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STURM–LIOUVILLE OPERATORS WITH MEASURE-VALUED COEFFICIENTS

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Abstract. We give a comprehensive treatment of Sturm–Liouville operators with measure-valued coefficients including, a full discussion of self-adjoint extensions and boundary conditions, resolvents, and Weyl–Titchmarsh theory. We avoid previous technical restrictions and, at the same time, extend all results to a larger class of operators. Our operators include classical Sturm–Liouville operators, Lax operators arising in the treatment of the Camassa–Holm equation, Jacobi operators, and Sturm–Liouville operators on time scales as special cases.

1. Introduction

Sturm–Liouville problems

$$-\left(p(x)y'\right)' + q(x)y = z\, r(x)y \tag{1.1}$$

have a long tradition (see, e.g., the textbooks [31], [37], [38] and the references therein) and so have their generalizations to measure-valued coefficients. In fact, extensions to the case

$$\frac{d}{d\varrho(x)}\left(-p(x)y' + \int^x y d\chi\right) = z\, y \tag{1.2}$$

date back at least to Feller [15] and were also advocated in the fundamental monograph by Atkinson [4]. Here the derivative on the left-hand side has to be understood as a Radon–Nikodym derivative. We refer to the book by Mingarelli [23] for a more detailed historical discussion.

However, while this generalization on the level of differential equations has been very successful (see e.g. [4], [23], [36] and the references therein) much less is known about the associated operators in an appropriate Hilbert space. First attempts were made by Feller and later complemented by Kac [19] (cf. also Langer [21] and Bennewitz [5]). Again, a survey of these results and further information can be found in the book of Mingarelli [23].

The case where only the potential is a measure is fairly well treated since it allows to include the case of point interactions which is an important model in physics (see, e.g., the monographs [1], [2] as well as the recent results [6] and the references therein). Moreover, recently Shavcuk and Shkalikov [25]–[28] were even able to cover the case where the potential is a derivative of an L^2 function. Similarly, the case where the weight is a measure is known as Krein string and has also attracted considerable interest recently [33]–[35].

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However, while the theory developed by Kac and extended by Mingarelli is quite general, it still does exclude some cases of interest. More precisely, the basic assumptions in Chapter 3 of Mingarelli [23] require that the corresponding measures have no weight at a finite boundary point. Unfortunately, this assumption excludes for example classical cases like Jacobi operators on a half-line. The reason for this assumption is the fact that otherwise the corresponding maximal operator will be multivalued and one has to work within the framework of multivalued operators. This nuisance is already visible in the case of half-line Jacobi operators where the underlying Hilbert space has to be artificially expanded in order to be able to formulate appropriate boundary conditions [30]. In our case there is no natural way of extending the Hilbert space and the intrinsic approach via multivalued operators is more natural. Moreover, this multivaluedness is not too severe and corresponds to an at most two dimensional space which can be removed to obtain a single-valued operator. Again, a fact well-known from Jacobi operators with finite end points.

Moreover, the fact that our differential equation is defined on a larger set than the support of the measure ϱ (which determines the underlying Hilbert space) also reflects requirements from the applications we have in mind. The most drastic example in this respect is the Sturm–Liouville operator

$$\frac{d}{d\varrho(x)}\left(-y' + \frac{1}{4}\int^x y dx\right) = z\, y \tag{1.3}$$

which arises in the Lax pair of the dispersionless Camassa–Holm equation [8], [9]. In the case of a peakon ϱ is single Dirac measure and the underlying Hilbert space is one-dimensional. However, the corresponding differential equations has to be investigated on all of $\mathbb{R}$, where the Camassa–Holm equation is defined. An appropriate spectral theory for this operator in the case where ϱ is a genuine measure (i.e. not absolutely continuous with respect to Lebesgue measure) seems missing and is one of the main motivations for the present paper.

Furthermore, there is of course another reason why Sturm–Liouville equations with measure-valued coefficients are of interest, namely, the unification of the continuous with the discrete case. While such a unification already was one of the main motivations in Atkinson [4] and Mingarelli [23], it has recently attracted enormous attention via the introduction of the calculus on time scales [7]. In fact, given a time scale $\mathbb{T} \subseteq \mathbb{R}$, the so-called associated Hilger (or delta) derivative is just the Radon–Nikodym derivative with respect to the measure $d\varrho$, where $\varrho(x) = \inf\{y \in \mathbb{T} | y > x\}$. We refer to [13] for further details and to a follow-up publication [14], where we will provide further details on this connection.

2. Notation

Let (a, b) be an arbitrary interval and μ be a locally finite complex Borel measure on (a, b). By $AC_{loc}((a, b); \mu)$ we denote the set of left-continuous functions, which are locally absolutely continuous with respect to μ. These are precisely the functions f that can be written in the form

$$f(x) = f(c) + \int_c^x h(s) d\mu(s), \quad x \in (a, b), \tag{2.1}$$

where $h \in L^1_{loc}((a,b);\mu)$ and the integral has to be read as

$$\int_c^x h(s)d\mu(s) = \begin{cases} \int_{[c,x)} h(s)d\mu(s), & \text{if } x > c, \\ 0, & \text{if } x = c, \\ -\int_{[x,c)} h(s)d\mu(s), & \text{if } x < c. \end{cases} \tag{2.2}$$

The function h is the Radon–Nikodym derivative of f with respect to μ. It is uniquely defined in $L^1_{loc}((a,b);\mu)$ and we write

$$\frac{df}{d\mu} = h. \tag{2.3}$$

Every μ-absolutely continuous function is locally of bounded variation, hence also the limits from the right exist everywhere. Furthermore, it can only be discontinuous in some point, if the mass of this point is non zero.

Given a measure μ we will use the same letter for its distribution function

$$\mu(x) = \mu(c) + \int_c^x d\mu. \tag{2.4}$$

In this respect we also recall the integration by parts formula ([18, Theorem 21.67])

$$\int_{[c,d)} \nu(x)d\varrho(x) = \nu(d)\varrho(d) - \nu(c)\varrho(c) - \int_{[c,d)} \varrho(x+)d\nu(x) \tag{2.5}$$

for two Borel measures ν, ϱ with $f(x\pm) = \lim_{\varepsilon\downarrow 0} f(x \pm \varepsilon)$.

3. Sturm–Liouville equations with measure-valued coefficients

Let (a,b) be an arbitrary interval and ϱ, ς, and χ be locally finite complex Borel measures on (a,b). We want to define a linear differential expression τ which is informally given by

$$\tau f = \frac{d}{d\varrho}\left(-\frac{df}{d\varsigma} + \int f d\chi\right).$$

Up to now the only additional assumptions on our measures is that ς is supported on the whole interval, i.e. $\operatorname{supp}(\varsigma) = (a,b)$.

The maximal domain $\mathfrak{D}_\tau$ of functions in $AC_{loc}((a,b);\varsigma)$ such that τf makes sense consists of all functions $f \in AC_{loc}((a,b);\varsigma)$ for which the function

$$-\frac{df}{d\varsigma}(x) + \int_c^x f d\chi, \quad x \in (a,b) \tag{3.1}$$

is locally absolutely continuous with respect to ϱ, i.e. there is some representative of this function lying in $AC_{loc}((a,b);\varrho)$. As a consequence of the assumption $\operatorname{supp}(\varsigma) = (a,b)$, this representative is unique. We then set $\tau f \in L^1_{loc}((a,b);\varrho)$ to be the Radon–Nikodym derivative of this function with respect to ϱ. One easily sees that this definition is independent of $c \in (a,b)$ since the corresponding functions (3.1) as well as their unique representatives only differ by an additive constant. As usual, we denote the Radon–Nikodym derivative with respect to ς of some function $f \in \mathfrak{D}_\tau$ by

$$f^{[1]} = \frac{df}{d\varsigma} \in L^1_{loc}((a,b);|\varsigma|).$$

The function $f^{[1]}$ is called the first quasi-derivative of f.

The definition of τ is consistent with classical theory: Indeed let ϱ, ς, and χ be locally absolutely continuous with respect to Lebesgue measure, and denote by r, p^{-1}, and q the respective densities i.e.

$$\varrho(B) = \int_B r(x)dx, \quad \varsigma(B) = \int_B \frac{1}{p(x)}dx, \quad \text{and} \quad \chi(B) = \int_B q(x)dx,$$

for each Borel set B. Then some function f is in $\mathfrak{D}_\tau$ if and only if f as well as the quasi-derivative $f^{[1]} = pf'$ are locally absolutely continuous (with respect to Lebesgue measure). In this case

$$\tau f(x) = \frac{1}{r(x)}\left(-\frac{d}{dx}\left(p(x)\frac{df(x)}{dx}\right) + q(x)f(x)\right), \quad x \in (a,b),$$

is the usual Sturm–Liouville differential expression.

Moreover, choosing

$$\varrho = \sum_n \delta_n, \quad \varsigma = \sum_n \frac{1}{p_{n-1}}\mathbb{1}_{(n-1,n)}(x)dx, \quad \text{and} \quad \chi = \sum_n q_n\delta_n$$

where $p_n \neq 0$, $q_n \in \mathbb{R}$ and δ_n is the unit Dirac measure at $n \in \mathbb{Z}$ we obtain the usual Jacobi difference expression. In fact, $\tau f(n)$ at some point n is equal to the jump of the function

$$-p_{n-1}\mathbb{1}_{(n-1,n)}(x)f'(x) + \sum_{n\leq x} q_n f(n)$$

in this point and hence

$$\tau f(n) = p_{n-1}(f(n) - f(n-1)) - p_n(f(n+1) - f(n)) + q_n f(n).$$

Theorem 3.1. *Fix some $g \in L^1_{loc}((a,b);\varrho)$. Then there is a unique solution $u \in \mathfrak{D}_\tau$ of*

$$(\tau - z)u = g \quad \textit{with} \quad u(c) = d_1 \quad \textit{and} \quad u^{[1]}(c) = d_2$$

for each $z \in \mathbb{C}$, $c \in (a,b)$ and d_1, $d_2 \in \mathbb{C}$ if and only if

$$\varrho(\{x\})\varsigma(\{x\}) = 0 \quad \textit{and} \quad \chi(\{x\})\varsigma(\{x\}) \neq 1 \tag{3.2}$$

for all $x \in (a,b)$. If, in addition, g, d_1, d_2, and z are real, then the solution is real.

Proof. Some function $u \in \mathfrak{D}_\tau$ is a solution of $(\tau - z)u = g$ with $u(c) = d_1$ and $u^{[1]}(c) = d_2$, if and only if

$$\begin{aligned} u(x) &= d_1 + \int_c^x u^{[1]}d\varsigma, \\ u^{[1]}(x) &= d_2 + \int_c^x u d\chi - \int_c^x (zu + g)\, d\varrho, \quad x \in (a,b). \end{aligned}$$

Now set $\omega = |\varsigma| + |\chi| + |\varrho|$ and let m_{12}, m_{21} and f_2 be the Radon–Nikodym derivatives of ς, $\chi - z\varrho$ and $g\varrho$ with respect to ω. Then these equations can be written as

$$\begin{pmatrix} u(x) \\ u^{[1]}(x) \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \int_c^x \begin{pmatrix} 0 & m_{12} \\ m_{21} & 0 \end{pmatrix}\begin{pmatrix} u \\ u^{[1]} \end{pmatrix} d\omega + \int_c^x \begin{pmatrix} 0 \\ f_2 \end{pmatrix} d\omega, \quad x \in (a,b).$$

Hence the claim follows from Theorem A.2 since (3.2) holds for all $x \in (a,b)$ if and only if

$$I + \omega(\{x\}) \begin{pmatrix} 0 & m_{12}(x) \\ m_{21}(x) & 0 \end{pmatrix} = \begin{pmatrix} 1 & \varsigma(\{x\}) \\ \chi(\{x\}) - z\varrho(\{x\}) & 1 \end{pmatrix}$$

is regular for all $z \in \mathbb{C}$ and $x \in (a,b)$. □

Note that if $g \in L^1_{loc}((a,b);\varrho)$, $z \in \mathbb{C}$, $c \in (a,b)$, d_1, $d_2 \in \mathbb{C}$ and (3.2) holds for each $x \in (a,b)$ then there is also a unique solution of the initial value problem

$$(\tau - z)u = g \quad \text{with} \quad u(c+) = d_1 \quad \text{and} \quad u^{[1]}(c+) = d_2,$$

by Corollary A.3.

Because of Theorem 3.1, in the following we will always assume that the measure ς has no point masses in common with ϱ or χ, i.e.

$$\varsigma(\{x\})\varrho(\{x\}) = \varsigma(\{x\})\chi(\{x\}) = 0 \tag{3.3}$$

for all $x \in (a,b)$. This assumption is stronger than the one needed in Theorem 3.1 but we will need it for the Lagrange identity below.

For $f, g \in \mathfrak{D}_\tau$ we define the Wronski determinant

$$W(f,g)(x) = f(x)g^{[1]}(x) - f^{[1]}(x)g(x), \quad x \in (a,b).$$

The Wronskian is locally absolutely continuous with respect to ϱ with derivative

$$\frac{dW(f,g)}{d\varrho} = g\tau f - f\tau g.$$

Indeed this follows from the following Lagrange identity.

Proposition 3.2. *For each $f, g \in \mathfrak{D}_\tau$ and α, $\beta \in (a,b)$, $\alpha < \beta$, we have*

$$\int_\alpha^\beta (g(x)\tau f(x) - f(x)\tau g(x))\, d\varrho(x) = W(f,g)(\beta) - W(f,g)(\alpha). \tag{3.4}$$

Proof. By definition g is a distribution function of the measure $g^{[1]}d\varsigma$. Furthermore, the function

$$f_1(x) = -f^{[1]}(x) + \int_\alpha^x f d\chi, \quad x \in (a,b)$$

is a distribution function of $\tau f d\varrho$. Hence one gets by integration by parts

$$\int_\alpha^\beta g(t)\tau f(t) d\varrho(t) = [f_1(t)g(t)]_{t=\alpha}^\beta - \int_\alpha^\beta f_1(t+)g^{[1]}(t)d\varsigma(t).$$

We can drop the right-hand limit in the integral since the discontinuities of f_1 are a null set with respect to ς by (3.3). Hence the integral becomes

$$\begin{aligned} \int_\alpha^\beta f_1(t)g^{[1]}(t)d\varsigma(t) &= \int_\alpha^\beta \int_\alpha^t f d\chi\, g^{[1]}(t)d\varsigma(t) - \int_\alpha^\beta f^{[1]}(t)g^{[1]}(t)d\varsigma(t) \\ &= \int_\alpha^\beta f d\chi\, g(\beta) - \int_\alpha^\beta g(t)f(t)d\chi(t) - \int_\alpha^\beta f^{[1]}(t)g^{[1]}(t)d\varsigma(t), \end{aligned}$$

where we performed another integration by parts (and used again (3.3)). Now verifying the identity is an easy calculation. □

As a consequence of the Lagrange identity one sees that the Wronskian $W(u_1,u_2)$ of two solutions $u_1,u_2\in\mathfrak{D}_\tau$ of $(\tau-z)u=0$ is constant. Furthermore, we have

$$W(u_1,u_2)\neq 0 \quad\Leftrightarrow\quad u_1,\ u_2 \text{ linearly independent.}$$

Indeed the Wronskian of two linearly dependent solutions vanishes obviously. Conversely $W(u_1,u_2)=0$ means that the vectors

$$\begin{pmatrix} u_1(x)\\ u_1^{[1]}(x)\end{pmatrix} \quad\text{and}\quad \begin{pmatrix} u_2(x)\\ u_2^{[1]}(x)\end{pmatrix}$$

are linearly dependent for each $x\in(a,b)$. But by the uniqueness of solutions this implies the linear dependence of u_1 and u_2.

For each $z\in\mathbb{C}$ we call two linearly independent solutions of $(\tau-z)u=0$ a fundamental system of $(\tau-z)u=0$. By the existence and uniqueness theorem and the properties of the Wronskian, one sees that fundamental systems always exist.

Proposition 3.3. *Let $z\in\mathbb{C}$ and u_1, u_2 be a fundamental system of $(\tau-z)u=0$. Furthermore, let $c\in(a,b)$, $d_1,d_2\in\mathbb{C}$, $g\in L^1_{loc}((a,b);\varrho)$. Then there exist c_1, $c_2\in\mathbb{C}$ such that the solution u of*

$$(\tau-z)f=g \quad\text{with}\quad f(c)=d_1,\ f^{[1]}(c)=d_2,$$

is given for each $x\in(a,b)$ by

$$f(x)=c_1u_1(x)+c_2u_2(x)+\frac{u_1(x)}{W(u_1,u_2)}\int_c^x u_2 g\,d\varrho-\frac{u_2(x)}{W(u_1,u_2)}\int_c^x u_1 g\,d\varrho,$$
$$f^{[1]}(x)=c_1u_1^{[1]}(x)+c_2u_2^{[1]}(x)+\frac{u_1^{[1]}(x)}{W(u_1,u_2)}\int_c^x u_2 g\,d\varrho-\frac{u_2^{[1]}(x)}{W(u_1,u_2)}\int_c^x u_1 g\,d\varrho.$$

If u_1, u_2 is the fundamental system with

$$u_1(c)=u_2^{[1]}(c)=1 \quad\text{and}\quad u_1^{[1]}(c)=u_2(c)=0,$$

then $c_1=d_1$ and $c_2=d_2$.

