# Holonomy Reductions of Cartan Geometries and Curved Orbit Decompositions

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# HOLONOMY REDUCTIONS OF CARTAN GEOMETRIES AND CURVED ORBIT DECOMPOSITIONS

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ABSTRACT. We develop a holonomy reduction procedure for general Cartan geometries. We show that, given a reduction of holonomy, the underlying manifold naturally decomposes into a disjoint union of initial submanifolds. Each such submanifold corresponds to an orbit of the holonomy group on the modelling homogeneous space and carries a canonical induced Cartan geometry. The result can therefore be understood as a 'curved orbit decomposition'. The theory is then applied to the study of several invariant overdetermined differential equations in projective, conformal and CR-geometry. This makes use of an equivalent description of solutions to these equations as parallel sections of a tractor bundle. In projective geometry we study a third order differential equation that governs the existence of a compatible Einstein metric. In CR-geometry we discuss an invariant equation that governs the existence of a compatible Kähler-Einstein metric.

## 1. INTRODUCTION

In differential geometry great gains can be achieved when apparently unrelated structures are shown to be concretely linked. Well known examples include: the *Fefferman metric* [Fef76] which associates to hypersurface type CR geometries a uniquely determined conformal geometry in one greater dimension; *Poincaré-Einstein geometries* [FG85], which realise a conformal manifold as the boundary (at infinity in a suitable sense) for a negative Einstein Riemannian manifold. Both structures have been the focus of considerable attention, cf. e.g. [GL91, Lee95, FG07, GQ10] and [Lee86, Gra87, Bau99, NS03, ČG10]. More recently there has been excitement surrounding the canonical association by Nurowski [Nur05] of conformal geometries to certain distributional structures (generic 2 distributions in dimension 5). These arise in the study certain ordinary differential equations linked to Cartan's 5-variable paper [Car10] and Bryant's natural contruction [Bry06] of a conformal split signature $(3, 3)$-structure from a given generic rank 3 distribution, cf. also [LN11, HS09, HS11]. While the three different constructions mentioned above appear at first sight to be unrelated, in fact they may be viewed as special cases of a single phenomenon. Namely they each can be understood as arising from a holonomy reduction of a certain connection. The connection involved is not on the tangent bundle but is on a prolonged structure, and is known as a Cartan connection.

In Riemannian geometry the study and application of holonomy reduction has a long history which includes Cartan's classification of symmetric spaces [Car26] and the de Rham decomposition theorem. The classification of possible holonomy groups of Riemannian and (torsion free) affine connections, as well as the construction of geometries realising these groups, forms one of the cornerstones of differential geometry [Ber53, Bry87, MS99, Bry00]. In this setting the local geometric implications of a given holonomy reduction, e.g. the existence of a compatible complex structure for a given Riemannian metric, can be readily read off from the nature of the group arising since the connections involved are on the tangent bundle. The notion of holonomy easily generalises to principal connections on principal bundles, in which case the holonomy group becomes a subgroup of the structure group of the principal bundle and in this generality the Ambrose-Singer Theorem [AS53] relates the holonomy group to the curvature form of the structure. In this case also the local geometric implications of reductions are evident.

Recently there been considerable interest in understanding holonomy questions for those Cartan connections arising naturally in *parabolic geometries*; the latter form a broad class of structures which includes conformal, CR, and projective geometries as special cases. For the Cartan connections of projective and conformal structures the possible holonomy groups have been studied, and partial classifications are available [Arm08b, Arm07, Lei05]. This aspect follows the treatment of principal connections. In contrast determining the geometric implications of reduced holonomy in this setting is far more subtle, since the connection which defines the holonomy lives on a prolonged bundle. Prior to the current work there has been no general approach for studying this problem. In cases where such geometric implications have been discussed, it was necessary to make certain non-degeneracy assumptions, which means that typically they apply only to a dense open subset of the original manifold. For instance, the *conformal de'Rham* theorem discussed in [Arm07] and [Lei05] yields a decomposition of a conformal structure with decomposoable holonomy on an open dense subset. Other results in [Arm08a] show that reduced projective holonomy yields certain geometric familiar structures on such subsets of the original manifold, like certain contact, or complex projective structures. The complement of such a non-degenerate open dense interior (when non-empty) carries geometric structure itself, and an interesting aspect is how this relates to the ambient structure. A prominent case is that of Poincaré-Einstein metrics, which correspond to a holonomy reduction of a conformal structure [Gov10]. Here the open dense subset carries an Einstein metric, and the singularity set forms a hypersurface with an induced conformal structure, that is the conformal infinity of the Einstein metric; the Poincaré-Einstein programme is precisely concerned with relating the Einstein structure to the geoemtry of the conformal hypersurface.

The purpose of this article is to develop a completely general approach to determining the geometric implications of any specific holonomy reduction of a Cartan connection. We find that the behaviour just described for Poincaré-Einstein manifolds is typical, at least of the simplest cases. In broad terms,

our results can be described as follows: Given a manifold equipped with a Cartan connection we show that a holonomy reduction of this connection determines a decomposition of the underlying manifold into a disjoint union of initial submanifolds, so yielding a form of stratification. Each such submanifold inherits a canonical geometry from the original data. What this geometry is varies according to the type of strata; we show how to determine this and key aspects of how it relates to the ambient structure. In many interesting cases this yields an open dense piece which is canonically equipped with an affine connection, and a stratification of the closed complement to this piece, which in turn is endowed with geometric structure that may be viewed as a "limit" of the structure on the distinguished open set. An important point here is that we treat all possible reductions of the connection, not simply a reduction to the mimimal holonomy group. This means that our results apply non-trivially to the homogenous model, and we show that there the decomposition coincides with an orbit decomposition with respect to the group arising in the reduction.

Remarkably, many of the nice properties of the orbit decomposition on the homogeneous model carry over to the curved cases without essential changes, so we call the decomposition a *curved orbit decomposition* in the general case. This was first observed in the special case of almost Einstein scales for conformal structures in [Gov10]. The basic tool to establish this in a general setting is a comparison map between a curved geometry and the homogenous model and we develop this here. (A version of this was introduced for projective structures in our article [ČGH10].) This proves that the curved orbits are always initial submanifolds. In addition each curved orbit is then seen to carry a natural Cartan geometry of the same type as the corresponding orbit on the homogeneous model.

To obtain more detailed information on the geometric structure of the curved orbits it is necessary to study the relation between the curvature of the induced Cartan subgeometry and the curvature of the original Cartan geometry. This is illustrated by several examples.

Specialising to parabolic geometries there is a strong connection between holonomy reductions, in the sense we treat here, and solutions to invariant overdetermined linear partial differential equations. This is one of the key motivations for our work. Parabolic geometries are canonically equipped with such equations; they arise as the equations of the first operator in certain invariant differential sequences known as Bernstein-Gelfand-Gelfand (BGG) sequences [ČSS01, CD01]. These sequences have strong links to symmetry and representation theory; on the model they resolve finite dimension representations and in the curved setting one such sequence controls deformation of structure and is connected with the existence of infinitesmal automorphisms [Čap08]. As we shall explain, certain special solutions of these "first BGG" equations are exactly equivalent to holonomy reductions in our sense. Thus our results may be recast as describing the geometric implications of the existence of such *normal solutions.* For example we can show that the zero locus of such a solution cannot have worse singularities on curved geometries than the zero loci of solutions on the homogeneous

model. In fact related much finer data is available. This vastly generalises
the known results on the possible form of zero sets for twistor spinors and
Einstein rescalings, cf. e.g. [Fri90, BFGK90, Hab94, KR94, Gov10].

We now give a brief outline of the article: The main result will be developed
in section 2. There we begin with a short review of general Cartan geometries
in 2.1, but refer the interested reader to the extensive treatments of these
structures that can be found in [Sha97, ČS09]. We introduce the notion of
holonomy of a general Cartan geometry in 2.2 and then discuss how a given
holonomy reduction canonically induces a decomposition of the manifold
in 2.4. Our main theorem 2.6 describes the structure of the curved orbit
decomposition and the relations of the induced Cartan-subgeometries with
the ambient Cartan geometry. A particularly interesting area of applications
is formed by the solutions of BGG-equations and the study of their zero-
locus, and the general principles for this are outlined in 2.7. In section 3 we
study several concrete BGG-equations. We begin in 3.1 with a third order
differential equation that governs the existence of of an Einstein metric,
whose Levi–Civita connection is projectively equivalent to a given affine
torsion-free connection. In 3.3 we treat an equation on complex projective
structures whose solutions describe (almost) Kähler metrics on the manifold.
In section 3.4 we discuss how our general holonomy reduction results can
be applied to Fefferman-type constructions. The relation with the original
Fefferman-construction is discussed in detail. Finally we discuss the equation
governing almost Einstein scales in conformal geometry in 3.5 and give an
interesting analog of that equation in CR-geometry, 3.6.

## 2. Holonomy reductions of Cartan geometries

2.1. **Cartan geometries.** Let $G$ be a Lie group and $P \subset G$ a closed sub-
group. The Lie algebras of $G$ and $P$ will be denoted $\mathfrak{g}$ and $\mathfrak{p}$, respectively.
We will always assume that $P$ meets each connected component of $G$, so the
homogeneous space $G/P$ is connected. Cartan geometries of type $(G, P)$
can be thought of as "curved analogs" of the $G$–homogeneous space $G/P$.

A *Cartan geometry* of type $(G, P)$ on a manifold $M$ is a $P$-principal bundle
$\mathcal{G} \to M$ endowed with a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Denote the
principal right action of $g \in P$ on $\mathcal{G}$ by $r^g$ and the fundamental vector field
generated by $Y \in \mathfrak{p}$ by $\zeta_Y$, i.e., $\zeta_Y(u) = \frac{d}{dt}|_{t=0}(u \cdot \exp(tY))$. Then $\omega$ being a
Cartan connection means that the following three properties hold:

(C.1) $\omega_{u \cdot p}(T_u r^p \xi) = \mathrm{Ad}(p^{-1})\omega_u(\xi)$ for all $p \in P$, $u \in \mathcal{G}$, and $\xi \in T_u \mathcal{G}$.
(C.2) $\omega(\zeta_Y) = Y$ for all $Y \in \mathfrak{p}$.
(C.3) $\omega_u : T_u \mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

The homogeneous model of Cartan geometries of type $(G, P)$ is the bundle
$G \to G/P$ with the left Maurer-Cartan form $\omega^{MC} \in \Omega^1(G, \mathfrak{g})$ as the Car-
tan connection. Indeed, the defining properties of a Cartan connection are
obvious weakenings of properties of the Maurer–Cartan form, which make
sense in the more general setting.

The curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of $\omega$ is defined by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)],$$

which is exactly the failure of $\omega$ to satisfy the Maurer-Cartan equation. In particular, the homogeneous model has vanishing curvature in this sense. It is a basic fact of Cartan geometries that $(\mathcal{G}, \omega)$ has vanishing curvature if and only if it is locally isomorphic to the homogeneous model $(G, \omega^{MC})$. It will sometimes be useful to work with the curvature function $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ of $\omega$, which is defined defined by

(1) $$\kappa(u)(X, Y) := K(\omega_u^{-1}(X), \omega_u^{-1}(Y))$$

for $u \in \mathcal{G}$ and $X, Y \in \mathfrak{g}$.

A Cartan geometry is called *torsion–free* if its curvature form $K$ has values in $\mathfrak{p} \subset \mathfrak{g}$, or equivalently if the curvature function satisfies $\kappa(u)(X, Y) \in \mathfrak{p}$ for a all $u \in \mathcal{G}$ and $X, Y \in \mathfrak{g}$.

2.2. **Holonomies of Cartan geometries and reductions.** The classical concept of holonomy can not be directly applied to a Cartan connection. Since a Cartan connection $\omega$ restricts to a linear isomorphism on each tangent space, there are no non–constant curves which are horizontal for $\omega$ in the usual sense. There is a simple way, however, to connect to the classical concept. A Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is easily seen to extend canonically to a $G$–principal connection $\hat{\omega} \in \Omega^1(\hat{\mathcal{G}}, \mathfrak{g})$ on the $G$–principal bundle $\hat{\mathcal{G}} := \mathcal{G} \times_P G$. The extension is characterised by the fact that $i^*\hat{\omega} = \omega$, where $i : \mathcal{G} \to \hat{\mathcal{G}}$ is the obvious inclusion. Hence for any point $\hat{u} \in \hat{\mathcal{G}}$ the holonomy group (based at $u$) of the $G$-principal connection $\hat{\omega}$ is a subgroup $\mathrm{Hol}_{\hat{u}}(\hat{\omega}) \subset G$. If we forget about the choice of the point, we obtain a conjugacy class of subgroups of $G$, which we denote by $\mathrm{Hol}(\hat{\omega})$. We simply define this to be the holonomy of the original Cartan connection, i.e. $\mathrm{Hol}(\omega) := \mathrm{Hol}(\hat{\omega})$.

