ESI The Erwin Schrödinger International Institute for Mathematical Physics
Boltzmanngasse 9
A-1090 Wien, Austria

On Topological Radon Transforms

Lars Ernström
Toru Ohmoto
Shoji Yokura

Vienna, Preprint ESI 231 (1995) June 19, 1995

ON TOPOLOGICAL RADON TRANSFORMATIONS

Lars Ernström, Toru Ohmoto and Shoji Yokura

0. Introduction

Recently many mathematicians have been working on Radon transformation (see [3,4,5,7,8] etc.). In [3] Brylinski deals with topological Radon transformations of constructible sheaves and in [7,8] Ernström deals with topological Radon transformations of constructible functions, both in the set-up of the following diagram or correspondence:

$$\begin{tikzcd} & I_k \arrow[dl, "p"'] \arrow[dr, "q"] & \\ P^N & & Gr_k(P^N) \end{tikzcd}$$

where $Gr_k(P^N)$ is the Grassmannian of k-dimensional planes of the projective space P^N, the point-k-plane incidence variety of $P^N \times Gr_k(P^N)$ is denoted I_k, and p and q are the restriction of the projections. In this paper we consider Radon transformations of constructible functions and of homology classes.

The Radon transformation treated in [7,8] is the homomorphism

$$\mathcal{F}(P^N) \to \mathcal{F}(Gr_k(P^N))$$

from the abelian group of constructible functions on P^N to that on $Gr_k(P^N)$, defined by the composite $q_* \circ p^*$ of the pull-back $p^* : \mathcal{F}(P^N) \to \mathcal{F}(I_k)$ and the pushforward $q_* : \mathcal{F}(I_k) \to \mathcal{F}(Gr_k(P^N))$. The functor $\mathcal{F}$, assigning to a variety X, the abelian group $\mathcal{F}(X)$ of constructible functions on X, is both covariant and contravariant. Namely, the functor $\mathcal{F}$ is covariant with respect to pushforward, and contravariant with respect to pull-back. The topological Radon transformation $q_* \circ p^*$ mixes both covariant and contravariant natures. In this paper we introduce a category of varieties, whose morphisms are isomorphism classes of pairs of maps $X \leftarrow M \to Y$, and we capture the topological Radon transformation as a covariant functor from this category to the category of abelian groups; i.e., we show the following :

Theorem A. *There is a category Div of compact complex algebraic varieties and there is a covariant functor $\mathcal{F}^{Rad} : Div \to Ab$ from the category Div to the category Ab of abelian groups such that*
(i) *for a variety X, $\mathcal{F}^{Rad}(X) = \mathcal{F}(X)$, is the abelian group of constructible functions on X,*

The third author is partially supported by Grant-in-Aid for Scientific Research (No.07640129), the Japanese Ministry of Education, Science and Culture.

(ii) *for a morphism* $f \in Hom_{Div}(X,Y)$, $\mathcal{F}^{Rad}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ *is the Radon transformation.*

Also, we introduce the notion of a homological Verdier-Radon transformation on smooth varieties, which is closely related to the topological Radon transformation via the Chern-Schwartz-MacPherson transformation C_* ([2,14]). In fact, we show the following result.

Theorem B. *For a divergent diagram* $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, *where* X, M *and* Y *are smooth and* $p : M \to X$ *is Euler, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "C_*"] \arrow[d, "\mathcal{F}^{Rad}(\alpha)"'] & H_*(X;\mathbb{Z}) \arrow[d, "H^{V-Rad}(\alpha)"] \\
\mathcal{F}(Y) \arrow[r, "C_*"'] & H_*(Y;\mathbb{Z})
\end{tikzcd}$$

Here $H^{V-Rad}(\alpha) : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ *is the homological Verdier-Radon transformation.*

We use Theorem B to generalize the Plücker formula in [7,8]. This formula can be interpreted as a formula expressing the $0-$dimensional component of the Chern-Mather class of the $k-$dual variety of tangent k-planes of a a variety X in P^N, denoted $X^{<k>}$ in $Gr_k(P^N)$, via the $0-$dimensional component of the Chern-Mather class of the variety X. Our generalization gives a formula expressing the total Chern-Mather class of the $k-$dual variety $X^{<k>}$ via the total Chern-Mather class of the source variety X.

Acknowledgement. Some of this work was done during the third author's stay at the Erwin Schrödinger Institute, Vienna, Austria. He would like to express his gratitude to the Institute, and in particular to Professor Peter W. Michor, for their hospitality.

1. The Divergent Category

In this section we introduce a category of algebraic varieties that will be used to generalize the Radon transformations of constructible functions in [7,8].

Definition (1.1). Given varieties X, Y and M, an orderered pair α of maps $p : M \to X$ and $q : M \to Y$

$$\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$$

is called *a divergent diagram* from X to Y. Sometimes the notation $\alpha = (p : M : q)$ will be used. If α is a divergent diagram from X to itself and $p = q$, then we call α *a symmetric divergent diagram.*

The term "divergent diagram" is used in the theory of dynamics (see [6]).

Remark (1.2). The reverse of the divergent diagram $\alpha := (p : M : q)$ is

$$-\alpha : Y \overset{q}{\leftarrow} M \overset{p}{\rightarrow} X.$$

The diagram $-\alpha := (q : M : p)$ is a divergent diagram from Y to X. It is different from the divergent diagram α.

Definition (1.3). Let $\alpha = (p : M : q)$ and $\alpha' = (p' : M' : q')$ be two divergent diagrams from X to Y. Then we say that the divergent diagram α is isomorphic to the divergent diagram α', denoted by $\alpha \sim \alpha'$, if there exists an isomorphism $h : M \to M'$ such that the following diagram commutes:

$$\begin{tikzcd}
 & M \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "h"] & \\
X & & Y \\
 & M' \arrow[ul, "p'"] \arrow[ur, "q'"'] &
\end{tikzcd}$$

i.e., $p = p' \circ h$ and $q = q' \circ h$.

Note that the relation $\sim$ is an equivalence relation. We denote the isomorphism class of a divergent diagram α by $[\alpha]$.