Proof. We set

$$h(x)=u_1(x)\int_c^x u_2 g\,d\varrho-u_2(x)\int_c^x u_1 g\,d\varrho,\quad x\in(a,b).$$

Integration by parts shows

$$\int_\alpha^\beta u_1^{[1]}(x)\int_c^x u_2 g\,d\varrho-u_2^{[1]}(x)\int_c^x u_1 g\,d\varrho\,d\varsigma(x)=$$
$$=\left[u_1(x)\int_c^x u_2 g\,d\varrho-u_2(x)\int_c^x u_1 g\,d\varrho\right]_{x=\alpha}^{\beta},$$

for all α, $\beta\in(a,b)$, $\alpha<\beta$, hence

$$h^{[1]}(x)=u_1^{[1]}(x)\int_c^x u_2 g\,d\varrho-u_2^{[1]}(x)\int_c^x u_1 g\,d\varrho,\quad x\in(a,b).$$

Using again integration by parts we get

$$\begin{aligned}\int_\alpha^\beta u_1(x)\int_c^x u_2 g\,d\varrho\,d\chi(x) - z\int_\alpha^\beta u_1(x)\int_c^x u_2 g\,d\varrho\,d\varrho(x) = \\ &= \left[\int_c^x u_2 g\,d\varrho\left(\int_c^x u_1 d\chi - z\int_c^x u_1 d\varrho\right)\right]_{x=\alpha}^\beta \\ &\qquad - \int_\alpha^\beta\left(\int_c^x u_1 d\chi - z\int_c^x u_1 d\varrho\right)u_2(x)g(x)d\varrho(x) \\ &= \left[\int_c^x u_2 g\,d\varrho\left(u_1^{[1]}(x) - u_1^{[1]}(c)\right)\right]_{x=\alpha}^\beta \\ &\qquad - \int_\alpha^\beta\left(u_1^{[1]}(x) - u_1^{[1]}(c)\right)u_2(x)g(x)d\varrho(x) \\ &= u_1^{[1]}(\beta)\int_c^\beta u_2 g\,d\varrho - u_1^{[1]}(\alpha)\int_c^\alpha u_2 g\,d\varrho - \int_\alpha^\beta u_2 u_1^{[1]} g\,d\varrho\end{aligned}$$

for all $\alpha,\beta\in(a,b)$ with $\alpha<\beta$. Now an easy calculation shows that

$$\int_\alpha^\beta h d\chi - \int_\alpha^\beta zh + W(u_1,u_2)g d\varrho = h^{[1]}(\beta) - h^{[1]}(\alpha).$$

Hence h is a solution of $(\tau - z)h = W(u_1,u_2)g$ and therefore the function f given in the claim is a solution of $(\tau - z)f = g$. Now if we choose

$$c_1 = \frac{W(f,u_2)(c)}{W(u_1,u_2)(c)} \quad\text{and}\quad c_2 = \frac{W(u_1,f)(c)}{W(u_1,u_2)(c)},$$

then f satisfies the initial conditions at c. □

Another important identity for the Wronskian is the following Plücker identity.

Proposition 3.4. *For each functions $f_1,\ f_2,\ f_3,\ f_4\in\mathfrak{D}_\tau$ we have*

$$0 = W(f_1,f_2)W(f_3,f_4) + W(f_1,f_3)W(f_4,f_2) + W(f_1,f_4)W(f_2,f_3).$$

Proof. The right-hand side is equal to the determinant

$$\frac{1}{2}\begin{vmatrix} f_1 & f_2 & f_3 & f_4 \\ f_1^{[1]} & f_2^{[1]} & f_3^{[1]} & f_4^{[1]} \\ f_1 & f_2 & f_3 & f_4 \\ f_1^{[1]} & f_2^{[1]} & f_3^{[1]} & f_4^{[1]}\end{vmatrix}.$$

□

We say τ is regular at a, if $|\varrho|((a,c])$, $|\varsigma|((a,c])$ and $|\chi|((a,c])$ are finite for one (and hence for all) $c\in(a,b)$. Similarly one defines regularity for the right endpoint b. Furthermore, we say τ is regular if τ is regular at both endpoints.

Theorem 3.5. *Let τ be regular at a, $z\in\mathbb{C}$, and $g\in L^1((a,c);\varrho)$ for each $c\in(a,b)$. Then for every solution f of $(\tau - z)f = g$ the limits*

$$f(a) := \lim_{x\downarrow a} f(x) \quad\textit{and}\quad f^{[1]}(a) := \lim_{x\downarrow a} f^{[1]}(x)$$

exist and are finite. For each $d_1,\ d_2\in\mathbb{C}$ there is a unique solution of

$$(\tau - z)f = g \quad\textit{with}\quad f(a) = d_1 \quad\textit{and}\quad f^{[1]}(a) = d_2.$$

Furthermore, if g, d_1, d_2, and z are real, then the solution is real. Similar results hold for the right endpoint b.

Proof. The first part of the theorem is an immediate consequence of Theorem A.4. All solutions of $(\tau - z)f = g$ are given by $f = c_1 u_1 + c_2 u_2 + \tilde{f}$, where c_1, $c_2 \in \mathbb{C}$, u_1, u_2 are a fundamental system of $(\tau - z)u = 0$ and $\tilde{f}$ is some solution of $(\tau - z)f = g$. Now since $W(u_1, u_2)(a) = u_1(a)u_2^{[1]}(a) - u_1^{[1]}(a)u_2(a) \neq 0$ there is exactly one choice for the coefficients $c_1, c_2 \in \mathbb{C}$ such that the solution satisfies the initial values at a. If g, d_1, d_2 and z are real then u_1, u_2, and $\tilde{f}$ can be chosen real and hence also c_1 and c_2 are real. □

Under the assumptions of Theorem 3.5 one sees that Proposition 3.3 remains valid even in the case when $c = a$ (resp. $c = b$) with essentially the same proof.

We now turn to analytic dependence of solutions on the spectral parameter $z \in \mathbb{C}$. These results will be needed in Section 9.

Theorem 3.6. *Let $g \in L^1_{loc}((a,b);\varrho)$, $c \in (a,b)$, $d_1, d_2 \in \mathbb{C}$ and for each $z \in \mathbb{C}$ let f_z be the unique solution of*

$$(\tau - z)f = g \quad \text{with} \quad f(c) = d_1 \quad \text{and} \quad f^{[1]}(c) = d_2.$$

Then $f_z(x)$ and $f_z^{[1]}(x)$ are entire functions of order $1/2$ in z for each $x \in (a,b)$. Moreover, for each α, $\beta \in (a,b)$ with $\alpha < \beta$ we have

$$|f_z(x)| + |f_z^{[1]}(x)| \leq C \mathrm{e}^{B\sqrt{|z|}}, \quad x \in [\alpha, \beta],\ z \in \mathbb{C},$$

for some constants C, $B \in \mathbb{R}$.

Proof. The analyticity part follows by applying Theorem A.5 to the equivalent system from the proof of Theorem 3.1. If we set for each $z \in \mathbb{C}$ with $|z| \geq 1$

$$v_z(x) = |z||f_z(x)|^2 + |f_z^{[1]}(x)|^2, \quad x \in (a,b),$$

an integration by parts shows that for each $x \in (a,b)$

$$v_z(x) = v_z(c) + |z| \int_c^x \left(f_z f_z^{[1]*} + f_z^{[1]} f_z^* \right) d\varsigma \\ + \int_c^x \left(f_z f_z^{[1]*} + f_z^{[1]} f_z^* \right) d\chi - \int_c^x \left(z f_z f_z^{[1]*} + z^* f_z^{[1]} f_z^* \right) d\varrho.$$

Because of the elementary estimate

$$2|f_z(x) f_z^{[1]}(x)| \leq \frac{|z||f_z(x)|^2 + |f_z^{[1]}(x)|^2}{\sqrt{|z|}} = \frac{v_z(x)}{\sqrt{|z|}}, \quad x \in (a,b),$$

we get an upper bound for v_z

$$v_z(x) \leq v_z(c) + \int_c^x v_z(t)\sqrt{|z|} d\omega(t), \quad x \in [c,b),$$

where $\omega = |\varsigma| + |\chi| + |\varrho|$, as in the proof of Theorem 3.1. Now an application of Lemma A.1 yields

$$v_z(x) \leq v_z(c) \mathrm{e}^{\int_c^x \sqrt{|z|} d\omega}, \quad x \in [c,b).$$

To the left-hand side of c we have

$$v_z(x+) \leq v_z(c) + \int_{x+}^c v_z(t)\sqrt{|z|} d\omega(t), \quad x \in (a,c),$$

and hence again by the Gronwall lemma A.1

$$v_z(x+) \leq v_z(c) \mathrm{e}^{\int_{x+}^{c} \sqrt{|z|} d\omega}, \quad x \in (a,c),$$

which is the required bound. □

Under the assumptions of Theorem 3.6 also the right-hand limits $f_z(x+)$ and $f_z^{[1]}(x+)$ are entire functions of order $1/2$ for each $x \in (a,b)$ with a corresponding bound. Moreover, the same analytic properties are true for the solutions f_z of the initial value problem

$$(\tau - z)f = g \quad \text{with} \quad f_z(c+) = d_1 \quad \text{and} \quad f_z^{[1]}(c+) = d_2.$$

Indeed this follows for example from the remark after the proof of Theorem A.5.

Furthermore, if, in addition to the assumptions of Theorem 3.6, τ is regular at a and g is integrable near a, then the limits $f_z(a)$ and $f_z^{[1]}(a)$ are entire functions of order $1/2$ and the bound in Theorem 3.6 holds for all $x \in [a,\beta]$. Indeed this follows since the entire functions $f_z(x)$ and $f_z^{[1]}(x)$, $x \in (a,c)$ are locally bounded, uniformly in $x \in (a,c)$. Moreover, in this case the assertions of Theorem 3.6 are valid even if we take $c = a$ and/or $\alpha = a$. This follows from the construction of the solution in the proof of Theorem 3.5, whereas the bound is proven as in the general case (note that ω is finite near a by assumption).

We gather the assumptions made on the coefficients so far and add some new which are needed in the sequel. Here we say that some interval $(\alpha,\beta) \subseteq \operatorname{supp}(\varrho)^{\mathsf{C}}$ is a gap of $\operatorname{supp}(\varrho)$ if the endpoints α and β lie in $\operatorname{supp}(\varrho)$.

Hypothesis 3.7.

(i) *The measure ϱ is positive.*
(ii) *The measure χ is real-valued.*
(iii) *The measure ς is real-valued and supported on the whole interval, i.e.*

$$\operatorname{supp}(\varsigma) = (a,b).$$

(iv) *The measure ς has no point masses in common with ϱ or χ, i.e.*

$$\varsigma(\{x\})\chi(\{x\}) = \varsigma(\{x\})\varrho(\{x\}) = 0.$$

(v) *For each gap (α,β) of $\operatorname{supp}(\varrho)$ and every function $f \in \mathfrak{D}_\tau$ with the outer limits $f(\alpha-) = f(\beta+) = 0$ we have $f(x) = 0$, $x \in (\alpha,\beta)$.*
(vi) *The measure ϱ is supported on more than one point, i.e. $|\operatorname{supp}(\varrho)| > 1$.*

As a consequence of the real-valuedness of the measures, τ is a real differential expression, i.e. $f \in \mathfrak{D}_\tau$ if and only if $f^* \in \mathfrak{D}_\tau$ and in this case $\tau f^* = (\tau f)^*$. Furthermore, ϱ has to be positive in order to obtain a definite inner product later. Moreover, condition (v) in Hypothesis 3.7 is crucial for Proposition 3.9 and Proposition 3.10 to hold. In fact, if $(0,2\pi)$ is a gap in the support of ϱ and we choose $d\varsigma = dx$, $d\chi = -dx$, then the function $f(x) = \sin(x)$ for $x \in (0,2\pi)$ and $f(x) = 0$ else is in $\mathfrak{D}_\tau$ with $\tau f = 0$. However, this condition is satisfied by a large class of measures as the next lemma shows.

Lemma 3.8. *Suppose that for each gap (α,β) of $\operatorname{supp}(\varrho)$ the measures $\varsigma|_{(\alpha,\beta)}$ and $\chi|_{(\alpha,\beta)}$ are of one and the same sign. Then* (v) *in Hypothesis* (3.7) *holds.*

Proof. Let (α,β) be a gap of $\operatorname{supp}(\varrho)$ and $f\in\mathfrak{D}_\tau$ with $f(\alpha-)=f(\beta+)=0$. As in the proof of Proposition 3.2 integration by parts yields

$$\begin{aligned} f(\beta)^*\tau f(\beta)\varrho(\{\beta\}) &= \int_\alpha^{\beta+} \tau f(x) f(x)^* d\varrho(x) \\ &= \int_\alpha^{\beta+} |f^{[1]}(x)|^2 d\varsigma(x) + \int_\alpha^{\beta+} |f(x)|^2\, d\chi(x). \end{aligned}$$

Now the left-hand side vanishes since either $\varrho(\{\beta\})=0$ or f is continuous in β, $f(\beta-)=f(\beta+)=0$. Hence $f^{[1]}$ vanishes almost everywhere with respect to ς, i.e. $f^{[1]}$ vanishes in (α,β) and f is constant in (α,β). Now since $f(\beta+)=f(\beta)+f^{[1]}(\beta)\varsigma(\{\beta\})$, we see that f vanishes in (α,β) □

The theory we are going to develop from now on is not applicable if the support of ϱ consists of not more than one point, since in this case $L^1_{loc}((a,b);\varrho)$ is only one-dimensional (and hence all solutions of $(\tau-z)u=0$ are linearly dependent). In particular, the essential Proposition 3.9 does not hold in this case. Hence we have to exclude this case from now on. Nevertheless this case is important, in particular for applications to the isospectral problem of the Camassa–Holm equation. Hence we will treat the case when $\operatorname{supp}(\varrho)$ consists of only one point separately in Appendix C.

We aim towards introducing linear operators in the Hilbert space $L^2((a,b);\varrho)$, induced by the differential expression τ. As a first step we define a linear relation T_{loc} of $L^1_{loc}((a,b);\varrho)$ into $L^1_{loc}((a,b);\varrho)$ by

$$T_{\mathrm{loc}} = \{(f,\tau f)\,|\, f\in\mathfrak{D}_\tau\} \subseteq L^1_{loc}((a,b);\varrho)\times L^1_{loc}((a,b);\varrho).$$

Now, in contrast to the classical case, in general $\mathfrak{D}_\tau$ is not embedded in $L^1_{loc}((a,b);\varrho)$, i.e. T_{loc} is multi-valued. Instead we have the following result, which is important for our approach. For later use, we introduce the quantities

$$\alpha_\varrho = \inf\operatorname{supp}(\varrho) \quad\text{and}\quad \beta_\varrho = \sup\operatorname{supp}(\varrho).$$

Proposition 3.9. *The linear map*

$$\begin{array}{ccc} \mathfrak{D}_\tau & \to & T_{\mathrm{loc}} \\ f & \mapsto & (f,\tau f) \end{array}$$

is bijective.

Proof. Clearly this mapping is linear and onto T_{loc} by definition. Now let $f\in\mathfrak{D}_\tau$ such that $f=0$ almost everywhere with respect to ϱ. We will show that f is of the form

$$f(x) = \begin{cases} c_a u_a(x), & \text{if } a<x\le\alpha_\varrho, \\ 0, & \text{if } \alpha_\varrho<x\le\beta_\varrho, \\ c_b u_b(x), & \text{if } \beta_\varrho<x<b, \end{cases} \tag{3.5}$$

where c_a, $c_b\in\mathbb{C}$ and u_a, u_b are the solutions of $\tau u=0$ with

$$u_a(\alpha_\varrho-) = u_b(\beta_\varrho+) = 0 \quad\text{and}\quad u_a^{[1]}(\alpha_\varrho-) = u_b^{[1]}(\beta_\varrho+) = 1.$$

Obviously we have $f(x)=0$ for all x in the interior of $\operatorname{supp}(\varrho)$ and points of mass of ϱ. Now if (α,β) is a gap of $\operatorname{supp}(\varrho)$, then since α, $\beta\in\operatorname{supp}(\varrho)$ at least we have $f(\alpha-)=f(\beta+)=0$ and hence $f(x)=0$, $x\in[\alpha,\beta]$ by Hypothesis 3.7. Hence all points $x\in(\alpha_\varrho,\beta_\varrho)$ for which possibly $f(x)\neq 0$, lie on the boundary of $\operatorname{supp}(\varrho)$

such that there are monotone sequences $x_{+,n}$, $x_{-,n} \in \operatorname{supp}(\varrho)$ with $x_{+,n} \downarrow x$ and $x_{-,n} \uparrow x$. Then for each $n \in \mathbb{N}$, either $f(x_{-,n}+) = 0$ or $f(x_{-,n}-) = 0$, hence

$$f(x-) = \lim_{n\to\infty} f(x_{-,n}-) = \lim_{n\to\infty} f(x_{-,n}+) = 0.$$

Similarly one shows that also $f(x+) = 0$. Now since f is a solution of $\tau u = 0$ outside $[\alpha, \beta]$, it remains to show that $f(\alpha_\varrho) = f(\beta_\varrho) = 0$. Therefore assume that f is not continuous in α_ϱ, i.e. $\varsigma(\{\alpha_\varrho\}) \neq 0$. Then $f^{[1]}$ is continuous in α_ϱ, hence $f^{[1]}(\alpha_\varrho) = 0$. But this yields $f(\alpha_\varrho-) = f(\alpha_\varrho+) - f^{[1]}(\alpha_\varrho)\varsigma(\{\alpha_\varrho\}) = 0$. Hence f is of the claimed form. Furthermore, a simple calculation yields

$$\tau f = c_a \mathbb{1}_{\{\alpha_\varrho\}} - c_b \mathbb{1}_{\{\beta_\varrho\}}. \tag{3.6}$$

Now in order to prove that our mapping is one-to-one let $f \in \mathfrak{D}_\tau$ be such that $f = 0$ and $\tau f = 0$ almost everywhere with respect to ϱ. By the existence and uniqueness theorem it suffices to prove that $f(c) = f^{[1]}(c) = 0$ at some point $c \in (a, b)$. But this is valid for all points between α_ϱ and β_ϱ. □

In the following we will always identify the elements of the relation T_{loc} with functions in $\mathfrak{D}_\tau$. Hence some element $f \in T_{\mathrm{loc}}$ is always a function $f \in \mathfrak{D}_\tau$, which is an $AC_{loc}((a, b); \varsigma)$ representative of the first component of f (as an element of T_{loc}) and $\tau f \in L^1_{loc}((a, b); \varrho)$ is the second component of f (again as an element of T_{loc}). In general the relation T_{loc} is multi-valued, i.e.

$$\operatorname{mul}(T_{\mathrm{loc}}) = \left\{ g \in L^1_{loc}((a, b); \varrho) \,|\, (0, g) \in T_{\mathrm{loc}} \right\} \neq \{0\}.$$

In view of the formulation of the next result we say that ϱ has no mass in a and b.

Proposition 3.10. *The multi-valued part of T_{loc} is given by*

$$\operatorname{mul}(T_{\mathrm{loc}}) = \operatorname{span}\left\{ \mathbb{1}_{\{\alpha_\varrho\}}, \mathbb{1}_{\{\beta_\varrho\}} \right\}.$$

In particular

$$\dim \operatorname{mul}(T_{\mathrm{loc}}) = \begin{cases} 0, & \text{if } \varrho \text{ has neither mass in } \alpha_\varrho \text{ nor in } \beta_\varrho, \\ 1, & \text{if } \varrho \text{ has either mass in } \alpha_\varrho \text{ or in } \beta_\varrho, \\ 2, & \text{if } \varrho \text{ has mass in } \alpha_\varrho \text{ and in } \beta_\varrho. \end{cases}$$

Hence T_{loc} is an operator if and only if ϱ has neither mass in α_ϱ nor in β_ϱ.