If the holonomy $\mathrm{Hol}(\omega)$ is not full, i.e., $\mathrm{Hol}(\hat{\omega}) \subsetneq G$, the extended connection $\hat{\omega}$ can be reduced: For every closed subgroup $H \subset G$ that contains (any conjugate of the) holonomy group $\mathrm{Hol}(\hat{\omega})$ there exists a reduction of structure group $\mathcal{H} \xrightarrow{j} \hat{\mathcal{G}}$ from $G$ to $H$ that preserves the connection. If $M$ is connected, then one simply chooses a point $\hat{u} \in \hat{\mathcal{G}}$ such that $\mathrm{Hol}_{\hat{u}}(\hat{\omega}) \subset H$ and defines $\mathcal{H}$ as the set of all points which can be written as $c(1) \cdot h$ for some $h \in H$ and some horizontal curve $c : [0, 1] \to \hat{\mathcal{G}}$ with $c(0) = \hat{u}$. One immediately verifies that this is a principal $H$–subbundle such that $\hat{\omega}$ restricts to an $H$–principal connection on $\mathcal{H}$; formulated in terms of the embedding $j$, this says that

$$j^*\hat{\omega} \in \Omega^1(\mathcal{H}, \mathfrak{h}),$$

with $\mathfrak{h}$ the Lie algebra of $H$. The holonomy group $\mathrm{Hol}(\hat{\omega})$ is the smallest subgroup of $G$ to which the connection $\hat{\omega}$ can be reduced.

By standard theory, a reduction $j : \mathcal{H} \hookrightarrow \hat{\mathcal{G}}$ can be equivalently described as a section of the associated fibre-bundle $\hat{\mathcal{G}} \times_G (G/H) = \hat{\mathcal{G}}/H$. The second description of this bundle shows that for any $x \in M$ the fibre of $\mathcal{H}$ over $x$

is mapped to a single point in the fibre of $\hat{\mathcal{G}} \times_G (G/H)$; this describes the section corresponding to $\mathcal{H}$. Conversely, the preimage of a section of $\hat{\mathcal{G}}/H$ under the natural projection $\hat{\mathcal{G}} \to \hat{\mathcal{G}}/H$ is an $H$–principal subbundle. Note further that $\hat{\omega}$ induces a (non–linear) connection on the associated bundle $\hat{\mathcal{G}} \times_G (G/H)$. It is easy to see that the section corresponding to $j : \mathcal{H} \hookrightarrow \hat{\mathcal{G}}$ is parallel if and only if $\hat{\omega}$ restricts to an $H$–principal connection on $\mathcal{H}$.

For our general definition in the setting of Cartan geometries, it will be very useful to avoid having a distinguished base point. Hence we work with an abstract $G$–homogeneous space $\mathcal{O}$ rather than with $G/H$. One can identify $\mathcal{G} \times_P \mathcal{O}$ with $\hat{\mathcal{G}} \times_G \mathcal{O}$ so this bundle carries a natural (non–linear) connection.

**Definition 2.2.** Let $(\mathcal{G}, \omega)$ be a Cartan geometry of type $(G, P)$ and let $\mathcal{O}$ be a homogeneous space of the group $G$. Then a *holonomy reduction of $G$–type $\mathcal{O}$* of the geometry $(\mathcal{G}, \omega)$ is a parallel section of the bundle $\mathcal{G} \times_P \mathcal{O}$.

*Remark 2.2.* We note here that our holonomy reductions need not be minimal: the minimal holonomy reduction of $(\mathcal{G}, \omega)$ is of type $G/\operatorname{Hol}(\omega)$, which reduces the structure group of the extended principal bundle connection $\hat{\omega}$ to $\operatorname{Hol}(\hat{\omega})$. Whenever $(\mathcal{G}, \omega)$ allows a holonomy reduction of type $\mathcal{O} = G/H$ one necessarily has that $\operatorname{Hol}(\omega) \subset H \subset G$.

### 2.3. Parallel sections of tractor bundles and corresponding holonomy reductions.
For a $G$-representation $V$ the associated bundle $\mathcal{V} = \hat{\mathcal{G}} \times_G V = \mathcal{G} \times_P V$ is a *tractor bundle* and the linear connection induced by $\hat{\omega}$ is called its *tractor connection*. Sections of $\mathcal{V}$ can be identified with $G$–equivariant smooth maps $\hat{\mathcal{G}} \to V$, and such a map $s : \hat{\mathcal{G}} \to V$ corresponds to a parallel section if and only if it is constant along any curve $c : [0,1] \to \mathcal{G}$ which is horizontal in the sense that $\hat{\omega}(\frac{d}{dt}(c(t))) = 0$.

**Lemma 2.3.** *The image of the map $s : \hat{\mathcal{G}} \to V$ corresponding to a parallel section of $\mathcal{V}$ over a connected manifold $M$ is a $G$–orbit $\mathcal{O} \subset V$.*

*Proof.* Since $M$ is assumed to be connected we can take a smooth curve joining two given points $x, x' \in M$ and lift it to a horizontal curve $c : [0,1] \to \hat{\mathcal{G}}$. By the usual formula for an associated connection, $s$ has to be constant along $c$, so $s(u) = s(u') \in V$, where $u := c(0) \in \hat{\mathcal{G}}_x$ and $u' := c(1) \in \hat{\mathcal{G}}_{x'}$. By $G$–equivariancy $s(\hat{\mathcal{G}}_x)$ coincides with the $G$–orbit of $s(u)$, and the same is true for $s(\hat{\mathcal{G}}_{x'})$. $\qquad\square$

Hence any parallel section of $\mathcal{V}$ canonically determines a $G$–orbit $\mathcal{O} \subset V$, which we will refer to as its *$G$–type*. Of course, $\mathcal{G} \times_P \mathcal{O}$ is then a subbundle of $\mathcal{V}$ and this inclusion is compatible with the natural connections. Thus a parallel section of $\mathcal{V}$ of $G$-type $\mathcal{O}$ is the same as a holonomy reduction of $(\mathcal{G}, \omega)$ of this $G$–type.

### 2.4. The canonical $P$-type decomposition induced by a holonomy reduction.
So far our description of holonomy reductions did not take into account that the principal $G$-bundle $\hat{\mathcal{G}}$ is the extended bundle of the Cartan

bundle $\mathcal{G}$. Since $\hat{\mathcal{G}} = \mathcal{G} \times_P \mathcal{G}$ is the quotient of $\mathcal{G} \times G$ by the $P$-right action

$$(u, g) \cdot p = (up, p^{-1}g), u \in \mathcal{G}, g \in G, p \in P,$$

there is a canonical embedding $\mathcal{G} \hookrightarrow \hat{\mathcal{G}}$ which maps $u \in \mathcal{G}$ to $(u, e) \cdot P \in \hat{\mathcal{G}}$. This canonical $P$-subbundle subbundle of $\hat{\mathcal{G}}$ gives rise to a pointwise invariant that is specific to holonomy reductions of Cartan geometries:

**Definition 2.4.** Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type $(G, P)$ together with a holonomy reduction of type $\mathcal{O}$ described by $s : \hat{\mathcal{G}} \to \mathcal{O}$. Then for a a point $x \in M$ the *$P$–type of $x$ with respect to $s$* is the $P$-orbit $s(\mathcal{G}_x) \subset \mathcal{O}$.

We denote by $P \backslash \mathcal{O}$ the set of $P$–orbits in $\mathcal{O}$. Then for a Cartan geometry $(\mathcal{G} \to M, \omega)$ endowed with a holonomy reduction of type $\mathcal{O}$, the base manifold $M$ decomposes into a disjoint union according to $P$–type as

$$M = \bigcup_{i \in P \backslash \mathcal{O}} M_i.$$

We term the components $M_i$ *curved orbits* for a reason that will shortly be obvious.

2.5. **The $P$-type decomposition of the homogeneous model.** We now study the $P$-type decomposition on the homogeneous model for a given holonomy reduction of $G/P$ of type $\mathcal{O}$. The extended bundle $\hat{\mathcal{G}} = G \times_P G$ can be canonically trivialised via

$$G/P \times G \to G \times_P G$$
(2)
$$(gP, g') \mapsto [g, g^{-1}g']_P.$$

Using this, the $G$-equivariant map $s : \hat{\mathcal{G}} \to \mathcal{O}$ defining a holonomy reduction corresponds to a smooth map $G/P \times G \to \mathcal{O}$, which by $G$–equivariancy is determined by its restriction to $(G/P) \times \{e\}$. Moreover, the extended principal connection $\hat{\omega}$ in this case is just the flat connection coming from this trivialisation. In particular, any curve $c : [0, 1] \to G/P \times G$ of the form $c(t) = (\bar{c}(t), e)$ is horizontal, so the map $s : G/P \times G \to \mathcal{O}$ has to be given by $s(gP, g') = (g')^{-1} \cdot \alpha$ for some fixed element $\alpha \in \mathcal{O}$. Fixing a holonomy reduction of $G/P$ (of type $\mathcal{O}$) thus amounts to fixing an element $\alpha = s(eP, e) \in \mathcal{O}$, and we denote $H = G_\alpha \subset G$ the isotropy group of this element. In particular, we can then identify $\mathcal{O}$ with $G/H$.

To determine the $P$-type of $x = gP \in G/P$ with respect to $s$, we observe that in the trivialisation (2) the fibre $\mathcal{G}_{gP} \subset \hat{\mathcal{G}}_{gP} = \{gP\} \times G$ is just $\{gP\} \times \{gb : b \in P\}$. It follows by equivariancy that $s(\mathcal{G}_x)$ is the $P$–orbit $Pg^{-1} \cdot s(eP, e) = Pg^{-1}H \in P \backslash G/H = P \backslash \mathcal{O}$. The map $G/P \to P \backslash \mathcal{O}$ which sends each point to its $P$–type thus factories to a bijection

(3)
$$H \backslash G/P \to P \backslash \mathcal{O} = P \backslash G/H,$$
$$HgP \mapsto Pg^{-1}H$$

of double coset spaces, compare with Proposition 2.13 of [ČGH10].

This shows that for $M = G/P$, we get $M_{Pg^{-1}H} = HgP/P = H \cdot (gP/P) \subset G/P$. Hence the decomposition of $G/P$ according to $P$–type with respect to the holonomy reduction determined by $\alpha \in \mathcal{O}$ coincides with the decomposition of $G/P$ into orbits under the action of the subgroup $H = G_\alpha \subset G$. Now it is clear that each of the $H$-orbits naturally shows up as the homogeneous homogeneous model of a Cartan geometry. The stabiliser of a point $gP \in G/P$ in $H$ of course is $H \cap gPg^{-1}$ and hence $HgP \cong H/(H \cap gPg^{-1}) \cong (g^{-1}Hg)/(g^{-1}Hg \cap P)$.

*Remark 2.5.* It looks as if there were many different holonomy reductions of type $\mathcal{O}$ of the homogeneous model. This is true, but they are all related by the action of $G$. For our purposes, the main difference between these reductions is the $P$–type the origin $eP \in G/P$. Up to $G$–action, there is only one holonomy reduction of type $\mathcal{O}$ of $G/P$, whence we will talk about *the* model of holonomy reductions of type $\mathcal{O}$.

2.6. **Curved orbit decomposition and induced Cartan geometries.**
Consider a $G$–homogeneous space $\mathcal{O}$ and two elements $\alpha, \alpha' \in \mathcal{O}$. If $\alpha' = g \cdot \alpha$, then the stabilisers are conjugate, so $G_{\alpha'} = gG_\alpha g^{-1}$. If we in addition assume that $\alpha$ and $\alpha'$ lie in the same $P$–orbit in $\mathcal{O}$, then we can choose $g \in P$, and thus $gPg^{-1} = P$. Consequently, putting $P_\alpha := G_\alpha \cap P$ and likewise for $\alpha'$, we see that $P_{\alpha'} = gP_\alpha g^{-1}$. Thus we see that $(G_\alpha, P_\alpha)$ is isomorphic to $(G_{\alpha'}, P_{\alpha'})$ as pair of a group endowed with a distinguished subgroup. In the formulation of our main result below, given a $P$–orbit $i \in P\backslash\mathcal{O}$, we will denote by $(H_i, P_i)$ an abstract representative of this isomorphism class of groups with a distinguished subgroup.