Definition (1.4). The composite $\beta \circ \alpha$ of two divergent diagrams $\alpha = (p : M : q) : X \leftarrow M \rightarrow Y$ and $\beta = (r : N : s) : Y \leftarrow N \rightarrow Z$ is defined using the following diagram:

$$\begin{tikzcd}
 & & M \times_Y N \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & & \\
 & M \arrow[dl, "p"'] \arrow[dr, "q"] & & N \arrow[dl, "r"'] \arrow[dr, "s"] & \\
X & & Y & & Z
\end{tikzcd}$$

where $M \times_Y N$ is the fiber product of $q : M \to Y$ and $r : N \to Y$, and pr_1 and pr_2 are the projections. Then the composite $\beta \circ \alpha$ is defined by:

$$\beta \circ \alpha := (p \circ pr_1 : M \times_Y N : s \circ pr_2)$$

Now we are ready to define the divergent category Div of compact complex algebraic varieties.

Proposition (1.5). *There is a category Div defined by the following data:*

(i) *the objects $Obj(Div)$ consist of all compact complex algebraic varieties,*

(ii) *for two objects X and Y, the morphisms $Hom_{Div}(X,Y)$ consist of the isomorphism classes of divergent diagrams from X to Y. For $[\alpha] \in Hom_{Div}(X,Y)$ and $[\beta] \in Hom_{Div}(Y,Z)$; we define the composite $[\beta] \circ [\alpha]$ by*

$$[\beta] \circ [\alpha] := [\beta \circ \alpha] \in Hom_{Div}(X,Z).$$

We shall call this category the divergent category of algebraic varieties.

Proof. We have to show the following three things: (the proofs are straightforward using the properties of a fiber product and therefore omitted).
(1)The composite $[\beta] \circ [\alpha]$ is well-defined; i.e., it is independent of the choice of the representatives in the isomorphism classes $[\alpha]$ and $[\beta]$; i.e., $\alpha \sim \alpha'$ and $\beta \sim \beta'$ implies that $\beta \circ \alpha \sim \beta' \circ \alpha'$.
(2) The composite operation is associative, i.e., $[\gamma] \circ ([\beta] \circ [\alpha]) = ([\gamma] \circ [\beta]) \circ [\alpha]$ for any divergent diagrams: $\alpha : X \xleftarrow{p} M \xrightarrow{q} Y$, $\beta : Y \xleftarrow{r} N \xrightarrow{s} Z$ and $\gamma : Z \xleftarrow{t} L \xrightarrow{u} W$.
(3) The composition law has units: let $1_X := [e_X]$ be the isomorphism class of the "identity" divergent diagram $e_X : X \xleftarrow{id} X \xrightarrow{id} X$. Then, for any morphism $[\alpha] \in \mathrm{Hom}_{Div}(X, Y)$, and for any morphism $[\beta] \in \mathrm{Hom}_{Div}(W, X)$, $[\alpha] \circ 1_X = [\alpha]$ and $1_X \circ [\beta] = [\beta]$, i.e., $\alpha \circ 1_X \sim \alpha$ and $1_X \circ \beta \sim \beta$. □

Remark (1.6). Correspondingly, for categories of topological spaces, real analytic varieties etc., we can define divergent categories, provided that the fiber product is closed in the category; i.e., if three objects X, Y and Z belong to the category then the fiber product $X \times_Z Y$ also belongs to the category.

Definition (1.7). Let $f : X \to Y$ be a morphism and let Γ_f be the graph of the morphism f. Then the divergent diagram $X \xleftarrow{p} \Gamma_f \xrightarrow{q} Y$, where p and q are the restrictions of the projections, is called the *graph divergent diagram of f*, and is denoted by γ_f.

Observation (1.8). We note that the graph divergent diagram γ_f of f is isomorphic to the divergent diagram $|f> := (id_X : X : f) : X \xleftarrow{id_X} X \xrightarrow{f} Y$.

Here we show that the usual category Var, of complex algebraic varieties, can be embedded into the divergent category Div.

Theorem (1.9). (The graph functor Γ) *Let $\Gamma : Var \to Div$ be the map defined as follows:*
(i) *for an object $X \in Obj(Var)$, define $\Gamma(X) := X$, and*
(ii) *for a morphism $f \in Hom_{Var}(X, Y)$, define $\Gamma(f)$ by*

$$\Gamma(f) := [\gamma_f] \in Hom_{Div}(X, Y),$$

where $[\gamma_f]$ is the isomorphism class of the graph divergent diagram γ_f.
Then the map $\Gamma : Var \to Div$ is a faithful covariant functor.

Proof. (1) To prove covariance we have to show that $\Gamma(f \circ g) = \Gamma(f) \circ \Gamma(g)$, given $g \in \mathrm{Hom}_{Var}(X, Y)$ and $f \in \mathrm{Hom}_{Var}(Y, Z)$. By definition, $\Gamma(f \circ g) = [\gamma_{f \circ g}]$ and $\Gamma(f) \circ \Gamma(g) = [\gamma_f] \circ [\gamma_g] = [\gamma_f \circ \gamma_g]$, hence we have to show that $\gamma_{f \circ g} \sim \gamma_f \circ \gamma_g$. We can show this directly, but using Observation (1.8) above, it suffices to show that

$|f\circ g>\sim |f>\circ |g>$. By the definition of $|f>\circ|g>$ we have the following diagram:

$$\begin{tikzcd}
& & X\times_Y Y \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & & \\
& X \arrow[dl, "id"'] \arrow[dr, "g"] & & Y \arrow[dl, "id"'] \arrow[dr, "f"] & \\
X & & Y & & Z
\end{tikzcd}$$

We note that $X\times_Y Y$ is the graph Γ_g of the map $g : X\to Y$. The map $pr_1 : \Gamma_g\to X$ is an isomorphism and the following diagram is clearly commutative:

$$\begin{tikzcd}
& \Gamma_g \arrow[dl, "pr_1"'] \arrow[dd, "pr_1"] \arrow[dr, "f\circ pr_2"] & \\
X & & Z \\
& X \arrow[ul, "id"] \arrow[ur, "f\circ g"'] &
\end{tikzcd}$$

