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Vienna, Preprint ESI 2300 (2011)

January 12, 2011

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available online at http://www.esi.ac.at

ON THE CONE OF KÄHLERIAN INFINITESIMAL DEFORMATIONS FOR COMPLEX TORI

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ABSTRACT. For any compact Kähler manifold M, the cone of Kählerian infinitesimal deformations of M, say KID(M), is by definition the subset of $H^1(M,\Theta)$ (the 1[st] cohomology of the holomorphic tangent sheaf of M) consisting of those elements in the images of Kodaira-Spencer maps for the germs of complex analytic families $(\boldsymbol{X}, M) \to (S, 0)$ such that $\boldsymbol{X}$ carries a Kähler metric. If M is a complex torus of dimension ≥ 2, KID(M) will be described in terms of the canonical matrix representation of $H^1(M,\Theta)$.

Introduction. In the theory of deformations of complex structures initiated by Kodaira and Spencer, a basic fact established by Kuranishi [Kur62] says that, given any compact complex manifold M, there exist a complex analytic subset S of a neighborhood of the origin 0 of $H^1(M,\Theta)$ and a complex analytic family $\pi : \boldsymbol{M} \to S$ with $\pi^{-1}(0) = M$, such that it is **complete** in the sense that any complex structure sufficiently close to M on the fixed underlying differentiable manifold of M arises as a fiber of π, and **effective** in the sense that the Kodaira-Spencer map is injective at 0. It was first noted by Schumacher [Sch84] that, given any Kähler metric say g_0 on M, there exists uniquely a maximal analytic subgerm of $(S, 0)$, say $(S', 0)$, such that the preimage of S' by π carries a Kähler metric g with $g|_M = g_0$. (See also [Fuj84] and [FS90].) By the cone of Kählerian infinitesimal deformations for M, denoted by KID(M) in short, we shall mean the collection of the images in $H^1(M,\Theta)$ of the (Zariski) tangent vectors of S' by Kodaira-Spencer maps for all possible S', as g_0 runs through the set of all Kähler metrics on M.

A recent paper of Berndtsson [Ber09] gives some information on the geometry of KID(M). Namely, given any complex analytic family $\pi' : \boldsymbol{X} \to S''$ such that $\boldsymbol{X}$ is Kähler and S'' is smooth, he has shown that the direct image by π' of the relative canonical sheaf $\omega_{\boldsymbol{X}/S''} (:= \omega_{\boldsymbol{X}} \otimes (\pi'^{*}\omega_{S''})^{*})$ is Nakano semipositive with respect to a canonically defined fiber metric, the so-called L^2 metric. This theorem of Berndtsson implies, in particular, that KID(M) does not coincide with the total space $H^1(M,\Theta)$ if M is a complex torus of dimension ≥ 2, because the L^2 metric is by definition the reciprocals of the diagonalized Bergman kernels of the fibers of π, whose curvature form is easily seen to be indefinite on S if M is a complex torus of dimension ≥ 2. (See §3.)

Therefore we are naturally led to study more on KID(M) to clarify the situation. The purpose of this note is to report the following observation.

Theorem. *Let M be a complex torus of dimension ≥ 2. Then, with respect to a canonical identification of $H^1(M,\Theta)$ with the space of $n \times n$ matrices (cf. §1), one has KID(M) $= \{\Xi \mid \Xi Y^{-1} H$ is symmetric for some positive Hermitian matrix $H\}$.*

2010 Mathematics Subject Classification. Primary 32G07 ; Secondary 53C40. Running head: KID for complex tori

Here Y denotes a real matrix Y with $\det Y > 0$ *such that $M \cong \mathbb{C}^n/\Gamma_Z$ holds for some real matrix X with $Z = X + \sqrt{-1}Y$. (For the notation Γ_Z, see §1 below.)*

Based on this result, it will be noted that, if M is a two dimensional torus, $\mathrm{KID}(M)$ is a proper subset of the cone defined as the positive directions with respect to the curvature form of the L^2 metric of $\pi_*\omega_{\boldsymbol{M}/S}$ (cf. §3).