Proof. Let $(f, \tau f) \in T_{\mathrm{loc}}$ with $f = 0$ almost everywhere with respect to ϱ. In the proof of Proposition 3.9 we saw that such an f is of the form (3.5) and τf is a linear combination of $\mathbb{1}_{\{\alpha_\varrho\}}$ and $\mathbb{1}_{\{\beta_\varrho\}}$ by (3.6). It remains to prove that $\operatorname{mul}(T_{\mathrm{loc}})$ indeed contains $\mathbb{1}_{\{\alpha_\varrho\}}$ if ϱ has mass in α_ϱ. Therefore consider the function

$$f(x) = \begin{cases} u_a(x), & \text{if } a < x \leq \alpha_\varrho, \\ 0, & \text{if } \alpha_\varrho < x < b. \end{cases}$$

One easily checks that f lies in $\mathfrak{D}_\tau$ and hence $(0, \mathbb{1}_{\{\alpha_\varrho\}}) = (f, \tau f) \in T_{\mathrm{loc}}$. Similarly one shows that $\mathbb{1}_{\{\beta_\varrho\}}$ lies in $\operatorname{mul}(T_{\mathrm{loc}})$ if ϱ has mass in β_ϱ. Furthermore, note that $\mathbb{1}_{\{\alpha_\varrho\}} = 0$ (resp. $\mathbb{1}_{\{\beta_\varrho\}} = 0$) if ϱ has no mass in α_ϱ (resp. in β_ϱ). □

In contrast to the classical case one can't define a proper Wronskian for elements in $\operatorname{dom}(T_{\mathrm{loc}})$, instead we define the Wronskian of two elements f, $g \in T_{\mathrm{loc}}$ as

$$W(f, g)(x) = f(x)g^{[1]}(x) - f^{[1]}(x)g(x), \quad x \in (a, b).$$

The Lagrange identity then takes the form

$$W(f,g)(\beta) - W(f,g)(\alpha) = \int_\alpha^\beta g(x)\tau f(x) - f(x)\tau g(x) d\varrho(x).$$

Furthermore, note that by the existence and uniqueness theorem we have

$$\text{ran}\,(T_{\text{loc}} - z) = L^1_{loc}((a,b);\varrho) \quad \text{and} \quad \dim\ker(T_{\text{loc}} - z) = 2, \quad z \in \mathbb{C}. \tag{3.7}$$

4. Sturm–Liouville relations

In this section we will restrict the differential relation T_{loc} in order to obtain a linear relation in the Hilbert space $L^2((a,b);\varrho)$ with scalar product

$$\langle f, g\rangle = \int_{(a,b)} g(x)^* f(x) d\varrho(x).$$

First we define the maximal relation $T_{\max}$ in $L^2((a,b);\varrho)$ by

$$T_{\max} = \left\{(f,\tau f) \in T_{\text{loc}} \,|\, f \in L^2((a,b);\varrho), \tau f \in L^2((a,b);\varrho)\right\}.$$

In general $T_{\max}$ is not an operator. Indeed we have

$$\text{mul}\,(T_{\max}) = \text{mul}\,(T_{\text{loc}}),$$

since all elements of $\text{mul}\,(T_{\text{loc}})$ are square integrable with respect to ϱ. In order to obtain a symmetric relation we restrict the maximal relation $T_{\max}$ to functions with compact support

$$T_0 = \{(f,\tau f) \,|\, f \in \mathfrak{D}_\tau, \text{supp}(f) \text{ compact in } (a,b)\}.$$

Indeed this relation T_0 is an operator as we will see later.

Since τ is a real differential expression, the relations T_0 and $T_{\max}$ are real with respect to the natural conjugation in $L^2((a,b);\varrho)$, i.e. if $f \in T_{\max}$ (resp. $f \in T_0$) then also $f^* \in T_{\max}$ (resp. $f^* \in T_0$) where the conjugation is defined componentwise.

We say some measurable function f lies in $L^2((a,b);\varrho)$ near a (resp. near b) if f lies in $L^2((a,c);\varrho)$ (resp. in $L^2((c,b);\varrho)$) for each $c \in (a,b)$. Furthermore, we say some $f \in T_{\text{loc}}$ lies in $T_{\max}$ near a (resp. near b) if f and τf both lie in $L^2((a,b);\varrho)$ near a (resp. near b). One easily sees that some $f \in T_{\text{loc}}$ lies in $T_{\max}$ near a (resp. b) if and only if f^* lies in $T_{\max}$ near a (resp. b).

Proposition 4.1. *Let τ be regular at a and f lie in $T_{\max}$ near a. Then both limits*

$$f(a) := \lim_{x\downarrow a} f(x) \quad \text{and} \quad f^{[1]}(a) := \lim_{x\downarrow a} f^{[1]}(x)$$

exist and are finite. Similar results hold at b.

Proof. Under this assumptions τf lies in $L^2((a,b);\varrho)$ near a and since ϱ is a finite measure near a we have $\tau f \in L^1((a,c);\varrho)$ for each $c \in (a,b)$. Hence the claim follows from Theorem 3.5. □

From the Lagrange identity we now get the following lemma.

Lemma 4.2. *If f and g lie in $T_{\max}$ near a, the limit*

$$W(f,g^*)(a) := \lim_{\alpha\downarrow a} W(f,g^*)(\alpha)$$

exists and is finite. A similar result holds at the endpoint b. If f, $g \in T_{\max}$ then

$$\langle \tau f, g\rangle - \langle f, \tau g\rangle = W(f,g^*)(b) - W(f,g^*)(a) =: W_a^b(f,g^*).$$

Proof. If f and g lie in $T_{\max}$ near a, the limit $\alpha \downarrow a$ of the left-hand side in equation (3.4) exists. Hence also the limit in the claim exists. Now the remaining part follows by taking the limits $\alpha \downarrow a$ and $\beta \uparrow b$. □

If τ is regular at a and f and g lie in $T_{\max}$ near a, then we clearly have

$$W(f, g^*)(a) = f(a)g^{[1]}(a)^* - f^{[1]}(a)g(a)^*.$$

In order to determine the adjoint of T_0

$$T_0^* = \left\{(f,g) \in L^2((a,b);\varrho) \times L^2((a,b);\varrho) \,|\, \forall (u,v) \in T_0 : \langle f, v\rangle = \langle g, u\rangle\right\},$$

as in the classical theory, we need the following lemma (see [31, Lemma 9.3]).

Lemma 4.3. *Let V be a vector space over $\mathbb{C}$ and $F_1, \dots, F_n, F \in V^*$, then*

$$F \in \operatorname{span}\{F_1, \dots, F_n\} \quad \Leftrightarrow \quad \bigcap_{i=1}^{n} \ker F_i \subseteq F.$$

Theorem 4.4. *The adjoint of T_0 is $T_{\max}$.*

Proof. From Lemma 4.2 one immediately gets $T_{\max} \subseteq T_0^*$. Indeed for each $f \in T_0$ and $g \in T_{\max}$ we have

$$\langle \tau f, g\rangle - \langle f, \tau g\rangle = \lim_{\beta\uparrow b} W(f,g^*)(\beta) - \lim_{\alpha \downarrow a} W(f, g^*)(\alpha) = 0$$

since $W(f, g^*)$ has compact support. Conversely let $(f, f_2) \in T_0^*$ and $\tilde{f}$ be a solution of $\tau\tilde{f} = f_2$. We expect that $(f - \tilde{f}, 0) \in T_{\text{loc}}$. To prove this we will invoke Lemma 4.3. Therefore we consider linear functionals

$$l(g) = \int_{(a,b)} \left(f(x) - \tilde{f}(x)\right)^* g(x) d\varrho(x), \qquad g \in L_c^2((a,b);\varrho),$$

$$l_j(g) = \int_{(a,b)} u_j(x)^* g(x) d\varrho(x), \qquad g \in L_c^2((a,b);\varrho),\ j = 1, 2,$$

where u_j are two solutions of $\tau u = 0$ with $W(u_1, u_2) = 1$ and $L_c^2((a,b);\varrho)$ is the space of square integrable functions with compact support. For these functionals we have $\ker l_1 \cap \ker l_2 \subseteq \ker l$. Indeed let $g \in \ker l_1 \cap \ker l_2$, then the function

$$u(x) = u_1(x) \int_a^x u_2(t)g(t)d\varrho(t) + u_2(x) \int_x^b u_1(t)g(t)d\varrho(t), \quad x \in (a,b),$$

is a solution of $\tau u = g$ by Proposition 3.3 and has compact support since g lies in the kernel of l_1 and l_2, hence $u \in T_0$. Then the Lagrange identity and the definition of the adjoint yields

$$\begin{aligned}\int_a^b \left(f(x) - \tilde{f}(x)\right)^* \tau u(x) d\varrho(x) &= \langle \tau u, f\rangle - \int_a^b \tilde{f}(x)^* \tau u(x) d\varrho(x) \\ &= \langle u, f_2\rangle - \int_a^b \tau\tilde{f}(x)^* u(x) d\varrho(x) = 0,\end{aligned}$$

hence $g = \tau u \in \ker l$. Now applying Lemma 4.3 there are c_1, $c_2 \in \mathbb{C}$ such that

$$\int_a^b \left(f(x) - \tilde{f}(x) + c_1 u_1(x) + c_2 u_2(x)\right)^* g(x) d\varrho(x) = 0, \tag{4.1}$$

for each $g \in L_c^2((a,b);\varrho)$. By definition of T_{loc} obviously $(\tilde{f} + c_1u_1 + c_2u_2, f_2) \in T_{\text{loc}}$. But the first component of this pair is equal to f, almost everywhere with respect to ϱ because of (4.1). Hence we also have $(f, f_2) \in T_{\mathrm{loc}}$ and therefore $(f, f_2) \in T_{\max}$. □

By the preceding theorem T_0 is symmetric. The closure $T_{\min}$ of T_0 is called the minimal relation,

$$T_{\min} = \overline{T_0} = T_0^{**} = T_{\max}^*.$$

In order to determine $T_{\min}$ we need the following lemma on functions of $T_{\max}$.

Lemma 4.5. *If f_a lies in $T_{\max}$ near a and f_b lies in $T_{\max}$ near b, then there exists $f \in T_{\max}$ such that $f = f_a$ near a and $f = f_b$ near b (as functions in $\mathfrak{D}_\tau$).*

Proof. Let u_1, u_2 be a fundamental system of $\tau u = 0$ with $W(u_1, u_2) = 1$ and let α, $\beta \in (a, b)$, $\alpha < \beta$ such that the functionals

$$F_j(g) = \int_\alpha^\beta u_i g d\varrho, \quad g \in L^2((a, b); \varrho), \ j = 1, 2,$$

are linearly independent. This is possible since otherwise u_1 and u_2 were linearly dependent in $L^2((a, b); \varrho)$ and hence also in $\mathfrak{D}_\tau$ by the identification in Lemma 3.9. First we show that for each d_1, d_2, d_3, $d_4 \in \mathbb{C}$ there is some $u \in \mathfrak{D}_\tau$ such that

$$u(\alpha) = d_1, \quad u^{[1]}(\alpha) = d_2, \quad u(\beta) = d_3 \quad \text{and} \quad u^{[1]}(\beta) = d_4.$$

Indeed let $g \in L^2((a, b); \varrho)$ and consider the solution u of $\tau u = g$ with initial conditions

$$u(\alpha) = d_1 \quad \text{and} \quad u^{[1]}(\alpha) = d_2.$$

With Proposition 3.3 one sees that u has the desired properties if

$$\begin{pmatrix} F_2(g) \\ F_1(g) \end{pmatrix} = \begin{pmatrix} \int_\alpha^\beta u_2 g d\varrho \\ \int_\alpha^\beta u_1 g d\varrho \end{pmatrix} = \begin{pmatrix} u_1(\beta) & -u_2(\beta) \\ u_1^{[1]}(\beta) & -u_2^{[1]}(\beta) \end{pmatrix}^{-1} \begin{pmatrix} d_3 - c_1 u_1(\beta) - c_2 u_2(\beta) \\ d_4 - c_1 u_1^{[1]}(\beta) - c_2 u_2^{[1]}(\beta) \end{pmatrix},$$

where c_1, $c_2 \in \mathbb{C}$ are the constants appearing in Proposition 3.3. But since the functionals F_1, F_2 are linearly independent, we may choose $g \in L^2((a, b); \varrho)$ such that this equation is valid. Now the function f defined by

$$f(x) = \begin{cases} f_a(x), & \text{if } x < \alpha, \\ u(x), & \text{if } \alpha \le x < \beta, \\ f_b(x), & \text{if } \beta \le x, \end{cases}$$

has the claimed properties. □

Theorem 4.6. *The minimal relation $T_{\min}$ is given by*

$$T_{\min} = \left\{ f \in T_{\max} \,|\, \forall g \in T_{\max} : W(f, g)(a) = W(f, g)(b) = 0 \right\}.$$

Furthermore, $T_{\min}$ is an operator, i.e. $\dim \operatorname{mul}(T_{\min}) = 0$.

Proof. If $f \in T_{\min} = T_{\max}^* \subseteq T_{\max}$ we have

$$0 = \langle \tau f, g \rangle - \langle f, \tau g \rangle = W(f, g^*)(b) - W(f, g^*)(a)$$

for each $g \in T_{\max}$. Given some $g \in T_{\max}$, there is a $g_a \in T_{\max}$ such that $g_a^* = g$ in a vicinity of a and $g_a = 0$ in a vicinity of b. Therefore $W(f, g)(a) = W(f, g_a^*)(a) - W(f, g_a^*)(a) = 0$. Similar one sees that $W(f, g)(b) = 0$ for each $g \in T_{\max}$.

Conversely if $f \in T_{\max}$ such that for each $g \in T_{\max}$, $W(f,g)(a) = W(f,g)(b) = 0$, then

$$\langle \tau f, g\rangle - \langle f, \tau g\rangle = W(f,g^*)(b) - W(f,g^*)(a) = 0,$$

hence $f \in T_{\max}^* = T_{\min}$.

In order to show that $T_{\min}$ is an operator, let $f \in T_{\min}$ with $f = 0$ almost everywhere with respect to ϱ. Assume $\alpha_\varrho > a$ and $\varrho(\{\alpha_\varrho\}) \neq 0$, then f is of the form (3.5). From what we already proved we know that $W(f,u_1)(a) = W(f,u_2)(a) = 0$ for each fundamental system u_1, u_2 of $\tau u = 0$. But $W(f,u_i)(x)$ is constant on (a,α_ϱ) and hence $f(\alpha_\varrho-) = f^{[1]}(\alpha_\varrho-) = 0$. From this we see that f vanishes on (a,α_ϱ). Similarly one proves that f also vanishes on (β_ϱ,b), hence $f = 0$. $\square$

For regular τ we may characterize the minimal operator by the boundary values of functions $f \in T_{\max}$.

Corollary 4.7. *If τ is regular at a and $f \in T_{\max}$ we have*

$$f(a) = f^{[1]}(a) = 0 \quad \Leftrightarrow \quad \forall g \in T_{\max} : W(f,g)(a) = 0.$$

A similar result holds at b.

Proof. The claim follows from $W(f,g)(a) = f(a)g^{[1]}(a) - f^{[1]}(a)g(a)$ and the fact that one finds $g \in T_{\max}$ with prescribed initial values at a. Indeed one can take g to coincide with some solution of $\tau u = 0$ near a. $\square$

If the measure ϱ has no weight near some endpoint, we get another characterization for functions in $T_{\min}$ in terms of their left-hand (resp. right-hand) limit at α_ϱ (resp. β_ϱ).

Corollary 4.8. *If $\alpha_\varrho > a$ and $f \in T_{\max}$ we have*

$$f(\alpha_\varrho-) = f^{[1]}(\alpha_\varrho-) = 0 \quad \Leftrightarrow \quad \forall g \in T_{\max} : W(f,g)(a) = 0.$$

A similar result holds at b.

Proof. The Wronskian of two functions f, g which lie in $T_{\max}$ near a is constant on (a,α_ϱ) by the Lagrange identity. Hence we have

$$W(f,g)(a) = \lim_{x\uparrow\alpha_\varrho} \Big(f(x)g^{[1]}(x) - f^{[1]}(x)g(x) \Big).$$

Now the claim follows since we may find some g which lies in $T_{\max}$ near a, with prescribed left-hand limits at α_ϱ. Indeed one may take g to be a suitable solution of $\tau u = 0$. $\square$

Note that all functions in $T_{\min}$ vanish outside of $(\alpha_\varrho,\beta_\varrho)$. In general the operator $T_{\min}$ is, because of

$$\operatorname{dom}(T_{\min})^{\perp} = \operatorname{mul}(T_{\min}^*) = \operatorname{mul}(T_{\max})$$

not densely defined. On the other side $\operatorname{dom}(T_{\max})$ is always dense in $L^2((a,b);\varrho)$ since

$$\operatorname{dom}(T_{\max})^{\perp} = \operatorname{mul}(T_{\max}^*) = \operatorname{mul}(T_{\min}) = \{0\}.$$

Next we will show that $T_{\min}$ always has self-adjoint extensions.

Theorem 4.9. *The deficiency indices* $n(T_{\min})$ *of the minimal relation* $T_{\min}$ *are equal and at most two, i.e.*

$$n(T_{\min}) = \dim \operatorname{ran} (T_{\min} - \mathrm{i})^{\perp} = \dim \operatorname{ran} (T_{\min} + \mathrm{i})^{\perp} \leq 2.$$

Proof. The fact that the dimensions are less than two, is a consequence of the inclusion

$$\operatorname{ran}(T_{\min} \pm \mathrm{i})^{\perp} = \ker(T_{\max} \mp \mathrm{i}) \subseteq \ker(T_{\mathrm{loc}} \mp \mathrm{i}).$$

Now since $T_{\min}$ is real with respect to the natural conjugation in $L^2((a,b);\varrho)$, we see that the natural conjugation is a conjugate-linear isometry from the kernel of $T_{\max} + \mathrm{i}$ onto the kernel of $T_{\max} - \mathrm{i}$, hence their dimensions are equal. □

5. Weyl's alternative

We say τ is in the limit-circle (l.c.) case at a, if for each $z \in \mathbb{C}$ all solutions of $(\tau - z)u = 0$ lie in $L^2((a,b);\varrho)$ near a. Furthermore, we say τ is in the limit-point (l.p.) case at a if for each $z \in \mathbb{C}$ there is some solution of $(\tau - z)u = 0$ which does not lie in $L^2((a,b);\varrho)$ near a. Similar one defines the l.c. and l.p. cases for the endpoint b. It is clear that τ is only either l.c. or l.p. at some boundary point. The next lemma shows that τ indeed is in one of these cases at each endpoint.