Thus we have the equivalence of $|f\circ g>\sim |f>\circ|g>$.
(2) To prove faithfullness we have to prove, given $f, g\in \mathrm{Hom}_{Var}(X,Y)$, that $\Gamma(f) = \Gamma(g)$ implies that $f=g$. By definition $\Gamma(f)=\Gamma(g)$ means that $[\gamma_f]=[\gamma_g]$,which means that $|f>\sim |g>$ via Observation (1.8). This means that there is an isomorphism $h : X\to X$ such that the folowing diagram commutes, i.e., that $id_X = id_X\circ h$ and $f = g\circ h$.

$$\begin{tikzcd}
& X \arrow[dl, "id_X"'] \arrow[dd, "h"] \arrow[dr, "f"] & \\
X & & Y \\
& X \arrow[ul, "id_X"] \arrow[ur, "g"'] &
\end{tikzcd}$$

This means that $h = id_X$ and hence $f = g$. □

Similarly we can show the following:

Theorem (1.10). (The reverse graph functor $-\Gamma$) *Let $-\Gamma : Var\to Div$ be a map defined as follows:*

(i) *for each object $X\in Obj(Var)$, $-\Gamma(X) = X$, and*

(ii) *for a morphism $f\in Hom_{Var}(X,Y)$, $-\Gamma(f)$ is defined by*

$$-\Gamma(f) := [-\gamma_f]\in Hom_{Var}(Y,X)$$

where $-\gamma_f$ is the reverse of the graph divergent diagram γ_f.

Then the map $-\Gamma : Var \to Div$ *is a faithful contravariant functor.*

Remark (1.11). Theorem (1.10) can be shown similarly as in the proof of Theorem (1.9), but we can easily see this because the map $- : Div \to Div$, taking the reverse, is a faithful contravariant functor.

Remark (1.12). Any divergent diagram $a = (f : M : g) : X \leftarrow M \to Y$ can be expressed as the composite $|g> \circ <f|$ of the divergent diagrams $|g>$ and $<f|$.

2. THE RADON FUNCTOR

We shall here define a covariant functor from the divergent category Div to the category Ab of abelian groups. We will denote the covariant functor of constructible functions with push-forward by $\mathcal{F}_*$, to emphasize the covariance. Given a morphism $f : X \to Y$ the pushforward is denoted $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$. With the pullback $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ the constructible functions become a contravariant functor, which will be denoted by $\mathcal{F}^*$, to emphasize the contravariance.

In this section, motivated by the Radon transformations of constructible sheaves [3], and constructible functions [7,8], we introduce the Radon functor $\mathcal{F}^{Rad} : Div \to Ab$. If the Radon functor $\mathcal{F}^{Rad}$ is restricted to the faithful subcategories $\Gamma(Var)$ and $-\Gamma(Var)$, it specializes to the covariant functor $\mathcal{F}_* : Var \to Ab$ and the contravariant functor $\mathcal{F}^* : Var \to Ab$, respectively.

Definition (2.1). (**Radon transformations**) For a divergent diagram

$$\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$$

we define the homomorphism $\mathcal{F}^{Rad}(\alpha) : \mathcal{F}(X) \to \mathcal{F}(Y)$ to be the composite $q_* \circ p^*$, where
(i) $p^* : \mathcal{F}(X) \to \mathcal{F}(M)$ is the pull-back, i.e., $p^*(\lambda) := \lambda \circ p$,
(ii) $q_* : \mathcal{F}(M) \to \mathcal{F}(X)$ is the pushforward, which is defined by

$$q_*(\lambda)(y) := \sum_W m_W \chi(q^{-1}(y) \cap W),$$

where $\lambda = \sum_W m_W 1_W$ and each W is a subvariety of M.
We call this homomorphism $\mathcal{F}^{Rad}(\alpha)$ *the Radon transformation associated to* α.
The reverse Radon transformation associated to α, is defined to be the Radon transformation $\mathcal{F}^{Rad}(-\alpha)$, where $-\alpha : Y \overset{q}{\leftarrow} M \overset{p}{\rightarrow} X$.

It is easy to see the following:

Lemma (2.2). *Let* $\Sigma : X \overset{p}{\leftarrow} M \overset{p}{\rightarrow} X$ *be a symmetric divergent diagram. Let* W *be a subvariety of* X. *Then we have*
(i)

$$\mathcal{F}^{Rad}(\Sigma)(1_W)(y) = \begin{cases} \chi(p^{-1}(y)) & \textit{if } y \in W \\ 0 & \textit{if } y \notin W \end{cases},$$

in particular

(ii) *if p is surjective and the Euler-Poincaré characteristic $\chi(p^{-1}(y))$ of each fiber $p^{-1}(y)$ is equal to 1, then $\mathcal{F}^{Rad}(\Sigma) = 1_{\mathcal{F}(X)}$ the identity homomorphism on the abelian group of constructible functions on X.*

Theorem (2.3). *The map $\mathcal{F}^{Rad} : Div \to Ab$ defined below, is a covariant functor.*
(i) For each object $X \in Obj(Div)$ set $\mathcal{F}^{Rad}(X) := \mathcal{F}(X)$, the abelian group of constructible functions on X,
(ii) For a morphism $[\alpha] \in Hom_{Div}(X, Y)$, set $\mathcal{F}^{Rad}([\alpha]) := \mathcal{F}^{Rad}(\alpha)$, the Radon transformation.
We call the functor $\mathcal{F}^{Rad} : Div \to Ab$ the Radon functor.

Proof. We have to show: (1) that $\mathcal{F}^{Rad}([\alpha])$ is well-defined, and (2) the functoriality of the map $\mathcal{F}^{Rad} : Div \to Ab$.

(1): Let $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\to} Y$ be isomorphic to $\alpha' : X \overset{p'}{\leftarrow} M' \overset{q'}{\to} Y$, i.e., there is an isomorphism $h : M \to M'$ such that $p = p' \circ h$ and $q = q' \circ h$. Then we have

$$\begin{aligned} \mathcal{F}^{Rad}(\alpha) &= q_* p^* \\ &= (q' \circ h)_* (p' \circ h)^* \\ &= (q')_* h_* h^* (p')^* \\ &= (q')_* (p')^* \text{ (since } h_* h^* = 1_{\mathcal{F}(M')} \text{ by Lemma (2.2) (ii))} \\ &= \mathcal{F}^{Rad}(\alpha') \end{aligned}$$

Thus the definition of $\mathcal{F}^{Rad}([\alpha]) := \mathcal{F}^{Rad}(\alpha)$ is independent of the choice of the representative from the isomorphism class $[\alpha]$, i.e., $\mathcal{F}^{Rad}([\alpha])$ is well-defined.