§1. $\mathbf{KID}((\mathbb{C}/(\mathbb{Z}+\sqrt{-1}\mathbb{Z}))^n)$. At first we shall describe $\mathrm{KID}(M)$ for the special case $M = (\mathbb{C}/(\mathbb{Z}+\sqrt{-1}\mathbb{Z}))^n$. Let Ω be the domain in $\mathrm{End}(\mathbb{C}^n)$ ($\cong \mathbb{C}^{n^2}$) defined as the set of $n\times n$ complex matrices whose imaginary parts have positive determinants. Let $\boldsymbol{e}_i$ denote the (column) vector in $\mathbb{C}^n$ whose j-th component is δ_{ij} (Kronecker's delta). For any $Z \in \Omega$, let Γ_Z denote the additive subgroup of $\mathbb{C}^n$ generated by $\boldsymbol{e}_i$ $(i = 1, 2, ..., n)$ and $Z\boldsymbol{e}_i$ $(i = 1, 2, ..., n)$. Let $\boldsymbol{z} = (z_1, \dots, z_n)$ be the coordinate of $\mathbb{C}^n$.

Then we put $\boldsymbol{T} = \{(Z, \Gamma_Z + \boldsymbol{w}) \mid Z \in \Omega \text{ and } \boldsymbol{w} \in \mathbb{C}^n\}$, let $\pi : \boldsymbol{T} \to \Omega$ be the map induced by the projection to the first component, and put $\boldsymbol{T}_Z = \pi^{-1}(Z)$. It is clear that $\boldsymbol{T}_{\sqrt{-1}I} \cong (\mathbb{C}/(\mathbb{Z}+\sqrt{-1}\mathbb{Z}))^n$, where I denotes the unit matrix, and that the family $\boldsymbol{T} \to \Omega$ is complete and effective at every point of Ω. For any $Z \in \Omega$, we shall naturally identify $H^1(\boldsymbol{T}_Z, \Theta)$ with $\mathrm{End}(\mathbb{C}^n)$.

Lemma 1. *Let Ξ be any $n \times n$ matrix with complex entries and let $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ be the $\mathbb{R}$-linear map satisfying $\Phi(\boldsymbol{e}_i) = 0$ $(1 \le i \le n)$ and $\Phi(\sqrt{-1}\boldsymbol{e}_i) = \sum_j \Xi_{ij}\boldsymbol{e}_j$ $(1 \le i \le n)$. Then, for any $n \times n$ positive definite Hermitian matrix $H = (H_{ij})$, the $(2,0)$-part of*

$$(\partial/\partial\varepsilon)\left((Id_{\mathbb{C}^n} + \varepsilon\Phi)^{-1}\right)^*\left(\sum_{i,j} H_{ij}dz_i \wedge d\bar{z}_j\right)\Big|_{\varepsilon=0}$$

is $-{}^t d\boldsymbol{z} \wedge (H^t\Xi d\boldsymbol{z})$.

Proof. We put

$$(\partial/\partial\varepsilon)\left(((Id_{\mathbb{C}^n} + \varepsilon\Phi)^{-1})^* dz_i\right)\big|_{\varepsilon=0} = \sum_j A_{ij}dz_j + \sum_j B_{ij}d\bar{z}_j$$

and

$$(\partial/\partial\varepsilon)\left(((Id_{\mathbb{C}^n} + \varepsilon\Phi)^{-1})^* d\bar{z}_i\right)\big|_{\varepsilon=0} = \sum_j C_{ij}dz_j + \sum_j D_{ij}d\bar{z}_j$$

Then we have $A_{ij} = C_{ij}$ and $B_{ij} = D_{ij}$ $(1 \le i, j \le n)$ because $\phi(\boldsymbol{e}_i) = 0$ $(1 \le i \le n)$. Combining these equalities with

$$\sum_j (A_{ij} + C_{ij})dz_j + \sum_j (B_{ij} + D_{ij})d\bar{z}_j = -2\sum_j \Xi_{ji}dz_j,$$

which is valid because $\Phi(\sqrt{-1}\boldsymbol{e}_i) = \sum_j \Xi_{ij}\boldsymbol{e}_j$, we obtain $A_{ij} = C_{ij} = -\Xi_{ji}$ and $B_{ij} = D_{ij} = 0$.