**Theorem 2.6.** *Let $(\mathcal{G}, \omega)$ be a Cartan geometry of type $(G, P)$ which is endowed with a holonomy reduction of type $\mathcal{O}$. Consider a $P$–orbit $i \in P\backslash\mathcal{O}$ such that the curved orbit $M_i$ is non–empty, and consider the corresponding groups $P_i \subset H_i$ as discussed above. Then*

(i) *Choose a representative $\alpha \in \mathcal{O}$ for the Orbit $i$, let $G_\alpha \in G$ be its stabiliser and consider the holonomy reduction of the homogeneous model $G/P$ determined by $\alpha$ as in Section 2.5. Then for each $x \in M_i$ there exist neighbourhoods $N$ of $x$ in $M$, and $N'$ of $eP$ in $G/P$ and a diffeomorphism $\varphi : N \to N'$ with $\varphi(x) = eP$ and $\varphi(M_i \cap N) = (G_\alpha \cdot eP) \cap N'$. In particular, $M_i$ is an initial submanifold of $M$.*

(ii) *$M_i$ carries a canonical Cartan geometry $(\mathcal{G}_i \to M_i, \omega_i)$ of type $(H_i, P_i)$.*
    *Choosing a representative $\alpha$ for $i \in P\backslash\mathcal{O}$ as in (i) and identifying $(H_i, P_i)$ with $(G_\alpha, P_\alpha)$, we obtain an embedding of principal bundles $\mathbf{j}_\alpha : \mathcal{G}_i \to \mathcal{G}|_{M_i}$ such that $\mathbf{j}_\alpha^* \omega = \omega_i$. Thus $(\mathcal{G}_i, \omega_i)$ can be realized as a $P_\alpha$–subbundle in $\mathcal{G}|_{M_i}$ on which $\omega$ restricts to a Cartan connection of type $(G_\alpha, P_\alpha)$.*

(iii) *For the embedding $\mathbf{j}_\alpha$ from (ii), the curvatures $K$ of $\omega$ and $K_i$ of $\omega_i$ are related as*

$$K_i = \mathbf{j}_\alpha^* K.$$

*In particular, if $\omega$ is torsion free, so is $\omega_i$.*

*Likewise, let $\kappa$ and $\kappa_i$ denote, respectively, the curvature functions of the two Cartan connections. Then $\kappa(\mathbf{j}_\alpha(u))$ maps $\Lambda^2(\mathfrak{g}_\alpha/(\mathfrak{g}_\alpha \cap \mathfrak{p})$ to $\mathfrak{g}_\alpha$ and its restriction to this subspace coincides with $\kappa_i(u)$.*

The proof of the theorem is based on the following comparison method.

**Lemma 2.6** (Comparison). *Let $(p : \mathcal{G} \to M, \omega)$ and $(p' : \mathcal{G}' \to M', \omega')$ be Cartan geometries of type $(G, P)$ that are endowed with holonomy reductions of type $\mathcal{O}$ described by $s : \hat{\mathcal{G}} \to \mathcal{O}$, respectively $s' : \hat{\mathcal{G}}' \to \mathcal{O}$. Assume that for some $\alpha \in \mathcal{O}$ with $P$–orbit $\bar{\alpha} \in P \backslash \mathcal{O}$ both curved orbits $M_{\bar{\alpha}}$ and $M'_{\bar{\alpha}}$ are non–empty. Then for points $x \in M_{\bar{\alpha}}$ and $x' \in M_{\bar{\alpha}}$ we obtain:*

- *A diffeomorphism $\phi : N \to N'$ from an open neighbourhood of $x$ in $M$ to an open neighbourhood of $x'$ in $M'$ such that $\phi(x) = x'$.*
- *A $P$-equivariant diffeomorphism $\Phi : p^{-1}(N) \to (p')^{-1}(N')$ which covers $\phi$ and satisfies*

(4)
$$s' \circ \Phi = s.$$

*In particular, it follows that*

(5)
$$\phi(M_i \cap N) = M'_i \cap N'$$

*for all $i \in P \backslash \mathcal{O}$.*

We remark that some of the intersections $M_i \cap N$ may be empty.

*Proof of lemma 2.6.* The proof is based on an adapted version of normal coordinates for Cartan geometries: For this we fix a linear subspace $\mathfrak{g}_- \subset \mathfrak{g}$ which is complementary to the subspace $\mathfrak{p} \subset \mathfrak{g}$. For $X \in \mathfrak{g}_-$, we denote by $\tilde{X} \in \mathfrak{X}(\mathcal{G})$ the vector field characterised by $\omega(\tilde{X}) = X$. Next, take a point $u \in \mathcal{G}_x$ such that $s(u) = \alpha$, and consider the flow $\mathrm{Fl}_1^{\tilde{X}}(u)$ of $\tilde{X}$ starting in $u$ up to time 1. This is defined for $X$ in a neighbourhood $W$ of zero in $\mathfrak{g}_-$ and, possibly shrinking $W$, $\Psi(X) := \mathrm{Fl}_1^{\tilde{X}}(u)$ defines a smooth map $W \to \mathcal{G}$ such that $\psi := p \circ \Psi$ is a diffeomorphism from $W$ onto an open neighbourhood $N$ of $x = p(u)$ in $M$. These are the local normal coordinates around $x$ determined by $u$.

Next, we define a local section $\tau : N \to \mathcal{G}$ by $\tau(\psi(X)) := \Psi(X)$, and an adapted local section $\hat{\tau} : N \to \hat{\mathcal{G}}$ by $\hat{\tau}(\psi(X)) := \Psi(X) \cdot \exp(-X)$.

Then $\hat{\tau}$ has the property that for fixed $X \in \mathfrak{g}_-$ the curve $c$ defined by

$$c(t) := \hat{\tau}(\psi(tX)) = \Psi(tX) \cdot \exp(tX) = \mathrm{Fl}_t^X(u) \cdot \exp(-tX)$$

(for sufficiently small $t$) is horizontal for the principal connection $\hat{\omega}$. Indeed, we have

$$\hat{\omega}(c'(t)) = \mathrm{Ad}(\exp(tX))X - X = 0.$$

But since $s : \hat{\mathcal{G}} \to \mathcal{O}$ is constant along horizontal curves we conclude that for $X \in W$ we get

$$\alpha = s(c(0)) = s(c(1)) = s(\hat{\tau}(X)) = s(\Psi(X) \cdot \exp(-X)),$$

and by $G$-equivariancy, we obtain

(6)
$$s(\Psi(X)) = \exp(-X) \cdot \alpha.$$

Now we can perform the same construction for $(p' : \mathcal{G}' \to M', \omega')$ and a point $u' \in \mathcal{G}'_{x'}$ such that $s'(u') = \alpha$. Shrinking the neighbourhoods of zero in $\mathfrak{g}_-$ appropriately, we may assume that $W' = W$, and put $\phi := \psi' \circ \psi^{-1} : N \to N'$, so $\phi(x) = x'$. Further there evidently is a unique $P$-equivariant diffeomorphism $\Phi : p^{-1}(N) \to (p')^{-1}(N')$ such that $\Phi \circ \tau = \tau'$ and by construction, this covers $\phi$.

Since $s'(\Psi'(X)) = \exp(-X) \cdot \alpha = s(\Psi(X))$ one immediately obtains (4). For the last claim, recall that by definition $y \in M_i$ is equivalent to the fact that $s(\mathcal{G}_y)$ is the orbit $i$. For $v \in \mathcal{G}_y$ put $v' = \Phi(v) \in \mathcal{G}'_{\varphi(y)}$ we then have by (4) that $s'(v') = s(v)$ and therefore $y$ and $\varphi(y)$ have the same $P$–type. $\qquad\square$

*Proof of theorem 2.6.* We choose a representative $\alpha$ for the orbit $i \in P \backslash \mathcal{O}$ and apply Lemma 2.6 to the case where $M' = G/P$ is the homogeneous model of type $(G, P)$ with the holonomy reduction determined by $\alpha \in \mathcal{O}$, so $eP \in M'_i$. Let $\phi : N \to N'$, $\Phi : p^{-1}(N) \to (\pi')^{-1}(N')$ be the maps constructed in the lemma for the given point $x \in M_i$ and $x' = eP \in (G/P)_i$.

Claim (i) immediately follows from the fact the $(G/P)_{\bar{\alpha}} = G_\alpha \cdot eP \subset G/P$ is a $G_\alpha$-orbit (recall (3)) and formula (5). The fact that orbits are initial submanifolds is well known, see e.g. Theorem 5.14 in [KMS93].

To prove (ii), observe first that via the inclusion $j : M_i \hookrightarrow M$, we can pull back $\mathcal{G}$ and $\hat{\mathcal{G}}$ to a principal $P$–bundle respectively a principal $G$–bundle over $M_i$. In view of the discussion in 2.2, the reduction of $\hat{\mathcal{G}}$ determined by $s$ can be described as the pre–image $\hat{\mathcal{G}}_\alpha = s^{-1}(\alpha) \subset \hat{\mathcal{G}}$, and we define

$$\mathcal{G}_\alpha := j^*(\hat{\mathcal{G}}_\alpha) \cap j^*\mathcal{G}.$$

We claim that this is a principal bundle with structure group $P_\alpha = G_\alpha \cap P$ over $M_i$. This is again proved by comparison to the homogeneous model $M' = G/P$ and its holonomy reduction determined by $\alpha$: In Lemma 2.6 we (locally) constructed a $P$-bundle map $\Phi : p^{-1}(N) \to (p')^{-1}(N')$ such that $s' \circ \Phi = s$. Hence it clearly suffices to prove that $\mathcal{G}'_\alpha$ is a $P_\alpha$–principal subbundle over $M'_{\bar{\alpha}}$.

But on the homogeneous model we can use the trivialisation $\hat{\mathcal{G}}' = G \times_P G \cong (G/P) \times G$, and there we simply have $\hat{\mathcal{G}}'_\alpha = (G/P) \times G_\alpha$. Therefore $(j')^*(\hat{\mathcal{G}}'_\alpha) = (G_\alpha/P_\alpha) \times G_\alpha$ and $(j')^*(\hat{\mathcal{G}}'_\alpha) \cap (j')^*\mathcal{G}' = (G_\alpha/P_\alpha) \times P_\alpha$. In particular this shows that the intersection $(j')^*(\hat{\mathcal{G}}'_\alpha) \cap (j')^*\mathcal{G}'$ is a $P_\alpha$-subbundle of the $P$-bundle $(j')^*\mathcal{G}'$ over $M'_{\bar{\alpha}} = G_\alpha \cdot eP$.

Therefore also $\mathcal{G}_\alpha$ is an $P_\alpha$–principal subbundle of $j^*\mathcal{G}$ over $M_i$. We next claim that $j^*\omega$ defines a Cartan connection of type $(G_\alpha, P_\alpha)$ on $\mathcal{G}_\alpha$. For this, first note that the extended $G$-principal connection form $\hat{\omega} \in \Omega^1(\hat{\mathcal{G}}, \mathfrak{g})$ has values in $\mathfrak{g}_\alpha$ on $\hat{\mathcal{G}}_\alpha$, and in particular $(\omega_\alpha)_u(T_u\mathcal{G}_\alpha) \subset \mathfrak{g}_\alpha$ for all $u \in \mathcal{G}_\alpha$. But since $(j^*\omega)_u$ is injective on $T_u(j^*\mathcal{G})$ a simple counting of dimensions shows that $(j^*\omega)_u : T_u\mathcal{G}_\alpha \to \mathfrak{g}_\alpha$ is a linear isomorphism, which yields (C.3). The necessary equivariance (C.1) and reproduction (C.2) properties follow immediately from those of $\omega$ by restriction.

