$):

$$T_{H_0} = \frac{1}{2} Id\,, \quad T_{G_0} = 0\,,$$

$$\begin{aligned}
T_{H_1} &= \frac{1}{2}\left(\tilde{P}_+ \otimes \tilde{P}_+ + \tilde{P}_0 \otimes \tilde{P}_0 + \tilde{P}_- \otimes \tilde{P}_-\right) \\
&\quad - \frac{1}{2}\left(\tilde{Q}_+ \otimes \tilde{Q}_+ + \tilde{Q}_0 \otimes \tilde{Q}_0 + \tilde{Q}_- \otimes \tilde{Q}_-\right)\,, \\
T_{G_1} &= -\tilde{P}_0 \otimes \tilde{Q}_0 - \tilde{P}_+ \otimes \tilde{Q}_+ - \tilde{P}_- \otimes \tilde{Q}_-\,,
\end{aligned}$$

$$\begin{aligned}
T_{H_2}|\tilde{\mathfrak{x}}_1^- &= \frac{1}{\sqrt{2}}\left(\tilde{P}_+ \otimes \tilde{P}_+ - \tilde{P}_- \otimes \tilde{P}_- + \tilde{Q}_+ \otimes \tilde{Q}_+ - \tilde{Q}_- \otimes \tilde{Q}_-\right)\,, \\
T_{G_2}|\tilde{\mathfrak{x}}_1^- &= \tilde{Q}_0 \otimes \tilde{Q}_- - \tilde{Q}_+ \otimes \tilde{Q}_0 + \tilde{P}_0 \otimes \tilde{P}_- - \tilde{P}_+ \otimes \tilde{P}_0\,,
\end{aligned}$$

$$T_{H_2}|\tilde{\mathfrak{x}}_+ = \frac{\sqrt{2}}{4} Id\,, \quad T_{H_2}|\tilde{\mathfrak{x}}_- = -\frac{\sqrt{2}}{4} Id\,,$$

$$T_{G_2}|\tilde{\mathfrak{x}}_+ = 0\,, \quad T_{G_2}|\tilde{\mathfrak{x}}_- = -\frac{\sqrt{2}}{2} J_1\,,$$

$$\begin{aligned}
T_{X_+} &= \tilde{P}_+ \otimes \tilde{X}_+ + \frac{1}{\sqrt{2}}\tilde{P}_0 \otimes \widetilde{JX}_+ + \widetilde{JX}_+ \otimes \tilde{Q}_- - \frac{1}{\sqrt{2}}\tilde{X}_+ \otimes \tilde{Q}_0\,, \\
T_{X_-} &= \tilde{P}_- \otimes \tilde{X}_- + \frac{1}{\sqrt{2}}\tilde{P}_0 \otimes \widetilde{JX}_- + \widetilde{JX}_- \otimes \tilde{Q}_+ - \frac{1}{\sqrt{2}}\tilde{X}_- \otimes \tilde{Q}_0\,.
\end{aligned}$$

(Here we identify covectors with vectors by means of the scalar product $\langle , \rangle$.) In particular, $T_{\mathfrak{f}_2}|\tilde{\mathfrak{x}}_1$ has weight decomposition $\tilde{\mathfrak{x}}_1 = \tilde{\mathfrak{x}}_+ + \tilde{\mathfrak{x}}_-$.

$\mathcal{T}(0) = SO_0(3,4)/(SO(3) \times SO(4))$ is the only symmetric space in the family of quaternionic Kähler manifolds $\mathcal{T}(p)$ associated to $\mathfrak{t}(p)$, $p = 0, 1, 2 \ldots$ We will always assume $p \neq 0$.

As usual, let $\mathfrak{t} = \mathfrak{a} + \mathfrak{n}$ denote the decomposition (1).

Lemma 2.1 *The table below lists all semidirect orthogonal decompositions* $\mathfrak{n} = \mathfrak{n}_1 \bar{\oplus} \mathfrak{n}_2$ *of the Lie algebra* $\mathfrak{n} = [\mathfrak{t}, \mathfrak{t}]$*, such that any root space* $\mathfrak{n}_\alpha$*,* $\alpha \in \mathcal{R}$*, (with respect to the Cartan subalgebra* $\mathfrak{a}$ *of* $\mathfrak{t} = \mathfrak{a} + \mathfrak{n}$*) is contained in* $\mathfrak{n}_1$ *or* $\mathfrak{n}_2$*:*

$\mathfrak{n}_1$	$\mathfrak{n}_2$
0	$\langle\langle G_2, \tilde{Q}_-, \mathfrak{x}_- \rangle\rangle = \mathfrak{n}$
$\ll G_2 \gg$	$\langle\langle \tilde{Q}_-, \mathfrak{x}_- \rangle\rangle$
$\ll \tilde{Q}_- \gg$	$\langle\langle G_2, \mathfrak{x}_- \rangle\rangle$
$\ll \mathfrak{x}_- \gg$	$\langle\langle G_2, \tilde{Q}_- \rangle\rangle$
$\ll G_2, \tilde{Q}_- \gg$	$\langle\langle \mathfrak{x}_- \rangle\rangle$
$\ll G_2, \mathfrak{x}_- \gg$	$\langle\langle \tilde{Q}_- \rangle\rangle$
$\ll \tilde{Q}_-, \mathfrak{x}_- \gg$	$\langle\langle G_2 \rangle\rangle$
$\ll G_2, \tilde{Q}_-, \mathfrak{x}_- \gg$	0

$\ll \cdots \gg = \mathfrak{n}_1$ *respectively* $\langle\langle \cdots \rangle\rangle = \mathfrak{n}_2$ *means that* $\cdots$ *generates the subalgebra* $\mathfrak{n}_1$ *respectively the ideal* $\mathfrak{n}_2$*. Since* $\mathfrak{n}_1 - \mathfrak{n}_2$ *it is enough to specify* $\mathfrak{n}_1$ *or* $\mathfrak{n}_2$*.*

Given a decomposition $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$ as in the lemma we canonically associate a decomposition $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$ by defining $\mathfrak{a}_2 := \{A \in \mathfrak{a} |\ ad_A \mathfrak{n}_1 = 0\}$, $\mathfrak{a}_1 - \mathfrak{a}_2$. Setting $\mathfrak{t}_i = \mathfrak{a}_i + \mathfrak{n}_i$, $i = 1, 2$, we obtain a decomposition $\mathfrak{t} = \mathfrak{t}_1 \bar{\oplus} \mathfrak{t}_2$. It is clear that every SR-decomposition of $\mathfrak{t}$ has to show up in this way. The converse is not true: $\mathfrak{n}_1 = \mathfrak{n}$, $\mathfrak{n}_2 = 0$ does not correspond to an SR-decomposition of $\mathfrak{t}(p)$, since we are assuming $p \neq 0$.

Remark 7: For $\mathfrak{n}_1 = \ll G_2, \mathfrak{x}_- \gg$ we obtain $\mathfrak{a}_1 + \mathfrak{n}_1 = \mathfrak{u}_0$ the principal part of the (totally geodesic) Kählerian subalgebra $\mathfrak{u} = \mathfrak{f}_0 + \mathfrak{u}_0$. As Lie algebras we have $\mathfrak{u}_0 \cong \mathfrak{i}(\mathfrak{so}(p+2, 2))$. Nevertheless, the *Riemannian* homogeneous space associated with the *metric* Lie algebra $\mathfrak{u}_0$ is not the hermitian symmetric space $SO_0(p+2,2)/(SO(p+2) \times SO(2))$; simply because the scalar product on $\mathfrak{u}_0$ is not symmetric. In all the other cases (excluding $\mathfrak{n}_1 = \mathfrak{n}$) $\mathfrak{a}_1 + \mathfrak{n}_1$ is a *symmetric* totally geodesic subalgebra of $\mathfrak{t}$. As we shall see, $\mathfrak{n}_1 = \ll G_2, \tilde{Q}_- \gg$ gives rise to the maximal suitable SR-decomposition of $\mathfrak{t}$. The subalgebra $\mathfrak{n}_1 = \ll G_2, \tilde{Q}_- \gg$ is maximal with the property that the full isometry algebra of the totally geodesic symmetric submanifold corresponding to $\mathfrak{a}_1 + \mathfrak{n}_1$ extends to an isometry algebra on the ambient quaternionic Kähler manifold $\mathcal{T}(p)$.

Proposition 2.2 *The maximal suitable SR-decomposition* $\mathfrak{t} = \mathfrak{t}_s + \mathfrak{t}_r$ *of* $\mathfrak{t} = \mathfrak{t}(p)$*,* $p \neq 0$*, is given as follows*

1) $\mathfrak{t}_s = \mathfrak{a}_s + \mathfrak{n}_s$, $\mathfrak{a}_s = span\{H_0 - H_1, H_2\}$, $\mathfrak{n}_s = \ll G_2, \tilde{Q}_- \gg = span\{G_2, \tilde{Q}_0, \tilde{Q}_+, \tilde{Q}_-\}$,

2) $\mathfrak{t}_r = \mathfrak{a}_r + \mathfrak{n}_r$, $\mathfrak{a}_r = span\{H_0 + H_1\}$, $\mathfrak{n}_r = \langle\langle \mathfrak{x}_- \rangle\rangle = span\{G_0, G_1, \tilde{P}_0, \tilde{P}_+, \tilde{P}_-\} + \mathfrak{x}_1 + \tilde{\mathfrak{x}}_1$,

3) $\mathfrak{t}_s = \mathfrak{i}(\mathfrak{s})$, $\mathfrak{s} = \mathfrak{so}(\mathfrak{v}_{3,2}) \cong \mathfrak{so}(3, 2)$, *where a scalar product* $(,)$ *of signature* $(3,2)$ *on the vector space* $\mathfrak{v}_{3,2} = span\{G_0,\ G_1,\ \tilde{P}_0,\ \tilde{P}_+,\ \tilde{P}_-\} = \mathfrak{zent}(\mathfrak{n}_r)$ *is defined by the formulas* ($i = 0,1$, $j = 0,+,-$)

$$(G_0, G_1) = (\tilde{P}_+, \tilde{P}_-) = (\tilde{P}_0, \tilde{P}_0) = 1\,;$$
$$(G_i, G_i) = (\tilde{P}_\pm, \tilde{P}_\pm) = (G_i, \tilde{P}_j) = 0\,.$$

We consider $\mathfrak{t}_s$ *as subalgebra of* $\mathfrak{s} = \mathfrak{so}(\mathfrak{v}_{3,2}) \cong \bigwedge^2 \mathfrak{v}_{3,2}$ *via the embedding*

$$\begin{cases} H_0 - H_1 \mapsto G_0 \hat{\wedge} G_1 \\ \sqrt{2}H_2 \mapsto \tilde{P}_+ \hat{\wedge} \tilde{P}_- \\ G_2 \mapsto -\tilde{P}_0 \hat{\wedge} \tilde{P}_+ \\ \tilde{Q}_0 \mapsto \tilde{P}_0 \hat{\wedge} G_0 \\ \tilde{Q}_+ \mapsto G_0 \hat{\wedge} \tilde{P}_+ \\ \tilde{Q}_- \mapsto G_0 \hat{\wedge} \tilde{P}_- \end{cases}$$

($(U\hat{\wedge}V)(W) = (V,W)U - (U,W)V$, $U,V,W \in \mathfrak{v}_{3,2}$). *The extension* $\rho : \mathfrak{s} \to \mathfrak{Der}(\mathfrak{t}_r)$ *of the representation* $ad : \mathfrak{t}_s \to \mathfrak{Der}(\mathfrak{t}_r)$ *is trivial on* $span\{H_0 + H_1\}$, *standard on* $\mathfrak{v}_{3,2}$ *and acts on* $\mathfrak{x}_1 + \tilde{\mathfrak{x}}_1$ *as a sum of* p *semi-spinor representations defined by the following formulas:*

$$\rho(\tilde{P}_0 \hat{\wedge} G_1)|\mathfrak{x}_1 = 0\,, \quad \rho(\tilde{P}_0 \hat{\wedge} G_1)|\tilde{\mathfrak{x}}_1 = \frac{1}{\sqrt{2}} J_2\,;$$

$$\rho(\tilde{P}_0 \hat{\wedge} \tilde{P}_-)|(\mathfrak{x}_+ + \tilde{\mathfrak{x}}_+) = \frac{1}{\sqrt{2}} J_1\,, \quad \rho(\tilde{P}_0 \hat{\wedge} \tilde{P}_-)|(\mathfrak{x}_- + \tilde{\mathfrak{x}}_-) = 0\,;$$

$$\rho(G_1 \hat{\wedge} \tilde{P}_-)|\tilde{\mathfrak{x}}_+ = J_3\,, \quad \rho(G_1 \hat{\wedge} \tilde{P}_-)|(\mathfrak{x}_1 + \tilde{\mathfrak{x}}_-) = 0\,;$$
$$\rho(\tilde{P}_+ \hat{\wedge} G_1)|\tilde{\mathfrak{x}}_- = -J_3\,, \quad \rho(\tilde{P}_+ \hat{\wedge} G_1)|(\mathfrak{x}_1 + \tilde{\mathfrak{x}}_+) = 0\,.$$

Proof. The embedding $\mathfrak{t}_s \hookrightarrow \mathfrak{s}$ shows that $\mathfrak{t}_s$ is a symmetric Iwasawa algebra of $\mathfrak{s}$. It is straightforward to check that the formulas given in the proposition define a representation ρ of $\mathfrak{s}$ on $\mathfrak{t}_r$ which extends $ad : \mathfrak{t}_s \to \mathfrak{Der}(\mathfrak{t}_r)$ and acts by derivations.