(2): Let $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\to} Y$ and $\beta : Y \overset{r}{\leftarrow} N \overset{s}{\to} Z$ be two divergent diagrams. We want to show that $\mathcal{F}^{Rad}([\beta] \circ [\alpha]) = \mathcal{F}^{Rad}([\beta]) \circ \mathcal{F}^{Rad}([\alpha])$. It follows from the definition of composition in Div that $\mathcal{F}^{Rad}([\beta] \circ [\alpha]) = \mathcal{F}^{Rad}([\beta \circ \alpha])$. By (1), it suffices to show that $\mathcal{F}^{Rad}(\beta \circ \alpha) = \mathcal{F}^{Rad}(\beta) \circ \mathcal{F}^{Rad}(\alpha)$. Recalling the definition of the composite $\beta \circ \alpha$ (using the same symbols as in Definition (1.4)) and the definition of $\mathcal{F}^{Rad}(\beta \circ \alpha)$, we have that

$$\begin{aligned} \mathcal{F}^{Rad}(\beta \circ \alpha) &= (s \circ pr_2)_* (p \circ pr_1)^* \\ &= s_* (pr_2)_* (pr_1)^* p^*. \end{aligned}$$

Here we recall the following lemma:

Lemma (2.3.1). ([8, Proposition 3.5]) *If the following square is a fiber square (or a Cartesian diagram), i.e., W is the fiber product of $f : Y \to X$ and $p : Z \to X$ and q and g are the projections,*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "q"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

then it holds that $p^* f_* = g_* q^*$.

Proof. Given $\lambda = 1_V$ where $V \to Y$ is a closed embedding, it suffices to prove that $p^* f_*(\lambda) = g_* q^*(\lambda)$. Fix $z \in Z$ and denote $x = p(z)$. Then by definition we have

$$(p^* f_*(\lambda))(z) = \chi(f^{-1}(x) \cap V) \quad \text{and} \quad (g_* q^*(\lambda))(z) = \chi(g^{-1}(z) \cap q^{-1}(V)).$$

Furthermore, since W is the fiber product $Y \times_X Z$, we have that $g^{-1}(z) = f^{-1}(x) \times z$. Then by a simple computation we can see that $g^{-1}(z) \cap q^{-1}(V) = (f^{-1}(x) \cap V) \times z$. Thus we have that $(p^* f_*(\lambda))(z) = (g_* q^*(\lambda))(z)$, i.e., $p^* f_*(\lambda) = g_* q^*(\lambda)$ since z is an arbitrary point of Z . □

We apply this lemma to the following fiber sqare,

$$\begin{tikzcd}
& M \times_Y N \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
M \arrow[dr, "q"'] & & N \arrow[dl, "r"'] \\
& Y &
\end{tikzcd}$$

we have that $(pr_2)_*(pr_1)^* = r^* q_*$. Thus we get that

$$\begin{aligned}
\mathcal{F}^{Rad}(\beta \circ \alpha) &= (s \circ pr_2)_*(p \circ pr_1)^* \\
&= s_* {pr_2}_* {pr_1}_* p^* \\
&= s_* r^* q_* p^* \\
&= \mathcal{F}^{Rad}(\beta) \circ \mathcal{F}^{Rad}(\alpha). \qquad \Box
\end{aligned}$$

Corollary (2.4). *The Radon functor* $\mathcal{F}^{Rad}$ *is related to the push-forward* $\mathcal{F}_*$ *and the pull-back* $\mathcal{F}^*$ *via the graph functor* Γ*:*
(i) $\mathcal{F}^{Rad} \circ \Gamma = \mathcal{F}_* : Var \to Ab$.
(ii) $\mathcal{F}^{Rad} \circ (-\Gamma) = \mathcal{F}^* : Var \to Ab$.

Next we give some specific examples of divergent diagrams.

Example (2.5).

(1) A divergent diagram $\Pi : X \overset{pr_1}{\leftarrow} X \times Y \overset{pr_2}{\to} Y$ is called *a product divergent diagram.* For this product divergent diagram $\mathcal{F}^{Rad}(\Pi)(1_W) = \chi(W) 1_Y$. Thus, $\mathcal{F}^{Rad}(\Pi)(\lambda) = \chi(\lambda) 1_Y$, where $\chi(\lambda) = \sum_W n_W \chi(W)$ if $\lambda = \sum_W n_W 1_W$.

(2) *A Grassmannian divergent diagram* is a diagram $\Gamma_k : P^N \overset{p}{\leftarrow} I_k \overset{q}{\to} Gr_k(P^N)$, where $Gr_k(P^N)$ is the Grassmanian of k-dimensional planes of the projective space P^N,

$$I_k := \{(x, l) \in P^N \times Gr_k(P^N) \mid x \in l\}$$

is the k-th incidence variety and p and q are the restrictions of the projections. A specific description of the topological Radon transformation $\mathcal{F}^{Rad}(G)$ of the Grassmannian divergent diagrams is one of the main results of Ernström [7,8]. We will get back to this topic in §4.

(3) Let X be a variety and let $\sigma : E \to F$ be a homomorphism of vector bundles E and F of ranks e and f over X. Let k be any integer such that $0 \leq k \leq \min(e, f)$. Then the degeneracy locus $D_k(\sigma)$ of the bundle homomorphism σ is defined to be (see Fulton's book [11]):

$$D_k(\sigma) := \{x \in X | \text{rank}\,(\sigma(x)) \leq k\}.$$

Let $\text{Hom}(E, F)$ be the vector space of all the homomorphisms from E to F. Since for any non-zero scalar $c \in \mathbb{C}$, the complex numbers, $D_k(\sigma) = D_k(c\sigma)$, we consider the projective space $P(\text{Hom}(E, F))$. Then we consider the following incidence variety:

$$I_k := \{(x, \sigma) \in X \times P(\text{Hom}(E, F)) \mid \text{rank}\,(\sigma(x)) \leq k\}.$$

A diagram of the following type is called a *degeneracy divergent diagram*:

$$\Delta_k(E, F) : X \overset{p}{\leftarrow} I_k \overset{q}{\rightarrow} P(Hom(E, F)).$$

It would be interesting to find a specific description of the topological Radon transformation $\mathcal{F}^{Rad}(\Delta_k(E, F))$. By the definition we see that

$$(\mathcal{F}^{Rad}(\Delta_k(E, F))(1_W))(\sigma) = \chi(W \cap D_r(\sigma)).$$

Thus the problem is equivalent to finding the Euler characteristic of the degeneracy loci of a homomorphism of vector bundles.