Therefore the $(2,0)$-part of $(\partial/\partial\varepsilon)\left((Id_{\mathbb{C}^n} + \varepsilon\Phi)^{-1}\right)^*\left(\sum_{i,j} H_{ij}dz_i \wedge d\bar{z}_j\right)\big|_{\varepsilon=0}$ is equal to

$$-\sum_{i,k}\sum_j H_{ij}\Xi_{kj}dz_i \wedge dz_k = -{}^t d\boldsymbol{z} H^t\Xi \wedge d\boldsymbol{z}.$$

□

From the Lemma, we immediately obtain

Proposition 1. *$\Xi \in KID(\boldsymbol{T}_{\sqrt{-1}I})$ if and only if there exists an $n \times n$ positive definite Hermitian matrix H such that ΞH is symmetric.*

Since the set of positive definite Hermitian matrices is open in the space of Hermitian matrices, obviously one can infer the following from Proposition 1.

Corollary 1. *Theorem holds if $M \cong (\mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z}))^n$ and $n \geq 2$.*

§2. Proof of Theorem. Let $Z \in \Omega$, let $Z = X + \sqrt{-1}Y$, where $\bar{X} = X$, $\bar{Y} = Y$ and $\det Y > 0$, and let Ψ be a biholomorphic automorphism of Ω defined by $\Psi(W) = X + WY$.

Then, for any Ξ and Φ as in Lemma 1, one has $\Phi(Z\boldsymbol{e}_i) = \Xi Y \boldsymbol{e}_i$, so that $\Xi Y \in \mathrm{KID}(\boldsymbol{T}_Z)$ if and only if $\Xi \in \mathrm{KID}(\boldsymbol{T}_{\sqrt{-1}I})$.

Combining this observation with Corollary 1, we conclude that Theorem holds true for any complex torus of dimension ≥ 2. □

§3. Concluding Remarks. For the family $\boldsymbol{T} \to \Omega$ as above, the diagonalized Bergman kernels of the fibers $\boldsymbol{T}_Z$ are $\det(\mathrm{Im}Z)^{-1}\omega^n/n!$, where ω denotes the fundamental form of the metric on $\boldsymbol{T}_Z$ induced from the euclidean metric of $\mathbb{C}^n$. Therefore, by Berndtsson's theorem (cf. Theorem 1.2 in [Ber09]), the complex Hessian of the function $\log\det(\mathrm{Im}Z)$ on Ω evaluated at $\Xi \in H^1(\boldsymbol{T}_Z, \Theta)$ must be nonpositive if $\Xi \in \mathrm{KID}(\boldsymbol{T}_Z)$.

Hence one has

$$\mathrm{KID}(\boldsymbol{T}_Z) \subseteq \{\Xi | \langle \sqrt{-1}\partial\bar{\partial}\log\det(\mathrm{Im}Z), \sqrt{-1}\Xi \wedge \Xi\rangle \leq 0\}.$$

Here $\langle\ ,\ \rangle$ denotes the canonical pairing. If $n = 2$, this inclusion relation can be directly verified at $Z = \sqrt{-1}I$ as follows.

Let

$$\Xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } H = \begin{pmatrix} 1 & \mu \\ \bar{\mu} & \lambda \end{pmatrix} \qquad (\lambda > 0 \text{ and } |\mu|^2 < \lambda).$$

Then the condition $H\Xi = {}^t(H\Xi)$ means

$$\bar{\mu}a + \lambda c = b + \mu d. \tag{1}$$

From (1) it is easy to see that

$$-\frac{\partial^2}{\partial z \partial \bar{z}} \log\det\left(\mathrm{Im}(\sqrt{-1}I + z\Xi)\right)\Big|_{z=0} = |a|^2 + |d|^2 + \bar{b}c + b\bar{c} \geq 0 \tag{2}$$

as desired.

However, letting $\Xi = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$, one has $|a|^2 + |d|^2 + \bar{b}c + b\bar{c} = c + \bar{c}$. Therefore, whenever the real part of c is nonnegative and the imaginary part of c is not zero, Ξ satisfies (2) and does not belong to $\mathrm{KID}(\boldsymbol{T}_{\sqrt{-1}I})$.

To generalize the result, it may be worthwhile to extend Theorem at first to K3 surfaces and complex symplectic manifolds of higher dimension.

$\mathrm{KID}(M)$ is the cone associated to the deformations of those complex structures which keeps some Kähler class invariant. Similarly, with respect to the deformations of symplectic structures, there are cones in the tangent spaces of the parameter space as the collection of the directions in which the complex structures are kept invariant. It turns out that they can be decided for the case of tori. The detail will appear elsewhere.

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