To see that $\mathfrak{k}^\rho := \rho(\mathfrak{s}) \cap \mathfrak{d}(\mathfrak{t}_r)$ is maximal compact in $\rho(\mathfrak{s})$ it is enough to note that $\mathfrak{k}_s$ is maximal compact in $\mathfrak{s}$, ρ is faithful and $\rho(\mathfrak{k}_s) \subset \mathfrak{d}(\mathfrak{t}_r)$. We prove this inclusion by a direct computation:

$$\mathfrak{k}_s = span\{\tilde{P}_0 \hat{\wedge} R_1^+,\ \tilde{P}_0 \hat{\wedge} R_2^+,\ R_1^+ \hat{\wedge} R_2^+\} \oplus span\{R_1^- \hat{\wedge} R_2^-\} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(2)\,,$$

where $R_1^\pm = G_0 \pm G_1$ and $R_2^\pm = \tilde{P}_+ \pm \tilde{P}_-$.

$$\rho(\tilde{P}_0 \hat{\wedge} R_1^+)|(\mathfrak{x}_1 + \tilde{\mathfrak{x}}_1) = \frac{1}{\sqrt{2}} J_2 \quad (3)$$

$$\rho(\tilde{P}_0 \hat{\wedge} R_2^+)|(\mathfrak{x}_1 + \tilde{\mathfrak{x}}_1) = \frac{1}{\sqrt{2}} J_1 \quad (4)$$

$$\rho(R_1^+ \hat{\wedge} R_2^+)|(\mathfrak{x}_1 + \tilde{\mathfrak{x}}_1) = J_3 \quad (5)$$

$$\rho(R_1^-\hat{\wedge}R_2^-)|(\mathfrak{x}_- + \tilde{\mathfrak{x}}_+) \;=\; J_3\,, \quad \rho(R_1^-\hat{\wedge}R_2^-)|(\mathfrak{x}_+ + \tilde{\mathfrak{x}}_-) \;=\; -J_3\,.$$

We see that $\rho(\mathfrak{k}_s)|(\mathfrak{x}_1+\tilde{\mathfrak{x}}_1)$ and therefore $\rho(\mathfrak{k}_s)$ consists of skew-symmetric endomorphisms. This shows that $\mathfrak{t}=\mathfrak{t}_s+\mathfrak{t}_r$ is a suitable SR-decomposition. It corresponds to the splitting $\mathfrak{n}=\mathfrak{n}_1+\mathfrak{n}_2$, $\mathfrak{n}_2=\langle\langle\mathfrak{x}_-\rangle\rangle$, in Lemma 2.1. One sees from the list that there is no other splitting $\mathfrak{n}=\mathfrak{n}_1'+\mathfrak{n}_2'$, with $\mathfrak{n}_1\subset\mathfrak{n}_1'\neq\mathfrak{n}$. This proves the maximality. □

Corollary 2.3 *Consider the Cartan decomposition $\mathfrak{s}=\mathfrak{k}_s+\mathfrak{p}$ of $\mathfrak{s}=\mathfrak{so}(\mathfrak{v}_{3,2})$ into its skew-symmetric part $\mathfrak{k}_s$ and its symmetric part $\mathfrak{p}$ with respect to the* Euclidean *scalar product $\langle,\rangle$ on $\mathfrak{v}_{3,2}$. Then $\rho(\mathfrak{k}_s)$ (respectively, $\rho(\mathfrak{p})$) consists of skew-symmetric (respectively, symmetric) endomorphisms.*

Proof. We know already that $\rho(\mathfrak{k}_s)$ consists of skew-symmetric endomorphisms. The fact that $\rho(\mathfrak{p})$ consists of symmetric endomorphisms is established similarly by computing $\rho(\mathfrak{p})|(\mathfrak{x}_1+\tilde{\mathfrak{x}}_1)$ starting from

$$\mathfrak{p} = span\{\tilde{P}_0\hat{\wedge}R_i^-,\ R_i^+\hat{\wedge}R_j^-;\quad i,j=1,2\}\,.\quad\Box$$

Corollary 2.4 *Denote by $\bar{\mathfrak{k}}_s\subset\mathfrak{q}+\mathfrak{z}(\mathfrak{q})\subset\mathfrak{so}(\mathfrak{t})$ the image of $\mathfrak{k}_s$ under the isotropy representation on $\mathfrak{s}/\mathfrak{k}_s+\mathfrak{t}_r\cong\mathfrak{t}$. Then the projection of $\bar{\mathfrak{k}}_s$ on $\mathfrak{q}$ is surjective.*

Proof. This follows immediately from the equations (3), (4) and (5) since $\mathfrak{x}_1+\tilde{\mathfrak{x}}_1$ is a quaternionic subspace of $\mathfrak{t}$. □

To formulate the final result we introduce the following notation. Let $V=\mathbb{H}$ denote the quaternions with standard scalar product $\langle q,q\rangle=q\bar{q}$ and by R_i, R_j, R_k, right multiplication by $i,j,k\in\mathbb{H}$. Then $\omega_0=\langle R_k\cdot,\cdot\rangle$ is a symplectic form on the real vector space $V\cong\mathbb{R}^4$. Now we define a metric Lie algebra $\mathfrak{t}_r$. It will be proved that it is isomorphic to $\mathfrak{t}_r$ described in Prop. 2.2. We put

$$\mathfrak{t}_r = V\otimes\mathbb{R}^p \stackrel{-}{+} \wedge^2 V$$

with natural scalar product $\langle,\rangle$ induced by $(V,\langle,\rangle)$ and the standard scalar product on $\mathbb{R}^p$. We normalize $\langle,\rangle$ on $\wedge^2V$ by $|v\wedge w|=1$ for orthonormal vectors v, w. Denote by $\omega^0=\omega_0^{-1}$ the element of $\wedge^2V$ which is inverse to $\omega_0\in\wedge^2V^*$. We can decompose

$$\wedge^2V=\wedge_0^2V+\mathbb{R}\omega^0\,,$$

where $\wedge_0^2V=ker\,\omega_0$, $\omega_0\in\wedge^2V^*$. We note that $|\omega^0|^2=2$ and $\omega^0-\wedge_0^2V$.

Now we define the Lie algebra structure

$$ad_{\omega^0}|V \otimes \mathbb{R}^p = \frac{1}{2}Id, \quad ad_{\omega^0}| \wedge_0^2 V = Id;$$

it follows that $[\wedge_0^2 V, V \otimes \mathbb{R}^p + \wedge_0^2 V] = 0$. It remains to define $[,] : (V \otimes \mathbb{R}^p) \times (V \otimes \mathbb{R}^p) \to \wedge_0^2 V$. It is given by

$$[v \otimes x, w \otimes y] = (v \wedge w)_0 \langle x, y \rangle ,$$

where the subindex 0 denotes the natural projection $\wedge^2 V \to \wedge_0^2 V$ and $\langle , \rangle$ the standard scalar product on $\mathbb{R}^p$. Denote by $\mathfrak{sp}(V, \omega_0)$ the symplectic Lie algebra of the symplectic vector space (V, ω_0). It acts naturally on $\mathfrak{t}_r$ by derivations. We define also an action of the Lie group (resp. algebra) $O(p)$ (resp. $\mathfrak{so}(p)$) on $\mathfrak{t}_r$ by orthogonal automorphisms (resp. skew-symmetric derivations) as follows. An element $\phi \in O(p)$ (resp. $\varphi \in \mathfrak{so}(p)$) acts on $\mathfrak{t}_r$ by

$$\phi(v \otimes x) = v \otimes \phi(x) \quad \phi| \wedge^2 V = Id ,$$

$$\varphi(v \otimes x) = v \otimes \varphi(x) \quad \varphi| \wedge^2 V = 0 .$$

So we defined on $\mathfrak{t}_r = V \otimes \mathbb{R}^p + \wedge_0^2 V + \mathbb{R}\omega^0$ the structure of a metric Lie algebra such that the Lie algebra $\mathfrak{sp}(V, \omega_0) \oplus \mathfrak{so}(p)$ acts on $\mathfrak{t}_r$ by derivations.

Theorem 2.5 *1) The quaternionic Kähler Lie algebra $\mathfrak{t}(p)$, $p \geq 1$, is isomorphic (as metric Lie algebra) to the metric Lie algebra $\mathfrak{i}(\mathfrak{sp}(V, \omega_0)) \bar{+\!\!\!\!\supset}\, \mathfrak{t}_r$, where $\mathfrak{i}(\mathfrak{sp}(V, \omega_0))$ is the symmetric Iwasawa algebra associated with $\mathfrak{s} = \mathfrak{sp}(V, \omega_0)$, s. Example 1.*

2) The full isometry algebra $\mathfrak{g}(\mathcal{T}(p))$ of the quaternionic Kähler manifold $\mathcal{T}(p)$, $p \geq 1$, is given by

$$\mathfrak{g}(\mathcal{T}(p)) = (\mathfrak{sp}(V, \omega_0) \oplus \mathfrak{so}(p)) +\!\!\!\!\supset \mathfrak{t}_r .$$

The adjoint representation of this Lie algebra on its ideal $\mathfrak{t}_r$ defines an embedding $\mathfrak{g}(\mathcal{T}(p)) \hookrightarrow \mathfrak{Der}(\mathfrak{t}_r)$.

Proof. The proof follows from Prop. 2.2. Indeed, the maximal semisimple subalgebra of non-compact type $\mathfrak{s} \cong \mathfrak{so}(3, 2)$ acts on $\mathfrak{x}_1 + \tilde{\mathfrak{x}}_1$ as a sum of p semi-spinor representations. It is known that the semi-spinor representation of $\mathfrak{so}(3, 2)$ can be identified with the standard representation of $\mathfrak{sp}(4, \mathbb{R}) \cong \mathfrak{so}(3, 2)$ on $V = \mathbb{R}^4$. Moreover, the induced representation of $\mathfrak{sp}(4, \mathbb{R})$ on $\wedge^2 V$ can be decomposed as $\wedge^2 V = \wedge_0^2 V + \mathbb{R}\omega^0$, where $\wedge_0^2 V$ is identified with the standard representation of $\mathfrak{so}(3,2)$. The natural $\mathfrak{s}$-invariant symplectic form ω_0 on $\mathfrak{x}_1 + \tilde{\mathfrak{x}}_1$ is given by

$$\omega_0 = \sum_{i=1}^{p} (X_-^i \wedge \widetilde{JX}_-^i + JX_-^i \wedge \tilde{X}_-^i),$$

where $(X_-^i)_{i=1,\dots,p}$ is an orthonormal basis of $\mathfrak{x}_-$. Now we can identify as $\mathfrak{s}$-modules $\mathfrak{x}_1 + \tilde{\mathfrak{x}}_1 \cong V \otimes \mathbb{R}^p$, $\mathfrak{v}_{3,2} \cong \wedge_0^2 V$ and $\mathfrak{a}_r \cong \mathbb{R}\omega^0$, where $H_0 - H_1 = \omega^0$.

It remains to determine the Lie algebra $\mathfrak{d}_0(\mathfrak{t}_r)$. For future application in section 3 we do more. We determine the corresponding (not necessarily connected) subgroup $A^s(\mathfrak{t}_r)$ of $I(\mathcal{T}(p))$ which consists of orthogonal automorphisms of $\mathfrak{t}_r$ commuting with the action of $\mathfrak{s}$. This will be done in the following lemma. □

Lemma 2.6 *$A^s(\mathfrak{t}_r) = O(p)$ with the action described before Theorem 2.5.*

Proof. It is evident that $O(p) \subset A^s(\mathfrak{t}_r)$. Conversely, let $\phi \in A^s(\mathfrak{t}_r)$. We may identify $\mathfrak{t}_r = V \otimes \mathbb{R}^p + \wedge_0^2 V + \mathbb{R}\omega^0$. Since ϕ commutes with $\mathfrak{s} = \mathfrak{sp}(V, \omega_0)$ it preserves the decomposition. Moreover, $\phi | V \otimes \mathbb{R}^p \in O(p)$, $\phi | \wedge_0^2 V = \epsilon Id$ and $\phi\omega^0 = \epsilon' \omega^0$ $(\epsilon, \epsilon' \in \{\pm 1\})$, due to Schurs lemma and orthogonality of ϕ. Commutator relations immediately imply $\epsilon = \epsilon' = +1$. This proves the lemma and Thm. 2.5. □

2.2 $\mathcal{W}$-Spaces

We are going to define $\mathcal{W}$- and $\mathcal{V}$-spaces simultaneously. The following concept is important.