Parunsiński and Pragacz [15] have proved a formula for the degeneracy loci $D_r(\varphi)$, under the assumption that φ is a k-general homomorphism of vector bundles. The notion of k-generality may be defined by requiring that the degeneracy locus has the expected dimension and that the singularities of $D_k(\varphi)$ is contained in $D_{k-1}(\varphi)$.

The formula for the degeneracy locus of an k-general homomorphisms of vector bundles is based on the theory of polynomials supported on the rth degeneracy locus, developed by Pragacz. These polynomials are homology classes, defined by polynomials in the Chern classes of E, F and X. However, to find an explicit formula for the Radon transformation of a degeneracy divergent diagram, it is necessary to find the Euler characteristic of the degeneracy locus of homomorphism that are possibly not k-general.

Remark (2.6). For a divergent diagram $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$ the composite $-\alpha \circ \alpha$ of α and its reverse $-\alpha$ is a symmetric divergent diagram from X to itself, which is in general far from the "identity" divergent diagram $e_X : X \overset{id}{\leftarrow} X \overset{id}{\rightarrow} X$ or any symmetric divergent diagram isomorphic to e_X . However, it is reasonable to pose the question of to what extent they are different from each other, or to try to give a specific description of the Radon transformation $\mathcal{F}^{Rad}(-\alpha \circ \alpha)$, i.e., the composite $\mathcal{F}^{Rad}(-\alpha) \circ \mathcal{F}^{Rad}(\alpha)$. In the case of the above Grassmannian divergent diagram

$$\Gamma_k : P^N \overset{p}{\leftarrow} I_k \overset{q}{\rightarrow} Gr_k(P^N),$$

Ernström [8, Proposition 3.6] solved this problem: For a constructible function $\lambda \in \mathcal{F}(P^N)$,

$$\mathcal{F}^{Rad}(-\Gamma_k \circ \Gamma_k)(\lambda) = (\binom{N}{k} - \binom{N-1}{k-1})\lambda + \binom{N-1}{k-1}\chi(\lambda)1_{P^N}.$$

In particular when $k = N-1$ we have

$$\mathcal{F}^{Rad}(-\Gamma_{N-1} \circ \Gamma_{N-1})(\lambda) = \lambda + (N-1)\chi(\lambda)1_{P^N}.$$

This was previously proved by Viro [17]. The above formulas are closely related to the problem of finding inversion formulas for Radon transforms (see Schapira [16]).

Remark (2.7). There is a dual notion of divergent diagram, which we denote convergent diagram. For a convergent diagram,

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

one considers the homomorphism $q^* \circ p_* : \mathcal{F}(X) \to \mathcal{F}(Y)$, which is also called a topological Radon transformation. The Radon transformation $q^* \circ p_*$ is equal to the Radon tramsformation associated to the divergent diagram $X \overset{pr_1}{\leftarrow} X \times_M Y \overset{pr_2}{\to} Y$ because of the following fiber square

$$\begin{tikzcd} & X \times_M Y \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

and by Lemma (2.3.1) : $q^*p_* = (pr_2)_*(pr_1)^* : \mathcal{F}(X) \to \mathcal{F}(Y)$. Thus, as far as we are concerned with Radon transformations of constructible functions, it suffices to consider divergent diagrams.

3. Verdier-type Riemann-Roch and homological Verdier-Radon transformations

The Chern-Schwartz-MacPherson transformation C_* is a transformation from the functor of constructible functions F to homology groups H_* with coefficients in $\mathbb{Z}$. Using the Radon transformations of constructible functions and the Chern-Schwartz-MacPherson transformation we will construct a homological Radon transformation.

Definition (3.1). A divergent diagram $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\to} Y$ is said to be *smooth* if X, M and Y are smooth and α is said to be *Euler* if furthermore the morphism $p : M \to X$ is Euler; i.e., the constructible function 1_M on M satisfies the local Euler condition (see [12] page 77). The latter means that 1_M is an element of the bivariant group of constructible functions $F(M \to X)$.

Definition (3.2). Given a smooth divergent diagram $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, the composite

$$H^{Rad}(\alpha) := q_* \circ p^! : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$$

is called *the homological Radon transformation.* Here $q_* : H_*(M;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ is the usual pushforward and $p^! : H_*(X;\mathbb{Z}) \to H_*(M;\mathbb{Z})$ is the usual Gysin homomorphism defined by $p^! = \mathcal{D}_M \cdot p^* \cdot \mathcal{D}_X^{-1}$, where $\mathcal{D}_Z : H^*(Z;\mathbb{Z}) \to H_*(Z;\mathbb{Z})$ is the Poincaré duality isomorphism for a smooth variety Z.

Remark (3.3). Our homological Radon transformation is a generalization of Friedlander-Mazur's "homological correspondence homomorphism"

$$\varphi_V : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$$

in the case when V is a smooth subvariety of the Cartesian product $X \times Y$ of smooth varieties X and Y (see [10, Chapter 4]).

We will show how these three transformations $\mathcal{F}^{Rad}, H^{Rad}$ and C_* are related. A naive guess would be that given a smooth divergent diagram $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "C_*"] \arrow[d, "\mathcal{F}^{Rad}(\alpha)"'] & H_*(X;\mathbb{Z}) \arrow[d, "H^{Rad}(\alpha)"] \\
\mathcal{F}(Y) \arrow[r, "C_*"'] & H_*(Y;\mathbb{Z})
\end{tikzcd}$$

But we can see that this is not always true; we consider the following case: let $X = pt$ be a one-point-variety, and let $M = Y$ be any compact complex smooth variety of dim > 0. Consider the following divergent diagram:

$$\delta : pt \overset{p}{\leftarrow} Y \overset{id_Y}{\rightarrow} Y.$$

Then we have that $(C_* \circ \mathcal{F}^{Rad}(\delta))(1_{pt}) = c(T_Y) \cap [Y]$ and $(H^{Rad}(\delta) \circ C_*)(1_{pt}) = [Y]$. Thus, in general, we have $C_* \circ \mathcal{F}^{Rad}(\delta) \neq H^{Rad}(\delta) \circ C_*$.