Now let $b \in P$ and $\alpha' = b \cdot \alpha$ be another point in $P \cdot \alpha = i \in P \backslash \mathcal{O}$. We know that $\mathcal{G}_\alpha$ and $\mathcal{G}_{\alpha'}$ are principal subbundles of $\mathcal{G}$ with structure group $P_\alpha$ and $P_{\alpha'}$, respectively to which $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ restricts nicely. Then one immediately checks that the restriction of the principal right action $r^{b^{-1}}$ induces an isomorphism between the two principal subbundles (which is equivariant over the isomorphism $P_\alpha \cong P_{\alpha'}$ induced by conjugation). Equivariancy of $\omega$ further implies that this isomorphism is compatible with the induced Cartan connections (where we identify $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{\alpha'}$ via the isomorphism induced by $\operatorname{Ad}(b^{-1})$). Hence we can view the result of our construction as a canonical Cartan geometry $(\mathcal{G}_i \to M_i, \omega_i)$ together with an inclusion $\mathbf{j}_\alpha$ induced by the choice of a representative $\alpha$ of $i \in P \backslash \mathcal{O}$ as claimed in (ii), which completes the proof of this part.

The first part of (iii) then follows immediately from the definition of the Cartan curvature $K = d\omega + \frac{1}{2}[\omega, \omega]$ of $\omega$ and pullback via $\mathbf{j}_\alpha$, while the second part is just the obvious restatement of this in terms of the curvature functions. $\qquad\square$

## 2.7. Parabolic geometries and normal solutions of BGG equations.
We now consider special holonomy reductions for Cartan geometries of type $(G, P)$ with $G$ a semisimple Lie group and $P \subset G$ a parabolic subgroup. In this case the Lie algebra $\mathfrak{g}$ has a natural grading

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k}_{\mathfrak{p}},$$

such that $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$. This class of Cartan geometries is particularly interesting from several points of view. First the class includes many structures already studied intensively, such as conformal geometry and (hypersurface type) CR geometry. Second, for every structure in the class one has canonical regularity and normality conditions on the Cartan connection $\omega$ which lead to Cartan geometries which are equivalent (in a categorical sense) to underlying geometric structures. Finally via the canonical Cartan connection and related calculus the structures in the class admit the application of efficient tools from representation theory to geometric problems; the relevant representation theory is far from trivial, but is very well studied. For extensive background on this class of geometries we refer to [ČS09].

For a parabolic geometry of type $(G, P)$, [ČSS01] introduced a construction for a natural sequence of linear differential operators that was then simplified in [CD01]. For each tractor bundle $\mathcal{V} = \mathcal{G} \times_P V$, with $V$ irreducible for $G$, one obtains the *generalised BGG-sequence*

$$\Gamma(\mathcal{H}_0) \overset{\Theta_0^V}{\to} \Gamma(\mathcal{H}_1) \overset{\Theta_1^V}{\to} \cdots \overset{\Theta_{n-2}^V}{\to} \Gamma(\mathcal{H}_{n-1}) \overset{\Theta_{n-1}^V}{\to} \Gamma(\mathcal{H}_n).$$

Here each $\mathcal{H}_k$ is a certain subquotient bundle of the bundle $\Lambda^k T^* M \otimes \mathcal{V}$ of $\mathcal{V}$–valued $k$–forms, and each $\Theta_i^V$ is a linear differential operator intrinsic to the given geometry.

We are mainly interested in the operator $\Theta_0^V$, which defines an overdetermined system and is closely related to the tractor connection $\nabla$ on $\mathcal{V}$. The

parabolic subgroup $P \subset G$ determines a filtration on $V$ by $P$–invariant sub-
spaces. We only need the largest non–trivial filtration component $V^0 \subset V$.
Then $\mathcal{H}_0$ is simply the quotient of $\mathcal{V}/\mathcal{V}^0$, and we denote by $\Pi : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{H}_0)$
the natural projection.

It turns out that the bundle map $\Pi$ can be used to identify parallel sections
of $\mathcal{V}$, with special solutions of the first BGG operator $\Theta_0^V$, which are then
called *normal solutions*. More precisely, one has:

**Theorem 2.7.** [ČSS01] *Let $V$ be a $G$-irreducible tractor bundle on $M$. The
the bundle map $\Pi$ induces an injection from the space of parallel sections of
$\mathcal{V}$ to a subspace of $\Gamma(\mathcal{H}_0)$ which is contained in the kernel of the first BGG
operator*

$$\Theta_0^V : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_1).$$

Let $U \subset V$ be a $P$–invariant subspace. Then the associated bundle $\mathcal{U}$ is a
subbundle of $\mathcal{V}$. For a normal solution $\sigma$ of $\Theta_0^V$ let $s \in \Gamma(\mathcal{V})$ be the parallel
section such that $\Pi(s) = \sigma$. Define $\mathcal{Z}^U(\sigma) := \{x \in M : s(x) \in \mathcal{U}_x \subset \mathcal{V}_x\}$. Of
course, this is just the zero set of the section of $\mathcal{V}/\mathcal{U}$ obtained by projecting
$s$ to the quotient. Note that for $U = V^0 \subset V$, the largest proper filtration
component, we get $\mathcal{Z}^{V^0}(\sigma) = \mathcal{Z}(\sigma)$, the zero set of $\sigma$. For other proper
filtration components $U \subset V$ we have $U \subset V^0$ and hence $\mathcal{Z}^U(\sigma) \subset \mathcal{Z}(\sigma)$
can be viewed as a space of "higher order zeros" of $\sigma$. This point of view
can be made precise using the fact that $s$ can be described as the image of
$\sigma$ under a linear differential operator [ČSS01, CD01]. More generally, for
$P$–invariant subspaces $U \subset U' \subset V$ one has $\mathcal{Z}^U(\sigma) \subset \mathcal{Z}^{U'}(\sigma)$. This typically
yields a stratification of the zero set of $\sigma$, examples of which were given in
[ČGH10].

The parallel section $s$ of $\mathcal{V}$ gives rise to a holonomy reduction of type $\mathcal{O}$
for some $G$–orbit $\mathcal{O} \subset V$, called the $G$–type of $s$. We will also refer to the
orbit $\mathcal{O} \subset V$ as the $G$-type of the normal solution $\sigma = \Pi(s)$. According to
Definition 2.4 the holonomy reduction provides a curved orbit decomposition
$M = \bigcup_{i \in P \backslash \mathcal{O}} M_i$. We will also refer to the $P$–type of $x \in M$ as the $P$–type
with respect to $\sigma$. For a $P$–invariant subspace $U \subset V$, the subspace $U \cap \mathcal{O}$
is $P$–invariant, so it is a union of $P$–orbits. Clearly, we have

$$(7) \qquad\qquad \mathcal{Z}^U(\sigma) = \bigcup_{i \in P \backslash (U \cap \mathcal{O})} M_i.$$

As for holonomy reductions, we can describe all normal solutions on the
homogeneous model $G/P$ for some given $G$–type $\mathcal{O} \subset V$: For any element
$v \in \mathcal{O}$, the $P$–equivariant function $G \to V$ defined by $g \mapsto g^{-1}v$ defines a
parallel section of $\mathcal{V}$. Via the trivialisation $\hat{G} = G \times_P G \cong G/P \times G$ it is
easy to see that every parallel section of $\mathcal{V}$ is obtained in that way, and it
turns out that the space of parallel sections surjects onto the kernel of $\Theta_0^V$
on the homogeneous model, i.e. all solutions are normal in this case.

Our results on curved orbit decompositions now easily imply that locally, all
possible forms of the sets $\mathcal{Z}^U(\sigma) \subset M$ already show up on the homogeneous
model $G/P$.

**Proposition 2.7.** *Let $\sigma$ be a normal solution of $\Theta_0^V$ on $(\mathcal{G} \to M, \omega)$ of G-type $\mathcal{O} \subset V$ and let $x \in M$ be any point. Then there is a (normal) solution $\sigma'$ on $(G \to G/P, \omega^{MC})$ for which $eP \in G/P$ has the same P–type with respect to $\sigma'$ that $x$ has with respect to $\sigma$. Further, there are open neighbourhoods $N$ of $x$ in $M$ an $N'$ of $eP$ in $G/P$ and there is a diffeomorphism $\varphi : N \to N'$, such that $\varphi(x) = \varphi(x')$ and $\varphi(\mathcal{Z}^U(\sigma) \cap N) = \mathcal{Z}^U(\sigma') \cap N'$ for any P–invariant subspace $U \subset V$.*

*Proof.* Consider the equivariant function $s : \mathcal{G} \to V$ corresponding to the parallel section of $\mathcal{V}$ which induces $\sigma$. Choose a point $u \in \mathcal{G}_x$ and put $v = s(u) \in \mathcal{O}$. Let $\sigma'$ be the normal solution on $G/P$ determined by the function $g \mapsto g^{-1} \cdot v$, the claim about P–types follows. Then the result follows immediately from theorem 2.6, since the set $\mathcal{Z}^U(\sigma)$ is a union of curved orbits in $M$, while $\mathcal{Z}^U(\sigma')$ is the union of the corresponding orbits in $G/P$. $\qquad\square$

## 3. Examples and applications

3.1. **Metrics on the projective standard tractor bundle.** Let $(M, [D])$ be an oriented smooth $n$-manifold endowed with a projective equivalence class of torsion-free affine connections. Hence the equivalence class of $D$ consists of all those torsion-free affine connections which have the same geodesics as $D$ up to parametrization. It is well known that $D$ and $\hat{D}$ are projectively equivalent if and only if there is a 1–form $\Upsilon$ such that

$$\hat{D}_a \varphi_b = D_a \varphi_b + \Upsilon_a \varphi_b + \Upsilon_b \varphi_a$$

for all $\varphi \in \Omega^1(M)$, see e.g. [Eas08], also for the notation.

An oriented projective structure can be equivalently described as a Cartan geometry $(\mathcal{G}, \omega)$ of type $(G, P)$, where $G = SL(n+1, \mathbb{R})$ and $P \subset G$ is the stabiliser of a ray $\mathbb{R}_+ X \in \mathbb{R}^{n+1}$. In particular, the homogeneous model is the projective $n$–sphere $S^n$, which is a 2-fold covering of projective $n$–space $\mathbb{R}P^n$. The bundle associated to the standard representation of $SL(n+1)$ is the *standard tractor bundle* $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{n+1}$. The ray stabilised by $P$ gives rise to a canonical oriented line subbundle $\mathcal{E}(-1) \subset \mathcal{T}$, whose sections are referred to as projective $(-1)$–densities.

The holonomy reduction we are interested in has been studied in section 3.3 of [ČGH10], and we consider it both for comparison and for the purpose of motivation. Namely, we want to consider the case that $\mathcal{T}$ is endowed with a parallel non–degenerate metric. This can be viewed as a parallel section of $S^2\mathcal{T}^*$, which as discussed in 2.3 has a G–type. Linear algebra shows that the decomposition of $S^2\mathbb{R}^{n+1*}$ into orbits of $SL(n+1)$ is described by rank and signature. Since we assume our metric to be non–degenerate, $\mathcal{O}$ consists of all inner products on $\mathbb{R}^{n+1}$ which have some fixed signature $(p, q)$ with $p + q = n + 1$. As shown in 2.3, a parallel tractor metric is the same as a holonomy reduction of $(\mathcal{G}, \omega)$ of type $\mathcal{O}$.

**Theorem 3.1.** *Let $(M, [D])$ be a projective structure endowed with a holonomy reduction of type $\mathcal{O}$ given by a parallel metric $\mathbf{h}$ of signature $(p, q)$ on the standard tractor bundle $\mathcal{T}$.*

*(1) The metric $\mathbf{h}$ determines a normal solution $\sigma$ of the first BGG operator acting on the line bundle $\mathcal{E}(2)$ of all metrics on $\mathcal{E}(-1) \subset \mathcal{T}$.*

*(2) The curved orbit decomposition has the form $M = M_+ \cup M_0 \cup M_-$, where $M_\pm \subset M$ are open and $M_0$ coincides with $\mathcal{Z}(\sigma)$ and (if non–empty) consists of embedded hypersurfaces.*

*(3) The induced Cartan geometry on $M_+$ (respectively $M_-$) is given by a pseudo–Riemannian metric $g_\pm$ of signature $(p-1, q)$ (respectively $(p, q-1)$) whose Levi–Civita connection lies in the projective class.*

*(4) If $M_0$ is non–empty then it naturally inherits a conformal structure of signature $(p-1, q-1)$ via the induced Cartan geometry.*

*Proof.* The space $\mathcal{O}$ splits into $P$-orbits as $\mathcal{O} = \mathcal{O}_+ \cup \mathcal{O}_0 \cup \mathcal{O}_-$ according to the restriction of an inner product to the distinguished ray $\mathbb{R}_+ X \in \mathbb{R}^{n+1}$. On the homogeneous model $S^n = G/P$, a parallel section of $S^2 \mathcal{T}^*$ is determined by an element of $S^2 \mathbb{R}^{(n+1)*}$, so for the given $G$–type, this is just an inner product $\langle \ , \ \rangle$ of signature $(p, q)$ on $\mathbb{R}^{n+1}$. It is easy to see (compare with section 3.3 of [ČGH10]) that the corresponding normal solution is the projective polynomial on $S^n$ induced by the homogeneous polynomial $\langle x, x \rangle$ of degree two. In particular, the zero set of this polynomial is a smooth embedded hypersurface and coincides with $(S^n)_0$. Via Theorems 2.6 and 2.7 this carries over to the curved case, which proves (1) and (2).