Definition 2.2 *Let $\mathfrak{x}$, $\mathfrak{y}$ and $\mathfrak{z}$ be Euclidean vector spaces. A bilinear mapping $\psi : \mathfrak{x} \times \mathfrak{z} \to \mathfrak{y}$ is said to be* **isometric**, *if*

$$\langle \psi(X,Z), \psi(X,Z) \rangle = \langle X, X \rangle \langle Z, Z \rangle$$

for all $X \in \mathfrak{x}$ and $Z \in \mathfrak{z}$. Isometric mappings $\psi : \mathfrak{x} \times \mathfrak{z} \to \mathfrak{y}$ and $\psi' : \mathfrak{x}' \times \mathfrak{z}' \to \mathfrak{y}'$ are said to be **equivalent**, *if there are isomorphisms $\sigma : \mathfrak{x} \to \mathfrak{x}'$, $\tau : \mathfrak{z} \to \mathfrak{z}'$ and $\upsilon : \mathfrak{y} \to \mathfrak{y}'$ of Euclidean vector spaces such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{x} \times \mathfrak{z} & \xrightarrow{\psi} & \mathfrak{y} \\ \downarrow \sigma \times \tau & & \downarrow \upsilon \\ \mathfrak{x}' \times \mathfrak{z}' & \xrightarrow{\psi'} & \mathfrak{y}' \end{array}$$

Accordingly, we say that the data (σ, τ, υ) define an **equivalence** *between ψ and ψ'. If $\psi = \psi'$ we speak of an* **autoequivalence**. *Two isometric* *mappings* $\psi : \mathfrak{x} \times \mathfrak{z} \to \mathfrak{y}$ *and* $\psi' : \mathfrak{x} \times \mathfrak{z}' \to \mathfrak{y}'$ *are said to be* **isomorphic***, if there are isomorphisms* $\tau : \mathfrak{z} \to \mathfrak{z}'$ *and* $\upsilon : \mathfrak{y} \to \mathfrak{y}'$ *of Euclidean vector spaces such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{x} \times \mathfrak{z} & \xrightarrow{\psi} & \mathfrak{y} \\ \downarrow {\scriptstyle Id \times \tau} & & \downarrow {\scriptstyle \upsilon} \\ \mathfrak{x} \times \mathfrak{z}' & \xrightarrow{\psi'} & \mathfrak{y}' \end{array}$$

We say that the data (τ, υ) *define an* **isomorphism** *between* ψ *and* ψ'*. If* $\psi = \psi'$ *we speak of an* **automorphism***. An isometric mapping* $\psi : \mathfrak{x} \times \mathfrak{z} \to \mathfrak{y}$ *with* $k = \dim \mathfrak{x} \neq 0$ *is said to be* **special***, if* $\dim \mathfrak{y} = \dim \mathfrak{z} \neq 0$*.* k *is called the* **order** *of the special isometric mapping.*

Let $\psi : \mathfrak{x} \times \mathfrak{z} \to \mathfrak{y}$ *be a special isometric mapping. The* **transpose** *of* ψ *is the special isometric mapping defined by (*$X \in \mathfrak{x}$*,* $Y \in \mathfrak{y}$*,* $Z \in \mathfrak{z}$*)*

$$\langle \psi^t(X, Y), Z \rangle := \langle Y, \psi(X, Z) \rangle .$$

Given a second special isometric mapping $\psi' : \mathfrak{x} \times \mathfrak{z}' \to \mathfrak{y}'$ *(over the same Euclidean vector space* $\mathfrak{x}$*) the* **sum** $\psi + \psi'$ *is the special isometric mapping* $\psi + \psi' : \mathfrak{x} \times (\mathfrak{z} + \mathfrak{z}') \to \mathfrak{y} + \mathfrak{y}'$ *defined by (*$X \in \mathfrak{x}$*,* $Z \in \mathfrak{z}$*,* $Z' \in \mathfrak{z}'$*)*

$$(\psi + \psi')(X, Z + Z') := \psi(X, Z) + \psi'(X, Z') .$$

We recall that special isometric mappings and $\mathbb{Z}_2$-graded Clifford modules are equivalent notions. The natural correspondence is induced by the following construction. To a special isometric mapping $\psi : \mathfrak{x} \times \mathfrak{z} \to \mathfrak{y}$ we can associate a $\mathbb{Z}_2$-graded module $M_\psi = M_0 \oplus M_1$ over the Clifford algebra $C\ell(\mathfrak{x})$. In fact, set $M_0 := \mathfrak{z}$, $M_1 := \mathfrak{y}$ and define $\Psi : \mathfrak{x} \to End(M_\psi)$ by

$$\Psi(X)Z := \psi(X, Z) , \quad \Psi(X)Y := -\psi^t(X, Y) \quad (X \in \mathfrak{x},\ Y \in \mathfrak{y},\ Z \in \mathfrak{z}) ,$$

then Ψ satisfies the relation $\Psi(X)^2 = -\langle X, X \rangle Id$.

Let $\mathfrak{x}_-$, $\mathfrak{z}_-$ and $\mathfrak{y}_-$ be Euclidean vector spaces. Every isometric mapping $\psi : \mathfrak{x}_- \times \mathfrak{z}_- \to \mathfrak{y}_-$ defines an admissible Kählerian Lie algebra $\mathfrak{u}(\psi) = (\mathfrak{k}_0 + \mathfrak{u}_0, J)$ by means of the following recipe

1. $\mathfrak{u}_0$ is a semidirect orthogonal sum $\mathfrak{u}_0 = (\mathfrak{k}_1 + \mathfrak{x}_1) + (\mathfrak{k}_2 + \mathfrak{x}_2) + \mathfrak{k}_3$ of elementary Kählerian Lie algebras with commuting key algebras with root 1.

2. $\mathfrak{x}_1$ admits a (J-invariant) decomposition $\mathfrak{x}_1 = \mathfrak{y} + \mathfrak{z}$ such that the following is true for $\mathfrak{x} := \mathfrak{x}_2$, $\mathfrak{y}$ and $\mathfrak{z}$: $ad_{\mathfrak{k}_3}|\mathfrak{y}$, $ad_{\mathfrak{k}_2}|\mathfrak{z}$ and $ad_{\mathfrak{k}_3}|\mathfrak{x}$ have weight decompositions $\mathfrak{y} = \mathfrak{y}_+ + \mathfrak{y}_-$, $\mathfrak{z} = \mathfrak{z}_+ + \mathfrak{z}_-$ and $\mathfrak{x} = \mathfrak{x}_+ + \mathfrak{x}_-$, where $\mathfrak{y}_+ = J\mathfrak{y}_-$, $\mathfrak{z}_+ = J\mathfrak{z}_-$ and $\mathfrak{x}_+ = J\mathfrak{x}_-$. Furthermore:

$$[\mathfrak{k}_1, \mathfrak{x}] = [\mathfrak{k}_2, \mathfrak{y}] = [\mathfrak{k}_3, \mathfrak{z}] = [\mathfrak{y}, \mathfrak{z}] = [\mathfrak{x}, \mathfrak{z}_+] = [\mathfrak{x}_+, \mathfrak{y}_+] = [\mathfrak{x}_-, \mathfrak{y}_-] = 0 .$$

3. The remaining Lie brackets are computed according to the rules ($X \in \mathfrak{x}$, $X_\pm \in \mathfrak{x}_\pm$, $Y_\pm \in \mathfrak{y}_\pm$ and $Z_\pm \in \mathfrak{z}_\pm$):

$$[X_-, Z_-] = \frac{1}{\sqrt{2}}\psi(X_-, Z_-)\,, \quad [JX, Z_-] = J[X, Z_-]\,,$$

$$[\mathfrak{x}_-, \mathfrak{y}_+] \subset \mathfrak{z}_+\,, \quad [X_+, Y_-] = [JX_+, JY_-] \quad \text{and}$$

$$\langle [X_-, Y_+], Z_+\rangle = -\frac{1}{\sqrt{2}}\langle JY_+, \psi(X_-, JZ_+)\rangle\,.$$

$\mathfrak{u}(\psi)$ admits a (unique) Q-representation if either

(i) $\mathfrak{x}_- = 0$ (hence $\psi = 0$) and $\mathfrak{u} = \mathfrak{u}(p, q) \cong \mathfrak{u}(q, p)$ is completely determined by the parameters $p = \dim \mathfrak{y}_-$ and $q = \dim \mathfrak{z}_-$ or

(ii) ψ is a special isometric mapping.

The corresponding quaternionic Kählerian Lie algebras (solvmanifolds) are denoted by $\mathfrak{w}(p, q)$ ($\mathcal{W}(p, q)$) and $\mathfrak{v}(\psi)$ ($\mathcal{V}(\psi)$) in the cases (i) and (ii) respectively. We recall that the Kählerian Lie subalgebra $\mathfrak{u} = \mathfrak{u}(\psi)$ is called principal Kählerian subalgebra. The corresponding subgroup $\mathcal{U}$ is a totally geodesic Kählerian submanifold of the quaternionic Kählerian solvmanifold.

The definition of the Q-representation T of $\mathfrak{u}(\psi)$ is given as follows. Set $\mathfrak{f} := \sum_{i=0}^{3} \mathfrak{f}_i$. The operators $T_{H_\alpha}|\tilde{\mathfrak{f}}$ and $T_{G_\alpha}|\tilde{\mathfrak{f}}$ are given, with respect to the orthonormal basis

$$\begin{aligned} \tilde{P}_0 &:= \frac{1}{2}\left(\tilde{H}_0 + \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3\right)\,, \\ \tilde{P}_\alpha &:= \frac{1}{2}\left(-\tilde{H}_0 - \tilde{H}_\alpha + \tilde{H}_\beta + \tilde{H}_\gamma\right)\,, \end{aligned}$$

$$\tilde{Q}_i := \hat{J}\tilde{P}_i\,, \quad (\{\alpha, \beta, \gamma\} = \{1, 2, 3\}\,, \quad i \in \{0, 1, 2, 3\})$$

of $\tilde{\mathfrak{f}}$, by the following formulas:

$$\begin{aligned} T_{H_\alpha}|span\left\{\tilde{P}_0, \tilde{P}_\alpha, \tilde{Q}_\beta, \tilde{Q}_\gamma\right\} &= \tfrac{1}{2}Id\,, \\ T_{H_\alpha}|span\left\{\tilde{Q}_0, \tilde{Q}_\alpha, \tilde{P}_\beta, \tilde{P}_\gamma\right\} &= -\tfrac{1}{2}Id\,, \end{aligned}$$

$$T_{G_\alpha} : \begin{cases} \tilde{Q}_0 \mapsto \tilde{P}_\alpha \mapsto 0\,, \\ \tilde{Q}_\alpha \mapsto \tilde{P}_0 \mapsto 0\,, \\ \tilde{P}_\beta \mapsto \tilde{Q}_\gamma \mapsto 0\,. \end{cases}$$

$T_{\mathfrak{f}_1}|\tilde{\mathfrak{x}}$, $T_{\mathfrak{f}_2}|\tilde{\mathfrak{y}}$ and $T_{\mathfrak{f}_3}|\tilde{\mathfrak{z}}$ have weight decompositions $\tilde{\mathfrak{x}} = \tilde{\mathfrak{x}}_+ + \tilde{\mathfrak{x}}_-$, $\tilde{\mathfrak{y}} = \tilde{\mathfrak{y}}_+ + \tilde{\mathfrak{y}}_-$ and $\tilde{\mathfrak{z}} = \tilde{\mathfrak{z}}_+ + \tilde{\mathfrak{z}}_-$ and

$$T_{\mathfrak{f}_1}(\tilde{\mathfrak{y}} + \tilde{\mathfrak{z}}) = T_{\mathfrak{f}_2}(\tilde{\mathfrak{x}} + \tilde{\mathfrak{z}}) = T_{\mathfrak{f}_3}(\tilde{\mathfrak{x}} + \tilde{\mathfrak{y}}) = 0\,.$$

In the following, let $X_\pm$, $Y_\pm$ and $Z_\pm$ denote arbitrary elements and $(X^i_\pm)_i$, $(Y^i_\pm)_i$ and $(Z^i_\pm)_i$ arbitrary orthonormal bases of $\mathfrak{x}_\pm$, $\mathfrak{y}_\pm$ and $\mathfrak{z}_\pm$ respectively. $U \circ V := 2\nabla_U V$ defines a bilinear mapping from the product of two of the spaces $\mathfrak{x}$, $\mathfrak{y}$ and $\mathfrak{z}$ into the third. With these conventions the expressions for the remaining operators T_U $(U \in \mathfrak{y} + \mathfrak{z} + \mathfrak{x})$ read

$$\begin{aligned}
T_{X_+} &= \tilde{P}_0 \otimes \tilde{X}_+ - \tilde{X}_+ \otimes \tilde{P}_1 + \widetilde{JX_+} \otimes \tilde{Q}_0 - \tilde{Q}_1 \otimes \widetilde{JX_+} \\
&\quad + \sum_i \widetilde{X_+ \circ Y^i_-} \otimes \tilde{Y}^i_- + \sum_i \widetilde{X_+ \circ Z^i_-} \otimes \tilde{Z}^i_- \,, \\
T_{Y_+} &= \tilde{P}_0 \otimes \tilde{Y}_+ - \tilde{Y}_+ \otimes \tilde{P}_2 + \widetilde{JY_+} \otimes \tilde{Q}_0 - \tilde{Q}_2 \otimes \widetilde{JY_+} \\
&\quad + \sum_i \widetilde{Y_+ \circ X^i_-} \otimes \tilde{X}^i_- + \sum_i \widetilde{Y_+ \circ Z^i_-} \otimes \tilde{Z}^i_- \,, \\
T_{Z_+} &= \tilde{P}_0 \otimes \tilde{Z}_+ - \tilde{Z}_+ \otimes \tilde{P}_3 + \widetilde{JZ_+} \otimes \tilde{Q}_0 - \tilde{Q}_3 \otimes \widetilde{JZ_+} \\
&\quad + \sum_i \widetilde{Z_+ \circ X^i_-} \otimes \tilde{X}^i_- + \sum_i \widetilde{Z_+ \circ Y^i_-} \otimes \tilde{Y}^i_- \,, \\
T_{X_-} &= -\tilde{P}_2 \otimes \tilde{X}_- - \tilde{X}_- \otimes \tilde{P}_3 - \widetilde{JX_-} \otimes \tilde{Q}_2 - \tilde{Q}_3 \otimes \widetilde{JX_-} \\
&\quad + \sum_i \widetilde{X_- \circ Y^i_-} \otimes \tilde{Y}^i_- + \sum_i \widetilde{X_- \circ Z^i_+} \otimes \tilde{Z}^i_+ \,, \\
T_{Y_-} &= -\tilde{P}_1 \otimes \tilde{Y}_- - \tilde{Y}_- \otimes \tilde{P}_3 - \widetilde{JY_-} \otimes \tilde{Q}_1 - \tilde{Q}_3 \otimes \widetilde{JY_-} \\
&\quad + \sum_i \widetilde{Y_- \circ X^i_-} \otimes \tilde{X}^i_- + \sum_i \widetilde{Y_- \circ Z^i_+} \otimes \tilde{Z}^i_+ \,, \\
T_{Z_-} &= -\tilde{P}_1 \otimes \tilde{Z}_- - \tilde{Z}_- \otimes \tilde{P}_2 - \widetilde{JZ_-} \otimes \tilde{Q}_1 - \tilde{Q}_2 \otimes \widetilde{JZ_-} \\
&\quad + \sum_i \widetilde{Z_- \circ X^i_-} \otimes \tilde{X}^i_- + \sum_i \widetilde{Z_- \circ Y^i_+} \otimes \tilde{Y}^i_+ \,.
\end{aligned}$$