Lemma 5.1. *If there is a* $z_0 \in \mathbb{C}$ *such that all solutions of* $(\tau - z_0)u = 0$ *lie in* $L^2((a,b);\varrho)$ *near* a*, then* τ *is in the l.c. case at* a*. A similar result holds at the endpoint* b*.*

Proof. Let $z \in \mathbb{C}$ and u be a solution of $(\tau - z)u = 0$. If u_1, u_2 are a fundamental system of $(\tau - z_0)u = 0$ with $W(u_1, u_2) = 1$, then u_1 and u_2 lie in $L^2((a,b);\varrho)$ near a by assumption. Therefore there is some $c \in (a,b)$ such that the function $v = |u_1| + |u_2|$ satisfies

$$|z - z_0| \int_a^c v^2 d\varrho \leq \frac{1}{2}.$$

Since u is a solution of $(\tau - z_0)u = (z - z_0)u$ we have for each $x \in (a,b)$

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + (z - z_0) \int_c^x (u_1(x)u_2(t) - u_1(t)u_2(x))\, u(t) d\varrho(t),$$

for some c_1, $c_2 \in \mathbb{C}$ by Proposition 3.3. Therefore we have with $C = \max(|c_1|, |c_2|)$

$$|u(x)| \leq C v(x) + |z - z_0| v(x) \int_x^c v(t)|u(t)| d\varrho(t), \quad x \in (a,c),$$

and furthermore, using Cauchy–Schwarz

$$|u(x)|^2 \leq 2C^2 v(x)^2 + 2|z - z_0|^2 v(x)^2 \int_x^c v(t)^2 d\varrho(t) \int_x^c |u(t)|^2 d\varrho(t).$$

Now an integration yields for each $s \in (a,c)$

$$\begin{aligned}\int_s^c |u|^2 d\varrho &\leq 2C^2 \int_a^c v^2 d\varrho + 2|z - z_0|^2 \left(\int_a^c v^2 d\varrho \right)^2 \int_s^c |u|^2 d\varrho \\ &\leq 2C^2 \int_a^c v^2 d\varrho + \frac{1}{2} \int_s^c |u|^2 d\varrho,\end{aligned}$$

and therefore

$$\int_s^c |u|^2 d\varrho \le 4C^2 \int_a^c v^2 d\varrho < \infty.$$

Since $s \in (a,c)$ was arbitrary, this yields the claim. □

Theorem 5.2 (Weyl's alternative)**.** *Each boundary point is either in the l.c. case or in the l.p. case.*

Proposition 5.3. *If τ is regular at a or if ϱ has no weight near a, then τ is in the l.c. case at a. Similar results hold at the endpoint b.*

Proof. If τ is regular at a each solution of $(\tau - z)u = 0$ can be continuously extended to a. Hence u is in $L^2((a,b);\varrho)$ near a, since ϱ is a finite measure near a. If ϱ has no weight near a, each solution lies in $L^2((a,b);\varrho)$ near a, since every solution is locally bounded. □

The set $\mathrm{r}(T_{\min})$ of points of regular type of $T_{\min}$ consists of all complex numbers $z \in \mathbb{C}$ such that $(T_{\min} - z)^{-1}$ is a bounded operator (not necessarily everywhere defined). Recall that $\dim \mathrm{ran}(T_{\min} - z)^{\perp}$ is constant on every connected component of $\mathrm{r}(T_{\min})$ ([37, Thm. 8.1]) and thus $\dim \mathrm{ran}(T_{\min} - z)^{\perp} = \dim \ker(T_{\max} - z^*) = n(T_{\min})$ for every $z \in \mathrm{r}(T_{\min})$.

Lemma 5.4. *For each $z \in \mathrm{r}(T_{\min})$ there is a non-trivial solution of $(\tau - z)u = 0$ which lies in $L^2((a,b);\varrho)$ near a. Similar result holds for the endpoint b.*

Proof. Let $z \in \mathrm{r}(T_{\min})$. First assume τ is regular at b. If there were no solution of $(\tau - z)u = 0$ which lies in $L^2((a,b);\varrho)$ near a, we had $\ker(T_{\max} - z) = \{0\}$ and hence $n(T_{\min}) = 0$, i.e. $T_{\min} = T_{\max}$. But since there is an $f \in T_{\max}$ with

$$f(b) = 1 \quad \text{and} \quad f^{[1]}(b) = 0,$$

this is a contradiction to Theorem 4.6.

In the general case we take some $c \in (a,b)$ and consider the minimal operator T_c in $L^2((a,c);\varrho)$ induced by $\tau|_{(a,c)}$. Then z is a point of regular type of T_c. Indeed we can extend each $f_c \in \mathrm{dom}\,(T_c)$ with zero and obtain a function $f \in \mathrm{dom}\,(T_{\min})$. For these functions and some positive constant C we have

$$\|(T_c - z)f_c\|_c = \|(T_{\min} - z)f\| \ge C\,\|f\| = C\,\|f_c\|_c\,,$$

where $\|\cdot\|_c$ is the norm on $L^2((a,c);\varrho)$. Now since the solutions of $(\tau|_{(a,c)} - z)u = 0$ are exactly the solutions of $(\tau - z)u = 0$ restricted to (a,c), the claim follows from what we already proved. □

Corollary 5.5. *If $z \in \mathrm{r}(T_{\min})$ and τ is in the l.p. case at a, then there is a (up to scalar multiples) unique non-trivial solution of $(\tau - z)u = 0$, which lies in $L^2((a,b);\varrho)$ near a. A similar result holds for the endpoint b.*

Proof. If there were two linearly independent solutions in $L^2((a,b);\varrho)$ near a, τ would be l.c. at a. □

Lemma 5.6. *τ is in the l.p. case at a if and only if*

$$W(f,g)(a) = 0, \quad f,\ g \in T_{\max}.$$

τ is in the l.c. case at a if and only if there is a $f \in T_{\max}$ such that

$$W(f,f^*)(a) = 0 \quad \textit{and} \quad W(f,g)(a) \ne 0 \quad \textit{for some } g \in T_{\max}.$$

Similar results hold at the endpoint b.

Proof. Let τ be in the l.c. case at a and u_1, u_2 be a real fundamental system of $\tau u = 0$ with $W(u_1, u_2) = 1$. Both, u_1 and u_2 lie in $T_{\max}$ near a. Hence there are f, $g \in T_{\max}$ with $f = u_1$ and $g = u_2$ near a and $f = g = 0$ near b. Then we have

$$W(f,g)(a) = W(u_1,u_2)(a) = 1 \neq 0$$

and

$$W(f,f^*)(a) = W(u_1,u_1^*)(a) = 0,$$

since u_1 is real.

Now assume τ is in the l.p. case at a and regular at b. Then $T_{\max}$ is a two-dimensional extension of $T_{\min}$, since $\dim\ker(T_{\max} - \mathrm{i}) = 1$ by Corollary 5.5. Let v, $w \in T_{\max}$ with $v = w = 0$ in a vicinity of a and

$$v(b) = w^{[1]}(b) = 1 \quad \text{and} \quad v^{[1]}(b) = w(b) = 0.$$

Then

$$T_{\max} = T_{\min} + \mathrm{span}\{v, w\},$$

since v and w are linearly independent modulo $T_{\min}$ and do not lie in $T_{\min}$. Then for each f, $g \in T_{\max}$ there are f_0, $g_0 \in T_{\min}$ such that $f = f_0$ and $g = g_0$ in a vicinity of a and therefore

$$W(f,g)(a) = W(f_0,g_0)(a) = 0.$$

Now if τ is not regular at b we take some $c \in (a,b)$. Then for each $f \in T_{\max}$, $f|_{(a,c)}$ lies in the maximal relation induced by $\tau|_{(a,c)}$ and the claim follows from what we already proved. □

Lemma 5.7. *Let τ be in the l.p. case at both endpoints and $z \in \mathbb{C}\backslash\mathbb{R}$. Then there is no non-trivial solution of $(\tau - z)u = 0$ in $L^2((a,b);\varrho)$.*

Proof. If $u \in L^2((a,b);\varrho)$ is a solution of $(\tau - z)u = 0$, then u^* is a solution of $(\tau - z^*)u = 0$ and both u and u^* lie in $T_{\max}$. Now the Lagrange identity yields

$$W(u,u^*)(\beta) - W(u,u^*)(\alpha) = (z - z^*)\int_\alpha^\beta uu^* d\varrho = 2\mathrm{i}\mathrm{Im}(z)\int_\alpha^\beta |u|^2 d\varrho.$$

If $\alpha \to a$ and $\beta \to b$, the left-hand side converges to zero by Lemma 5.6 and the right-hand side converges to $2\mathrm{i}\mathrm{Im}(z)\|u\|^2$, hence $\|u\| = 0$. □

Theorem 5.8. *The deficiency indices of the minimal relation $T_{\min}$ are given by*

$$n(T_{\min}) = \begin{cases} 0, & \textit{if } \tau \textit{ is l.c. at no boundary point}, \\ 1, & \textit{if } \tau \textit{ is l.c. at exactly one boundary point}, \\ 2, & \textit{if } \tau \textit{ is l.c. at both boundary points}. \end{cases}$$

Proof. If τ is in the l.c. case at both endpoints, all solutions of $(\tau - \mathrm{i})u = 0$ lie in $L^2((a,b);\varrho)$ and hence in $T_{\max}$. Therefore $n(T_{\min}) = \dim\ker(T_{\max} - \mathrm{i}) = 2$.

In the case when τ is in the l.c. case at exactly one endpoint, there is (up to scalar multiples) exactly one non-trivial solution of $(\tau - \mathrm{i})u = 0$ in $L^2((a,b);\varrho)$, by Corollary 5.5.

Now if τ is in the l.p. case at both endpoints, we have $\ker(T_{\max} - \mathrm{i}) = \{0\}$ by Lemma 5.7 and hence $n(T_{\min}) = 0$. □

6. Self-adjoint relations

We are interested in the self-adjoint restrictions of $T_{\max}$ (or equivalent the self-adjoint extensions of $T_{\min}$). To this end we introduce the convenient short-hand notation

$$W_a^b(f,g) = W(f,g)(b) - W(f,g(a) \tag{6.1}$$

Theorem 6.1. *Some relation S is a self-adjoint restriction of $T_{\max}$ if and only if*

$$S = \left\{ f \in T_{\max} \,|\, \forall g \in S : W_a^b(f,g^*) = 0 \right\}.$$

Proof. We denote the right-hand side by S_0. First assume S is a self-adjoint restriction of $T_{\max}$. If $f \in S$ then

$$0 = \langle \tau f, g\rangle - \langle f, \tau g\rangle = W_a^b(f,g^*)$$

for each $g \in S$. Now if $f \in S_0$, then

$$0 = W_a^b(f,g^*) = \langle \tau f, g\rangle - \langle f, \tau g\rangle$$

for each $g \in S$, hence $f \in S^* = S$.

Conversely assume $S = S_0$ then S is symmetric since we have $\langle \tau f, g\rangle = \langle f, \tau g\rangle$ for each f, $g \in S$. Now let $f \in S^* \subseteq T_{\min}^* = T_{\max}$, then

$$0 = \langle \tau f, g\rangle - \langle f, \tau g\rangle = W_a^b(f,g^*),$$

for each $g \in S$ and hence $f \in S_0 = S$. □

The aim of this section is, to determine all self-adjoint restrictions of $T_{\max}$. If both endpoints are in the l.p. case this is an immediate consequence of Theorem 5.8.

Theorem 6.2. *If τ is in the l.p. case at both endpoints then $T_{\min} = T_{\max}$ is a self-adjoint operator.*

Next we turn to the case when one endpoint is in the l.c. case and the other is in the l.p. case. But before we do this, we need some more properties of the Wronskian.

Lemma 6.3. *Let $v \in T_{\max}$ such that $W(v,v^*)(a) = 0$ and suppose there is an $h \in T_{\max}$ with $W(h,v^*)(a) \neq 0$. Then for each f, $g \in T_{\max}$ we have*

$$W(f,v^*)(a) = 0 \quad \Leftrightarrow \quad W(f^*,v^*)(a) = 0 \tag{6.2}$$

and

$$W(f,v^*)(a) = W(g,v^*)(a) = 0 \quad \Rightarrow \quad W(f,g)(a) = 0. \tag{6.3}$$

Similar results hold at the endpoint b.

Proof. Choosing $f_1 = v$, $f_2 = v^*$, $f_3 = h$ and $f_4 = h^*$ in the Plücker identity, we see that also $W(h,v)(a) \neq 0$. Now let $f_1 = f$, $f_2 = v$, $f_3 = v^*$ and $f_4 = h$, then the Plücker identity yields (6.2), whereas $f_1 = f$, $f_2 = g$, $f_3 = v^*$ and $f_4 = h$ yields (6.3). □

Theorem 6.4. *Suppose τ is in the l.c. case at a and in the l.p. case at b. Then some relation S is a self-adjoint restriction of $T_{\max}$ if and only if there is a $v \in T_{\max} \backslash T_{\min}$ with $W(v,v^*)(a) = 0$ such that*

$$S = \left\{ f \in T_{\max} \,|\, W(f,v^*)(a) = 0 \right\}.$$

A similar result holds if τ is in the l.c. case at b and in the l.p. case at a.

Proof. Because of $n(T_{\min}) = 1$ the self-adjoint extensions of $T_{\min}$ are precisely the one-dimensional, symmetric extensions of $T_{\min}$. Hence some relation S is a self-adjoint extension of $T_{\min}$ if and only if there is an $v \in T_{\max}\backslash T_{\min}$ with $W(v,v^*)(a) = 0$ such that

$$S = \operatorname{span}\{T_{\min} \cup \{v\}\}.$$

Hence we have to prove that

$$\operatorname{span}\{T_{\min} \cup \{v\}\} = \{f \in T_{\max} \,|\, W(f,v^*)(a) = 0\}.$$

The subspace on the left-hand side is included in the right one because of Theorem 4.6 and $W(v,v^*)(a) = 0$. But if the subspace on the right-hand side were larger, it were equal to $T_{\max}$ and hence would imply $v \in T_{\min}$. □

Two self-adjoint restrictions are distinct if and only if the corresponding functions v are linearly independent modulo $T_{\min}$. Furthermore, v can always be chosen such that v is equal to some real solution of $(\tau - z)u = 0$ with $z \in \mathbb{R}$ in some vicinity of a. By Lemma 6.3 one sees that all these self-adjoint restrictions are real with respect to the natural conjugation in $L^2((a,b);\varrho)$.

In contrast to the classical theory, not all of this self-adjoint restrictions of $T_{\max}$ are operators. We will determine which of them are multi-valued in the next section.

It remains to consider the case when both endpoints are in the l.c. case.

Theorem 6.5. *Suppose τ is in the l.c. case at both endpoints. Then some relation S is a self-adjoint restriction of $T_{\max}$ if and only if there are v, $w \in T_{\max}$, linearly independent modulo $T_{\min}$, with*

$$W_a^b(v,v^*) = W_a^b(w,w^*) = W_a^b(v,w^*) = 0 \tag{6.4}$$

such that

$$S = \left\{f \in T_{\max} \,|\, W_a^b(f,v^*) = W_a^b(f,w^*) = 0\right\}.$$

Proof. Since $n(T_{\min}) = 2$ the self-adjoint extensions of $T_{\min}$ are precisely the two-dimensional, symmetric extensions of $T_{\min}$. Hence a relation S is a self-adjoint restriction of $T_{\max}$ if and only if there are v, $w \in T_{\max}$, linearly independent modulo $T_{\min}$, with (6.4) such that

$$S = \operatorname{span}\{T_{\min} \cup \{v,w\}\}.$$

Therefore we have to prove that

$$\operatorname{span}\{T_{\min} \cup \{v,w\}\} = \left\{f \in T_{\max} \,|\, W_a^b(f,v^*) = W_a^b(f,w^*) = 0\right\} = T,$$

where we denote the subspace on the right-hand side by T. Indeed the subspace on the left-hand side is contained in T by Theorem 4.6 and (6.4). In order to prove that it is also not larger, consider the linear functionals F_v, F_w on $T_{\max}$ defined by

$$F_v(f) = W_a^b(f,v^*) \quad \text{and} \quad F_w(f) = W_a^b(f,w^*) \quad \text{for } f \in T_{\max}.$$

The intersection of the kernels of these functionals is precisely T. Furthermore, these functionals are linearly independent. Indeed assume c_1, $c_2 \in \mathbb{C}$ and $c_1 F_v + c_2 F_w = 0$, then for all $f \in T_{\max}$ we have

$$0 = c_1 F_v(f) + c_2 F_w(f) = c_1 W_a^b(f,v^*) + c_2 W_a^b(f,w^*) = W_a^b(f, c_1 v^* + c_2 w^*).$$

But by Lemma 4.5 this yields

$$W(f, c_1 v^* + c_2 w^*)(a) = W(f, c_1 v^* + c_2 w^*)(b) = 0$$

for all $f \in T_{\max}$ and hence $c_1 v^* + c_2 w^* \in T_{\min}$. Now since v, w are linearly independent modulo $T_{\min}$ we get $c_1 = c_2 = 0$. Now from Lemma 4.3 we infer that

$$\ker F_v \not\subseteq \ker F_w \quad \text{and} \quad \ker F_w \not\subseteq \ker F_v.$$

Hence there exist f_v, $f_w \in T_{\max}$ such that $W_a^b(f_v, v^*) = W_a^b(f_w, w^*) = 0$ but $W_a^b(f_v, w^*) \neq 0$ and $W_a^b(f_w, v^*) \neq 0$. Both, f_v and f_w do not lie in T and are linearly independent, hence T is at most a two-dimensional extension of $T_{\min}$. □

In the case when τ is in the l.c. case at both endpoints, we may divide the self-adjoint restrictions of $T_{\max}$ into two classes. Indeed we say some relation is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions if it is of the form

$$S = \{ f \in T_{\max} \,|\, W(f, v^*)(a) = W(f, w^*)(b) = 0 \}$$

for some v, $w \in T_{\max} \backslash T_{\min}$ with $W(v, v^*)(a) = W(w, w^*)(b) = 0$. Conversely each relation of this form is a self-adjoint restriction of $T_{\max}$ by Theorem 6.5 and Lemma 4.5. The remaining self-adjoint restrictions are called self-adjoint restrictions of $T_{\max}$ with coupled boundary conditions.

From Lemma 6.3 one sees that all self-adjoint restrictions of $T_{\max}$ with separate boundary conditions are real with respect to the natural conjugation in $L^2((a,b);\varrho)$. In the case of coupled boundary conditions in general this is not the case Again we will determine the self-adjoint restrictions which are multi-valued in the next section.