However, the drawback of the above (non-commutative) diagram can be remedied by the following *Verdier-type Riemann-Roch* theorem.

Theorem (3.4). (Verdier-type Riemann-Roch) *Let $f : M \to X$ be an Euler morphism of smooth varietes M and X. Then the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "C_*"] \arrow[d, "f^*"'] & H_*(X;\mathbb{Z}) \arrow[d, "c(T_f)\cap f^!"] \\
\mathcal{F}(M) \arrow[r, "C_*"'] & H_*(M;\mathbb{Z})
\end{tikzcd}$$

Here $c(T_f)$ is the total Chern class of the virtual relative tangent bundle $T_f := TM - f^*TX$.

Definition (3.5). Given a morphism $f : X \to Y$ of compact smooth manifolds X and Y, the homomorphism

$$c(T_f) \cap f^! : H_*(Y; Z) \to H_*(X; \mathbb{Z})$$

is called *the Verdier-Gysin homomorphism* and denoted by $f^{!!}$ with a double shriek.

Remark (3.6). It is straightforward to see that just like the Gysin homomorphism the above Verdier-Gysin homomorphism is also functorial, i.e., for morphisms $f : X \to Y$ and $g : Y \to Z$,

$$(gf)^{!!} = f^{!!} g^{!!}.$$

For this we need to observe that for a cohomology class β of Y,

$$f^!(\beta \cap g^!) = f^*\beta \cap (gf)^!.$$

This functoriality and Theorem (3.4) imply that C_* is a natural transformation from the contravariant functor F^* to the "contravariant" homology functor with the Verdier-Gysin homomorphism, when it is restricted to the category of Euler morphisms of smooth varieties.

To prove Theorem (3.4), we use Fulton-MacPherson's bivariant theory [12]. For details of the bivariant theory see [12]. Here we cite the necessary parts of the theory for the proof of the above theorem. The key result is the following bivariant version of Chern-Schwartz-MacPherson transformation C_*, the existence of which was conjectured by Fulton and MacPherson [12, §10.4 Chern classes], and proved by J.-P. Brasselet [1]:

Theorem (3.7). (Bivariant Chern classes) (Brasselet [1])

There exists a Grothendieck transformation $\gamma : \mathbb{F} \to \mathbf{H}$ *from the bivariant theory* $\mathbb{F}$ *of constructible functions to the bivariant homology theory* $\mathbf{H}$ *such that if* X *is smooth, then* $\gamma(1_X) = c(TX).[X]$

$$\begin{tikzcd}
& X \arrow[dr, "{[X]}"] & \\
X \arrow[ur, "c(T_X)", "\mathrm{id}"'] \arrow[rr, "\gamma(1_X)"'] & & pt
\end{tikzcd}$$

where $c(TX)$ *is the usual Chern cohomology class of the tangent bundle* TX *and* $[X]$ *is the fundamental class of* X.

First, from this Brasselet's theorem we can get the following:

Proposition (3.8). *Let* M *and* X *be smooth compact complex algebraic varieties and let* $f : M \to X$ *be an Euler morphism. Then the following equality holds:* $\gamma(1_f) = c(T_f).[f]$,

$$\begin{tikzcd}
& M \arrow[dr, "{[f]}", "f"'] & \\
M \arrow[ur, "c(T_f)", "\mathrm{id}"'] \arrow[rr, "f", "\gamma(1_f)"'] & & X
\end{tikzcd}$$

where 1_f *is the canonical orientation for the bivariant constructible function group* $\mathbb{F}(M \to X)$, *the class* $[f]$ *is the canonical orientation for the bivariant homology group* $\mathbf{H}(M \to X)$ *and* $T_f := TM - f^*TX$ *is the virtual relative tangent bundle of the morphism* f.

Proof. (cf. [12, §6, Proposition 6A]) We consider the following diagram, which is commutative except possibly at the bottom triangle, whose commutativity is to be proved.

$$\begin{tikzcd}
M \arrow[rr, "{[f]}"] \arrow[dr, "{[M]}"] \arrow[ddd, swap, "f^*(c(T_X)^{-1})"] & & X \arrow[dl, swap, "{[X]}"] \arrow[ddd, "c(T_X)^{-1}"] \\
& \mathrm{pt} & \\
& M \arrow[uul, "c(T_M)"] \arrow[u, "\gamma(1_M)"] \arrow[dl, "c(T_f)"] \arrow[dr, "\gamma(1_f)"] & \\
M \arrow[rr, swap, "{[f]}"] & & X \arrow[uul, swap, "\gamma(1_X)"]
\end{tikzcd}$$

Then the proof goes as follows:

$$\begin{aligned}
c(T_f).[f].\gamma(1_X) &= c(TM).f^*c(TX)^{-1}.[f].\gamma(1_X) (\text{since } c(T_f) = c(TM).f^*c(TX)^{-1}) \\
&= c(TM).[f].c(TX)^{-1}.\gamma(1_X) \quad (\text{since } f^*c(TX)^{-1}.[f] = [f].c(TX)^{-1}) \\
&= c(TM).[f].[X] \quad (\text{since } \gamma(1_X) = c(TX).[X]) \\
&= c(TM).[M] \quad (\text{since } [f].[X] = [M]\ ) \\
&= \gamma(1_M) \\
&= \gamma(1_f).\gamma(1_X) \quad (\text{since } 1_M = 1_f.1_X\ )
\end{aligned}$$

Thus we get that $c(T_f).[f].\gamma(1_X) = \gamma(1_f).\gamma(1_X)$. Now, since the operation $(\ ).\gamma(1_X)$ is an isomorphism, we can conclude that $c(T_f).[f] = \gamma(1_f)$. □

Now, it is easy to see

Proof of Theorem (3.4). (cf. [12, §2.7 Grothendieck transformations]) By a general theory of Grothendieck transformations, the Grothendieck transformation $\gamma : \mathbb{F} \to \mathbf{H}$ induces the following commutative diagram (see [12, §2.7 , p.30]):