For the convenience of the reader we remark that the bilinear mapping "$\circ$" introduced above is completely determined by the following relations, which are obtained from the Koszul formula. The first three formulas are, as usual, to be read selecting either all the upper or all the lower signs and $\epsilon \in \{-,+\}$ and we used the standard rule of multiplication of signs: $-- = +$ etc.

$$\begin{aligned}
X_\epsilon \circ Y_{\pm\epsilon} &= \pm Y_{\pm\epsilon} \circ X_\epsilon \in \mathfrak{z}_\mp \,, \\
X_\epsilon \circ Z_{\pm\epsilon} &= \pm\epsilon Z_{\pm\epsilon} \circ X_\epsilon \in \mathfrak{y}_\mp \,, \\
Y_\epsilon \circ Z_{\pm\epsilon} &= Z_{\pm\epsilon} \circ Y_\epsilon \in \mathfrak{x}_\mp \,,
\end{aligned}$$

$$\begin{aligned}
\langle X_+ \circ Y_+, Z_- \rangle &= -\frac{1}{\sqrt{2}} \langle JY_+, \psi(JX_+, Z_-) \rangle \,, \\
\langle X_- \circ Y_-, Z_- \rangle &= -\frac{1}{\sqrt{2}} \langle Y_-, \psi(X_-, Z_-) \rangle \,,
\end{aligned}$$

$$\begin{aligned}
\langle X_+ \circ Y_-, Z_+\rangle &= \frac{1}{\sqrt{2}}\langle Y_-, \psi(JX_+, JZ_+)\rangle\,,\\
\langle X_- \circ Y_+, Z_+\rangle &= -\frac{1}{\sqrt{2}}\langle JY_+, \psi(X_-, JZ_+)\rangle\,,\\
\langle X_+ \circ Z_+, Y_-\rangle &= -\frac{1}{\sqrt{2}}\langle Y_-, \psi(JX_+, JZ_+)\rangle\,,\\
\langle X_- \circ Z_-, Y_-\rangle &= \frac{1}{\sqrt{2}}\langle Y_-, \psi(X_-, Z_-)\rangle\,,\\
\langle X_+ \circ Z_-, Y_+\rangle &= \frac{1}{\sqrt{2}}\langle JY_+, \psi(JX_+, Z_-)\rangle\,,\\
\langle X_- \circ Z_+, Y_+\rangle &= \frac{1}{\sqrt{2}}\langle JY_+, \psi(X_-, JZ_+)\rangle\,,\\
\langle Y_+ \circ Z_+, X_-\rangle &= -\frac{1}{\sqrt{2}}\langle JY_+, \psi(X_-, JZ_+)\rangle\,,\\
\langle Y_- \circ Z_-, X_-\rangle &= \frac{1}{\sqrt{2}}\langle Y_-, \psi(X_-, Z_-)\rangle\,,\\
\langle Y_+ \circ Z_-, X_+\rangle &= \frac{1}{\sqrt{2}}\langle JY_+, \psi(JX_+, Z_-)\rangle\,,\\
\langle Y_- \circ Z_+, X_+\rangle &= \frac{1}{\sqrt{2}}\langle Y_-, \psi(JX_+, JZ_+)\rangle\,.
\end{aligned}$$

$\mathcal{W}(p,0) \cong \mathcal{W}(0,p) = SO_0(p+4,4)/(SO(p+4)\times SO(4))$ are the only symmetric $\mathcal{W}$-spaces. The multiplication in the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ defines special isometric mappings of order 1, 2, 4 and 8 respectively. The corresponding $\mathcal{V}$-spaces are the only symmetric $\mathcal{V}$-spaces. For the remaining of this paper the $\mathcal{W}$- and $\mathcal{V}$-spaces under consideration will always be assumed to be non-symmetric.

First we repeat the analysis of the previous subsection for the $\mathcal{W}$-spaces. $\mathcal{V}$-spaces will be treated in the next subsection.

Consider $\mathfrak{w} = \mathfrak{w}(p,q)$ and let $\mathfrak{w} = \mathfrak{a} + \mathfrak{n}$ denote the decomposition (1).

Lemma 2.7 *The semidirect orthogonal decompositions $\mathfrak{n} = \mathfrak{n}_1 \bar{\oplus} \mathfrak{n}_2$ of the Lie algebra $\mathfrak{n} = [\mathfrak{w},\mathfrak{w}]$, such that any root space $\mathfrak{n}_\alpha$, $\alpha \in \mathcal{R}$, (with respect to the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{w} = \mathfrak{a}+\mathfrak{n}$) is contained in $\mathfrak{n}_1$ or $\mathfrak{n}_2$, are given by (cf. Lemma 2.1):*

$$\mathfrak{n}_1 = 0, \ll G_2 \gg, \ll G_3 \gg, \ll \tilde{Q}_0 \gg,$$

$$\ll \mathfrak{y}_- \gg, \ll \mathfrak{z}_- \gg, \ll G_2, G_3 \gg, \ll G_2, \tilde{Q}_0 \gg,$$

$$\gg G_2, \mathfrak{y}_- \gg, \ll G_2, \mathfrak{z}_- \gg, \ll G_3, \tilde{Q}_0 \gg,$$

$$\ll G_3, \mathfrak{z}_- \gg, \ \ll \tilde{Q}_0, \mathfrak{y}_- \gg, \ \ll \tilde{Q}_0, \mathfrak{z}_- \gg,$$

$$\ll \mathfrak{y}_-, \mathfrak{z}_- \gg, \ \ll G_2, G_3, \tilde{Q}_0 \gg, \ \ll G_2, \tilde{Q}_0, \mathfrak{y}_- \gg,$$

$$\ll G_3, \tilde{Q}_0, \mathfrak{z}_- \gg, \ \ll \tilde{Q}_0, \mathfrak{y}_-, \mathfrak{z}_- \gg, \ \ll G_2, G_3, \mathfrak{y}_-, \mathfrak{z}_- \gg,$$

$$\ll G_2, G_3, \mathfrak{y}_-, \mathfrak{z}_-, \tilde{P}_0 \gg \quad or \quad \mathfrak{n}\,.$$

Proposition 2.8 *The maximal suitable SR-decomposition* $\mathfrak{w} = \mathfrak{w}_s + \mathfrak{w}_r$ *of* $\mathfrak{w} = \mathfrak{w}(p,q)$*,* $p \neq 0$ *and* $q \neq 0$*, is as follows*

1) $\mathfrak{w}_s = \mathfrak{a}_s + \mathfrak{n}_s$*,* $\mathfrak{a}_s = span\{H_0 - H_1, H_2, H_3\}$*,* $\mathfrak{n}_s = \ll G_2, G_3, \tilde{Q}_0 \gg = span\{G_2, G_3, \tilde{P}_2, \tilde{P}_3, \tilde{Q}_0, \tilde{Q}_1\}$*,*

2) $\mathfrak{w}_r = \mathfrak{a}_r + \mathfrak{n}_r$*,* $\mathfrak{a}_r = span\{H_0 + H_1\}$*,* $\mathfrak{n}_r = \langle\langle \mathfrak{y}_-, \mathfrak{z}_- \rangle\rangle = span\{G_0,\ G_1,\ \tilde{P}_0,\ \tilde{P}_1,\ \tilde{Q}_2,\ \tilde{Q}_3\} + \mathfrak{x}_1 + \tilde{\mathfrak{x}}_1$*,* $\mathfrak{x}_1 = \mathfrak{y} + \mathfrak{z}$*,*

3) $\mathfrak{w}_s = \mathfrak{i}(\mathfrak{s})$*,* $\mathfrak{s} = \mathfrak{so}(\mathfrak{v}_{3,3}) \cong \mathfrak{so}(3,3)$*,*

where a scalar product $(,)$ *of signature* $(3,3)$ *on* $\mathfrak{v}_{3,3} = span\{G_0,\ G_1,\ \tilde{P}_0,\ \tilde{P}_1,\ \tilde{Q}_2,\ \tilde{Q}_3\} = \mathfrak{zent}(\mathfrak{n}_r)$ *is defined by the formulas* $(i,j = 0,1,\ k = 2,3)$

$$(G_0, G_1) = (-\tilde{P}_0, \tilde{P}_1) = (\tilde{Q}_2, \tilde{Q}_3) = 1\,;$$

$$(G_i, G_i) = (\tilde{P}_i, \tilde{P}_i) = (\tilde{Q}_k, \tilde{Q}_k) = (G_i, \tilde{P}_j) = (G_i, \tilde{Q}_k) = (\tilde{P}_j, \tilde{Q}_k) = 0\,.$$

We consider $\mathfrak{w}_s$ *as subalgebra of* $\mathfrak{s} = \mathfrak{so}(\mathfrak{v}_{3,3}) \cong \bigwedge^2 \mathfrak{v}_{3,3}$ *via the embedding*

$$\begin{aligned}
H_0 - H_1 &\mapsto G_0 \hat{\wedge} G_1\\
H_2 &\mapsto -(\tilde{P}_0 \hat{\wedge} \tilde{P}_1 + \tilde{Q}_2 \hat{\wedge} \tilde{Q}_3)/2\\
H_3 &\mapsto -(\tilde{P}_0 \hat{\wedge} \tilde{P}_1 - \tilde{Q}_2 \hat{\wedge} \tilde{Q}_3)/2\\
G_2 &\mapsto \tilde{P}_0 \hat{\wedge} \tilde{Q}_3\\
G_3 &\mapsto \tilde{P}_0 \hat{\wedge} \tilde{Q}_2\\
\tilde{P}_2 &\mapsto G_0 \hat{\wedge} \tilde{Q}_3\\
\tilde{P}_3 &\mapsto G_0 \hat{\wedge} \tilde{Q}_2\\
\tilde{Q}_0 &\mapsto G_0 \hat{\wedge} \tilde{P}_1\\
\tilde{Q}_1 &\mapsto G_0 \hat{\wedge} \tilde{P}_0\,,
\end{aligned}$$

where the notation $\hat{\wedge}$ *was introduced in Prop. 2.2. The extension* $\rho : \mathfrak{s} \to \mathfrak{Der}(\mathfrak{w}_r)$ *of the representation ad* $: \mathfrak{w}_s \to \mathfrak{Der}(\mathfrak{w}_r)$ *is trivial on* $span\{H_0 + H_1\}$*, standard on* $\mathfrak{v}_{3,3}$ *and acts on* $\mathfrak{x}_1 + \tilde{\mathfrak{x}}_1$ *as a sum of* $p + q$ *semi-spinor representations defined by the following formulas:*

$$\rho(-\tilde{P}_0 \hat{\wedge} G_1)|(\tilde{\mathfrak{y}}_- + \tilde{\mathfrak{z}}_-) = -J_3\,, \quad \rho(-\tilde{P}_0 \hat{\wedge} G_1)|(\mathfrak{x}_1 + \tilde{\mathfrak{y}}_+ + \tilde{\mathfrak{z}}_+) = 0\,;$$

$$\rho(\tilde{Q}_3 \hat{\wedge} G_1)|(\tilde{\mathfrak{y}}_- + \tilde{\mathfrak{z}}_+) = -J_2\,, \quad \rho(\tilde{Q}_3 \hat{\wedge} G_1)|(\mathfrak{x}_1 + \tilde{\mathfrak{y}}_+ + \tilde{\mathfrak{z}}_-) = 0\,;$$

$$\rho(\tilde{Q}_3\hat{\wedge}\tilde{P}_1)|(\mathfrak{y}_+ + \tilde{\mathfrak{z}}_+) = -J_1\,, \quad \rho(\tilde{Q}_3\hat{\wedge}\tilde{P}_1)|(\mathfrak{y}_- + \mathfrak{z} + \tilde{\mathfrak{z}}_- + \tilde{\mathfrak{y}}) = 0\,;$$
$$\rho(G_1\hat{\wedge}\tilde{P}_1)|(\tilde{\mathfrak{y}}_+ + \tilde{\mathfrak{z}}_+) = J_3\,, \quad \rho(G_1\hat{\wedge}\tilde{P}_1)|(\mathfrak{x}_1 + \tilde{\mathfrak{y}}_- + \tilde{\mathfrak{z}}_-) = 0\,;$$
$$\rho(G_1\hat{\wedge}\tilde{Q}_2)|(\tilde{\mathfrak{y}}_+ + \tilde{\mathfrak{z}}_-) = J_2\,, \quad \rho(G_1\hat{\wedge}\tilde{Q}_2)|(\mathfrak{x}_1 + \tilde{\mathfrak{y}}_- + \tilde{\mathfrak{z}}_+) = 0\,;$$
$$\rho(\tilde{P}_1\hat{\wedge}\tilde{Q}_2)|(\mathfrak{z}_+ + \tilde{\mathfrak{y}}_+) = J_1\,, \quad \rho(\tilde{P}_1\hat{\wedge}\tilde{Q}_2)|(\mathfrak{y} + \mathfrak{z}_- + \tilde{\mathfrak{y}}_- + \tilde{\mathfrak{z}}) = 0\,.$$

More precisely, $\mathfrak{y} + \tilde{\mathfrak{y}}$ *(respectively,* $\mathfrak{z} + \tilde{\mathfrak{z}}$*) consists of* p *(respectively, of* q*) equivalent semi-spinor representations and the irreducible summands of* $\mathfrak{y} + \tilde{\mathfrak{y}}$ *are not equivalent to the summands of* $\mathfrak{z} + \tilde{\mathfrak{z}}$*.*

Proof. The proof is completely analogous to the proof of Prop. 2.2. We only remark that the semi-spinor representations on $\mathfrak{y} + \tilde{\mathfrak{y}}$ and $\mathfrak{z} + \tilde{\mathfrak{z}}$ are precisely related by the outer automorphism ξ of $\mathfrak{so}(\mathfrak{v}_{3,3})$ given in Prop. 3.1. □

Corollary 2.9 *Consider the Cartan decomposition* $\mathfrak{s} = \mathfrak{k}_s + \mathfrak{p}$ *of* $\mathfrak{s} = \mathfrak{so}(\mathfrak{v}_{3,3})$ *into its skew-symmetric part* $\mathfrak{k}_s$ *and its symmetric part* $\mathfrak{p}$ *with respect to the* Euclidean *scalar product* $\langle , \rangle$ *on* $\mathfrak{v}_{3,3}$*. Then* $\rho(\mathfrak{k}_s)$ *(respectively,* $\rho(\mathfrak{p})$*) consists of skew-symmetric (respectively, symmetric) endomorphisms.*

Corollary 2.10 *Denote by* $\bar{\mathfrak{k}}_s \subset \mathfrak{q} + \mathfrak{z}(\mathfrak{q}) \subset \mathfrak{so}(\mathfrak{w})$ *the image of* $\mathfrak{k}_s$ *under the isotropy representation on* $\mathfrak{s}/\mathfrak{k}_s + \mathfrak{w}_r \cong \mathfrak{w}$*. Then the projection of* $\bar{\mathfrak{k}}_s$ *on* $\mathfrak{q}$ *is surjective.*

We reformulate the results using the following notation. Let $V = (\mathbb{R}^4, \langle , \rangle)$ denote the standard Euclidean vector space. Note that the volume form vol on V defines an indefinite scalar product $(,)$ on $\wedge^2 V$ by

$$\sigma \wedge \eta = (\sigma, \eta)\, vol\,.$$

We identify $\wedge^2 V \cong \wedge^2 V^*$ using $(,)$.