7. Boundary conditions

In this section let w_1, $w_2 \in T_{\max}$ with

$$W(w_1, w_2^*)(a) = 1 \quad \text{and} \quad W(w_1, w_1^*)(a) = W(w_2, w_2^*)(a) = 0, \tag{7.1a}$$

if τ is in the l.c. case at a and

$$W(w_1, w_2^*)(b) = 1 \quad \text{and} \quad W(w_1, w_1^*)(b) = W(w_2, w_2^*)(b) = 0, \tag{7.1b}$$

if τ is in the l.c. case at b. We will describe the self-adjoint restrictions of $T_{\max}$ in terms of the linear functionals BC_a^1, BC_a^2, BC_b^1 and BC_b^2 on $T_{\max}$, defined by

$$BC_a^1(f) = W(f, w_2^*)(a) \quad \text{and} \quad BC_a^2(f) = W(w_1^*, f)(a) \quad \text{for } f \in T_{\max},$$

if τ is in the l.c. case at a and

$$BC_b^1(f) = W(f, w_2^*)(b) \quad \text{and} \quad BC_b^2(f) = W(w_1^*, f)(b) \quad \text{for } f \in T_{\max},$$

if τ is in the l.c. case at b.

If τ is in the l.c. case at some endpoint, functions with (7.1a) (resp. with (7.1b)) always exist. Indeed one may take them to coincide near the endpoint with some real solutions of $(\tau - z)u = 0$ with $W(u_1, u_2) = 1$ for some $z \in \mathbb{R}$ and use Lemma 4.5.

In the regular case these functionals may take the form of point evaluations of the function and its quasi-derivative at the boundary point.

Proposition 7.1. *Suppose τ is regular at a. Then there are w_1, $w_2 \in T_{\max}$ with* (7.1a) *such that the corresponding linear functionals BC_a^1 and BC_a^2 satisfy*

$$BC_a^1(f) = f(a) \quad \text{and} \quad BC_a^2(f) = f^{[1]}(a) \quad \text{for } f \in T_{\max}.$$

A similar result holds at the endpoint b.

Proof. Take w_1, $w_2 \in T_{\max}$ to coincide near a with the real solutions u_1, u_2 of $\tau u = 0$ with

$$u_1(a) = u_2^{[1]}(a) = 1 \quad \text{and} \quad u_1^{[1]}(a) = u_2(a) = 0.$$

□

Also if ϱ has no weight near some endpoint we may choose special functionals.

Proposition 7.2. *If $\alpha_\varrho > a$ then there are w_1, $w_2 \in T_{\max}$ with* (7.1a) *such that the corresponding linear functionals BC_a^1 and BC_a^2 satisfy*

$$BC_a^1(f) = f(\alpha_\varrho-) \quad \textit{and} \quad BC_a^2(f) = f^{[1]}(\alpha_\varrho-) \quad \textit{for } f \in T_{\max}.$$

A similar result holds at the endpoint b.

Proof. Take w_1, $w_2 \in T_{\max}$ to coincide near a with the real solutions u_1, u_2 of $\tau u = 0$ with

$$u_1(\alpha_\varrho-) = u_2^{[1]}(\alpha_\varrho-) = 1 \quad \text{and} \quad u_1^{[1]}(\alpha_\varrho-) = u_2(\alpha_\varrho-) = 0.$$

Then since the Wronskian is constant in (a, α_ϱ), we get

$$BC_a^1(f) = W(f, u_2)(\alpha_\varrho-) = f(\alpha_\varrho-)$$

and

$$BC_a^2(f) = W(u_1, f)(\alpha_\varrho-) = f^{[1]}(\alpha_\varrho-),$$

for each $f \in T_{\max}$. □

Using the Plücker identity one easily obtains the equality

$$W(f, g)(a) = BC_a^1(f)BC_a^2(g) - BC_a^2(f)BC_a^1(g), \quad f,\ g \in T_{\max}.$$

Furthermore, for each $v \in T_{\max}\backslash T_{\min}$ with $W(v, v^*)(a) = 0$ one may show that there is a $\varphi_\alpha \in [0, \pi)$ such that

$$W(f, v^*)(a) = 0 \quad \Leftrightarrow \quad BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0, \quad f \in T_{\max}. \tag{7.2}$$

Conversely if some $\varphi_\alpha \in [0, \pi)$ is given, then there is some $v \in T_{\max}\backslash T_{\min}$ with $W(v, v^*)(a) = 0$ such that

$$W(f, v^*)(a) = 0 \quad \Leftrightarrow \quad BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0, \quad f \in T_{\max}. \tag{7.3}$$

Using this, Theorem 6.4 immediately yields the following characterization of the self-adjoint restrictions of $T_{\max}$ in terms of the boundary functionals.

Theorem 7.3. *Suppose τ is in the l.c. case at a and in the l.p. case at b. Then some relation S is a self-adjoint restriction of $T_{\max}$ if and only if*

$$S = \left\{ f \in T_{\max} \,\middle|\, BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0 \right\},$$

for some $\varphi_\alpha \in [0, \pi)$. A similar results holds if τ is in the l.c. case at b and in the l.p. case at a.

Now we will determine which self-adjoint restrictions of $T_{\max}$ are multi-valued. Of course we only have to consider the case when $\alpha_\varrho > a$ and ϱ has mass in α_ϱ.

Corollary 7.4. *Suppose ϱ has mass in α_ϱ and τ is in the l.p. case at b. Then some self-adjoint restriction S of $T_{\max}$ as in Theorem 7.3 is an operator if and only if*

$$\cos\varphi_\alpha w_2(\alpha_\varrho-) + \sin\varphi_\alpha w_1(\alpha_\varrho-) \neq 0. \tag{7.4}$$

A similar result holds for the endpoint b.

Proof. Assume (7.4) does not hold, then consider for each $c \in \mathbb{C}$ the functions

$$f_c(x) = \begin{cases} cu_a(x), & \text{if } a < x \leq \alpha_\varrho \\ 0, & \text{if } \alpha_\varrho < x < b, \end{cases} \tag{7.5}$$

where u_a is a solution of $\tau u = 0$ with $u_a(\alpha_\varrho) = 0$ and $u_a^{[1]}(\alpha_\varrho) = 1$. These functions lie in S with $\tau f_c \neq 0$, hence S is multi-valued. Conversely assume (7.4) holds and let $f \in S$ such that $f = 0$ and $\tau f = 0$ almost everywhere with respect to ϱ. Then f is of the form (7.5). But because of the boundary condition

$$c = f^{[1]}(\alpha_\varrho) = f(\alpha_\varrho)\frac{\cos\varphi_\alpha w_2^{[1]}(\alpha_\varrho)^* + \sin\varphi_\alpha w_1^{[1]}(\alpha_\varrho)^*}{\cos\varphi_\alpha w_2(\alpha_\varrho)^* + \sin\varphi_\alpha w_1(\alpha_\varrho)^*} = 0,$$

i.e. $f = 0$. □

Note that in this case there is precisely one multi-valued, self-adjoint restriction S of $T_{\max}$. In terms of the boundary functionals from Proposition 7.2 it is precisely the one with $\varphi_\alpha = 0$. That means that in this case each function in S vanishes in α_ϱ. Now since ϱ has mass in this point one sees that the domain of S is not dense and hence S is not an operator. However if we excluding the linear span of $\mathbb{1}_{\alpha_\varrho}$ from $L^2((a,b);\varrho)$ by setting

$$\mathfrak{D} = \overline{\operatorname{dom}(S)} = L^2((a,b);\varrho) \ominus \operatorname{span}\{\mathbb{1}_{\{\alpha_\varrho\}}\},$$

the linear relation $S_{\mathfrak{D}}$ in the Hilbert space $\mathfrak{D}$, given by

$$S_{\mathfrak{D}} = S \cap (\mathfrak{D} \times \mathfrak{D}),$$

is a self-adjoint operator (see (B.4) in Appendix B). Also note that if $\tilde{\tau}$ is obtained from τ by removing the point measure in α_ϱ from the measure ϱ, then $S_{\mathfrak{D}}$ is a self-adjoint restriction of the maximal relation corresponding to $\tilde{\tau}$.

Next we will give a characterization of the self-adjoint restrictions of $T_{\max}$, if τ is in the l.c. case at both endpoints.

Theorem 7.5. *Suppose τ is in the l.c. case at both endpoints. Then some relation S is a self-adjoint restriction of $T_{\max}$ if and only if there are matrices B_a, $B_b \in \mathbb{C}^{2\times 2}$ with*

$$\operatorname{rank}(B_a|B_b) = 2 \quad \textit{and} \quad B_a J B_a^* = B_b J B_b^* \quad \textit{with} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{7.6}$$

such that

$$S = \left\{ f \in T_{\max} \,\middle|\, B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} \right\}.$$

Proof. If S is a self-adjoint restriction of $T_{\max}$, there exist v, $w \in T_{\max}$, linearly independent modulo $T_{\min}$, with

$$W_a^b(v,v^*) = W_a^b(w,w^*) = W_a^b(v,w^*) = 0,$$

such that

$$S = \left\{ f \in T_{\max} \,\middle|\, W_a^b(f,v^*) = W_a^b(f,w^*) = 0 \right\}.$$

Let B_a, $B_b \in \mathbb{C}^{2\times 2}$ be defined by

$$B_a = \begin{pmatrix} BC_a^2(v^*) & -BC_a^1(v^*) \\ BC_a^2(w^*) & -BC_a^1(w^*) \end{pmatrix} \quad \text{and} \quad B_b = \begin{pmatrix} BC_b^2(v^*) & -BC_b^1(v^*) \\ BC_b^2(w^*) & -BC_b^1(w^*) \end{pmatrix}.$$

Then a simple computation shows that

$$B_a J B_a^* = B_b J B_b^* \quad \Leftrightarrow \quad W_a^b(v,v^*) = W_a^b(w,w^*) = W_a^b(v,w^*) = 0.$$

In order to prove rank $(B_a|B_b) = 2$, let c_1, $c_2 \in \mathbb{C}$ and

$$0 = c_1 \begin{pmatrix} BC_a^2(v^*) \\ -BC_a^1(v^*) \\ BC_b^2(v^*) \\ -BC_b^1(v^*) \end{pmatrix} + c_2 \begin{pmatrix} BC_a^2(w^*) \\ -BC_a^1(w^*) \\ BC_b^2(w^*) \\ -BC_b^1(w^*) \end{pmatrix} = \begin{pmatrix} BC_a^2(c_1v^* + c_2w^*) \\ -BC_a^1(c_1v^* + c_2w^*) \\ BC_b^2(c_1v^* + c_2w^*) \\ -BC_b^1(c_1v^* + c_2w^*) \end{pmatrix}.$$

Hence the function $c_1v^* + c_2w^*$ lies in the kernel of BC_a^1, BC_a^2, BC_b^1 and BC_b^2, therefore $W(c_1v^*+c_2w^*, f)(a) = 0$ und $W(c_1v^*+c_2w^*, f)(b) = 0$ for each $f \in T_{\max}$. This means that $c_1v^*+c_2w^* \in T_{\min}$ and hence $c_1 = c_2 = 0$, since v^*, w^* are linearly independent modulo $T_{\min}$. This proves that $(B_a|B_b)$ has rank two. Furthermore, a calculation yields that for $f \in T_{\max}$

$$W_a^b(f,v^*) = W_a^b(f,w^*) = 0 \quad \Leftrightarrow \quad B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix},$$

which proves that S is given as in the claim.

Conversely let B_a, $B_b \in \mathbb{C}^{2\times 2}$ with the claimed properties be given. Then there are v, $w \in T_{\max}$ such that

$$B_a = \begin{pmatrix} BC_a^2(v^*) & -BC_a^1(v^*) \\ BC_a^2(w^*) & -BC_a^1(w^*) \end{pmatrix} \quad \text{and} \quad B_b = \begin{pmatrix} BC_b^2(v^*) & -BC_b^1(v^*) \\ BC_b^2(w^*) & -BC_b^1(w^*) \end{pmatrix}.$$

In order to prove that v and w are linearly independent modulo $T_{\min}$, let c_1, $c_2 \in \mathbb{C}$ and $c_1v + c_2w \in T_{\min}$, then

$$0 = \begin{pmatrix} BC_a^2(c_1^*v^* + c_2^*w^*) \\ -BC_a^1(c_1^*v^* + c_2^*w^*) \\ BC_b^2(c_1^*v^* + c_2^*w^*) \\ -BC_b^1(c_1^*v^* + c_2^*w^*) \end{pmatrix} = c_1^* \begin{pmatrix} BC_a^2(v^*) \\ -BC_a^1(v^*) \\ BC_b^2(v^*) \\ -BC_b^1(v^*) \end{pmatrix} + c_2^* \begin{pmatrix} BC_a^2(w^*) \\ -BC_a^1(w^*) \\ BC_b^2(w^*) \\ -BC_b^1(w^*) \end{pmatrix}.$$

Now the rows of $(B_a|B_b)$ are linearly independent, hence $c_1 = c_2 = 0$. Since again we have

$$B_a J B_a^* = B_b J B_b^* \quad \Leftrightarrow \quad W_a^b(v,v^*) = W_a^b(w,w^*) = W_a^b(v,w^*) = 0,$$

the functions v, w satisfy the assumptions of Theorem 6.5. As above one sees again that for $f \in T_{\max}$

$$B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} \quad \Leftrightarrow \quad W_a^b(f,w^*) = W_a^b(f,w^*) = 0.$$

Hence S is a self-adjoint restriction of $T_{\max}$ by Theorem 6.5. □

As in the preceding section, if τ is in the l.c. case at both endpoints, we may divide the self-adjoint restrictions of $T_{\max}$ into two classes.

Theorem 7.6. *Suppose τ is in the l.c. case at both endpoints. Then some relation S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions if and only if there are φ_α, $\varphi_\beta \in [0,\pi)$ such that*

$$S = \left\{ f \in T_{\max} \;\middle|\; \begin{array}{c} BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0 \\ BC_b^1(f)\cos\varphi_\beta - BC_b^2(f)\sin\varphi_\beta = 0 \end{array} \right\}. \tag{7.7}$$

Furthermore, S is a self-adjoint restriction of $T_{\max}$ with coupled boundary conditions if and only if there are $\varphi \in [0,\pi)$ and $R \in \mathbb{R}^{2\times 2}$ with $\det R = 1$ such that

$$S = \left\{ f \in T_{\max} \,\middle|\, \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = e^{\mathrm{i}\varphi} R \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} \right\}. \tag{7.8}$$

Proof. Using (7.2) and (7.3) one easily sees that the self-adjoint restrictions of $T_{\max}$ are precisely the ones given in (7.7). Hence we only have to prove the second claim. Let S be a self-adjoint restriction of $T_{\max}$ with coupled boundary conditions and B_a, $B_b \in \mathbb{C}^{2\times 2}$ matrices as in Theorem 7.5. Then by (7.6) either both of them have rank one or both have rank two. In the first case we had

$$B_a z = c_a^T z w_a \quad \text{and} \quad B_b z = c_b^T z w_b$$

for some c_a, c_b, w_a, $w_b \in \mathbb{C}^2 \backslash \{(0,0)\}$. Since the vectors w_a and w_b are linearly independent by $\operatorname{rank}(B_a|B_b) = 2$ we have

$$B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} \quad \Leftrightarrow \quad B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = 0$$

In particular

$$B_a J B_a^* = B_b J B_b^* \quad \Leftrightarrow \quad B_a J B_a^* = B_b J B_b^* = 0.$$

Now let $v \in T_{\max}$ with $BC_a^2(v^*) = c_1$ and $BC_a^1(v^*) = -c_2$. A simple calculation yields

$$\begin{aligned} 0 = B_a J B_a^* &= W(w_1, w_2)(a)(BC_a^1(v)BC_a^2(v^*) - BC_a^2(v)BC_a^1(v^*)) w_a w_a^{*T} \\ &= W(w_1,w_2)(a) W(v, v^*)(a) w_a w_a^{*T}. \end{aligned}$$

Hence $W(v,v^*)(a) = 0$ and since $(BC_a^1(v), BC_a^2(v)) = (c_2, c_1) \neq 0$, $v \notin T_{\min}$. Furthermore, for each $f \in T_{\max}$ we have

$$B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = (BC_a^1(f)BC_a^2(v^*) - BC_a^2(f)BC_a^1(v^*)) w_a = W(f, v^*)(a) w_a.$$

Similarly one gets a function $f \in T_{\max} \backslash T_{\min}$ with $W(w,w^*)(b) = 0$ and

$$B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = W(f, w^*)(b) w_b, \quad f \in T_{\max}.$$

But this shows that S were a self-adjoint restriction with separate boundary conditions.

Hence both matrices, B_a and B_b have rank two. If we set $B = B_b^{-1} B_a$, then $B = J(B^{-1})^* J^*$ and therefore $|\det B| = 1$, hence $\det B = e^{2i\varphi}$ for some $\varphi \in [0,\pi)$. If we set $R = e^{-i\varphi} B$, one sees from the equation

$$\begin{aligned} B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} &= J(B^{-1})^* J^* = e^{2i\varphi} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_{22}^* & -b_{21}^* \\ -b_{12}^* & b_{11}^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= e^{2i\varphi} \begin{pmatrix} b_{11}^* & b_{12}^* \\ b_{21}^* & b_{22}^* \end{pmatrix}, \end{aligned}$$

that $R \in \mathbb{R}^{2\times 2}$ with $\det R = 1$. Now because we have for each $f \in T_{\max}$

$$B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = e^{\mathrm{i}\varphi} R \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix},$$

S has the claimed representation.

Conversely if S is of the form (7.8), then Theorem 7.5 shows that it is a self-adjoint restriction of $T_{\max}$. Now if S were a self-adjoint restriction with separate boundary conditions, we had an $f \in S\backslash T_{\min}$, vanishing in some vicinity of a. By the boundary condition we also have $BC_b^1(f) = BC_b^2(f) = 0$, i.e. $f \in T_{\min}$. Hence S can't be a self-adjoint restriction with separate boundary conditions. $\square$

Now we will again determine the self-adjoint restrictions of $T_{\max}$ which are multi-valued. In the case of separate boundary conditions these are determined by whether

$$\cos\varphi_\alpha w_2(\alpha_\varrho-) + \sin\varphi_\alpha w_1(\alpha_\varrho-) \neq 0, \tag{7.9a}$$

$$\cos\varphi_\beta w_2(\beta_\varrho+) + \sin\varphi_\beta w_1(\beta_\varrho+) \neq 0, \tag{7.9b}$$

hold or not. Note that if one takes the functionals from Proposition 7.2, then (7.9a) (resp. (7.9b)) is equivalent to $\varphi_\alpha = 0$ (resp. $\varphi_\beta = 0$). We start with the case when $\dim \mathrm{mul}\,(T_{\max}) = 1$.