Hence we turn to the induced Cartan geometries on the curved orbits. According to Theorem 2.6 they have type $(H, H \cap P)$, where $H$ is the stabiliser of some element in the orbit in question. Let us start with the case $h_+ \in \mathcal{O}_+$, i.e. $h_+(X, X) > 0$. Then of course $H = G_{h_+}$ is isomorphic to $SO(p, q)$ and $H \cap P$ is the stabiliser of a positive ray. But elements of $H$ preserve norms, so any element of $H \cap P$ has to preserve any vector in the positive ray. Hence $H \cap P$ is the isotropy group $H_X \cong SO(p-1, q)$ of a unit vector. A Cartan geometry of type $(SO(p, q), SO(p-1, q))$ is well known to be equivalent to a pseudo–Riemannian metric of signature $(p-1, q)$ together with a metric connection, see sections 1.1.1 and 1.1.2 of [ČS09]. Since the canonical Cartan connection associated to a projective structure is always torsion free, part (iii) of Theorem 2.6 implies that the induced Cartan geometries are torsion free. Hence in each case the corresponding metric connection in question is torsion free, and hence is the Levi-Civita connection. Since the induced Cartan geometries are simply obtained by restricting the projective Cartan connection, it follows that this Levi-Civita connection lies in the projective class. The description of $\mathcal{O}_-$ is completely parallel, so the proof of (3) is complete.

(4): Here we again have $H \cong SO(p, q)$, but $H \cap P \subset H$ now is the stabiliser of an isotropic ray in the standard representation. This is a parabolic subgroup

$\underline{P}$ of $SO(p, q)$ and Cartan geometries of this type correspond to pseudo–Riemannian conformal structures of signature $(p - 1, q - 1)$, see also section 3.4 below. $\qquad\square$

3.2. **Consequences of normality.** Most of the analysis of a parallel metric on the projective standard tractor bundle in 3.1 is valid for an arbitrary Cartan geometry of type $(SL(n + 1), P)$. Only in the last part we used that torsion freeness of the Cartan geometry implies that the induced Cartan geometries on the open orbits produce Levi-Civita connections of the induced metric, and not just any metric connection. In the next step, we will use the fact that we are dealing with the normal Cartan geometries associated to the underlying projective structure, so an additional normalisation condition on the Cartan curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{sl}(n + 1))$, respectively on the corresponding curvature function $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ is available.

This normalisation condition on the one hand requires $\omega$ to be torsion free, i.e., $\kappa$ to have values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$. Then for $X_1, X_2 \in \mathfrak{g}/\mathfrak{p}$ and $Y \in \mathfrak{p} \subset \mathfrak{sl}(n + 1)$ we have that $[\kappa(u)(X_1, X_2), Y] \in \mathfrak{p}$, and therefore $\kappa(u)(X_1, X_2)$ factors to a linear map $\kappa_0(u)(X_1, X_2) : \mathfrak{g}/\mathfrak{p} \to \mathfrak{g}/\mathfrak{p}$. Via the identification $\mathfrak{g}/\mathfrak{p} \cong \mathbb{R}^n$ we can view $\kappa_0(u)$ as an element of $\Lambda^2\mathbb{R}^{n*} \otimes L(\mathbb{R}^n, \mathbb{R}^n)$. Now the second part of the normalisation condition on $\omega$ implies that $\kappa_0$ is completely trace-free. This says that $\kappa_0(u) \in \Lambda^2\mathbb{R}^{n*} \otimes \mathfrak{sl}(\mathbb{R}^n)$ and also the Ricci-type contraction of $\kappa_0(u)$ vanishes, i.e.,

$$\mathrm{tr}(W \mapsto \kappa_0(u)(W, Y)Z) = 0$$

for all $Y, Z \in \mathbb{R}^n$. Now we can analyse the consequences for the induced Cartan geometries.

**Proposition 3.2.** *(1) If the open orbit $M_+$ (respectively $M_-$) is non–empty, then the induced metric $g_+$ (respectively $g_-$) is Einstein with positive (respectively negative) Einstein constant, i.e. $\mathrm{Ric}(g_+)$ is positive multiple of $g_+$ while $\mathrm{Ric}(g_-)$ is a negative multiple of $g_-$.*

*(2) If the closed curved orbit $M_0$ is non–empty, then the induced Cartan geometry of type $(SO(p, q), \underline{P})$ is normal.*

*Proof.* (1) We consider the case of $M_+$ and indicate the necessary changes for $M_-$ in the end. Throughout the proof, we work in a point $u \in \mathcal{G}$ which is contained in the reduced Cartan subbundle. Since we are dealing with an open orbit we get $\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{p}) \cong \mathfrak{g}/\mathfrak{p}$, so part (iii) of theorem 2.6 shows that the value $\kappa(u)$ of the curvature function $\kappa$ of $\omega$ coincides with the value of the curvature function of the induced Cartan connection. This also shows that torsion freeness of $\omega$ implies that $\kappa(u) \in \Lambda^2\mathbb{R}^{n*} \otimes \mathfrak{h}$.

Since $\mathcal{T}$ is associated to a principal $SL(n + 1, \mathbb{R})$–bundle, it has a distinguished volume form, and rescaling the tractor metric by a constant, we may assume that orthonormal bases have unit volume. We can thus work in matrix representations with respect to orthonormal bases. Then $\mathfrak{so}(p, q)$ has the form

$$\left\{ \begin{pmatrix} 0 & -Y^t\mathbb{I}_{p-1,q} \\ Y & A \end{pmatrix} : Y \in \mathbb{R}^n, A \in \mathfrak{so}(p - 1, q) \right\},$$

where $\mathbb{I}_{p-1,q}$ is diagonal with $p-1$ entries equal to 1 and $q$ entries equal to $-1$. Since the subspaces $\mathbb{R}^n$ and $\mathfrak{so}(p-1,q)$ in $\mathfrak{h}$ are invariant under $SO(p-1,q)$, the components of the Cartan connection in the two subspaces are individually equivariant. The $\mathbb{R}^n$–component $\theta$ defines a soldering form on the bundle which is used to carry over the inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ corresponding to $\mathbb{I}_{p-1,q}$ to the metric $g_+$ on the tangent spaces of $M_+$. The $\mathfrak{so}(p-1,q)$–component $\gamma$ is a principal connection, which induces a metric connection on the tangent bundle, and by torsion freeness, this is the Levi–Civita connection of $g_+$.

From the definition of the curvature of a Cartan connection it follows that the curvature $K_+$ of the induced Cartan connection is given by

$$(8) \qquad\qquad K_+(u)(\xi,\eta) = R(u)(\xi,\eta) + [\theta(\xi),\theta(\eta)],$$

where $R$ is the curvature of $\gamma$, and hence the Riemann curvature, and the last bracket is in $\mathfrak{so}(p,q)$. Now one immediately computes that for $Y_1, Y_2, Z \in \mathbb{R}^n \subset \mathfrak{h}$, we get

$$(9) \qquad\qquad [[Y_1,Y_2],Z] = \langle Y_1,Z\rangle Y_2 - \langle Y_2,Z\rangle Y_1$$

Using this, one easily calculates that the Ricci type contraction of $Y_1, Y_2 \mapsto [Y_1,Y_2]$ is given by $-(n-1)$ times the inner product $\langle\ ,\ \rangle$. Since the left hand side of (8) has vanishing Ricci type contraction by normality, we conclude that the Ricci type contraction $\mathrm{Ric}(g_+)$ of $R$ equals $(n-1)g_+$, so $g_+$ is positive Einstein.

In the case of $M_-$, the first basis vector used to define the matrix representation must be chosen to be negative. But then the entries of a matrix in $\mathbb{R}^n \subset \mathfrak{so}(p,q)$ must be $Y$ and $Y^t\mathbb{I}_{p,q-1}$. This causes a sign change in formula (9) and hence in the Ricci–type contraction, so on obtains $\mathrm{Ric}(g_-) = -(n-1)g_-$.

(2) Again we work in a point $u \in \mathcal{G}$ which lies in the reduced Cartan bundle over the curved orbit, which means that we work in a basis for $\mathcal{T}_x$, which is adapted to the tractor metric. We choose this basis in such a way that the first basis vector $X$ spans the distinguished line (which is isotropic for the tractor metric in this point), the last basis vector is isotropic and pairs to one with $X$ under the tractor metric. Then we choose an orthonormal basis for the orthocomplement of the plane spanned by these two vectors to complete our basis. The normalisation condition on the projective Cartan curvature implies torsion freeness and that its $\mathfrak{g}_0$–component has values in $\mathfrak{sl}(n)$. Moreover, it has to be skew symmetric with respect to the tractor metric, so altogether it must be of the form

$$(10) \qquad\qquad \kappa(\xi,\eta) = \begin{pmatrix} 0 & Z(\xi,\eta) & 0 \\ 0 & A(\xi,\eta) & -\mathbb{I}(Z(\xi,\eta))^t \\ 0 & 0 & 0 \end{pmatrix}.$$

Here the blocks are of size 1, $n-1$, and 1 and $\mathbb{I} = \mathbb{I}_{p-1,q-1}$. Finally the normalisation condition tells us, that the Ricci type contraction over the lower right $n \times n$–block has to vanish. This coincides with the Ricci type contraction of $A$ taken over $X^\perp/\mathbb{R}X$.

Now we know that restricted to the tangent space of the reduced Cartan bundle, the projective Cartan connection restricts to the reduced Cartan connection. In particular, its soldering form must have values in $X^\perp/\mathbb{R}X$, so this corresponds to $\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{p}) \subset \mathfrak{g}/\mathfrak{p}$ and thus represents the tangent spaces to $M_0$. Moreover, by part (iii) of Theorem 2.6, (10) coincides with the curvature of the reduced Cartan connection. Since the normalisation condition on a conformal Cartan connection is torsion freeness plus vanishing of the Ricci–type contraction of the $\mathfrak{g}_0$–component of the curvature function (which is represented by $A(\xi, \eta)$), normality of the induced Cartan geometry follows. $\qquad\square$

### 3.3. Hermitian metrics on complex projective standard tractors.

There is an almost complex version of metrics on the projective standard tractor bundle. This is significantly more complicated than the real version in several respects. Therefore, we will only derive some basic facts here, and study it in more detail elsewhere.

Almost complex classical projective structures can be equivalently described as parabolic geometries of type $(G, P)$, where $G = SL(n+1, \mathbb{C})$ and $P \subset G$ is the stabiliser of a line in the standard representation $\mathbb{C}^{n+1}$ of $G$. However, $G$ and $P$ are viewed as real Lie groups and likewise one has to consider their Lie algebras as real Lie algebras. Doing this, one obtains a geometry which is much more general than just the obvious holomorphic analog of a classical projective structure. As far as we know, the general theory of this geometry is not developed in detail in the literature, a brief account can be found in section 4.6 of [Čap05].

Explicitly, one has to consider manifolds $M$ of real dimension $2n$ endowed with an almost complex structure $J : TM \to TM$. Then it is well known that for a linear connection $D$ on $TM$ such that $DJ = 0$, the $(0,2)$–component of the torsion of $D$ is (up to a nonzero factor) given by the Nijenhuis tensor of $J$. On the other hand, one can always find a connection $D$ for which the torsion is of type $(0, 2)$.