Now we define a metric Lie algebra $\mathfrak{w}_r$, which as we will show is isomorphic to $\mathfrak{w}_r$ from Prop. 2.8. We set

$$\mathfrak{w}_r = V \otimes \mathbb{R}^p \mathbin{\bar{+}} V^* \otimes \mathbb{R}^q \mathbin{\bar{+}} \wedge^2 V \mathbin{\bar{+}} \mathbb{R}h\,,$$

with the scalar product $\langle , \rangle$ induced on $V \otimes \mathbb{R}^p + V^* \otimes \mathbb{R}^q + \wedge^2 V$ by the standard scalar product on V and $\mathbb{R}$; in addition $\langle h, h\rangle = 2$.

The Lie brackets are defined by

$$ad_h|(V \otimes \mathbb{R}^p + V^* \otimes \mathbb{R}^q) = \frac{1}{2} Id\,, \quad ad_h| \wedge^2 V = Id\,;$$

this implies $[\wedge^2 V, V \otimes \mathbb{R}^p + V^* \otimes \mathbb{R}^q] = 0$. Moreover $[V \otimes \mathbb{R}^p, V^* \otimes \mathbb{R}^q] = 0$ and

$$[v \otimes x, w \otimes y] = v \wedge w\, \langle x, y\rangle$$

for $v, w \in V$ or V^*. In the latter case we used the identification $\wedge^2 V \cong \wedge^2 V^*$ via $(\,,)$. Denote by $\mathfrak{sl}(V) \cong \mathfrak{sl}(4,\mathbb{R})$ the Lie algebra of endomorphisms of V which annihilate vol. It naturally acts by derivations on $\mathfrak{w}_r$ with trivial action on $\mathbb{R}h$. We have a natural action of $O(p)\times O(q)$ by orthogonal automorphisms on $\mathfrak{w}_r$, similar to the case of $\mathcal{T}$-spaces. We denote by $\mathfrak{so}(p) \oplus \mathfrak{so}(q)$ the corresponding Lie algebra of skew-symmetric derivations.

So we defined on $\mathfrak{w}_r = V \otimes \mathbb{R}^p + V^* \otimes \mathbb{R}^q + \wedge^2 V + \mathbb{R}h$ the structure of a metric Lie algebra such that the Lie algebra $\mathfrak{sl}(V) \oplus \mathfrak{so}(p) \oplus \mathfrak{so}(q)$ acts on $\mathfrak{w}_r$ by derivations.

Theorem 2.11 *1) The quaternionic Kähler Lie algebra $\mathfrak{w}(p,q)$, $p,q \geq 1$, is isomorphic (as metric Lie algebra) to the metric Lie algebra $\mathfrak{i}(\mathfrak{sl}(V)) \bar{+\!\!\!\!\supset} \mathfrak{w}_r$, where $\mathfrak{i}(\mathfrak{sl}(V))$ is the symmetric Iwasawa algebra associated with $\mathfrak{s} = \mathfrak{sl}(V)$, s. Example 1.*

2) The full isometry algebra $\mathfrak{g}(\mathcal{W}(p,q))$ of the quaternionic Kähler manifold $\mathcal{W}(p,q)$, $p,q \geq 1$, is given by

$$\mathfrak{g}(\mathcal{W}(p,q)) = (\mathfrak{sl}(V) \oplus \mathfrak{so}(p) \oplus \mathfrak{so}(q)) +\!\!\!\!\supset \mathfrak{w}_r \,.$$

The adjoint representation of this Lie algebra on its ideal $\mathfrak{w}_r$ defines an embedding $\mathfrak{g}(\mathcal{W}(p,q)) \hookrightarrow \mathfrak{Der}(\mathfrak{w}_r)$.

Proof. The proof reduces to a reformulation of Prop. 2.8 if we remark that the semi-spinor representation of $\mathfrak{so}(3,3)$ is the standard representation of $\mathfrak{sl}(V) \cong \mathfrak{so}(3,3)$ and the induced representation of $\mathfrak{sl}(V)$ on $\wedge^2 V$ is exactly the standard representation of $\mathfrak{so}(3,3)$. To identify the notations of Prop. 2.8 and Thm. 2.11 we have to identify

$$\mathfrak{h} + \tilde{\mathfrak{h}} \cong V \otimes \mathbb{R}^p, \quad \mathfrak{z} + \tilde{\mathfrak{z}} \cong V^* \otimes \mathbb{R}^q$$

$$\mathfrak{v}_{3,3} \cong \wedge^2 V, \quad \mathfrak{a}_r \cong \mathbb{R}h, \quad h = H_0 - H_1 \,.$$

To finish the proof we must check that $\mathfrak{so}(p) \oplus \mathfrak{so}(q) = \mathfrak{d}_0(\mathfrak{w}_r)$. This follows from the next lemma. □

Lemma 2.12 *$A^s(\mathfrak{w}_r) = O(p) \times O(q)$, where $A^s(\mathfrak{w}_r)$ is the group of orthogonal automorphisms of $\mathfrak{w}_r$ which commute with the action of $\mathfrak{s} = \mathfrak{sl}(V)$. The Lie algebra of $A^s(\mathfrak{w}_r)$ is $\mathfrak{d}_0(\mathfrak{w}_r) = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$.*

Proof. The proof is the same as for $\mathcal{T}$-spaces (cf. Lemma 2.6) since V, V^*, $\wedge^2 V$ and $\mathbb{R}h$ are non-equivalent irreducible $\mathfrak{sl}(V)$-modules. □

2.3 $\mathcal{V}$-Spaces

The definition of the non-symmetric $\mathcal{V}$-spaces was given in subsection 2.2. Let $\mathfrak{v} = \mathfrak{a} + \mathfrak{n}$ denote the decomposition (1) for the quaternionic Kähler Lie algebra $\mathfrak{v} = \mathfrak{v}(\psi)$ associated with a special isometric mapping ψ.

Lemma 2.13 *The semidirect orthogonal decompositions $\mathfrak{n} = \mathfrak{n}_1 \bar{\oplus} \mathfrak{n}_2$ of the Lie algebra $\mathfrak{n} = [\mathfrak{v}, \mathfrak{v}]$, such that any root space $\mathfrak{n}_\alpha$, $\alpha \in \mathcal{R}$, (with respect to the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{v} = \mathfrak{a} + \mathfrak{n}$) is contained in $\mathfrak{n}_1$ or $\mathfrak{n}_2$, are given by (cf. Lemma 2.1):*

$$\mathfrak{n}_1 \;=\; 0,\; \ll G_3 \gg,\; \ll \tilde{Q}_0 \gg,\; \ll \mathfrak{z}_- \gg,$$
$$\ll \mathfrak{x}_- \gg,\; \ll G_3, \tilde{Q}_0 \gg,\; \ll G_3, \mathfrak{z}_- \gg,\; \ll G_3, \mathfrak{x}_- \gg,$$
$$\ll \mathfrak{x}_-, \mathfrak{z}_- \gg,\; \ll \tilde{Q}_0, \mathfrak{z}_- \gg,\; \ll \tilde{Q}_0, \mathfrak{x}_- \gg,$$
$$\ll G_3, \tilde{Q}_0, \mathfrak{z}_- \gg,\; \ll G_3, \tilde{Q}_0, \mathfrak{x}_- \gg,$$
$$\ll \tilde{Q}_0, \mathfrak{x}_-, \mathfrak{z}_- \gg,\; \ll G_3, \mathfrak{x}_-, \mathfrak{z}_- \gg \quad \textit{or} \quad \mathfrak{n}\,.$$

Proposition 2.14 *The maximal suitable SR-decomposition $\mathfrak{v} = \mathfrak{v}_s + \mathfrak{v}_r$ of $\mathfrak{v} = \mathfrak{v}(\psi)$, $\mathfrak{v}$ non-symmetric, is given by*

1) $\mathfrak{v}_s = \mathfrak{a}_s + \mathfrak{n}_s$, $\mathfrak{a}_s = span\{H_0 - H_1, H_2, H_3\}$, $\mathfrak{n}_s = \ll G_3, \tilde{Q}_0, \mathfrak{x}_- \gg = span\{G_2, G_3, \tilde{P}_2, \tilde{P}_3, \tilde{Q}_0, \tilde{Q}_1\} + \mathfrak{x} + \tilde{\mathfrak{x}}_-$,

2) $\mathfrak{v}_r = \mathfrak{a}_r + \mathfrak{n}_r$, $\mathfrak{a}_r = span\{H_0 + H_1\}$, $\mathfrak{n}_r = \langle\langle \mathfrak{z}_- \rangle\rangle = span\{G_0,\ G_1,\ \tilde{P}_0,\ \tilde{P}_1,\ \tilde{Q}_2,\ \tilde{Q}_3\} + \mathfrak{x}_1 + \tilde{\mathfrak{x}}_1$, $\mathfrak{x}_1 = \mathfrak{y} + \mathfrak{z}$,

3) $\mathfrak{v}_s = \mathfrak{i}(\mathfrak{s})$, $\mathfrak{s} = \mathfrak{so}(\mathfrak{v}_{3,3+k}) \cong \mathfrak{so}(3, 3+k)$,

where a scalar product $(,)$ of signature $(3, 3+k)$ is defined on $\mathfrak{v}_{3,3+k} = \mathfrak{v}_{3,3} + \tilde{\mathfrak{x}}_+ = \mathfrak{zent}(\mathfrak{n}_r)$ by the requirement that $\mathfrak{v}_{3,3}$ and $\tilde{\mathfrak{x}}_+$ are $(,)$-orthogonal, that the scalar product coincides with $-\langle,\rangle$ on $\tilde{\mathfrak{x}}_+$ and is the same as in Prop. 2.8 on $\mathfrak{v}_{3,3} = span\{G_0, G_1, \tilde{P}_0, \tilde{P}_1, \tilde{Q}_2, \tilde{Q}_3\}$. We consider $\mathfrak{v}_s$ as subalgebra of $\mathfrak{s} = \mathfrak{so}(\mathfrak{v}_{3,3+k}) \cong \bigwedge^2 \mathfrak{v}_{3,3+k}$ via the embedding ($X_\pm \in \mathfrak{x}_\pm$):

$$\begin{aligned} X_+ &\mapsto \tilde{P}_0 \hat{\wedge} \tilde{X}_+ \\ X_- &\mapsto -\tilde{Q}_3 \hat{\wedge} \widetilde{JX}_- \\ \tilde{X}_- &\mapsto -G_0 \hat{\wedge} \widetilde{JX}_-\,, \end{aligned}$$

s. Prop. 2.2 for the notation $\hat{\wedge}$. The embedding of the subalgebra $\mathfrak{v}_s \cap (\mathfrak{x} + \tilde{\mathfrak{x}}_-)^- \cong \mathfrak{w}_s$ is given by the same formulas as in Prop. 2.8.