Corollary 7.7. *Suppose τ is in the l.c. case at both endpoints and ϱ has mass in α_ϱ but not in β_ϱ, i.e. $\dim \mathrm{mul}\,(T_{\max}) = 1$. Then for each self-adjoint restriction S of $T_{\max}$ with separate boundary conditions as in Theorem 7.6 we have*

$$\mathrm{mul}\,(S) = \begin{cases} \{0\}, & \text{if (7.9a) holds,} \\ \mathrm{span}\left\{\mathbb{1}_{\{\alpha_\varrho\}}\right\}, & \text{if (7.9a) does not hold.} \end{cases}$$

Furthermore, each self-adjoint restriction of $T_{\max}$ with coupled boundary conditions is an operator. Similar results hold if ϱ has mass in β_ϱ and no mass in α_ϱ.

Proof. If S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions then the claim follows as in the proof of Corollary 7.4.

Now let S be a self-adjoint restriction of $T_{\max}$ with coupled boundary conditions as in Theorem 7.6 and $f \in S$ with $f = 0$ and $\tau f = 0$ almost everywhere with respect to ϱ. Then again f is of the form (7.5). But because of the boundary condition this shows that $BC_a^1(f) = BC_a^2(f) = 0$, hence f vanishes everywhere. $\square$

The remark after Corollary 7.4 also holds literally here under the assumptions of Corollary 7.7. It remains to determine the self-adjoint restrictions of $T_{\max}$ which are multi-valued in the case when ϱ has mass in α_ϱ and in β_ϱ.

Corollary 7.8. *Suppose ϱ has mass in α_ϱ and in β_ϱ, i.e. $\dim \mathrm{mul}\,(T_{\max}) = 2$. If S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions as in Theorem 7.6, then*

$$\mathrm{mul}\,(S) = \begin{cases} \{0\}, & \text{if (7.9a) and (7.9b) hold,} \\ \mathrm{span}\left\{\mathbb{1}_{\{\alpha_\varrho\}}\right\}, & \text{if (7.9b) holds and (7.9a) does not,} \\ \mathrm{span}\left\{\mathbb{1}_{\{\beta_\varrho\}}\right\}, & \text{if (7.9a) holds and (7.9b) does not,} \\ \mathrm{span}\left\{\mathbb{1}_{\{\alpha_\varrho\}}, \mathbb{1}_{\{\beta_\varrho\}}\right\}, & \text{if neither (7.9a) nor (7.9b) holds.} \end{cases}$$

If S is a self-adjoint restriction of $T_{\max}$ with coupled boundary conditions as in Theorem 7.6 and

$$\tilde{R} = \begin{pmatrix} w_2^{[1]}(\beta_\varrho+)^* & -w_2(\beta_\varrho+)^* \\ -w_1^{[1]}(\beta_\varrho+)^* & w_1(\beta_\varrho+)^* \end{pmatrix}^{-1} R \begin{pmatrix} w_2^{[1]}(\alpha_\varrho-)^* & -w_2(\alpha_\varrho-)^* \\ -w_1^{[1]}(\alpha_\varrho-)^* & w_1(\alpha_\varrho-)^* \end{pmatrix},$$

then

$$\operatorname{mul}(S) = \begin{cases} \{0\}, & \text{if } \tilde{R}_{12} \neq 0, \\ \operatorname{span}\left\{ \mathbb{1}_{\{\alpha_\varrho\}} + e^{\mathrm{i}\varphi} \tilde{R}_{22} \mathbb{1}_{\{\beta_\varrho\}} \right\}, & \text{if } \tilde{R}_{12} = 0. \end{cases}$$

Proof. If S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions, the claim follows as in the proof of Corollary 7.4.

In order to prove the second part let S be a self-adjoint restriction of $T_{\max}$ with coupled boundary conditions. The boundary condition can be written as

$$\begin{pmatrix} f(\beta_\varrho+) \\ f^{[1]}(\beta_\varrho+) \end{pmatrix} = e^{\mathrm{i}\varphi} \tilde{R} \begin{pmatrix} f(\alpha_\varrho-) \\ f^{[1]}(\alpha_\varrho-) \end{pmatrix}, \quad f \in S.$$

Now assume $\tilde{R}_{12} \neq 0$ and $f \in S$ with $f = 0$ almost everywhere with respect to ϱ. Then because of the boundary condition we have $f^{[1]}(\alpha_\varrho-) = f^{[1]}(\beta_\varrho+) = 0$, i.e. $f = 0$. If we assume $\tilde{R}_{12} = 0$ then the boundary condition becomes

$$f^{[1]}(\beta_\varrho+) = e^{\mathrm{i}\varphi} \tilde{R}_{22} f^{[1]}(\alpha_\varrho-), \quad f \in S.$$

Hence all functions $f \in S$ with $f = 0$ almost everywhere with respect to ϱ are of the form

$$f(x) = \begin{cases} c_a u_a(x), & \text{if } a < x \leq \alpha_\varrho, \\ 0, & \text{if } \alpha_\varrho < x \leq \beta_\varrho, \\ e^{\mathrm{i}\varphi} \tilde{R}_{22} c_a u_b(x), & \text{if } \beta_\varrho < x < b. \end{cases}$$

Conversely all functions of this form lie in S, which yields the claim. □

Note that if one uses the boundary functionals of Proposition 7.2, then $\tilde{R} = R$. In contrast to Corollary 7.7, in this case there is a multitude of multi-valued, self-adjoint restrictions S of $T_{\max}$. However, if we again restrict S to the closure $\mathfrak{D}$ of the domain of S by

$$S_{\mathfrak{D}} = S \cap (\mathfrak{D} \times \mathfrak{D}),$$

we obtain a self-adjoint operator in the Hilbert space $\mathfrak{D}$.

8. Spectrum and resolvent

In this section we will compute the resolvent of the self-adjoint restrictions S of $T_{\max}$. The resolvent set $\rho(S)$ is the set of all $z \in \mathbb{C}$ such that

$$R_z = (S - z)^{-1} = \left\{ (g, f) \in L^2((a,b);\varrho) \times L^2((a,b);\varrho) \,|\, (f, g) \in S \right\}$$

is an everywhere defined operator in $L^2((a,b);\varrho)$, i.e. $\operatorname{dom}(R_z) = L^2((a,b);\varrho)$ and $\operatorname{mul}(R_z) = \{0\}$. According to Theorem B.1, the resolvent set $\rho(S)$ is a non-empty, open subset of $\mathbb{C}$ and the resolvent $z \mapsto R_z$ is an analytic function of $\rho(S)$ into the space of bounded linear operators on $L^2((a,b);\varrho)$. Note that in general the operators R_z, $z \in \rho(S)$ are not injective, indeed we have

$$\ker(R_z) = \operatorname{mul}(S) = \operatorname{dom}(S)^\perp = \operatorname{ran}(R_z)^\perp, \quad z \in \rho(S). \tag{8.1}$$

First we deal with the case, when both endpoints are in the l.c. case.

Theorem 8.1. *Suppose τ is in the l.c. case at both endpoints and S is a self-adjoint restriction of $T_{\max}$. Then for each $z \in \rho(S)$ the resolvent R_z is an integral operator*

$$R_z f(x) = \int_a^b G_z(x,y) f(y) d\varrho(y), \quad x \in (a,b), \; f \in L^2((a,b);\varrho),$$

with a square integrable kernel G_z. For any two given linearly independent solutions u_1, u_2 of $(\tau - z)u = 0$, there are coefficients $m_{ij}^{\pm}(z) \in \mathbb{C}$, i, $j \in \{1,2\}$, such that the kernel is given by

$$G_z(x,y) = \begin{cases} \sum_{i,j=1}^2 m_{ij}^+(z) u_i(x) u_j(y), & \text{if } y < x, \\ \sum_{i,j=1}^2 m_{ij}^-(z) u_i(x) u_j(y), & \text{if } y \geq x. \end{cases}$$

Proof. Let u_1, u_2 be two linearly independent solutions of $(\tau - z)u = 0$ with $W(u_1,u_2) = 1$. If $g \in L_c^2((a,b);\varrho)$, then $(R_z g, g) \in (S - z)$, hence there is some $f \in \mathfrak{D}_\tau$ satisfying the boundary conditions with $f = R_z g$ and $(\tau - z)f = g$. From Proposition 3.3 we get for suitable constants c_1, $c_2 \in \mathbb{C}$

$$f(x) = u_1(x)\left(c_1 + \int_a^x u_2 g \, d\varrho\right) + u_2(x)\left(c_2 - \int_a^x u_1 g \, d\varrho\right), \quad x \in (a,b). \tag{8.2}$$

Furthermore, since u satisfies the boundary conditions, we have

$$B_a \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} = B_b \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix},$$

for some suitable matrices B_a, $B_b \in \mathbb{C}^{2\times 2}$ as in Theorem 7.5. Now because g has compact support, we have

$$\begin{aligned} \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} &= \begin{pmatrix} c_1 BC_a^1(u_1) + c_2 BC_a^1(u_2) \\ c_1 BC_a^2(u_1) + c_2 BC_a^2(u_2) \end{pmatrix} = \begin{pmatrix} BC_a^1(u_1) & BC_a^1(u_2) \\ BC_a^2(u_1) & BC_a^2(u_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= M_\alpha \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \end{aligned}$$

as well as

$$\begin{aligned} \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} &= \begin{pmatrix} \left(c_1 + \int_a^b u_2 g \, d\varrho\right) BC_b^1(u_1) \\ \left(c_1 + \int_a^b u_2 g \, d\varrho\right) BC_b^2(u_1) \end{pmatrix} + \begin{pmatrix} \left(c_2 - \int_a^b u_1 g \, d\varrho\right) BC_b^1(u_2) \\ \left(c_2 - \int_a^b u_1 g \, d\varrho\right) BC_b^2(u_2) \end{pmatrix} \\ &= \begin{pmatrix} BC_b^1(u_1) & BC_b^1(u_2) \\ BC_b^2(u_1) & BC_b^2(u_2) \end{pmatrix} \begin{pmatrix} c_1 + \int_a^b u_2 g \, d\varrho \\ c_2 - \int_a^b u_1 g \, d\varrho \end{pmatrix} \\ &= M_\beta \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + M_\beta \begin{pmatrix} \int_a^b u_2 g \, d\varrho \\ -\int_a^b u_1 g \, d\varrho \end{pmatrix}. \end{aligned}$$

Hence we have

$$(B_a M_\alpha - B_b M_\beta) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B_b M_\beta \begin{pmatrix} \int_a^b u_2 g \, d\varrho \\ -\int_a^b u_1 g \, d\varrho \end{pmatrix}.$$

Now if $B_a M_\alpha - B_b M_\beta$ were not invertible, we had

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \mathbb{C}^2 \backslash \{(0,0)\} \quad \text{with} \quad B_a M_\alpha \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = B_b M_\beta \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

Then the function $d_1u_1 + d_2u_2$ were a solution of $(\tau - z)u = 0$ satisfying the boundary conditions of S, hence an eigenvector with eigenvalue z. But since this were a contradiction to $z \in \rho(S)$, $B_aM_\alpha - B_bM_\beta$ has to be invertible. Because of

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (B_aM_\alpha - B_bM_\beta)^{-1} B_bM_\beta \begin{pmatrix} \int_a^b u_2 g \, d\varrho \\ -\int_a^b u_1 g \, d\varrho \end{pmatrix},$$

the constants c_1 and c_2 may be written as linear combinations of

$$\int_a^b u_2 g \, d\varrho \quad \text{and} \quad \int_a^b u_1 g \, d\varrho,$$

where the coefficients are independent of g. Now using equation (8.2) one sees that f has an integral-representation with a function G_z as claimed. The function G_z is square-integrable, since the solutions u_1 and u_2 lie in $L^2((a,b);\varrho)$ by assumption. Finally since the operator K_z

$$K_z g(x) = \int_a^b G_z(x,y) g(y) d\varrho(y), \quad x \in (a,b), \; g \in L^2((a,b);\varrho),$$

on $L^2((a,b);\varrho)$, as well as the resolvent R_z are bounded, the claim follows since they coincide on a dense subspace. □

As in the classical case, the compactness of the resolvents implies discreteness of the spectrum.

Corollary 8.2. *Suppose τ is in the l.c. case at both endpoints and S is a self-adjoint restriction of $T_{\max}$. Then S has purely discrete spectrum, i.e. $\sigma(S) = \sigma_d(S)$. Moreover,*

$$\sum_{\substack{\lambda \in \sigma(S) \\ \lambda \neq 0}} \frac{1}{\lambda^2} < \infty \quad \textit{and} \quad \dim \ker (S - \lambda) \leq 2, \quad \lambda \in \sigma(S).$$

Proof. Since the resolvent is compact, Theorem B.2 shows that the spectrum of S consists of isolated eigenvalues. Furthermore, the sum converges since the resolvent is Hilbert–Schmidt. Finally their multiplicity is at most two because of (3.7). □

If S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions or if not both endpoints are in the l.c. case, the resolvent has a simpler form.

Theorem 8.3. *Suppose S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions (if τ is in the l.c. at both endpoints) and $z \in \rho(S)$. Furthermore, let u_a and u_b be non-trivial solutions of $(\tau - z)u = 0$, such that*

$$u_a \begin{cases} \textit{satisfies the boundary condition at } a \textit{ if } \tau \textit{ is in the l.c. case at } a, \\ \textit{lies in } L^2((a,b);\varrho) \textit{ near } a \textit{ if } \tau \textit{ is in the l.p. case at } a, \end{cases}$$

and

$$u_b \begin{cases} \textit{satisfies the boundary condition at } b \textit{ if } \tau \textit{ is in the l.c. case at } b, \\ \textit{lies in } L^2((a,b);\varrho) \textit{ near } b \textit{ if } \tau \textit{ is in the l.p. case at } b. \end{cases}$$

Then the resolvent R_z is given by

$$R_z g(x) = \int_a^b G_z(x,y) g(y) d\varrho(y), \quad x \in (a,b), \; g \in L^2((a,b);\varrho), \tag{8.3}$$

where

$$G_z(x,y) = \frac{1}{W(u_b,u_a)} \begin{cases} u_a(y)u_b(x), & \text{if } y < x, \\ u_a(x)u_b(y), & \text{if } y \geq x. \end{cases} \tag{8.4}$$

Proof. The functions u_a, u_b are linearly independent, since otherwise they were an eigenvector of S with eigenvalue z. Hence they form a fundamental system of $(\tau - z)u = 0$. Now for each $f \in L^2((a,b);\varrho)$ we define a function f_g by

$$f_g(x) = W(u_b,u_a)^{-1} \left(u_b(x) \int_a^x u_a g \, d\varrho + u_a(x) \int_x^b u_b g \, d\varrho \right), \quad x \in (a,b).$$

If $f \in L^2_c((a,b);\varrho)$, then f_g is a solution of $(\tau - z)f = g$ by Proposition 3.3. Moreover, f_g is a scalar multiple of u_a near a and a scalar multiple of u_b near b. Hence the function f_g satisfies the boundary conditions of S and therefore $(f_g, \tau f_g - z f_g) = (f_g, g) \in (S - z)$, i.e. $R_z g = f_g$. Now if $g \in L^2((a,b);\varrho)$ is arbitrary and $g_n \in L^2_c((a,b);\varrho)$ is a sequence with $g_n \to g$ as $n \to \infty$, we have, since the resolvent is bounded $R_z g_n \to R_z g$. Furthermore, f_{g_n} converges pointwise to f_g, hence $R_z g = f_g$. □

If τ is in the l.p. case at some endpoint, then Corollary 5.5 shows that there is always a, unique up to scalar multiples, non-trivial solution of $(\tau - z)u = 0$, lying in $L^2((a,b);\varrho)$ near this endpoint. Also if τ is in the l.c. case at some endpoint, there exists a, unique up to scalar multiples, non-trivial solution of $(\tau - z)u = 0$, satisfying the boundary condition at this endpoint. Hence functions u_a and u_b, as in Theorem 8.3 always exist.

Corollary 8.4. *If S is a self-adjoint restriction of $T_{\max}$ with separate boundary conditions (if τ is in the l.c. at both endpoints) then all eigenvalues of S are simple.*

Proof. Suppose $\lambda \in \mathbb{R}$ is an eigenvalue and $u_i \in S$ with $\tau u_i = \lambda u_i$ for $i = 1, 2$, i.e. they are solutions of $(\tau - \lambda)u = 0$. If τ is in the l.p. case at some endpoint, then clearly the Wronskian $W(u_1,u_2)$ vanishes. Otherwise, since both functions satisfy the same boundary conditions this follows using the Plücker identity. □

According to Theorem B.7 the essential spectrum of self-adjoint restrictions is independent of the boundary conditions, i.e. all self-adjoint restrictions of $T_{\max}$ have the same essential spectrum. We conclude this section by proving that the essential spectrum of the self-adjoint restrictions of $T_{\max}$ is determined by the behavior of the coefficients in some arbitrarily small neighborhood of the endpoints. In order to state this result we need some notation. Fix some $c \in (a,b)$ and denote by $\tau|_{(a,c)}$ (resp. by $\tau|_{[c,b)}$) the differential expression on (a,b) corresponding to the coefficients ς, χ and $\varrho|_{(a,c)}$ (resp. $\varrho|_{[c,b)}$). Furthermore, let $S_{(a,c)}$ (resp. $S_{[c,b)}$) be some self-adjoint extension of $\tau|_{(a,c)}$ (resp. of $\tau|_{[c,b)}$).

Theorem 8.5. *For each $c \in (a,b)$ we have*

$$\sigma_e(S) = \sigma_e\left(S_{(a,c)}\right) \cup \sigma_e\left(S_{[c,b)}\right).$$

Proof. If one identifies $L^2((a,b);\varrho)$ with the orthogonal sum

$$L^2((a,b);\varrho) = L^2((a,b);\varrho|_{(a,c)}) \oplus L^2((a,b);\varrho|_{[c,b)}),$$

the linear relation

$$S_c = S_{(a,c)} \oplus S_{[c,b)},$$

is self-adjoint in $L^2((a,b);\varrho)$. Now since S and S_c both are finite dimensional extensions of the symmetric linear relation

$$T_c = \left\{ f \in T_{\min} \,\middle|\, f(c) = f^{[1]}(c) = 0 \right\},$$

an application of Theorem B.7 and Theorem B.8 yields the claim.

□

As an immediate corollary one sees that the essential spectrum only depends on the behavior of the coefficients in some neighborhood of the endpoints.