The notion of projective equivalence that is used here is based on the transformation formula
$$\tilde{D}_\xi \eta = D_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi,$$
for all vector fields $\xi$, $\eta$ and a fixed one–form $\Upsilon$, and where the 1-form field $\Upsilon \in \Omega^1(M)$ is complex linear (with respect to $J$). Note that this does not imply projective equivalence in the real sense, since complex linear combinations of $\xi$ and $\eta$ occur in the right hand side. It is still true, however, that connections which are equivalent in this sense have the same torsion. The use of complex linear combinations leads to the fact that $DJ = 0$ implies that $\tilde{D}J = 0$ for any equivalent connection.

An almost complex projective structure is then given by an almost complex manifold $(M, J)$ together with an equivalence class $[D]$ of connections such that $DJ = 0$ and the torsion of $D$ is of type $(0, 2)$. As we have noted above, both these conditions are independent of the choice of representative in the equivalence class.

Now the holonomy reduction we want to consider in this case is related to the homogeneous space $\mathcal{O}$ of $SL(n+1, \mathbb{C})$ which consists of all Hermitian inner products on $\mathbb{C}^{n+1}$ which are non–degenerate with some fixed signature $(p+1, q+1)$, where $p+q = n-1$. Starting from an almost complex projective structure, one forms the corresponding normal parabolic geometry of type $(G, P)$. Forming the associated bundle to the Cartan bundle with respect to (the restriction to $P$ of) the standard representation $\mathbb{C}^{n+1}$ of $G$, one obtains the standard tractor bundle $\mathcal{T}$ of the almost complex projective structure. By construction, this is a complex vector bundle of complex rank $n+1$ and the complex line, in the standard representation, stabilised by $P$ gives rise to a complex line subbundle $\mathcal{E} \subset \mathcal{T}$. A holonomy reduction of type $\mathcal{O}$ is equivalent to a Hermitian bundle metric $\mathbf{h}$ on $\mathcal{T}$ which is non–degenerate of signature $(p+1, q+1)$ and parallel for the canonical connection on $\mathcal{T}$.

**Theorem 3.3.** *Let $(M, J, [D])$ be an almost complex projective structures and suppose that we have given a holonomy reduction of type $\mathcal{O}$ described by a parallel Hermitian metric $\mathbf{h}$ of signature $(p+1, q+1)$ on the standard tractor bundle $\mathcal{T}$. Then we have:*

*(1) The metric $\mathbf{h}$ determines a normal solution $\sigma$ of the first BGG operator acting on sections of the real line bundle of Hermitian metrics on the (complex) density bundle $\mathcal{E} \subset \mathcal{T}$.*

*(2) The curved orbit decomposition has the form $M = M_- \cup M_0 \cup M_+$, where the first and last curved orbits are open and given by those points where the metric defined by $\sigma$ is negative definite, respectively positive definite. If non–empty, the curved orbit $M_0$ is an embedded hypersurface which coincides with $\mathcal{Z}(\sigma)$.*

*(3) On $M_\pm$ one obtains induced Hermitian metrics of signature $(p, q+1)$, respectively $(p+1, q)$, together with metric connections. If the initial structure is torsion free, then the metric connections are the Levi–Civita connections, so one actually obtains Kähler structures on $M_\pm$.*

*(4) If $M_0 \neq \emptyset$, then it inherits a Cartan geometry of type $(SU(p+1, q+1), \underline{P})$, where $\underline{P}$ is the stabiliser of an isotropic line. If the initial structure is torsion free, then this induces an (integrable) CR structure of signature $(p, q)$ on $M_0$.*

*Proof.* First note that the space of all Hermitian bilinear forms on $\mathbb{C}^{n+1}$ which vanish on the complex line stabilised by $P$ is evidently $P$–invariant and has codimension one, so it must be the maximal $P$–invariant subspace. Thus part (1) immediately follows from the general theory discussed in 2.7.

The stabiliser of any Hermitian inner product from $\mathcal{O}$ is a subgroup of $G$ conjugate to $SU(p+1, q+1) \subset SL(n+1, \mathbb{C})$. The homogeneous model $G/P$ of the geometry is simply the complex projective space $\mathbb{C}P^n$, so according to 2.4 we can determine $P$–types by looking at $SU(p+1, q+1)$–orbits on the space of complex lines in $\mathbb{C}^{n+1}$. These orbits are determined by the signature of the restriction of the Hermitian inner product to a line, so this looks exactly as in the real case, and we have $\mathcal{O} = \mathcal{O}_+ \cup \mathcal{O}_0 \cup \mathcal{O}_-$. This also gives us the basic form of the curved orbit decomposition in (2). Identifying the stabiliser $H$ of the inner product with $SU(p+1, q+1)$ the subgroups $H \cap P$ in the

three cases are conjugate to $S(U(1) \times U(p, q+1)) \cong U(p, q+1)$, a parabolic subgroup $\underline{P} \subset SU(p+1, q+1)$, and $S(U(p+1, q) \times U(1)) \cong U(p+1, q)$, respectively.

For the $P$–type defined by $\mathcal{O}_+$, the standard way to present $Q := S(U(1) \times U(p, q+1)) \subset SU(p+1, q+1) =: H$ is as matrices of the form $\begin{pmatrix} \det(A)^{-1} & 0 \\ 0 & A \end{pmatrix}$ with $A \in U(p, q+1)$. On the level of Lie algebras, we have

$$\mathfrak{h} = \left\{ \begin{pmatrix} -\operatorname{tr}(B) & -Z^* \mathbb{I}_{p,q+1} \\ Z & B \end{pmatrix} : Z \in \mathbb{C}^n, B \in \mathfrak{u}(p, q+1) \right\},$$

where $\mathbb{I}_{p,q+1}$ is the diagonal matrix of size $p + q + 1$ with first $p$ entries equal to 1 and last $q + 1$ entries equal to $-1$. The Lie algebra $\mathfrak{q}$ of $Q \subset H$ corresponds to the block diagonal part. Hence $\mathfrak{h}/\mathfrak{q}$ can be identified with $\mathbb{C}^n$ with the representation of $Q$ on this space given by $A \cdot Z = \det(A)AZ$. In particular, the obvious complex structure on $\mathfrak{h}/\mathfrak{q}$ as well as the standard inner product of signature $(p, q + 1)$ on this space are invariant under the action of $Q$. Consequently, a Cartan geometry of type $(H, Q)$ on a smooth manifold $M$ gives rise to an almost complex structure $J$ and a Hermitian (with respect to $J$) metric $g$ of signature $(p, q + 1)$. Finally, the Cartan geometry also gives rise to a principal connection, which can be equivalently encoded as a linear connection $\nabla$ on the tangent bundle which is compatible both with $J$ and with $g$. Together, $J$, $g$, and $\nabla$ completely determine the Cartan geometry. For the orbit $\mathcal{O}_-$, the situation is completely parallel, with the only difference that $g$ has signature $(p+1, q)$ rather than $(p, q+1)$. This completes the proof of the first part of (3).

For $\mathcal{O}_0$, we get the stabiliser of an isotropic line as the subgroup in $H$, and it is well known that Cartan geometries of the corresponding type are related to partially integrable almost CR structures of signature $(p, q)$, see also 3.6 below. This also implies the first part of (4).

On the homogeneous model $\mathbb{C}P^n$, a parallel metric on the standard tractor bundle corresponds to a fixed Hermitian inner product $\underline{\mathbf{h}}$ of signature $(p+1, q+1)$ on $\mathbb{C}^{n+1}$. The orbit decomposition is just given by the signature of the restriction of $\underline{\mathbf{h}}$ to the complex line determined by a point in $\mathbb{C}P^n$ as described above. It is well known that the spaces of positive, respectively negative, lines are open and they are the complex hyperbolic spaces of signature $(p, q+1)$ and of signature$(p+1, q)$, respectively. The space of isotropic lines is a quadric and in particular as smooth embedded hypersurface, which is the homogeneous model of (partially integrable almost) CR structures of hypersurface type, which are non–degenerate of signature $(p, q)$. Since $\mathcal{O}_0$ is exactly the zero set of the normal solution $\underline{\sigma}$ determined by $\underline{\mathbf{h}}$, we obtain (2).

To complete the proof, let us assume that the initial Cartan geometry $(p : \mathcal{G} \to M, \omega)$ is torsion free (which is equivalent to the fact that we deal with a holomorphic projective structure, see [Čap05]). Then by part (iii) of theorem 2.6 also the induced Cartan geometries are torsion free. It is well known that torsion free Cartan geometries of type $(SU(p+1, q+1), \underline{P})$ are

equivalent to (integrable) non–degenerate CR structures of signature $(p, q)$, so the proof of (4) is complete.

For the curved orbits $M_\pm$ torsion freeness of the induced Cartan geometries implies that the induced metric connections are torsion free as in the real case. Since these connections preserve the Hermitian metrics, these metrics actually must be Kähler.                                    □

Analysing the consequences of normality is similar to the real case, but significantly more complicated, in particular if one drops the assumption of torsion freeness. Thus we just indicate some basic facts here, and pick up the detailed discussion elsewhere. First one has to analyse the relation between the Cartan curvatures of the initial Cartan connection of type $(G, P)$ and the induced Cartan connections. For the open orbits, this is similar to the discussion in the proof of Proposition 3.2, and one verifies that actually gets Kähler–Einstein metrics on $M_\pm$. Moreover, parallel to [ČGH10, Theorem 3.3] one shows that these metrics are complete if one starts from a complex projective structure on a compact manifold. Hence in the torsion free case, one obtains a compactification of a complete Kähler–Einstein manifold by adding a CR structure at infinity.

If one does not assume the original structure to be torsion free, the induced connection on $M_\pm$ will differ (in a controlled way) from the Levi-Civita connection, and also the normalisation condition for the Cartan connection of type $(G, P)$ becomes significantly more involved. To describe the induced geometry on $M_0$, one first has to check when the induced Cartan connection of type $(SU(p + 1, q + 1), \underline{P})$ is regular, since then it induces a partially integrable almost CR structure on $M_0$.

3.4. **Fefferman-type constructions.** We next outline examples of holonomy reductions for conformal structures. Let $M$ be a smooth manifold of dimension $n \geq 3$. Then a conformal structure of signature $(p, q)$ on $M$ is given by an equivalence class $[g]$ of pseudo–Riemannian metrics of signature $(p, q)$ on $M$. Here two metrics $g$ and $\hat{g}$ are considered equivalent if there is a positive smooth function $f : M \to \mathbb{R}$ such that $\hat{g} = fg$. It is a classical result going back to E. Cartan that an oriented conformal structure can be equivalently described as a parabolic geometry of type $(G, P)$, where $G = SO(p + 1, q + 1)$ and $P \subset G$ is the stabiliser of an isotropic ray in the standard representation $\mathbb{R}^{p+1,q+1}$ of $G$. The corresponding grading on the Lie algebra $\mathfrak{g}$ of $G$ has the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with $\mathfrak{g}_0 = \mathfrak{co}(p, q)$ (the Lie algebra of the conformal group of signature $(p, q)$), $\mathfrak{g}_{-1} \cong \mathbb{R}^{p,q}$ and $\mathfrak{g}_1 \cong (\mathbb{R}^{p,q})^*$ as representations of $\mathfrak{g}_0$.

The first group of examples we discuss is one in which the curved orbit decomposition is trivial, since there is only one $P$–type in this case. This does not mean that our results are vacuous in these cases, however. On the one hand, we may conclude from just looking at the homogeneous model that also in curved cases the solutions obtained from such a restriction cannot have any zeroes. On the other, we immediately see that in this case we get an reduced Cartan geometry over the whole manifold.

The most classical example of this situation comes up when both $p$ and $q$ are odd, say $p = 2p' + 1$ and $q = 2q' + 1$. Then we have the subgroup $U(p' + 1, q' + 1) \subset SO(2p' + 2, 2q' + 2)$, which looks like a good candidate for a holonomy group. In the language of section 2, we have to look at the homogeneous space $\mathcal{O}$ of $G$ which consists of all complex structures $J$ on $\mathbb{R}^{(2p'+2, 2q'+2)}$, which are orthogonal (or equivalently skew–symmetric) with respect to the standard inner product. This exactly means that the standard inner product can be extended to a Hermitian inner product with respect to $J$.