The extension $\rho : \mathfrak{s} \to \mathfrak{der}(\mathfrak{v}_r)$ of the representation $ad : \mathfrak{v}_s \to \mathfrak{der}(\mathfrak{w}_r)$ is trivial on $span\{H_0 + H_1\}$, standard on $\mathfrak{v}_{3,3+k}$ and its action on $\mathfrak{x}_1 + \tilde{\mathfrak{x}}_1$ is as *follows. The action of* $\mathfrak{so}(3,3)$ *on* $\mathfrak{x}_1+\tilde{\mathfrak{x}}_1$ *is given by the same formulas as in Prop. 2.8. For all* $X_- \in \mathfrak{x}_-$ *we have:*

$$\rho(G_1\hat{\wedge}\widetilde{JX}_-)|(\tilde{\mathfrak{y}}_+ + \tilde{\mathfrak{z}}_-) = J_2X_-\circ\,,$$

$$\rho(G_1\hat{\wedge}\widetilde{JX}_-)|(\tilde{\mathfrak{y}}_- + \tilde{\mathfrak{z}}_+) = -J_2X_-\circ\,,$$

$$\rho(G_1\hat{\wedge}\widetilde{JX}_-)|\mathfrak{x}_1 = 0\,;$$

$$\rho(\tilde{P}_1\hat{\wedge}\widetilde{JX}_-)|(\mathfrak{y}_+ + \tilde{\mathfrak{z}}_+) = -J_1X_-\circ\,,$$

$$\rho(\tilde{P}_1\hat{\wedge}\widetilde{JX}_-)|(\mathfrak{z}_+ + \tilde{\mathfrak{y}}_+) = J_1X_-\circ\,,$$

$$\rho(\tilde{P}_1\hat{\wedge}\widetilde{JX}_-)|(\mathfrak{y}_- + \mathfrak{z}_- + \tilde{\mathfrak{y}}_- + \tilde{\mathfrak{z}}_-) = 0\,;$$

$$\rho(\tilde{Q}_2\hat{\wedge}\widetilde{JX}_-)|(\mathfrak{y}_- + \mathfrak{z}_+ + \tilde{\mathfrak{y}}_+ + \tilde{\mathfrak{z}}_-) = X_-\circ\,,$$

$$\rho(\tilde{Q}_2\hat{\wedge}\widetilde{JX}_-)|(\mathfrak{y}_+ + \mathfrak{z}_- + \tilde{\mathfrak{y}}_- + \tilde{\mathfrak{z}}_+) = 0\,;$$

$$\rho(\widetilde{JX}_-\hat{\wedge}\widetilde{JX'}_-)|(\mathfrak{x}_1+\tilde{\mathfrak{x}}_1) = \frac{1}{2}(X_-\circ X'_- - X'_-\circ X_-)\circ\,,$$

where $X_-\circ$ *denotes the* $\mathfrak{q}$*-linear endomorphism of* $\mathfrak{x}_1+\tilde{\mathfrak{x}}_1$ *obtained from the map (s. p. 22)* $U \mapsto X_-\circ U$, $U \in \mathfrak{y}+\mathfrak{z}$, *by* J_2*-linear extension and* $J_\alpha X_-\circ$ *is the composition of the endomorphisms* $J_\alpha|(\mathfrak{x}_1+\tilde{\mathfrak{x}}_1)$ *and* $X_-\circ$.

Corollary 2.15 *Consider the Cartan decomposition* $\mathfrak{s} = \mathfrak{k}_s + \mathfrak{p}$ *of the Lie algebra* $\mathfrak{s} = \mathfrak{so}(\mathfrak{v}_{3,3+k})$ *into its skew-symmetric part* $\mathfrak{k}_s$ *and its symmetric part* $\mathfrak{p}$ *with respect to the* Euclidean *scalar product* $\langle,\rangle$ *on* $\mathfrak{v}_{3,3+k}$. *Then* $\rho(\mathfrak{k}_s)$ *(respectively,* $\rho(\mathfrak{p})$*) consists of skew-symmetric (respectively, symmetric) endomorphisms.*

Corollary 2.16 *Denote by* $\bar{\mathfrak{k}}_s \subset \mathfrak{q}+\mathfrak{z}(\mathfrak{q}) \subset \mathfrak{so}(\mathfrak{v})$ *the image of* $\mathfrak{k}_s$ *under the isotropy representation on* $\mathfrak{s}/\mathfrak{k}_s + \mathfrak{v}_r \cong \mathfrak{v}$. *Then the projection of* $\bar{\mathfrak{k}}_s$ *on* $\mathfrak{q}$ *is surjective.*

We recall that the isometry classes of $\mathcal{V}$-spaces are in one-to-one correspondence with the equivalence classes of special isometric mappings. Furthermore, every special isometric mapping can be decomposed into a sum of irreducible ones. Two special isometric mappings of order $k \not\equiv 0$ (4) are equivalent if and only if they have the same number $l = 1,2,3\ldots$ of irreducible summands. Every special isometric mapping ψ of order $k \equiv 0$ (4) admits a decomposition $\psi = p\psi^+ + q\psi^-$, where $\psi^\pm$ are two non-isomorphic special isometric mappings and $p,q \in \{0,1,2\ldots\}$, $l = p+q \geq 1$; $p\psi^+ + q\psi^-$ and $p'\psi^+ + q'\psi^-$ are equivalent if and only if $\{p,q\} = \{p',q'\}$.

For the final description of the full isometry algebra $\mathfrak{g}(\mathcal{V}(\psi))$ of $\mathcal{V}(\psi)$ we introduce further notation. Denote by $V = \mathbb{R}^{3,3+k}$ the vector representation of $\mathfrak{s} = \mathfrak{so}(3, 3+k)$ and by S a (real) spinorial representation of lowest dimension, i.e. spinor representation if it is irreducible and semi-spinor otherwise. In other words, S is spinor if $k \equiv 1, 2, 3, 5$ (8) and semi-spinor if $k \equiv 0, 4, 6, 7$ (8). For $k \equiv 0$ (4) we denote the two non-equivalent semi-spinor representations by S^+ and S^-. In order to describe the principal part $\mathfrak{s} \oplus \mathfrak{v}_r$ of the full isometry algebra $\mathfrak{g}(\mathcal{V}(\psi)) = (\mathfrak{s} \oplus \mathfrak{d}_0(\mathfrak{v}_r)) \oplus \mathfrak{v}_r$ of $\mathcal{V}(\psi)$ in more invariant terms we shall make use of the following facts concerning the spinorial representation S.

Proposition 2.17 *1) There exists a $Spin(V)$-invariant isomorphism $S \cong S^*$ if $k \not\equiv 0$ (4) and $S^- \cong (S^+)^*$ if $k \equiv 0$ (4).*

2) $\wedge^2 S$ contains a unique $Spin(V)$-submodule isomorphic to V.

3) Assume $k \equiv 0$ (4) and denote the unique vector submodules of $\wedge^2 S^+$ and $\wedge^2 S^-$ by V and V' resp. Then V and V' are canonically *isomorphic.*

Remark 8: The explicit description of $Spin(V)$-invariant structures on the spinor representation such as invariant bilinear forms is part of a general program which will be presented in a forthcomming paper [A-C]. As result of this program we obtain the explicit construction of embeddings of the vector representation into the (exterior or symmetric) square of the spinor or semi-spinor representation. This is used to construct Lie algebra and Lie superalgebra extensions of the (generalized) Poincaré algebra.

Proof. 1) Follows from the existence of a non-trivial $Spin(V)$-invariant bilinear form on S for $k \not\equiv 0$ (4) and on the spinor module $S^+ + S^-$ for $k \equiv 0$ (4); $S^\pm$ being isotropic in the last case (s. [A-C]).

2) The explicit construction of the embedding $V \hookrightarrow \wedge^2 S$ is given in [A-C], cf. Remark 9 below. The uniqueness can be extracted from [O-V] Table 1.

3) By 1) and 2) $V' \subset \wedge^2 S^- \cong (\wedge^2 S^+)^*$ is canonically identified with V^*. Since V^* is in turn canonically isomorphic to V via $(\,,)$ this gives a canonical isomorphism $V \cong V'$. $\Box$

We remark that only 3) and the existence statement in 2) will be used in the sequel. These can be easily derived from Prop. 2.14 without refering to [A-C]. We have not done so for expository reasons.

Remark 9: The explicit description of the embedding $V \hookrightarrow \wedge^2 S$ can be given in terms of Clifford multiplication μ and an appropriate (symmetric or skew-symmetric) $Spin(V)$-invariant bilinear form β. For example, if $k \equiv 1, 2, 3, 5$ (8) it is given by

$$v \mapsto \beta(\mu(v)\cdot, \cdot), \quad v \in V\,,$$

where $\mu : V \otimes S \to S$ and β is a $Spin(V)$-invariant skew-symmetric bilinear form on S.

With the notations above we associate to each $\mathcal{V}$-space $\mathcal{V}(\psi)$ a Lie algebra $\mathfrak{g} = \mathfrak{so}(3, 3+k) +\!\!\!\!\supset \mathfrak{v}_r$. We will prove later that $\mathfrak{g}$ is exactly the principal part $\mathfrak{s} +\!\!\!\!\supset \mathfrak{v}_r$ of the full isometry algebra $\mathfrak{g}(\mathcal{V}(\psi))$.

First we define an $\mathfrak{so}(3, 3+k)$-module $\mathfrak{v}_r$ by

$$\mathfrak{v}_r = S \otimes \mathbb{R}^l + V + \mathbb{R}h\,, \quad \text{if } k \not\equiv 0 \ (4); \tag{6}$$

$$\mathfrak{v}_r = S^+ \otimes \mathbb{R}^p + S^- \otimes \mathbb{R}^q + V + \mathbb{R}h\,, \quad \text{if } k \equiv 0 \ (4); \tag{7}$$

where the modules $\mathbb{R}h$, $\mathbb{R}^l$, $\mathbb{R}^p$ and $\mathbb{R}^q$ are trivial.

Now we define the Lie algebra structure on $\mathfrak{v}_r$

$$ad_h|S \otimes \mathbb{R}^l = \frac{1}{2} Id\,, \quad \text{if } k \not\equiv 0 \ (4),$$

$$ad_h|(S^+ \otimes \mathbb{R}^p + S^- \otimes \mathbb{R}^q) = \frac{1}{2} Id\,, \quad \text{if } k \equiv 0 \ (4),$$
$$ad_h|V = Id\,,$$

it follows that $[V, S \otimes \mathbb{R}^l] = 0$ (resp. $[V, S^+ \otimes \mathbb{R}^p + S^- \otimes \mathbb{R}^q] = 0$). The Lie bracket on $S \otimes \mathbb{R}^l$ ($k \not\equiv 0 \ (4)$) is defined by

$$[v \otimes x, w \otimes y] = pr(v \wedge w)\langle x, y\rangle\,, \quad v, w \in S\,,$$

where pr is the natural projection of $\wedge^2 S$ onto its unique vector submodule (s. Prop. 2.17 2)). In case $k \equiv 0$ (4) the same formula defines the Lie bracket on $S^+ \otimes \mathbb{R}^p$ and on $S^- \otimes \mathbb{R}^q$, using the identification of the vector submodules in $\wedge^2 S^+$ and $\wedge^2 S^-$ given in Prop. 2.17 3). Finally, we put $[S^+ \otimes \mathbb{R}^p, S^- \otimes \mathbb{R}^q] = 0$.

It is clear that the representation of $\mathfrak{so}(3, 3+k)$ on $\mathfrak{v}_r$ defined by the decompositions (6) and (7) acts by derivations of the Lie algebra structure just defined.

Theorem 2.18 *1) The quaternionic Kähler Lie algebra $\mathfrak{v}(\psi)$ of the non-symmetric quaternionic Kähler $\mathcal{V}$-space $\mathcal{V}(\psi)$ is isomorphic to the Lie algebra $\mathfrak{i}(\mathfrak{so}(3, 3+k)) +\!\!\!\!\supset \mathfrak{v}_r$, where $\mathfrak{i}(\mathfrak{so}(3, 3+k))$ denotes the Iwasawa algebra of $\mathfrak{s} = \mathfrak{so}(3, 3+k)$.*

2) The full isometry algebra $\mathfrak{g}(\mathcal{V}(\psi))$ of $\mathcal{V}(\psi)$ is given by

$$\mathfrak{g}(\mathcal{V}(\psi)) = (\mathfrak{so}(3, 3+k) \oplus \mathfrak{d}_0(\mathfrak{v}_r)) +\!\!\!\!\supset \mathfrak{v}_r\,,$$

where the Lie algebra $\mathfrak{d}_0(\mathfrak{v}_r) = Lie\, A^s(\mathfrak{v}_r)$ is given in the next lemma.

Proof. Consider the $\mathfrak{s}$-Module ($\mathfrak{s} = \mathfrak{so}(3, 3+k)$) $\mathfrak{v}_r = \mathfrak{x}_1 + \tilde{\mathfrak{x}}_1 + \mathfrak{v}_{3,3+k} + \mathfrak{a}_r$ described in Prop. 2.14. Using the formulas given there we can identify

$$\mathfrak{x}_1 + \tilde{\mathfrak{x}}_1 \cong S \otimes \mathbb{R}^l\,, \quad \text{if } k \not\equiv 0 \ (4),$$

$$\mathfrak{x}_1 + \tilde{\mathfrak{x}}_1 \cong S^+ \otimes \mathbb{R}^p + S^- \otimes \mathbb{R}^q\,, \quad \text{if } k \equiv 0 \ (4),$$

$$\mathfrak{v}_{3,3+k} \cong V\,, \quad \mathfrak{a}_r \cong \mathbb{R}h\,.$$

It is easy to see that this isomorphism of $\mathfrak{s}$-modules is also a Lie algebra isomorphism, if we identify $h = H_0 - H_1$. □

Lemma 2.19 *$A^s(\mathfrak{v}_r)$ is isomorphic to*

- *$O(l)$ if $k \equiv 1, 7 \ (8)$,*
- *$U(l)$ if $k \equiv 2, 6 \ (8)$,*
- *$Sp(l)$ if $k \equiv 3, 5 \ (8)$,*
- *$Sp(p) \times Sp(q)$ if $k \equiv 4 \ (8)$,*
- *$O(p) \times O(q)$ if $k \equiv 0 \ (8)$.*

Proof. Let $\phi \in A^s(\mathfrak{v}_r)$. We can consider ϕ as orthogonal automorphism of the metric Lie algebra $\mathfrak{v}(\psi)$ acting trivially on $\mathfrak{v}_s$. It was shown in [C] (s. Prop. II.23) that orthogonal automorphism of $\mathfrak{v}(\psi)$ correspond exactly to *autoequivalences* of ψ in the sense of Def. 2.2. Since ϕ acts as identity on $\mathfrak{v}_s \supset \mathfrak{x}_-$ it is an *automorphism* of ψ in the sense of Def. 2.2. Using the natural correspondence between special isometric mappings of order k, $\mathbb{Z}_2$-graded $C\ell_k$-modules and (ungraded) $C\ell_{k-1}$-modules, now the proof follows from next lemma and the fact that $C\ell_{k-1}$ is a