Corollary 8.6. *For each α, $\beta \in (a,b)$ we have*

$$\sigma_e(S) = \sigma_e\left(S_{(a,\alpha)}\right) \cup \sigma_e\left(S_{[\beta,b)}\right).$$

9. Weyl–Titchmarsh m-function

In this section let S be a self-adjoint restriction of $T_{\max}$ with separate boundary conditions (if τ is in the l.c. case at both endpoints). Our aim is to define a singular Weyl–Titchmarsh function as introduced recently in [16], [20]. To this end we need a real entire fundamental system θ_z, ϕ_z of $(\tau - z)u = 0$ with $W(\theta_z, \phi_z) = 1$, such that ϕ_z lies in S near a, i.e. ϕ_z lies in $L^2((a,b);\varrho)$ near a and satisfies the boundary condition at a if τ is in the l.c. case at a.

Hypothesis 9.1. *There is a real entire fundamental system θ_z, ϕ_z of $(\tau - z)u = 0$ with $W(\theta_z, \phi_z) = 1$, such that ϕ_z lies in S near a.*

Under the assumption of Hypothesis 9.1 we may define a function $M : \rho(S) \to \mathbb{C}$ by requiring that the solutions

$$\psi_z = \theta_z + M(z)\phi_z, \quad z \in \rho(S),$$

lie in S near b, i.e. they lie in $L^2((a,b);\varrho)$ near b and satisfy the boundary condition at b, if τ is in the l.c. case at b. This function M is well-defined (use Corollary 5.5 if τ is in the l.p. case at b) and called the singular Weyl–Titchmarsh function of S. The solutions ψ_z, $z \in \rho(S)$ are called the Weyl solutions of S.

Theorem 9.2. *The singular Weyl–Titchmarsh function M is analytic on $\rho(S)$ and satisfies*

$$M(z) = M(z^*)^*, \quad z \in \rho(S). \tag{9.1}$$

Proof. Let c, $d \in (a,b)$ with $c < d$. From Theorem 8.3 and the equation

$$W(\psi_z, \phi_z) = W(\theta_z, \phi_z) + M(z)W(\phi_z, \phi_z) = 1, \quad z \in \rho(S),$$

we get for each $z \in \rho(S)$ and $x \in [c,d)$

$$\begin{aligned} R_z \mathbb{1}_{[c,d)}(x) &= \psi_z(x) \int_c^x \phi_z d\varrho + \phi_z(x) \int_x^d \psi_z d\varrho \\ &= (\theta_z(x) + M(z)\phi_z(x)) \int_c^x \phi_z d\varrho + \phi_z(x) \int_x^d \theta_z + M(z)\phi_z d\varrho \\ &= M(z)\phi_z(x) \int_c^d \phi_z(y) d\varrho(y) + \int_c^d \tilde{G}_z(x,y) d\varrho(y), \end{aligned}$$

where

$$\tilde{G}_z(x,y) = \begin{cases} \phi_z(y)\theta_z(x), & \text{if } y < x, \\ \phi_z(x)\theta_z(y), & \text{if } y \geq x, \end{cases}$$

and hence

$$\langle R_z \mathbb{1}_{[c,d)}, \mathbb{1}_{[c,d)} \rangle = M(z) \left(\int_c^d \phi_z(y) d\varrho(y) \right)^2 + \int_c^d \int_c^d \tilde{G}_z(x,y) d\varrho(y)\, d\varrho(x).$$

The left-hand side of this equation is analytic in $\rho(S)$ since the resolvent is. Furthermore, the integrals are analytic in $\rho(S)$ as well, since the integrands are analytic and locally bounded by Theorem 3.6. Hence M is analytic if for each $z_0 \in \rho(S)$, there are some c, $d \in (a,b)$ such that

$$\int_c^d \phi_{z_0}(y) d\varrho(y) \neq 0.$$

But this is true since otherwise ϕ_{z_0} would vanish almost everywhere with respect to ϱ. Moreover, equation (9.1) is valid since the function

$$\theta_{z^*} + M(z)^* \phi_{z^*} = \left(\theta_z + M(z)\phi_z\right)^*,$$

lies in S near b by Lemma 6.3. □

As an immediate consequence of Theorem 9.2 we see that $\psi_z(x)$ and $\psi_z^{[1]}(x)$ are analytic functions in $z \in \rho(S)$ for each $x \in (a,b)$.

Remark 9.3. *Suppose $\tilde{\theta}_z$, $\tilde{\phi}_z$ is some other real entire fundamental system of $(\tau - z)u = 0$ with $W(\tilde{\theta}_z, \tilde{\phi}_z) = 1$, such that $\tilde{\phi}_z$ lies in S near a. Then*

$$\tilde{\theta}_z = e^{-g(z)}\theta_z - f(z)\phi_z, \quad \text{and} \quad \tilde{\phi}_z = e^{g(z)}\phi_z, \quad z \in \mathbb{C},$$

for some real entire functions f, g. The corresponding singular Weyl–Titchmarsh functions are related via

$$\tilde{M}(z) = e^{-2g(z)}M(z) + e^{-g(z)}f(z), \quad z \in \rho(S).$$

In particular, the maximal domain of holomorphy or the structure of poles and singularities do not change.

We continue with the construction of a real entire fundamental system in the case when τ is in the l.c. case at a.

Theorem 9.4. *Suppose τ is in the l.c. case at a. Then there exists a real entire fundamental system θ_z, ϕ_z of $(\tau - z)u = 0$ with $W(\theta_z, \phi_z) = 1$, such that ϕ_z lies in S near a,*

$$W(\theta_{z_1}, \phi_{z_2})(a) = 1 \quad \text{and} \quad W(\theta_{z_1}, \theta_{z_2})(a) = W(\phi_{z_1}, \phi_{z_2})(a) = 0, \quad z_1,\ z_2 \in \mathbb{C}.$$

Proof. Let θ, ϕ be a real fundamental system of $\tau u = 0$ with $W(\theta,\phi) = 1$ such that ϕ lies in S near a. Now fix some $c \in (a,b)$ and for each $z \in \mathbb{C}$ let $u_{z,1}$, $u_{z,2}$ be the fundamental system of

$$(\tau - z)u = 0 \quad \text{with} \quad u_{z,1}(c) = u_{z,2}^{[1]}(c) = 1 \quad \text{and} \quad u_{z,1}^{[1]}(c) = u_{z,2}(c) = 0.$$

Then by the existence and uniqueness theorem we have $u_{z^*,i} = u_{z,i}^*$, $i = 1,2$. If we introduce

$$\theta_z(x) = W(u_{z,1}, \theta)(a) u_{z,2}(x) - W(u_{z,2}, \theta)(a) u_{z,1}(x), \qquad x \in (a,b),$$
$$\phi_z(x) = W(u_{z,1}, \phi)(a) u_{z,2}(x) - W(u_{z,2}, \phi)(a) u_{z,1}(x), \qquad x \in (a,b),$$

then the functions ϕ_z lie in S near a since

$$W(\phi_z, \phi)(a) = W(u_{z,1}, \phi)(a) W(u_{z,2}, \phi)(a) - W(u_{z,2}, \phi)(a) W(u_{z_1}, \phi)(a) = 0.$$

Furthermore, a direct calculation shows that $\theta_{z^*} = \theta_z^*$ and $\phi_{z^*} = \phi_z^*$. The remaining equalities follow using the Plücker identity several times. It remains to prove that the functions $W(u_{z,1},\theta)(a)$, $W(u_{z,2},\theta)(a)$, $W(u_{z,1},\phi)(a)$ and $W(u_{z,2},\phi)(a)$ are entire in z. Indeed we have by the Lagrange identity

$$W(u_{z,1},\theta)(a) = W(u_{z,1},\theta)(c) - z \lim_{x\downarrow a} \int_x^c \theta(t)u_{z,1}(t)d\varrho(t).$$

Now the integral on the right-hand side is analytic by Theorem 3.6 and in order to prove that the limit is also analytic we need to show that the integral is bounded as $x \downarrow a$, locally uniformly in z. But the proof of Lemma 5.1 shows that for each $z_0 \in \mathbb{C}$ we have

$$\left| \int_x^c \theta(t)u_{z,1}(t)d\varrho(t) \right| \leq K \int_a^c |\theta|^2 \, d\varrho \int_a^c \left(|u_{z_0,1}| + |u_{z_0,2}|\right)^2 d\varrho,$$

for some constant $K \in \mathbb{R}$ and all z in some neighborhood of z_0. Analyticity of the other functions is proved similar. □

If τ is even regular at a then one may take θ_z, ϕ_z to be the solutions of $(\tau - z)u = 0$ with the initial values

$$\theta_z(a) = \phi_z^{[1]}(a) = \cos\varphi_\alpha \quad \text{and} \quad -\theta_z^{[1]}(a) = \phi_z(a) = \sin\varphi_\alpha,$$

for some suitable $\varphi_\alpha \in [0,\pi)$. Furthermore, in the case when ϱ has no weight near a, one may take for θ_z, ϕ_z the solutions of $(\tau - z)u = 0$ with the initial values

$$\theta_z(\alpha_\varrho-) = \phi_z^{[1]}(\alpha_\varrho-) = \cos\varphi_\alpha \quad \text{and} \quad -\theta_z^{[1]}(\alpha_\varrho-) = \phi_z(\alpha_\varrho-) = \sin\varphi_\alpha,$$

for some $\varphi_\alpha \in [0,\pi)$.

Corollary 9.5. *Suppose τ is in the l.c. case at a and θ_z, ϕ_z is a real entire fundamental system of $(\tau - z)u = 0$ as in Theorem 9.4. Then the corresponding singular Weyl–Titchmarsh function M is a Herglotz function.*

Proof. In order to prove the Herglotz property, we show that

$$0 < \|\psi_z\|^2 = \frac{\operatorname{Im}(M(z))}{\operatorname{Im}(z)}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{9.2}$$

Indeed if z_1, $z_2 \in \rho(S)$, then

$$\begin{aligned} W(\psi_{z_1},\psi_{z_2})(a) &= W(\theta_{z_1},\theta_{z_2})(a) + M(z_2)W(\theta_{z_1},\phi_{z_2})(a) \\ &\qquad + M(z_1)W(\phi_{z_1},\theta_{z_2})(a) + M(z_1)M(z_2)W(\phi_{z_1},\phi_{z_2})(a) \\ &= M(z_2) - M(z_1). \end{aligned}$$

If τ is in the l.p. case at b, then furthermore we have

$$W(\psi_{z_1},\psi_{z_2})(b) = 0,$$

since clearly ψ_{z_1}, $\psi_{z_2} \in T_{\max}$. This also holds if τ is in the l.c. case at b, since then ψ_{z_1} and ψ_{z_2} satisfy the same boundary condition at b. Now using the Lagrange identity yields

$$\begin{aligned} (z_1 - z_2) \int_a^b \psi_{z_1}(t)\psi_{z_2}(t)d\varrho(t) &= W(\psi_{z_1},\psi_{z_2})(b) - W(\psi_{z_1},\psi_{z_2})(a) \\ &= M(z_1) - M(z_2). \end{aligned}$$

In particular for $z\in\mathbb{C}\backslash\mathbb{R}$, using $M(z^*)=M(z)^*$ as well as

$$\psi_{z^*}=\theta_{z^*}+M(z^*)\phi_{z^*}=\psi_z^*$$

we get

$$||\psi_z||^2=\int_a^b\psi_z(t)\psi_{z^*}(t)d\varrho(t)=\frac{M(z)-M(z^*)}{z-z^*}=\frac{\mathrm{Im}(M(z))}{\mathrm{Im}(z)}.$$

Since ψ_z is a non-trivial solution, we furthermore have $0<||\psi_z||^2$. □

We conclude this section with a necessary and sufficient condition for Hypothesis 9.1 to hold. To this end for each $c\in(a,b)$ let $S^D_{(a,c)}$ be the self-adjoint operator associated to $\tau|_{(a,c)}$ with a Dirichlet boundary condition at c and the same boundary condition as S at a. The proofs are the same as those for Schrödinger operators given in [20, Lemma 2.2 and Lemma 2.4].

Theorem 9.6. *The following properties are equivalent:*

(i) *Hypothesis 9.1.*
(ii) *There is a real entire solution ϕ_z of $(\tau-z)u=0$ which lies in S near a.*
(iii) *The spectrum of $S^D_{(a,c)}$ is purely discrete for some $c\in(a,b)$.*

10. Spectral transformation

In this section let S be a self-adjoint restriction of $T_{\max}$ with separate boundary conditions as in the preceding section. Furthermore, we assume that there is a real entire fundamental system θ_z, ϕ_z of $(\tau-z)u=0$ with $W(\theta_z,\phi_z)=1$ such that ϕ_z lies in S near a. By M we denote the corresponding singular Weyl–Titchmarsh function and by ψ_z the Weyl solutions of S.

Recall that by Lemma B.4 for all functions f, $g\in L^2((a,b);\varrho)$ there is a unique complex measure $E_{f,g}$ such that

$$\langle R_zf,g\rangle=\int_{\mathbb{R}}\frac{1}{\lambda-z}dE_{f,g}(\lambda),\quad z\in\rho(S).$$

Indeed these measures are obtained by applying a variant of the spectral theorem to the operator part

$$S_{\mathfrak{D}}=S\cap(\mathfrak{D}\times\mathfrak{D}),\qquad \mathfrak{D}=\overline{\mathrm{dom}\,(S)}=\mathrm{mul}\,(S)^{\perp},$$

of S (see Lemma B.4 in Appendix B).

In order to obtain a spectral transformation we define for each $f\in L^2_c((a,b);\varrho)$ the transform of f

$$\hat{f}(z)=\int_a^b\phi_z(x)f(x)d\varrho(x),\quad z\in\mathbb{C}. \tag{10.1}$$

Next we can use this to associate a measure with $M(z)$ by virtue of the Stieltjes–Livšić inversion formula following literally the proof of [20, Lem. 3]:

Lemma 10.1. *There is a unique Borel measure $d\mu$ defined via*

$$\mu((\lambda_1,\lambda_2])=\lim_{\delta\downarrow0}\lim_{\varepsilon\downarrow0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta}\mathrm{Im}(M(\lambda+i\varepsilon))d\lambda, \tag{10.2}$$

for each λ_1, $\lambda_2\in\mathbb{R}$ with $\lambda_1<\lambda_2$, such that

$$dE_{f,g}=\hat{f}\hat{g}^*d\mu,\qquad f,g\in L^2_c((a,b);\varrho). \tag{10.3}$$

In particular,

$$\langle R_z f, g\rangle = \int_{\mathbb{R}} \frac{\hat{f}(\lambda)\hat{g}(\lambda)^*}{\lambda - z} d\mu(\lambda), \quad z \in \rho(S).$$

In particular the preceding lemma shows that the mapping $f \mapsto \hat{f}$ is an isometry from $L^2_c((a,b);\varrho) \cap \mathfrak{D}$ into $L^2(\mathbb{R};\mu)$. Indeed for each $f \in L^2_c((a,b);\varrho) \cap \mathfrak{D}$ we have

$$\|\hat{f}\|^2 = \int_{\mathbb{R}} \hat{f}(\lambda)\hat{f}(\lambda)^* d\mu(\lambda) = \int_{\mathbb{R}} dE_{f,f} = \|f\|^2.$$

Hence we may extend this mapping uniquely to a isometric linear operator $\mathcal{F}$ on the Hilbert space $\mathfrak{D}$ into $L^2(\mathbb{R};\mu)$ by

$$\mathcal{F}f(\lambda) = \lim_{\alpha\downarrow a} \lim_{\beta\uparrow b} \int_\alpha^\beta \phi_\lambda(x) f(x) d\varrho(x), \quad \lambda \in \mathbb{R},\ f \in \mathfrak{D},$$

where the limit on the right-hand side is a limit in the Hilbert space $L^2(\mathbb{R};\mu)$. Using this linear operator $\mathcal{F}$, it is quiet easy to extend the result of Lemma 10.1 to functions f, $g \in \mathfrak{D}$. Indeed one gets that $dE_{f,g} = \mathcal{F}f\,\mathcal{F}g^* d\mu$, i.e.

$$\langle R_z f, g\rangle = \int_{\mathbb{R}} \frac{\mathcal{F}f(\lambda)\mathcal{F}g(\lambda)^*}{\lambda - z} d\mu(\lambda), \quad z \in \rho(S).$$

We will see below that $\mathcal{F}$ is not only isometric, but also onto, i.e. $\operatorname{ran}(\mathcal{F}) = L^2(\mathbb{R};\mu)$. In order to compute the inverse and the adjoint of $\mathcal{F}$, we introduce for each function $g \in L^2_c(\mathbb{R};\mu)$ the transform

$$\check{g}(x) = \int_{\mathbb{R}} \phi_\lambda(x) g(\lambda) d\mu(\lambda), \quad x \in (a,b).$$

For arbitrary α, $\beta \in (a,b)$ with $\alpha < \beta$ we have

$$\begin{aligned}\int_\alpha^\beta |\check{g}(x)|^2 \, d\varrho(x) &= \int_\alpha^\beta \check{g}(x) \int_{\mathbb{R}} \phi_\lambda(x) g(\lambda)^* d\mu(\lambda)\, d\varrho(x) \\ &= \int_{\mathbb{R}} g(\lambda)^* \int_\alpha^\beta \phi_\lambda(x)\check{g}(x) d\varrho(x)\, d\mu(\lambda) \\ &\leq \|g\|_\mu \left\| \mathcal{F}\left(\mathbb{1}_{[\alpha,\beta)}\check{g}\right)\right\|_\mu \\ &\leq \|g\|_\mu \sqrt{\int_\alpha^\beta |\check{g}(x)|^2 \, d\varrho(x)}.\end{aligned}$$

Hence $\check{g}$ lies in $L^2((a,b);\varrho)$ with $\|\check{g}\| \leq \|g\|_\mu$ and we may extend this mapping uniquely to a bounded linear operator $\mathcal{G}$ on $L^2(\mathbb{R};\mu)$ into $\mathfrak{D}$.

If F is a Borel measurable function on $\mathbb{R}$, then we denote by M_F the maximally defined operator of multiplication with F in $L^2(\mathbb{R};\mu)$.