We can also deduce immediately what a holonomy reduction of type $\mathcal{O}$ means in the curved case. Given the Cartan bundle determined by a conformal structure, we can form the associated bundle with respect to the (restriction to $P$ of the) standard representation $\mathbb{R}^{2p'+2, 2q'+2}$ of $G$. This is the *standard tractor bundle* and it inherits a canonical linear connection from the conformal Cartan connection. By construction, this carries a natural bundle metric $h$ of signature $(2p' + 2, 2q' + 2)$, the tractor metric, and the isotropic line stabilised by $P$ gives rise to a line subbundle $\mathcal{T}^1 \subset \mathcal{T}$, whose fibres are isotropic with respect to $h$. The orthogonal spaces to these preferred isotropic lines fit together to form a subbundle $\mathcal{T}^0 \subset \mathcal{T}$ of corank one, so we obtain a filtration $\mathcal{T}^1 \subset \mathcal{T}^0 \subset \mathcal{T}$ of the tractor bundle. Now a holonomy reduction of type $\mathcal{O}$ is equivalent to the choice of a complex structure $\mathcal{J}$ on the bundle $\mathcal{T}$, which is skew symmetric with respect to $h$ and parallel with respect to (the connection induced by) the standard tractor connection.

The best way to view $\mathcal{J}$ is as a section of the bundle $\mathfrak{so}(\mathcal{T}M)$. This is the *adjoint tractor bundle* $\mathcal{A}M$, which is induced by (the restriction to $P$ of) the adjoint representation $\mathfrak{so}(2p' + 2, 2q' + 2)$ of $G$. The natural quotient $\mathcal{H}_0$ in this case is the tangent bundle $TM$, and the corresponding first BGG operator is the conformal Killing operator, whose kernel consists of all infinitesimal conformal isometries. In this case, also the condition of being a normal solution is particularly easy for formulate, it just means that the conformal Killing field has to insert trivially into the Weyl curvature and into the Cotton–York tensor, compare with [Čap08].

The homogeneous model $G/P$ of conformal geometry is the space of isotropic rays in $\mathbb{R}^{2p'+2, 2q'+2}$, which is diffeomorphic to $S^{2p'+1} \times S^{2q'+1}$. Since the group $U(p' + 1, q' + 1)$ evidently acts transitively on the space of real isotropic rays in $\mathbb{C}^{p'+1, q'+1}$, we conclude that indeed there is only one $P$–type in this case. In particular, this implies that the conformal Killing vector field underlying $\mathcal{J}$ is nowhere vanishing (since it cannot be identically zero). We also conclude that on all of $M$, we obtain an induced Cartan geometry of type $(U(p' + 1, q' + 1), P'_U)$, where $P'_U$ is the stabiliser of an isotropic real ray in $\mathbb{C}^{p'+1, q'+1}$.

On could now go on an study this Cartan geometry in more detail, but the geometric meaning of this holonomy reduction has already been completely clarified in [ČG10]. First of all, it turns our that the holonomy always reduces further to $SU(p' + 1, q' + 1)$, at least on an infinitesimal level. Thus, one is already close to the setting of CR–structures, see section 3.6 below.

Indeed, it turns out that one may consider a local leaf space for the one–dimensional foliation defined by the nowhere vanishing conformal Killing field underlying $\mathcal{J}$. One can then descend the Cartan geometry to this local leaf space to obtain a geometry of type $(SU(p'+1, q'+1), P_{SU})$, where $P_{SU}$ is the stabiliser of a complex null line. This Cartan geometry turns out to be torsion free, thus inducing a CR structure of hypersurface type, which is non–degenerate of signature $(p', q')$. Indeed, it is even normal and hence the canonical Cartan geometry associated to this CR structures. The original space is then locally conformally isometric to the Fefferman space of the CR structure on the local leaf space. The fact that the canonical Cartan connections associated to a CR structure agrees with the one associated to its Fefferman space has rather strong consequences, see [ČG08].

There are variants of this situation in which one obtains similar results. For example, let us consider conformal structures of signature $(2, 3)$ in dimension five. Then the basic group governing this geometry is $SO(3, 4)$. It is well known that letting this group act on the space $\Lambda^3 \mathbb{R}^{3,4}$ of three forms, there is a unique open orbit and the stabiliser of an element in this orbit has a Lie algebra which is a split real form of the exceptional Lie algebra of type $G_2$. In particular, this orbit defines a class of holonomy reductions for split signature conformal structures in dimension five. Such a reduction is given by a section of the third exterior power of the standard tractor bundle, which has the right $G$–type (i.e. lies in the open orbit) in each point. As before it is well known that the split real form of $G_2$ acts transitively on the space of isotropic lines in its seven–dimensional representation, so there is only one $P$–type in this case.

Our results imply that such a holonomy reduction gives rise to a Cartan geometry of type $(G_2, Q)$ over all of $M$, where $Q$ is the stabiliser of an isotropic line in the seven–dimensional irreducible representation of $G_2$. Analysing the curvatures (see [HS09]) one concludes that this geometry is regular, thus giving rise to a generic rank two distribution on $M$. This Cartan geometry coincides with the geometry introduced in Cartan's famous "five–variables–paper" [Car10]. The initial conformal structure then is the one associated to the generic distribution via Nurowski's construction, see [Nur05] and [ČS07]. Again the fact that the canonical Cartan connections coincide has strong consequences, see [HS09].

A very similar construction applies to split signature conformal structures in dimension 6, where the relevant subgroup is $Spin(8) \subset SO(4, 4)$, and one has to consider the fourth exterior power of the standard tractor bundle rather than the third one. A holonomy reduction then gives rise to a generic distribution of rank three, to which the initial conformal structure is canonically associated as first shown in R. Bryant's thesis, see [Bry06, HS11] for recent accounts.

3.5. **Almost Einstein scales.** We next study the simplest example of a holonomy reduction for conformal structures, namely the existence of a parallel section $s$ of the standard tractor bundle $\mathcal{T}$ of a conformal structure $(M, [g])$. This was actually the first case in which the zeroes of a solution

of a first BGG operator were studied using parabolic geometry methods, see [Gov10] and this motivated many of the developments that led to this article. We treat this case here to illustrate how it fits into the more general machinery (the latter also simplifying many aspects).

Recall from Section 3.4 above that for a conformal structure of signature $(p, q)$ the standard tractor bundle has rank $p + q + 2$ and is endowed with a canonical parallel metric $h$ of signature $(p + 1, q + 1)$, as well as a line subbundle $\mathcal{T}^1 \subset \mathcal{T}$ whose fibres are isotropic with respect to $h$. Following a standard convention in conformal geometry, we shall write $\mathcal{E}[1] := (\mathcal{T}^1)^*$. Note that for a parallel section $s$ of $\mathcal{T}$, the function $h(s, s)$ must be constant, and up to a constant rescaling of $s$, the possible $G$ types in this case are distinguished by the fact that $h(s, s)$ is positive, zero, or negative, respectively.

**Theorem 3.5.** *Suppose that $(M, [g])$ is an oriented conformal pseudo–Riemannian structure of signature $(p, q)$ and that $s$ is a parallel section of the standard tractor bundle $\mathcal{T}$ of $M$. Then $s$ projects onto a normal solution $\sigma \in \Gamma(\mathcal{E}[1])$ of a first BGG operator.*

*(1) Suppose that $h(s, s) > 0$ (respectively $h(s, s) < 0$). Then the curved orbit decomposition has the form $M = M_+ \cup M_0 \cup M_-$, where $M_+$ and $M_-$ are open and $M_0$ coincides with the zero locus of $\sigma$ and (if non–empty) consists of embedded hypersurfaces. Moreover, on $M_\pm$, there is an Einstein metric in the conformal class, whose Einstein constant is negative if $h(s, s) > 0$ and positive if $h(s, s) < 0$. The curved orbit $M_0$ is empty if $p = 0$ (respectively $q = 0$), otherwise it inherits a conformal structure of signature $(p - 1, q)$ (respectively $(p, q - 1)$). The induced Cartan geometry on $M_0$ is the normal Cartan geometry determined by this conformal structure.*

*(2) Suppose that $h(s, s) = 0$. Then the curved orbit decomposition has the form $M = M_1^+ \cup M_1^- \cup M_2 \cup M_3^+ \cup M_3^-$, where $M_1^\pm$ are open, $M_2 \cup M_3^\pm = \mathcal{Z}(\sigma)$, $M_2$ (if non–empty) consists of smoothly embedded hypersurfaces and $M_3^\pm$ (if non–empty) consists of isolated points. If $p = 0$ or $q = 0$, then $M_2$ must be empty, so $\sigma$ can only have isolated zeros in this case. Otherwise, if $M_3^+$ or $M_3^-$ is non–empty, then also $M_2$ has to be non–empty.*

*On $M_1^\pm$ there is a Ricci–flat metric in the conformal class. If non–empty, the curved orbit $M_2$ locally naturally fibres with one–dimensional fibres over a smooth manifold $N$ which inherits a natural conformal structure of signature $(p - 1, q - 1)$.*

*Proof.* (1) Let us assume that $h(s, s) > 0$. Then the stabiliser $H$ of $s$ is isomorphic to $SO(p, q + 1) \subset SO(p + 1, q + 1)$. There are three possible $P$–types in this case, defined by the fact that the inner product of $s(x) \in \mathcal{T}_x$ with a generator of the distinguished isotropic ray is positive, zero, or negative. (Note that $s(x)$ cannot lie in the distinguished isotropic line.) The irreducible quotient of $\mathcal{T}$ for conformal structures is a density bundle usually denoted by $\mathcal{E}[1]$ which is realized as $\mathcal{T}/(\mathcal{T}^1)^\perp$. Hence we conclude that the curved orbit defined by $s(x) \perp \mathcal{T}_x^1$ is exactly the zero set of the induced normal solution $\sigma = \Pi(s)$. On the homogeneous model, $s$ is determined by a fixed positive vector in $\mathbb{R}^{p+1, q+1}$ so the orbit in question is the subspace

of isotropic lines contained in a hyperplane, and thus a smoothly embedded hypersurface in $G/P$. This gives the description of curved orbits in (1). If $p = 0$, then the restriction of $h$ to $s(x)^\perp$ is negative definite, and hence this subspace does not contain any isotropic lines, so $M_0 = \emptyset$.

To describe the induced Cartan geometry on $M_\pm$, we have to understand the stabiliser of the isotropic ray in $H \cong SO(p, q + 1)$, and we know that this ray is transversal to the hyperplane $s(x)^\perp$ stabilised by $H$ and different from the line $\mathbb{R} \cdot s(x)$. If we project $\mathcal{T}_x^1$ orthogonally into $s(x)^\perp$ we thus obtain a line. This line has to be negative, since together with the positive line $\mathbb{R} \cdot s(x)$ it spans a plane which contains the isotropic line $\mathcal{T}_x^1$. A moment of thought shows that the stabiliser of this negative line in $H$ coincides with the stabiliser of the isotropic ray in that group, so $H \cap P = SO(p, q) \subset SO(p, q + 1)$.

Thus we are in the same situation as in the example of the parallel metric on the projective standard tractor bundle. The induced Cartan geometry on the open curved orbit is equivalent to a pseudo–Riemannian metric of signature $(p, q)$ together with a metric connection on the tangent bundle. Using torsion freeness of the initial conformal Cartan connection, one concludes that the metric connection must be the Levi–Civita connection. From the normality of the conformal Cartan connection one then deduces that the induced metric is Einstein, with the sign of the Einstein constant determined by $h(s, s)$.

The type of the induced Cartan geometry on $M_0$ is even easier to determine, since here $H \cap P$ simply is the stabiliser of an isotropic ray in the standard representation of $H = SO(p, q + 1)$. Hence the induced Cartan geometry on the closed curved orbit determines an oriented conformal structure of signature $(p - 1, q)$. It is straightforward to prove that normality of the initial conformal Cartan connection implies that also the induced Cartan connection over the closed curved orbit is normal. Since the discussion for $h(s, s) < 0$ is completely parallel, this completes the proof of (1).

(2) If $h(s, s) = 0$, then $s$ spans an isotropic line subbundle in the standard tractor bundle. In this case, the stabiliser $H$ is the stabiliser of an isotropic vector in $\mathbb{R}^{p+1,q+1}$ and thus isomorphic to $SO(p, q) \ltimes \mathbb{R}^{p,q}$ (so it is isomorphic to a codimension one subgroup in the parabolic subgroup $P$). It is also clear that there are five possible $P$–types, according to the cases that $s(x)$ lies in the preferred isotropic ray or its negative, respectively that $s(x)$ is not in the preferred line and the inner product of $s(x)$ with a generator of the preferred ray is zero, positive, and negative respectively. Notice however, that if either $p = 0$ or $q = 0$, the initial vector space $\mathbb{R}^{p+1,q+1}$ is Lorentzian and hence does not contain two perpendicular isotropic lines. Thus in this special case only the first two and the last two $P$–types are there.