- simple real matrix algebra if $k \equiv 1, 7 \ (8)$,
- simple complex matrix algebra if $k \equiv 2, 6 \ (8)$,
- simple quaternionic matrix algebra if $k \equiv 3, 5 \ (8)$,
- sum of two simple quaternionic matrix algebras if $k \equiv 4 \ (8)$,
- sum of two simple real matrix algebras if $k \equiv 0 \ (8)$. □

Lemma 2.20 *1) Let $\mathbb{K}(n)$ denote the simple matrix algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $V = l\,\mathbb{K}^n = \mathbb{K}^n \otimes \mathbb{R}^l$ a sum of l copies of the standard representation considered as real vector space with action of $\mathbb{K}(n) \subset End(V)$. Then*

$$Z_{GL(V)}(\mathbb{K}(n)) = \mathbb{K}^* \otimes GL(l, \mathbb{R}) = GL(l, \mathbb{K})$$

and

$$Z_{O(V)}(\mathbb{K}(n)) = \begin{cases} O(l) & \text{if} \quad \mathbb{K} = \mathbb{R} \\ U(l) & \text{if} \quad \mathbb{K} = \mathbb{C} \\ Sp(l) & \text{if} \quad \mathbb{K} = \mathbb{H}, \end{cases}$$

where $O(V)$ is the orthogonal group with respect to some positive definite scalar product on V.
2) Let $A = \mathbb{K}(n) \oplus \mathbb{K}(n)'$ be the direct sum of two copies of $\mathbb{K}(n)$ and denote by $\mathbb{K}^n$ and $\mathbb{K}^{n\prime}$ the two non-equivalent irreducible representations of A, with trivial action of $\mathbb{K}(n)'$ and $\mathbb{K}(n)$ respectively. Consider $V = p\mathbb{K}^n \oplus q\mathbb{K}^{n\prime} = (\mathbb{K}^n \otimes \mathbb{R}^p) \oplus (\mathbb{K}^{n\prime} \otimes \mathbb{R}^q)$ as real vector space and $A \subset End(V)$. Then

$$Z_{GL(V)}(A) = GL(p, \mathbb{K}) \times GL(q, \mathbb{K})$$

and

$$Z_{O(V)}(A) = \begin{cases} O(p) \times O(q) & \text{if} \quad \mathbb{K} = \mathbb{R} \\ U(p) \times U(q) & \text{if} \quad \mathbb{K} = \mathbb{C} \\ Sp(p) \times Sp(q) & \text{if} \quad \mathbb{K} = \mathbb{H}. \end{cases}$$

Recall that the subgroup $\mathcal{U}(\psi) \subset \mathcal{V}(\psi)$ corresponding to the principal Kählerian subalgebra $\mathfrak{u}(\psi)$ is a totally geodesic Kählerian submanifold. Now we describe the Lie algebra $\mathfrak{g}_\mathfrak{u}$ of the group of isometries of $\mathcal{V}(\psi)$ which preserve $\mathcal{U}(\psi)$. Denote by $\mathfrak{so}(2, 2+k)$ the subalgebra of $\mathfrak{so}(3, 3+k) = \mathfrak{so}(\mathfrak{v}_{3,3+k})$ which acts trivially on the two-dimensional subspace $\mathfrak{v}_{1,1} = span\{G_0, G_1\} \subset \mathfrak{v}_{3,3+k}$ and by $\mathfrak{v}_{2,2+k}$ the orthogonal complement of $\mathfrak{v}_{1,1}$ in $\mathfrak{v}_{3,3+k}$.

Lemma 2.21 *1) The $\mathfrak{so}(2, 2+k)$-module S decomposes as direct sum of two irreducible submodules $S_\mathfrak{u}$, $\tilde{S}_\mathfrak{u}$ of the same dimension. Moreover, we may assume that*

$$S_\mathfrak{u} \otimes \mathbb{R}^l = \mathfrak{x}_1 \quad \text{and} \quad \tilde{S}_\mathfrak{u} \otimes \mathbb{R}^l = \tilde{\mathfrak{x}}_1\,,$$

in case $k \not\equiv 0$ (4). In the case $k \equiv 0$ (4) we have $S = S^\pm = S^\pm_\mathfrak{u} \oplus \tilde{S}^\pm_\mathfrak{u}$ and may assume

$$S^+_\mathfrak{u} \otimes \mathbb{R}^p + S^-_\mathfrak{u} \otimes \mathbb{R}^q = \mathfrak{x}_1 \quad \text{and} \quad \tilde{S}^+_\mathfrak{u} \otimes \mathbb{R}^p + \tilde{S}^-_\mathfrak{u} \otimes \mathbb{R}^q = \tilde{\mathfrak{x}}_1\,.$$

2) The natural projection $pr : \wedge^2 S \to V = \mathfrak{v}_{3,3+k}$ maps $\wedge^2 S_\mathfrak{u}$ and $\wedge^2 \tilde{S}_\mathfrak{u}$ onto trivial $\mathfrak{so}(2, 2+k)$-submodules of V and $S_\mathfrak{u} \otimes \tilde{S}_\mathfrak{u}$ onto the $\mathfrak{so}(2, 2+k)$-vector submodule $\mathfrak{v}_{2,2+k}$. More precisely, from our identifications it follows that

$$pr(\wedge^2 S_\mathfrak{u}) = \mathbb{R}G_1\,, \quad pr(\wedge^2 \tilde{S}_\mathfrak{u}) = \mathbb{R}G_0\,.$$

3) For $k \not\equiv 0$ (4)

$$\mathfrak{u}_r = S_\mathfrak{u} \otimes \mathbb{R}^l + pr(\wedge^2 S_\mathfrak{u}) + pr(\wedge^2 \tilde{S}_\mathfrak{u}) + \mathbb{R}h$$

is a subalgebra of $\mathfrak{v}_r$. For $k \equiv 0$ (4) the same is true for

$$\mathfrak{u}_r = S_\mathfrak{u}^+ \otimes \mathbb{R}^p + S_\mathfrak{u}^- \otimes \mathbb{R}^q + pr(\wedge^2 S_\mathfrak{u}) + pr(\wedge^2 \tilde{S}_\mathfrak{u}) + \mathbb{R}h$$

($pr(\wedge^2 S_\mathfrak{u})$ and $pr(\wedge^2 \tilde{S}_\mathfrak{u})$ do not dependent on the choice $S_\mathfrak{u} = S_\mathfrak{u}^\pm$). Moreover,

$$\mathfrak{u}_r = \mathfrak{x}_1 + span\{G_0, G_1\} + \mathfrak{a}_r = \mathfrak{u} \cap \mathfrak{v}_r \,.$$

Proof. The lemma follows from the formulas in Prop. 2.14. □

Theorem 2.22 *1) The maximal suitable SR-decomposition of the non-symmetric quaternionic Kähler Lie algebra $\mathfrak{v}(\psi)$ induces a (not maximal) suitable SR-decomposition of the principal Kähler subalgebra $\mathfrak{u} = \mathfrak{u}(\psi)$, i.e. $\mathfrak{u} = \mathfrak{u}_s + \mathfrak{u}_r$, where $\mathfrak{u}_s = \mathfrak{u} \cap \mathfrak{v}_s$ is the Iwasawa subalgebra of $\mathfrak{so}(2, 2+k)$ and $\mathfrak{u}_r = \mathfrak{u} \cap \mathfrak{v}_r$.*
2) The Lie algebra $\mathfrak{g}_\mathfrak{u}$ is given by

$$\mathfrak{g}_\mathfrak{u} = (\mathfrak{so}(2, 2+k) \oplus \mathfrak{d}_0(\mathfrak{v}_r)^{\mathfrak{u}_r}) \,+\!\!\!\!\supset \mathfrak{u}_r \,,$$

where $\mathfrak{d}_0(\mathfrak{v}_r)^{\mathfrak{u}_r} = \{\varphi \in \mathfrak{d}_0(\mathfrak{v}_r) |\, \varphi \mathfrak{u}_r \subset \mathfrak{u}_r\}$.

Proof. One can check immediately using Prop. 2.14 that

$$\mathfrak{u} \cap \mathfrak{v}_s = \mathfrak{a}_s + span\{G_2, G_3\} + \mathfrak{x} \,,$$

$$\mathfrak{u} \cap \mathfrak{v}_r = \mathfrak{a}_r + span\{G_0, G_1\} + \mathfrak{x}_1 \,.$$

This implies 1). To prove 2) we remark that $\mathfrak{g}_\mathfrak{u} = \mathfrak{k}_\mathfrak{u} + \mathfrak{u}$, where $\mathfrak{k}_\mathfrak{u} = \mathfrak{k} \cap \mathfrak{u}$ is a compact subalgebra, hence $\mathfrak{k}_\mathfrak{u} \cap \mathfrak{v}_r = 0$ and $\mathfrak{u}_r = \mathfrak{g}_\mathfrak{u} \cap \mathfrak{v}_r$. This implies that $\mathfrak{u}_r$ is an ideal in $\mathfrak{g}_\mathfrak{u}$, because $\mathfrak{v}_r$ is an ideal in $\mathfrak{g}(\mathcal{V})$. This shows that $\mathfrak{g}_\mathfrak{u} \subset \mathfrak{n}_{\mathfrak{g}(\mathcal{V})}(\mathfrak{u}_r)$. On the other hand, one can check that

$$\mathfrak{n}_{\mathfrak{g}(\mathcal{V})}(\mathfrak{u}_r) = (\mathfrak{so}(2, 2+k) \oplus \mathfrak{d}_0(\mathfrak{v}_r)^{\mathfrak{u}_r}) \,+\!\!\!\!\supset \mathfrak{u}_r \subset \mathfrak{g}_\mathfrak{u} \,. \quad □$$

3 Full Isometry Group of Quaternionic Kählerian Solvmanifolds

3.1 Automorphism Group of $\mathfrak{s} = \mathfrak{so}(3, 3+k)$

In section 2 we proved that the maximal semisimple subalgebra $\mathfrak{s}$ of noncompact type in the full isometry algebra $\mathfrak{g}(\mathcal{L})$ of the quaternionic Kähler solvmanifolds $(\mathcal{L}, g)$ is isomorphic to $\mathfrak{so}(3, 3+k)$, where $k = -1, 0, 1, \ldots$ is the order of the isometric mapping defining $(\mathcal{L}, g)$. We will apply Cor. 1.9 to construct the full (not necessarily connected) isometry group $I(\mathcal{L}, g)$. For this we need a description of the group $Aut(\mathfrak{s})$ of all automorphisms of the Lie algebra $\mathfrak{s} = \mathfrak{so}(3, 3+k)$. It is given in the following proposition.

Proposition 3.1 *1) $Aut(\mathfrak{s}) = Int(\mathfrak{s}) \cup \xi Int(\mathfrak{s})$ for $\mathfrak{s} = \mathfrak{so}(3, 3+k)$, if $0 \neq k \geq -1$;*

2) $Aut(\mathfrak{s}) = Int(\mathfrak{s}) \cup \xi Int(\mathfrak{s}) \cup \eta Int(\mathfrak{s}) \cup \xi\eta Int(\mathfrak{s})$ for $\mathfrak{s} = \mathfrak{so}(3, 3)$ $(k = 0)$, where $\xi = Ad_D$, $\eta = Ad_C$ and

$$D = \begin{pmatrix} \mathbf{1}_{k+5} & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \mathbf{1}_3 \\ \mathbf{1}_3 & 0 \end{pmatrix}.$$

Proof. The proof will split into several lemmas. We denote by $\mathfrak{k}_s = \mathfrak{so}(3) \oplus \mathfrak{so}(3+k)$ the maximal compact subalgebra of $\mathfrak{s} = \mathfrak{so}(3, 3+k)$ and by $\mathfrak{s}^{\mathbb{C}} = \mathfrak{so}(6+k, \mathbb{C})$ the complexification of $\mathfrak{s}$. We will consider $\mathfrak{s}$ and $\mathfrak{s}^{\mathbb{C}}$ as linear Lie algebras of the vector spaces $V = \mathbb{R}^{6+k}$ and $V^{\mathbb{C}} = \mathbb{C}^{6+k}$ respectively.

Lemma 3.2 *[He]*

$$Aut(\mathfrak{s}^{\mathbb{C}}) = \begin{cases} Int(\mathfrak{s}^{\mathbb{C}}) & \text{if } k \text{ is even} \\ Int(\mathfrak{s}^{\mathbb{C}}) \cup \xi Int(\mathfrak{s}^{\mathbb{C}}) & \text{if } k \text{ is odd,} \end{cases}$$

where $\xi = Ad_D$, $D = diag(\mathbf{1}_{k+5}, -1)$. The automorphism ξ preserves $\mathfrak{s}$ and modulo $Int(\mathfrak{s})$ is equivalent to $Ad_E : X \mapsto -X^t$, $E = diag(\mathbf{1}_3, -\mathbf{1}_{3+k})$.

Denote by $Aut_e(\mathfrak{k}_s)$ the group of all automorphisms of $\mathfrak{k}_s = \mathfrak{so}(3) \oplus \mathfrak{so}(3+k)$ which can be extended to an automorphism of $\mathfrak{s}$.

Lemma 3.3 *1) $Aut(\mathfrak{k}_s) = Int(\mathfrak{k}_s)$ for k even $\neq 0$,*

2) $Aut(\mathfrak{k}_s) = Int(\mathfrak{k}_s) \cup \xi Int(\mathfrak{k}_s)$ for k odd $\neq 1$,

3) $Aut_e(\mathfrak{k}_s) = Int(\mathfrak{k}_s) \cup \xi Int(\mathfrak{k}_s)$ for $k = 1$,

4) $Aut(\mathfrak{k}_s) = Int(\mathfrak{k}_s) \cup \eta Int(\mathfrak{k}_s)$ for $k = 0$, where $\xi = Ad_D$ and $\eta = Ad_C$ with matrices D and C given in Prop. 3.1.