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "C_*"] \arrow[d, swap, "(1_f)^*"] & H_*(X;\mathbb{Z}) \arrow[d, "(\gamma(1_f))^*"] \\
\mathcal{F}(M) \arrow[r, swap, "C_*"] & H_*(M;\mathbb{Z})
\end{tikzcd}$$

Here $(1_f)^*$ and $(\gamma(1_f))^*$ are the Gysin homomorphisms (defined in [12, §2.5]) for the orientations 1_f and $\gamma(1_f)$. Then the Gysin homomorphism $(1_f)^*$ is the pull-back of constructible functions $f^* : \mathcal{F}(M) \to \mathcal{F}(X)$ and the Gysin homomorphism $(\gamma(1_f)^* = (c(T_f).[f])^*$ is nothing but the Verdier-Gysin homomorphism $c(T_f)f^! : H_*(X;\mathbb{Z}) \to H_*(M;\mathbb{Z})$. □

Definition (3.9). (A homological Verdier-Radon transformation) For a smooth divergent diagram $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, the composite

$$H^{V-Rad}(\alpha) := q_* \circ p^{!!} : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$$

shall be called *the homological Verdier-Radon transformation.*

Corollary (3.10). *For an Euler divergent diagram* $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$ *the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "C_*"] \arrow[d, "\mathcal{F}^{Rad}(\alpha)"'] & H_*(X;\mathbb{Z}) \arrow[d, "H^{V-Rad}(\alpha)"] \\
\mathcal{F}(Y) \arrow[r, "C_*"'] & H_*(Y;\mathbb{Z})
\end{tikzcd}$$

Remark (3.11). We will here define a compatible pair of Radon transforms of constructible functions and homology groups. The condition on the divergent diagram for which this is possible will be weaker than the Euler condition above. Suppose that $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$ is a divergent diagram, where p is not necessarily Euler. If there is a nonzero element b_p in the bivariant group of constructible functions $\mathbb{F}(M \overset{p}{\rightarrow} X)$ then there is a nontrivial bivariant transformation

$$\overline{b}_p : \mathcal{F}(X) \to \mathcal{F}(M)$$

induced by b_p. We define a Radon transform of constructible functions as the composition:

$$\mathcal{F}^{Rad}_{b_p}([\alpha]) : \mathcal{F}(X) \xrightarrow{\overline{b}_p} \mathcal{F}(M) \xrightarrow{q_*} \mathcal{F}(Y)$$

This Radon transform is of course dependent on the element b_p. However, for a certain class of morphisms, named "Sans éclatement en codimension 0" in French by Henry, Merle and Sabbah [13], there is a unique contructible function b_p in $\mathbb{F}(M \overset{p}{\rightarrow} X)$ such that there is an Zariski open dense set of M over which b_p is equal to one. If p is Euler then $b_p = 1_p$ and $\mathcal{F}^{Rad}_{b_p} = \mathcal{F}^{Rad}$.

We say that a morphism $M \overset{p}{\rightarrow} X$ is *micro locally equidimensional* if it satisfies the condition of "Sans éclatement en codimension 0", i.e. the fibers of the relative conormal morphism $C_p(M) \to X$ are all of the same dimension.

Next, we consider the Grothendieck transformation of bivariant theories γ introduced above. The image $\gamma(b_p)$ in $\mathbf{H}(M \overset{p}{\rightarrow} X)$ induces a homomorphism

$$\overline{\gamma(b_p)} : H_*(X;\mathbb{Z}) \to H_*(M;\mathbb{Z})$$

compatible with $\overline{b}_p$. Define a Radon transform of homology groups as the composition:

$$H^{Rad}_{b_p}([\alpha]) : H_*(X) \xrightarrow{\overline{\gamma(b_p)}} H_*(M) \xrightarrow{q_*} H_*(Y)$$

The following diagram is commutative because of the axioms of Grothendieck transformations [12, §2]:

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "C_*"] \arrow[d, "\overline{b_p}"'] & H_*(X) \arrow[d, "\overline{\gamma(b_p)}"] \\
\mathcal{F}(M) \arrow[r, "C_*"] \arrow[d, "q_*"'] & H_*(M) \arrow[d, "q_*"] \\
\mathcal{F}(Y) \arrow[r, "C_*"] & H_*(Y)
\end{tikzcd}$$

Therefore the two Radon transformations $\mathcal{F}^{Rad}_{b_p}$ and $H^{Rad}_{b_p}$ are compatible via C_*.

4. A formula for the Grassmannian homological Radon transform of the Chern-Mather class.

As mentioned in the introduction, Ernström [7,8] studied the topological Radon transformation of the Grassmannian divergent diagram $\Gamma_k : P^N \xleftarrow{p} I_k \xrightarrow{q} Gr_k(P^N)$. He gave an explicit description of the image of the local Euler obstruction $Eu_X \in \mathcal{F}(P^N)$ under the topological Radon transformation $\mathcal{F}^{Rad}(\Gamma_k)$, where X is a reduced subvariety of P^N. The k-dual variety $X^{<k>}$ is defined as the closure in $Gr_k(P^N)$ of the following set

$$\{L \in Gr_k(P^N) | \text{ there is a point } x \in X_{smooth} \cap L, \\ \text{and a hyperplane } H \supset TX_x,\ H \supset L\ \}.$$

Let $e^{<k>}$ be the generic value of $\mathcal{F}^{Rad}(Eu_X)$, i.e., $e^{<k>} = \chi(X \cap L)$ for a generic k-plane L. Set $n = \dim X$ and $n^{<k>} = \dim(X^{<k>})$. The main result is the following formula.

Theorem (4.1). ([8, Theorem (3.2)])

$$\mathcal{F}^{Rad}(Eu_X) = e^{<k>} 1_{Gr_k(P^N)} + (-1)^{n+k(N-k)-n^{<k>}} Eu_{X^{<k>}}$$

In the special case when $k = N-1$ the above Ernström's formula gives an affirmative solution to a conjecture due to Viro [17, 6D, p.132].