In the homogeneous model $G/P$, our parallel section is determined by an isotropic vector in $\mathbb{R}^{p+1,q+1}$, and it is evident that the first two and the last two of the five $P$–orbits consist of isolated points, and open subsets, respectively. For the middle $P$–type, we observe that taking the tractor inner product of $s$ with the preferred ray defines the solution $\sigma \in \Gamma(\mathcal{E}[1])$

underlying $s$. The orbit under consideration consists of the zero set of this section except for the single point where $s$ lies in $\mathcal{T}^1$. But it is evident that this single point is the only one in which the hypersurface orthogonal to $s$ is not transversal to the tangent space of the null–cone, which implies that our orbit is a smooth hypersurface in the homogeneous model. Hence we obtain the claimed form of the curved orbit decomposition.

To discuss the induced Cartan geometry on $M_1^{\pm}$, we have seen that $H \cong SO(p,q) \ltimes \mathbb{R}^{p,q}$, and $H \cap P$ is the stabiliser of an isotropic line in there, which is not perpendicular to the vector stabilised by $H$. A moment of thought shows that the isomorphism $H \cong SO(p,q) \ltimes \mathbb{R}^{p,q}$ can be chosen in such a way that $H \cap P = SO(p,q)$ viewed as a subgroup in the obvious way. Hence it is again clear that a Cartan geometry of type $(H, H \cap P)$ is a pseudo–Riemannian metric of signature $(p,q)$ together with a metric connection. In contrast to the situation in the proof of theorem 3.1 the Cartan curvature here simply agrees with torsion and curvature of that connection. Torsion freeness then implies that the connection is the Levi–Civita connection of the induced metric, and normality of the initial conformal Cartan connection shows that we actually get a Ricci flat metric in the conformal class.

Hence it remains to discuss the induced Cartan geometry on $M_2$. Here it is easiest to describe the Lie algebras $\mathfrak{h}$ and $\mathfrak{h} \cap \mathfrak{p}$ in a basis which starts with the distinguished isotropic vector $v$, next a perpendicular isotropic vector spanning the line stabilised by $P$ and then completing this appropriately to a basis. Then we get

$$\left\{ \begin{pmatrix} 0 & 0 & Z & z' & 0 \\ 0 & a & W & 0 & -z' \\ 0 & 0 & A & -\mathbb{I}W^t & -\mathbb{I}Z^t \\ 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} 0 & z & Z & z' & 0 \\ 0 & a & W & 0 & -z' \\ 0 & X & A & -\mathbb{I}W^t & -\mathbb{I}Z^t \\ 0 & 0 & -X^t\mathbb{I} & -a & -z \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Here $X, Z^t, W^t \in \mathbb{R}^{p-1,q-1}$, $a, z, z' \in \mathbb{R}$, and $A \in \mathfrak{so}(p-1, q-1)$ and $\mathbb{I} = \mathbb{I}_{p-1,q-1}$. Hence $\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{p})$, which models the tangent space, can be identified with $\mathbb{R} \oplus \mathbb{R}^{p-1,q-1}$, with the summands spanned by $z$ and $X$, respectively. The line spanned by $z$ is invariant under $\mathfrak{h}$ and will thus give rise to a natural line subbundle. Forming the quotient by this subspace one obtains $\mathbb{R}^{p-1,q-1}$ with the standard action of the conformal Lie algebra defined by $a$ and $A$. Hence locally $M_2$ fibres as claimed. $\square$

*Remark 3.5.* (1) The fact that the orbit $M_2$ fibres as claimed in the theorem is nicely visible on the homogeneous model. Here the orbit consists of isotropic lines contained in $v^{\perp}$ and different from the line spanned by $v$. Now the quotient $v^{\perp}/\mathbb{R}v$ inherits a natural inner product of signature $(p,q)$ and we can project our isotropic lines to isotropic lines in this quotient with one–dimensional fibres.

(2) It is worth noticing that the natural metrics over the open orbits which show up in the theorem are determined by the normal solution $\sigma \in \Gamma(\mathcal{E}[1])$ underlying the parallel tractor $s$ in a rather simple way. Recall that a conformal class of metrics can be viewed as a canonical section $\mathbf{g}$ of the bundle $S^2 T^*M \otimes \mathcal{E}[2]$. Hence outside the zero set of $\sigma$, one obtains a metric in

the conformal class as $\frac{1}{\sigma^2}\mathbf{g}$ and these are the Einstein metrics over the open curved orbits.

(3) If $h(s, s) \neq 0$, then it is shown in [Gov10] that locally around $M_0$, one actually obtains a Poincaré–Einstein metric, and any Poincaré–Einstein metric arises in this way.

(4) We want to point out here that both a parallel metric on a projective standard tractor bundle and a parallel section of a conformal standard tractor bundle give rise to Einstein metrics on open curved orbits and conformal structures on closed curved orbits, which are embedded hypersurfaces. In particular, if one considers either of the two cases on a compact manifold with boundary for which the closed curved orbit coincides with the boundary, one obtains a compactification of a (non–compact complete) Einstein manifold by adding a conformal structure at infinity. However, this leads to two different types of compactifications, see [ČGH10] for more details on this.

3.6. **A CR–analog.** We conclude this article by looking at a complex analog of almost Einstein scales in the realm of CR geometry. Since this case is significantly more complicated than the conformal one, we restrict to elementary aspects of the description here, and we will take this topic up in more detail elsewhere. We discuss this example on the one hand because it gives rise to a CR version of the Einstein condition, which is of intrinsic interest. On the other hand there are strong indications that it will lead to a notion of compactifying a non–compact complete Kähler–Einstein manifold by adding an infinity carrying a CR structure, which is different from the one discussed in Section 3.3.

We have already briefly discussed the description of CR structures as parabolic geometries in Section 3.3. The basic group here is $G = SU(p+1, q+1)$ and the parabolic subgroup $P \subset G$ is the stabiliser of an isotropic (complex) line in the standard representation $\mathbb{C}^{p+q+2}$. Regular normal parabolic geometries of this type turn out to be equivalent to partially integrable almost CR structures of hypersurface type, which are non–degenerate of signature $(p, q)$ together with the choice of a certain root of the canonical bundle, compare with [ČG10]. Forming the associated bundle corresponding to the standard representation, one obtains the *standard tractor bundle*. This is a complex vector bundle $\mathcal{T}$ of rank $p + q + 2$ endowed with a canonical Hermitian metric $h$ of signature $(p + 1, q + 1)$, a complex line–subbundle with isotropic fibres, and a canonical Hermitian connection. We want to study holonomy reductions determined by a parallel section of the standard tractor bundle. As in 3.5, for a parallel section $s$ of this bundle the function $h(s, s)$ is constant, and up to constant rescalings of $s$, the basic $G$–types are distinguished by the sign of $h(s, s)$. We will only analyse the case $h(s, s) < 0$ here, the case $h(s, s) > 0$ is closely parallel, and these are the cases related to compactifications as discussed above.

The homogeneous model $G/P$ is the space of isotropic lines in $\mathbb{C}^{p+q+2}$, and the parallel standard tractor is determined by a negative vector $v \in \mathbb{C}^{p+q+2}$. The stabiliser $H$ of $v$ in $G$ is isomorphic to $SU(p+1, q)$ via the action on the

orthocomplement $v^\perp$. Evidently, there are two $H$–orbits in $G/P$, consisting of the lines contained in $v^\perp$ and the lines transversal to $v^\perp$, respectively. The latter orbit is clearly open, while the former forms a smooth embedded submanifold of real codimension two if $q > 0$ and is empty if $q = 0$. The normal solution $\sigma$ corresponding to $v$ is obtained by interpreting the inner product with $v$ as a homogeneous function on the null cone in $\mathbb{C}^{p+q+2}$, and thus as a section of a (complex) density bundle on the space $G/P$ of isotropic lines. Consequently, the closed $H$–orbit coincides with the zero set $\mathcal{Z}(\sigma)$.

Via Theorem 2.6 this description readily carries over to a parallel standard tractor $s$ with $h(s,s) < 0$ on general curved geometries. The curved orbit decomposition has the form $M = M_+ \cup M_0$, where $M_+$ is open, and $M_0$ coincides with the zero set of the underlying normal solution and, if non–empty, consists of smoothly embedded submanifolds of real codimension two. Note that $M_0$ must be empty if $q = 0$.

Let us next describe the induced Cartan geometries. In the case of $M_0$, the distinguished isotropic line is contained in $v^\perp$, so we can simply identify $H \cap P$ with the stabiliser of a complex isotropic line in the standard representation $\mathbb{C}^{p+q+1}$ of $H \cong SU(p+1,q)$. It is straightforward to verify directly that this induced Cartan geometry is automatically regular, thus giving rise to a partially integrable almost CR–structure on $M_0$ of hypersurface type, which is non–degenerate of signature $(p, q-1)$.

In the case of $M_+$, we have to determine the stabiliser in $H$ of an isotropic line $\ell$ which is transversal to $v^\perp$. Now elementary linear algebra shows that there is a unique vector $w \in v^\perp$ such that $v + w \in \ell$. Clearly, any element of $H$ which stabilises $\ell$, also has to stabilise $w$, and the converse also holds. Since $v$ is negative and $v + w$ is null, $w$ must be positive, and we see that $H \cap P \cong SU(p,q)$. Passing to the Lie algebra level, we get a similar matrix presentation to that in Section 3.3:

$$H \cap P = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} ix & -Z^*\mathbb{I} \\ Z & A - \frac{ix}{p+q}\,\mathrm{id} \end{pmatrix} \right\} = H,$$

where $x \in \mathbb{R}$, $Z \in \mathbb{C}^{p+q}$, $A \in \mathfrak{su}(p,q)$, and $\mathbb{I} = \mathbb{I}_{p,q}$. It is easy to see that the component in $\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{p})$ determined by $Z$ corresponds the contact distribution $HM_+$ (equipped with the complex structure), while projecting to the component determined by $x$ gives rise to a distinguished contact form. This also determines a Hermitian metric on $HM_+$ which is extended to a Riemannian metric on $TM_+$. In addition, the induced Cartan geometry of type $(H, H \cap P)$ on $M_+$ determines a linear connection on $TM_+$ which is compatible with all these structures. However, the discussion of consequences of normality is much more complicated here than in the real case, in particular if the initial structure has torsion, and we will not go into this.

Compared to the real case, there is an entirely new feature here, however. Namely, the parallel section $s \in \Gamma(\mathcal{T})$ determines a parallel section of the bundle $\mathfrak{su}(\mathcal{T})$, which is given by the tracefree part of $\tilde{s} \mapsto h(\tilde{s}, s)Js$. Now $\mathfrak{su}(\mathcal{T})$ is the adjoint tractor bundle associated to the CR structure, so a parallel section of this bundle determines a normal infinitesimal automorphism

of the geometry. It is easy to see, that on $M_+$, this infinitesimal automorphism is nowhere vanishing, and it is even transversal to the contact distribution there. Hence the flow lines of this infinitesimal automorphism determine a foliation of $M_+$ with leaves of real dimension one, and one can form local spaces of leaves.

Now one can show that $M_0$ can be interpreted as a CR infinity for the local leaf spaces. Moreover, since the leaves of the foliation are transversal to the contact distribution, any tangent space of such a leaf space can be identified with the contact subspaces along the leaf. It turns out that one gets an almost complex structure on the leaf space. Again, all that can be dealt with by comparison to the homogeneous model using Theorem 2.6. Indeed, it turns out that on each local leaf space, one gets an induced Cartan geometry of type $(SU(p+1,q), S(U(1) \times U(p,q)))$. This can then be analysed in a similar way to the structure in Section 3.3. If the initial Cartan geometry is torsion free (i.e. if the initial structure is CR), then away from $M_0$ this is precisely the setting considered by [Lei07] and so it be expected that in fact this recovers the Kähler–Einstein metric found there. A detailed discussion of this example will entail considerably more work, and we expect this to yield an alternative notion of compactifying a Kähler–Einstein manifold by adding an infinity which carries a CR structure. It is anticipated that this will be a genuinely different compactification to that found in Section 3.3 (cf. the discussion of Klein-Einstein structures in [ČGH10]).

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