Corollary 3.4 *For $k \neq 1$, $Aut(\mathfrak{k}_s) = Aut_e(\mathfrak{k}_s)$.*

Proof. The proof of 1), 2), 4) follows from [He]. To prove 3) it is sufficient to show that an automorphism φ of $\mathfrak{k}_s = \mathfrak{so}(3) \oplus \mathfrak{so}(4)$ can be extended to an automorphism of $\mathfrak{s} = \mathfrak{so}(3,4)$ if and only if it preserves the ideal $\mathfrak{so}(3)$ (hence, also $\mathfrak{so}(4)$). Since any automorphism of $\mathfrak{s}^{\mathbb{C}} = \mathfrak{so}(7,\mathbb{C})$ is inner, $\varphi = Ad_A$ for some $A \in SO(7,\mathbb{C})$. Hence, the decomposition of the vector space $V^{\mathbb{C}} = \mathbb{C}^7$ into irreducible components with respect to the Lie algebras $\mathfrak{so}(3)$ and $\varphi(\mathfrak{so}(3)) = A\mathfrak{so}(3)A^{-1}$ are isomorphic. This implies that $\varphi(\mathfrak{so}(3)) = \mathfrak{so}(3)$, since $\mathfrak{so}(3)$ acts trivially on a 4-dimensional subspace of $V^{\mathbb{C}}$ and both $\mathfrak{so}(3)$-ideals of $\mathfrak{so}(4)$ act trivially only on a 3-dimensional subspace of $V^{\mathbb{C}}$. □

Lemma 3.5 *The only non-trivial automorphism of* $\mathfrak{s}$ *which acts trivially on* $\mathfrak{k}_s$ *is* Ad_E, $E = diag(\mathbf{1}_3, -\mathbf{1}_{3+k})$.

Proof. By Lemma 3.2 we may assume that the automorphism has the form $Ad_A|\mathfrak{s}$ for some $A \in SO(6+k,\mathbb{C})$; where Ad_A preserves $\mathfrak{s}$ and acts trivially on $\mathfrak{k}_s$. The last condition means that $[A, \mathfrak{k}_s] = 0$. The linear Lie algebra $\mathfrak{k}_s = \mathfrak{so}(3) \oplus \mathfrak{so}(3+k) \subset \mathfrak{gl}(V^{\mathbb{C}})$ has two non-equivalent invariant irreducible subspaces. By Schur's Lemma $A = diag(\lambda\mathbf{1}_3, \mu\mathbf{1}_{3+k})$, $\lambda, \mu \in \mathbb{C}$. Since $A \in SO(6+k,\mathbb{C})$, we have $\lambda^2 = \mu^2 = 1$. Hence, $A = \pm E$ and $Ad_A = Ad_E$. □

The proof of Proposition 3.1 now follows from Lemmas 3.3 and 3.5

3.2 $\mathcal{T}$-Spaces

Theorem 3.6 *The (full) isometry group of the quaternionic Kähler manifolds* $\mathcal{T}(p)$, $p \geq 1$, *is given by*

$$I(\mathcal{T}(p)) = \mathcal{G}^s \cup \varphi\mathcal{G}^s \hookrightarrow Aut(\mathfrak{t}_r)\,,$$

$\mathcal{G}^s = (\mathcal{S} \times A^s(\mathfrak{t}_r)) \ltimes \mathcal{T}_r$, *where* $\mathcal{S} = Sp(4,\mathbb{R}) = Spin_0(3,2)$ *and* $\mathcal{T}_r$ *are the linear subgroups of* $Aut(\mathfrak{t}_r)$ *generated by the linear Lie algebras* $\mathfrak{s} = \mathfrak{sp}(4,\mathbb{R}), \mathfrak{t}_r \subset \mathfrak{Der}(\mathfrak{t}_r)$ *(s. Thm. 2.5) and* $A^s(\mathfrak{t}_r) = O(p)$ *is given in Lemma 2.6. The automorphism* $\varphi \in Aut(\mathfrak{t}_r)$ *is defined in terms of the decomposition* $\mathfrak{t}_r = V \otimes \mathbb{R}^p + \wedge_0^2 V + \mathbb{R}\omega^0$ *as follows:*

$$\varphi|V \otimes \mathbb{R}^p = B \otimes Id, \quad B = diag(1,1,-1,-1)\,,$$

$$\varphi(v \wedge w)_0 = (\varphi v \wedge \varphi w)_0 \quad \text{and} \quad \varphi\omega^0 = \omega^0\,.$$

Proof. According to Cor. 1.9 and Prop. 3.1 it is sufficient to prove that the automorphism φ of $\mathfrak{t}_r$ normalizes $\mathfrak{s} = \mathfrak{so}(3,2) = \mathfrak{sp}(4,\mathbb{R})$ and induces an outer automorphism on $\mathfrak{s}$. Since $B\omega_0 = -\omega_0$, conjugation with B will be an outer automorphism of $\mathfrak{sp}(4,\mathbb{R})$ which is extended by φ to an automorphism of $\mathfrak{s} \oplus \mathfrak{t}_r$. □

3.3 $\mathcal{W}$-Spaces

Theorem 3.7 *The (full) isometry group of the quaternionic Kähler manifolds $\mathcal{W}(p,q)$, $p,q \geq 1$, is given by*

$$I(\mathcal{W}(p,q)) = \mathcal{G}^s \cup \chi\mathcal{G}^s \hookrightarrow Aut(\mathfrak{w}_r) \quad \textit{if} \quad p \neq q\,;$$

$$I(\mathcal{W}(p,p)) = \mathcal{G}^s \cup \chi\mathcal{G}^s \cup \varphi\mathcal{G}^s \cup (\chi\varphi)\mathcal{G}^s \hookrightarrow Aut(\mathfrak{w}_r)\,;$$

$\mathcal{G}^s = (\mathcal{S} \times A^s(\mathfrak{w}_r)) \ltimes \mathcal{W}_r$, where $\mathcal{S} = \widetilde{SL(4)} = Spin_0(3,3)$ and $\mathcal{W}_r$ are the linear subgroups of $Aut(\mathfrak{w}_r)$ generated by the linear Lie algebras $\mathfrak{s} = \mathfrak{sl}(4,\mathbb{R}), \mathfrak{w}_r \subset \mathfrak{der}(\mathfrak{w}_r)$ (s. Thm. 2.11) and $A^s(\mathfrak{w}_r) = O(p) \times O(q)$ is described in Lemma 2.12. In terms of the decomposition $\mathfrak{w}_r = V \otimes \mathbb{R}^p + V^* \otimes \mathbb{R}^q + \wedge^2 V + \mathbb{R}h$, χ is induced by the automorphism $diag(-1,1,1,1)$ of V putting $\chi h = h$. For $p = q$ the automorphism φ of $\mathfrak{w}_r$ is induced by the canonical isomorphisms $V \to V^*$ and $V^* \to V$ via $\langle , \rangle$ putting $\varphi h = h$ (recall that we have a canonical $\mathfrak{sl}(V)$-invariant identification $\wedge^2 V \cong \wedge^2 V^*$; note that $\varphi | \wedge^2 V = *$ (Hodge-star)).

Proof. χ is an orthogonal automorphism of $\mathfrak{w}_r$ and conjugation by the matrix $diag(-1,1,1,1)$ in $\mathfrak{s} = \mathfrak{sl}(4,\mathbb{R})$ is an outer automorphism of $\mathfrak{sl}(4,\mathbb{R}) = \mathfrak{so}(3,3)$. Moreover, this automorphism interchanges the simple summands of the maximal compact subalgebra $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and modulo inner automorphisms is equivalent to η in Prop. 3.1 2). According to Cor. 1.9 we have proved $\mathcal{G}^s \cup \chi\mathcal{G}^s \subset I(\mathcal{W}(p,q))$.

Let us now consider the outer automorphism $\varphi : X \mapsto -X^t$ of $\mathfrak{s} = \mathfrak{sl}(4,\mathbb{R})$.

Lemma 3.8 *φ can be extended to an automorphism of $\mathfrak{s} \oplus \mathfrak{w}_r$ if and only if $p = q$. The extension $(p = q)$ is orthogonal on $\mathfrak{w}_r$.*

Proof. Assume φ can be extended. Then we have

$$A\varphi v = -\varphi A^t v, \quad A \in \mathfrak{sl}(4,\mathbb{R}), \quad v \in V \otimes e \cong V\,,$$

which shows that the $\mathfrak{sl}(4,\mathbb{R})$-module $\varphi(V \otimes e) \cong V^*$. This implies that the multiplicitiy p of V in $\mathfrak{w}_r$ equals the multiplicitiy q of V^*. If $p = q$ then φ is canonically extended by φ given in Theorem 3.7. □

Using Cor. 1.9 we immediately derive Theorem 3.7 from Prop. 3.1 2) and the lemma. □

3.4 $\mathcal{V}$-Spaces

Theorem 3.9 *The (full) isometry group of the non-symmetric quaternionic Kähler manifolds $\mathcal{V}(\psi)$ is given by*

$$I(\mathcal{V}(\psi)) = \mathcal{G}^s \hookrightarrow Aut(\mathfrak{v}_r)\,, \quad \textit{if } k \textit{ is even}\,,$$

$$I(\mathcal{V}(\psi)) = \mathcal{G}^s \cup \varphi\mathcal{G}^s \hookrightarrow Aut(\mathfrak{v}_r)\,, \quad \textit{if } k \textit{ is odd}\,.$$

The automorphism $\varphi \in Aut(\mathfrak{v}_r)$ is defined by

$$\varphi|(\mathfrak{z} + \tilde{\mathfrak{z}} + \tilde{\mathfrak{x}}_+) = -Id\,, \quad \varphi|(\mathfrak{y} + \tilde{\mathfrak{y}} + \mathfrak{v}_{3,3} + \mathfrak{a}_r) = Id$$

and $\mathcal{G}^s = (\mathcal{S} \times A^s(\mathfrak{v}_r)) \ltimes \mathcal{V}_r$, where $\mathcal{S} = Spin_0(3, 3+k)$, $\mathcal{V}_r$ are the linear subgroups generated by the linear algebras $\mathfrak{s} = \mathfrak{so}(3, 3+k), \mathfrak{v}_r \subset \mathfrak{Der}(\mathfrak{v}_r)$ and $A^s(\mathfrak{v}_r)$ was determined in Lemma 2.19.

Proof. Consider first the case when k is odd. The outer automorphism Ad_{D_o}, $D_o|\mathfrak{v}_{3,3} = Id$, $D_o|\tilde{\mathfrak{x}}_+ = -Id$, of $\mathfrak{so}(\mathfrak{v}_{3,3+k}) \cong \mathfrak{so}(3, 3+k)$ is extended by φ to an automorphism of $\mathfrak{s} \oplus \mathfrak{v}_r$, which is orthogonal on $\mathfrak{v}_r$. By Cor. 1.9 and Prop. 3.1 this implies the theorem in case k is odd.

If k is even we consider the outer automorphism $\theta = Ad_{D_e}$, $D_e\tilde{Q}_2 = -\tilde{Q}_3$, $D_e\tilde{Q}_3 = -\tilde{Q}_2$, $D_e|(span\{G_0, G_1, \tilde{P}_0, \tilde{P}_1\} + \tilde{\mathfrak{x}}_+) = Id$, of $\mathfrak{so}(\mathfrak{v}_{3,3+k}) \cong \mathfrak{so}(3, 3+k)$. By the formulas for the embedding $\mathfrak{v}_s \hookrightarrow \mathfrak{s}$ (s. Prop. 2.14) we know that θ preserves the Cartan subalgebra $\mathfrak{a}_s$ of $\mathfrak{v}_s$ and $\theta|\mathfrak{a}_s = Id$. Assume θ has been extended to $\theta \in Aut(\mathfrak{s} \oplus \mathfrak{v}_r)$. Since $\theta|\mathfrak{a}_s = Id$, it follows e.g. that $\theta\mathfrak{y}_- = \mathfrak{y}_-$, because $\mathfrak{y}_-$ is the root space of $\mathfrak{a}_s$ with root $-(H_0 - H_1 + 2H_3)/4$. Now we check Lie brackets between $\mathfrak{s} = \mathfrak{so}(\mathfrak{v}_{3,3+k}) = \wedge^2\mathfrak{v}_{3,3+k}$ and $\mathfrak{v}_r \supset \mathfrak{y}_-$using the formulas in Prop. 2.14:

$$[\theta(-\tilde{Q}_3 \hat{\wedge} \widetilde{JX}_-), \theta Y_-] = [\tilde{Q}_2 \hat{\wedge} \widetilde{JX}_-, \theta Y_-]$$
$$= \rho(\tilde{Q}_2 \hat{\wedge} \widetilde{JX}_-)\theta Y_- = X_- \circ \theta Y_- \neq 0\,,$$

if $X_- \in \mathfrak{x}_- - \{0\}$ and $Y_- \in \mathfrak{y}_- - \{0\}$. On the other hand:

$$\theta[-\tilde{Q}_3 \hat{\wedge} \widetilde{JX}_-, Y_-] = \theta\rho(-\tilde{Q}_3 \hat{\wedge} \widetilde{JX}_-)Y_- = \theta[X_-, Y_-] = 0\,.$$

This shows that θ cannot be extended to an automorphism of $\mathfrak{s} \oplus \mathfrak{v}_r$ and proves the theorem, thanks to Cor. 1.9 and Prop. 3.1. □

3.5 Conclusion

A remarkable consequence of our analysis is the following.

Theorem 3.10 *The (connected) isometry group of the quaternionic Kählerian solvmanifolds acts transitively on the twistor space and on the SO_3-principal bundle associated with the quaternionic structure.*

Proof. The result follows from Corollaries 2.4, 2.10 and 2.16. □

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