Definition (4.2). For a constructible function α on a variety X the integer $\chi(X, \alpha)$ shall be called *the topological Euler characteristic* of the constructible function α:

$$\chi(X, \alpha) := \int_X C_*(\alpha), \qquad \text{i.e., the 0-th component of } C_*(\alpha).$$

Here $C_* : \mathcal{F}(X) \to H_*(X; \mathbb{Z})$ is the Chern-Schwartz-MacPherson homomorphism.

In particular, $\chi(X, Eu_X)$ is nothing but the degree (or the 0-th component) of the Chern-Mather class $C_M(X)$ of the variety X.

Lemma (4.3). ([7, Proposition 4.13]) *Let α be a constructible function on P^N. Then*

$$\chi(Gr_k(P^N), \mathcal{F}^{Rad}(\alpha)) = \binom{N}{k}\chi(P^N, \alpha).$$

The following formula (a generalized Plücker relation) follows from Theorem (4.1) and Lemma (4.3).

Theorem (4.4). ([7,Theorem (4.14)]) *Let X be a closed subvariety of P^N. Then we have*

$$\binom{N}{k}\chi(X, Eu_X) = e^{<k>}\binom{N+1}{k+1} + (-1)^{n+k(N-k)-n^{<k>}}\chi(X^{<k>}, Eu_{X^{<k>}}).$$

By the definition of the topological Euler characteristic of constructible functions, the above Plücker formula may be written as follows:

$$\binom{N}{k}(C_M(X))_0 = e^{<k>}\binom{N+1}{k+1} + (-1)^{n+k(N-k)-n^{<k>}}(C_M(X^{<k>}))_0,$$

or

$$(C_M(X^{<k>}))_0 = (-1)^{n+k(N-k)-n^{<k>}}\{\binom{N}{k}(C_M(X))_0 - e^{<k>}\binom{N+1}{k+1}\},$$

where $C_M(Z)_0$ denotes the 0-th component (or the degree) of the Chern-Mather class $C_M(Z)$. This means that the degree of the Chern-Mather class of the k-th dual variety $X^{<k>}$ can be described via the 0-th component of the Chern-Mather class of the source variety X. What we want to do is to describe the total Chern-Mather class of the k-th dual variety $X^{<k>}$ via the total Chern-Mather class of the source variety X. This naive wish was the very start of the present work. Now that we have Corollary (3.10), the solution for this problem follows immediately:

Theorem (4.5). *Let X be an n-dimensional reduced subvariety of P^N. Then we can describe the image of the total Chern-Mather class of the k-dual variety $X^{<k>}$ in the Grassmannian $Gr_k(P^N)$ as follows:*

$$(i_{X^{<k>}})_*C_M(X^{<k>}) = \\ (-1)^{n+k(N-k)-n^{<k>}}\{q_*(c(T_p)\cap p^!(i_X)_*C_M(X)) - e^{<k>}C_M(Gr_k(P^N))\},$$

where $i_{X^{<k>}}: X^{<k>} \to Gr_k(P^N)$ and $i_X : X \to P^N$ are the inclusion maps.

Proof. It follows from Theorem (4.1) and Corollary (3.10). Note that $p : I_k \to P^N$ is Euler, because it is a locally trivial fibration with non-singular fibers. □

Remark (4.6). Theorem (4.4) is a special case of Theorem (4.5). This follows by considering the 0-th component of the formula in Theorem(4.5), and the fact that the top Chern class of T_p is equal to the Euler characteristic of the fibers of p; that is $\binom{N}{k}$.

References

1. J.P.Brasselet, *Existence des classes de Chern en théorie bivariante*, Astérisque (1981), 7–22.
2. J.P.Brasselet and M.H.Schwartz, *Sur les classes de Chern d'une ensemble analytique complexe*, Astérisque **82–83** (1981), 93–148.
3. J.L.Brylinsk, *Transformations canonique, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques*, Astérisque **140–141** (1986), 3–134.
4. A.D'Agnolo et P.Schapira, *La transformée de Radon-Penrose des D-modules*, C. R. Acad. Sci. Paris t.319, Série I (1994), 461–466.
5. ______, *Quantification de Leray de la dualité projective*, C. R. Acad. Sci. Paris t.319, Série I (1994), 461–466.
6. J.P.Dufour, *Sur la stabilité des diagrammes d'applications différentiables*, Ann. Scient. Ann. Sci. Éc. Norm. Sup. t.10 (1977), 153–174.
7. L.Ernström, *Duality for the Local Euler Obstruction with Applications to Real and Complex Singularities*, Ph. D. Thesis (MIT) (1993).
8. ______, *Topological Radon transforms and the local Euler obstruction*, Duke Math. J. **76** (1994), 1–21.
9. ______, *Bivariant Schwartz-MacPherson classes with values in Chow groups*, preprint (1994).
10. E.M.Friedlander and B.Mazur, *Filtrations on the homology of algebraic varieties*, Memoirs of the A. M. S. **529** (1994).
11. W.Fulton, *Intersection Theory* (1981), Springer-Verlag.
12. W.Fulton and R.MacPherson, *Categorical framework for the study of singular spaces*, Memoirs of the A. M. S. **243** (1981).
13. J.P.Henry, M.Merle et C.Sabbah, *Sur la condition de Thom stricte pour un morphisme analytique complexe*, Ann. Scient. Éc. Norm. Sup. t.17 (1984), 227–268.
14. R.MacPherson, *Chern classes for singular algebraic varieties*, Ann.of Math. **100** (1974), 423–432.
15. A.Parusiński and P.Pragacz,, *Chern-Schwarz-MacPherson classes and the Euler Characteristic of degeneracy loci and special divisors*, Submitted to JAMS (1994).
16. P.Schapira, *Tomography of constructible functions*, preprint (1995).
17. O.Y.Viro, *Some integral calculus based on Euler characteristic*, Topology and Geometry-Rohlin Seminar, Springer Lecture Notes in Math. **1346** (1988), 127–138.

Department of Mathematics, Ohio State University at Mansfield, Mansfield, Ohio 44906, USA

E-mail address: ernstrom@math.ohio-state.edu

Department of Mathematics, Faculty of Science, University of Kagoshima, Kagoshima 890, Japan

E-mail address: ohmoto@sci.kagoshima-u.ac.jp

Department of Mathematics, College of Liberal Arts, University of Kagoshima, Kagoshima 890, Japan

E-mail address: yokura@cla.kagoshima-u.ac.jp