Lemma 10.2. *The operator $\mathcal{F}$ is a surjective isometry with $\mathcal{F}^{-1} = \mathcal{G}$ and adjoint*

$$\mathcal{F}^* = \left\{(g,f) \in L^2(\mathbb{R};\mu) \times L^2((a,b);\varrho) \,|\, \mathcal{G}g - f \in \operatorname{mul}(S)\right\}.$$

Proof. In order to prove $\operatorname{ran}(\mathcal{G}) \subseteq \mathfrak{D}$, let $g \in L^2_c(\mathbb{R};\mu)$. If $\mathbb{1}_{\{\alpha_\varrho\}} \in \operatorname{mul}(S)$, then the solutions ϕ_z, $z \in \mathbb{C}$ vanish in α_ϱ, hence also

$$\check{g}(\alpha_\varrho) = \int_{\mathbb{R}} \phi_\lambda(\alpha_\varrho) g(\lambda) d\mu(\lambda) = 0.$$

Furthermore, if $\mathbb{1}_{\{\beta_\varrho\}} \in \mathrm{mul}\,(S)$, then the spectrum of S is discrete and the solutions ϕ_λ, $\lambda \in \sigma(S)$ vanish in β_ϱ. Now since μ is supported on $\sigma(S)$, we also have

$$\check{g}(\beta_\varrho) = \int_{\sigma(S)} \phi_\lambda(\beta_\varrho) g(\lambda) d\mu(\lambda) = 0.$$

From this one sees that $\check{g} \in \mathrm{mul}\,(S)^\perp = \mathfrak{D}$, i.e. $\mathrm{ran}(\mathcal{G}) \subseteq \mathfrak{D}$.

Next we prove $\mathcal{G}\mathcal{F}f = f$ for each $f \in \mathfrak{D}$. Indeed if f, $g \in L^2_c((a,b);\varrho) \cap \mathfrak{D}$, then

$$\begin{aligned}\langle f, g\rangle &= \int_{\mathbb{R}} dE_{f,g} = \int_{\mathbb{R}} \hat{f}(\lambda)\hat{g}(\lambda)^* d\mu(\lambda) \\ &= \lim_{n\to\infty} \int_{(-n,n]} \hat{f}(\lambda) \int_a^b \phi_\lambda(x) g(x)^* d\varrho(x)\, d\mu(\lambda) \\ &= \lim_{n\to\infty} \int_a^b g(x)^* \int_{(-n,n]} \hat{f}(\lambda)\phi_\lambda(x) d\mu(\lambda)\, d\varrho(x) \\ &= \lim_{n\to\infty} \langle \mathcal{G}\mathrm{M}_{\mathbb{1}_{(-n,n]}}\mathcal{F}f, g\rangle = \langle \mathcal{G}\mathcal{F}f, g\rangle.\end{aligned}$$

Now since $\mathrm{ran}(\mathcal{G}) \subseteq \mathfrak{D}$ and $L^2_c((a,b);\varrho)\cap\mathfrak{D}$ is dense in $\mathfrak{D}$ we infer that $\mathcal{G}\mathcal{F}f = f$ for all $f \in \mathfrak{D}$. In order to prove that $\mathcal{G}$ is the inverse of $\mathcal{F}$, it remains to show that $\mathcal{F}$ is surjective, i.e. $\mathrm{ran}(\mathcal{F}) = L^2(\mathbb{R};\mu)$. Therefore let f, $g \in \mathfrak{D}$ and F, G be bounded measurable functions on $\mathbb{R}$. Since $E_{f,g}$ is the spectral measure of the operator part $S_{\mathfrak{D}}$ of S (see the proof of Lemma B.4) we get

$$\langle \mathrm{M}_G \mathcal{F} F(S_{\mathfrak{D}}) f, \mathcal{F}g\rangle_\mu = \langle G(S_{\mathfrak{D}})F(S_{\mathfrak{D}})f, g\rangle = \langle \mathrm{M}_G \mathrm{M}_F \mathcal{F}f, \mathcal{F}g\rangle_\mu.$$

Now if we set $h = F(S_{\mathfrak{D}})f$, we get from this last equation

$$\int_{\mathbb{R}} G(\lambda)\mathcal{F}g(\lambda)^* \left(\mathcal{F}h(\lambda) - F(\lambda)\mathcal{F}f(\lambda)\right) d\mu(\lambda) = 0.$$

Since this holds for each bounded measurable function G, we infer

$$\mathcal{F}g(\lambda)^* \left(\mathcal{F}h(\lambda) - F(\lambda)\mathcal{F}f(\lambda)\right) = 0,$$

for almost all $\lambda \in \mathbb{R}$ with respect to μ. Furthermore, for each $\lambda_0 \in \mathbb{R}$ we can find a $g \in L^2_c((a,b);\varrho) \cap \mathfrak{D}$ such that $\hat{g} \neq 0$ in a vicinity of λ_0. Hence we even have $\mathcal{F}h = F\mathcal{F}f$ almost everywhere with respect to μ. But this shows that $\mathrm{ran}(\mathcal{F})$ contains all characteristic functions of intervals. Indeed let $\lambda_0 \in \mathbb{R}$ and choose $f \in L^2_c((a,b);\varrho) \cap \mathfrak{D}$ such that $\hat{f} \neq 0$ in a vicinity of λ_0. Then for each interval J, whose closure is contained in this vicinity one may choose

$$F(\lambda) = \begin{cases} \hat{f}(\lambda)^{-1}, & \text{if } \lambda \in J, \\ 0, & \text{if } \lambda \in \mathbb{R}\backslash J, \end{cases}$$

and gets $\mathbb{1}_J = \mathcal{F}h \in \mathrm{ran}(\mathcal{F})$. Thus we have obtained $\mathrm{ran}(\mathcal{F}) = L^2(\mathbb{R};\mu)$. Finally the fact that the adjoint is given as in the claim follows from the equivalence

$$\mathcal{G}g - f \in \mathrm{mul}\,(S) \quad \Leftrightarrow \quad \forall u \in \mathfrak{D} : 0 = \langle \mathcal{G}g - f, u\rangle = \langle g, \mathcal{F}u\rangle_\mu - \langle f, u\rangle$$

for every $f \in L^2((a,b);\varrho)$ and $g \in L^2(\mathbb{R};\mu)$. □

Note that $\mathcal{F}$ is a unitary map from $L^2((a,b);\varrho)$ onto $L^2(\mathbb{R};\mu)$ if and only if S is an operator.

Theorem 10.3. *The self-adjoint relation S is given by $S = \mathcal{F}^*\mathrm{M}_{\mathrm{id}}\mathcal{F}$.*

Proof. First note that for each $f \in \mathfrak{D}$ we have

$$\begin{aligned} f \in \operatorname{dom}(S) \quad &\Leftrightarrow \quad \int_{\mathbb{R}} |\lambda|^2 dE_{f,f}(\lambda) < \infty \quad \Leftrightarrow \quad \int_{\mathbb{R}} |\lambda|^2 |\mathcal{F}f(\lambda)|^2 d\mu(\lambda) < \infty \\ &\Leftrightarrow \quad \mathcal{F}f \in \operatorname{dom}(\mathrm{M}_{\mathrm{id}}) \qquad \Leftrightarrow \quad f \in \operatorname{dom}(\mathcal{F}^* \mathrm{M}_{\mathrm{id}} \mathcal{F}). \end{aligned}$$

Furthermore, if $(f, f_\tau) \in S$, then from Lemma B.4 and Lemma 10.1 we infer

$$\begin{aligned} \langle f_\tau, g \rangle &= \int_{\mathbb{R}} \lambda dE_{f,g}(\lambda) = \int_{\mathbb{R}} \lambda \mathcal{F}f(\lambda) \mathcal{F}g(\lambda)^* d\mu(\lambda) = \int_{\mathbb{R}} \mathrm{M}_{\mathrm{id}} \mathcal{F}f(\lambda) \mathcal{F}g(\lambda)^* d\mu(\lambda) \\ &= \langle \mathcal{G} \mathrm{M}_{\mathrm{id}} \mathcal{F}f, g \rangle, \quad g \in \mathfrak{D}, \end{aligned}$$

and hence $\mathcal{G}\mathrm{M}_{\mathrm{id}}\mathcal{F}f = Pf_\tau$, where P is the orthogonal projection onto $\mathfrak{D}$. This and Lemma 10.2 show that $(\mathrm{M}_{\mathrm{id}}\mathcal{F}f, f_\tau) \in \mathcal{F}^*$, which is equivalent to $(f, f_\tau) \in \mathcal{F}^*\mathrm{M}_{\mathrm{id}}\mathcal{F}$. Now if we conversely assume that $(g, g_\tau) \in \mathcal{F}^*\mathrm{M}_{\mathrm{id}}\mathcal{F}$, then $(\mathrm{M}_{\mathrm{id}}\mathcal{F}g, g_\tau) \in \mathcal{F}^*$ (note that $g \in \operatorname{dom}(S)$). Hence we have $\mathcal{G}\mathrm{M}_{\mathrm{id}}\mathcal{F}g - g_\tau \in \operatorname{mul}(S)$ and since $(g, \mathcal{G}\mathrm{M}_{\mathrm{id}}\mathcal{F}g) \in S$, we also get $(g, g_\tau) \in S$. □

Note that the self-adjoint operator $S_{\mathfrak{D}}$ is unitarily equivalent to the operator of multiplication M_{id}. In fact, $\mathcal{F}$ is unitary as an operator from $\mathfrak{D}$ onto $L^2(\mathbb{R};\mu)$ and maps $S_{\mathfrak{D}}$ onto multiplication with the identity. Now the spectrum can be read off from the boundary behavior of the singular Weyl–Titchmarsh function M in the usual way.

Corollary 10.4. *The spectrum of S is given by*

$$\sigma(S) = \sigma(S_{\mathfrak{D}}) = \operatorname{supp}(\mu) = \overline{\{\lambda \in \mathbb{R} \,|\, 0 < \limsup_{\varepsilon \downarrow 0} \operatorname{Im}(M(\lambda + i\varepsilon))\}}.$$

Moreover,

$$\begin{aligned} \sigma_p(S_{\mathfrak{D}}) &= \{\lambda \in \mathbb{R} \,|\, 0 < \lim_{\varepsilon \downarrow 0} \varepsilon \operatorname{Im}(M(\lambda + i\varepsilon))\}, \\ \sigma_{ac}(S_{\mathfrak{D}}) &= \overline{\{\lambda \in \mathbb{R} \,|\, 0 < \limsup_{\varepsilon \downarrow 0} \operatorname{Im}(M(\lambda + i\varepsilon)) < \infty\}}^{ess}, \end{aligned}$$

where $\overline{\Omega}^{ess} = \{\lambda \in \mathbb{R} \,|\, |(\lambda - \varepsilon, \lambda + \varepsilon) \cap \Omega| > 0 \text{ for all } \varepsilon > 0\}$, is the essential closure of a Borel set $\Omega \subseteq \mathbb{R}$, and

$$\Sigma_s = \{\lambda \in \mathbb{R} \,|\, \limsup_{\varepsilon \downarrow 0} \operatorname{Im}(M(\lambda + i\varepsilon)) = \infty\}$$

is a minimal support for the singular spectrum (singular continuous plus pure point spectrum) of $S_{\mathfrak{D}}$.

Proof. Since the operator part $S_{\mathfrak{D}}$ of S is unitary equivalent to M_{id} we infer from Lemma B.3 that $\sigma(S) = \sigma(M_{\mathrm{id}}) = \operatorname{supp}(\mu)$. □

Proposition 10.5. *If $\lambda \in \sigma(S)$ is an eigenvalue, then*

$$\mu(\{\lambda\}) = \|\phi_\lambda\|^{-2}.$$

Proof. Under this assumptions ϕ_λ is an eigenvector, i.e. $(\phi_\lambda, \lambda\phi_\lambda) \in S$. Hence we get from the proof of Theorem 10.3 that $\mathrm{M}_{\mathrm{id}}\mathcal{F}\phi_\lambda = \lambda\mathcal{F}\phi_\lambda$. But this shows that

$\mathcal{F}\phi_\lambda(z)$ vanishes for almost all $z \neq \lambda$ with respect to μ. Now from this we get

$$\begin{aligned}\|\phi_\lambda\|^2 = \|\mathcal{F}\phi_\lambda\|_\mu^2 &= \int_{\{\lambda\}} |\mathcal{F}\phi_\lambda(z)|^2 \, d\mu(z) \\ &= \mu(\{\lambda\}) \left(\int_a^b \phi_\lambda(x)^2 d\varrho(x) \right)^2 = \mu(\{\lambda\}) \|\phi_\lambda\|^4 .\end{aligned}$$

□

By P we denote the orthogonal projection from $L^2((a,b);\varrho)$ onto $\mathfrak{D}$. If S is an operator, P is simply the identity.

Lemma 10.6. *For every $z \in \rho(S)$ and all $x \in (a,b)$ the transform of the Green function $G_z(x,\cdot)$ and its derivative $\partial_x G_z(x,\cdot)$ are given by*

$$\mathcal{F}PG_z(x,\cdot)(\lambda) = \frac{\phi_\lambda(x)}{\lambda - z} \quad \textit{and} \quad \mathcal{F}P\partial_x G_z(x,\cdot)(\lambda) = \frac{\phi_\lambda^{[1]}(x)}{\lambda - z}, \quad \lambda \in \mathbb{R}.$$

Proof. First note that $G_z(x,\cdot)$ and $\partial_x G_z(x,\cdot)$ both lie in $L^2((a,b);\varrho)$. Then using Lemma 10.1 we get for each $f \in L^2_c((a,b);\varrho)$ and $g \in L^2_c(\mathbb{R};\mu)$

$$\langle R_z \check{g}, f \rangle = \int_{\mathbb{R}} \frac{g(\lambda)\hat{f}(\lambda)^*}{\lambda - z} d\mu(\lambda) = \int_a^b \int_{\mathbb{R}} \frac{\phi_\lambda(x)}{\lambda - z} g(\lambda) d\mu(\lambda) \, f(x)^* d\varrho(x).$$

Hence we have

$$R_z \check{g}(x) = \int_{\mathbb{R}} \frac{\phi_\lambda(x)}{\lambda - z} g(\lambda) d\mu(\lambda)$$

for almost all $x \in (a,b)$ with respect to ϱ. Using Theorem 8.3 one gets

$$\langle \mathcal{F}PG_z(x,\cdot), g^* \rangle_\mu = \langle G_z(x,\cdot), \check{g}^* \rangle = \int_{\mathbb{R}} \frac{\phi_\lambda(x)}{\lambda - z} g(\lambda) d\mu(\lambda),$$

for almost all $x \in (a,b)$ with respect to ϱ. Since all three terms are absolutely continuous with respect to ς, this equality is true for all $x \in (a,b)$, which proves the first part of the claim. The second equality follows from

$$\langle \mathcal{F}P\partial_x G_z(x,\cdot), g^* \rangle_\mu = \langle \partial_x G_z(x,\cdot), \check{g}^* \rangle = R_z \check{g}^{[1]}(x) = \int_{\mathbb{R}} \frac{\phi_\lambda^{[1]}(x)}{\lambda - z} g(\lambda) d\mu(\lambda).$$

□

Note that $\mathcal{F}P$ is the unique extension on $L^2((a,b);\varrho)$ of the bounded linear mapping defined in (10.1) on $L^2_c((a,b);\varrho)$.

Lemma 10.7. *Suppose τ is in the l.c. case at a and θ_z, ϕ_z is a fundamental system as in Theorem 9.4. Then for each $z \in \rho(S)$ the transform of the Weyl solution ψ_z is given by*

$$\mathcal{F}P\psi_z(\lambda) = \frac{1}{\lambda - z}, \quad \lambda \in \mathbb{R}.$$

Proof. From Lemma 10.6 we obtain for each $x \in (a,b)$

$$\mathcal{F}P\tilde{\psi}_z(x,\cdot)(\lambda) = \frac{W(\theta_z, \phi_\lambda)(x)}{\lambda - z}, \quad \lambda \in \mathbb{R},$$

where

$$\tilde{\psi}_z(x,y) = \begin{cases} \psi_z(y), & \text{if } y \geq x, \\ M(z)\phi_z(y), & \text{if } y < x. \end{cases}$$

Now the claim follows by letting $x \downarrow a$, using Theorem 9.4. □

Under the assumptions of Lemma 10.7, M is a Herglotz function. Hence we have

$$M(z) = c_1 + c_2 z + \int_{\mathbb{R}} \frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} d\mu(\lambda), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{10.4}$$

where the constants c_1, c_2 are given by

$$c_1 = \mathrm{Re}(M(i)) \quad \text{and} \quad c_2 = \lim_{\eta\uparrow\infty} \frac{M(i\eta)}{i\eta} \geq 0.$$

Corollary 10.8. *Suppose τ is in the l.c. case at a and θ_z, ϕ_z is a fundamental system as in Theorem 9.4. Then the second constant in* (10.4) *is given by*

$$c_2 = \lim_{\eta\uparrow\infty} \frac{M(i\eta)}{i\eta} = \begin{cases} \theta_z(\alpha_\varrho)^2 \varrho(\{\alpha_\varrho\}), & \text{if } \mathbb{1}_{\{\alpha_\varrho\}} \in \mathrm{mul}\,(S)\,, \\ 0, & \text{else.} \end{cases}$$

Proof. Taking imaginary parts in (10.4) yields for each $z \in \mathbb{C}\backslash\mathbb{R}$

$$\mathrm{Im}(M(z)) = c_2 \mathrm{Im}(z) + \int_{\mathbb{R}} \mathrm{Im}\left(\frac{1}{\lambda - z}\right) d\mu(\lambda) = c_2 \mathrm{Im}(z) + \int_{\mathbb{R}} \frac{\mathrm{Im}(z)}{|\lambda - z|^2} d\mu(\lambda).$$

From this we get, using Lemma 10.7 and (9.2)

$$\begin{aligned} c_2 + \int_{\mathbb{R}} \frac{1}{|\lambda - z|^2} d\mu(\lambda) = \frac{\mathrm{Im}(M(z))}{\mathrm{Im}(z)} &= \|\psi_z\|^2 = \|(I-P)\psi_z\|^2 + \|\mathcal{F}P\psi_z\|_\mu^2 \\ &= \|(I-P)\psi_z\|^2 + \int_{\mathbb{R}} \frac{1}{|\lambda - z|^2} d\mu(\lambda). \end{aligned}$$

Hence we have (note that $\psi_z(\beta_\varrho) = 0$ if $\mathbb{1}_{\{\beta_\varrho\}} \in \mathrm{mul}\,(S)\backslash\{0\}$)

$$c_2 = \|(I-P)\psi_z\|^2 = \begin{cases} |\psi_z(\alpha_\varrho)|^2 \varrho(\{\alpha_\varrho\}), & \text{if } \mathbb{1}_{\{\alpha_\varrho\}} \in \mathrm{mul}\,(S)\,, \\ 0, & \text{else.} \end{cases}$$

Now assume $\mathbb{1}_{\{\alpha_\varrho\}} \in \mathrm{mul}\,(S)\backslash\{0\}$, then $\phi_z(\alpha_\varrho) = 0$. Hence in this case we have

$$c_2 = |\theta_z(\alpha_\varrho) + M(z)\phi_z(\alpha_\varrho)|^2 \varrho(\{\alpha_\varrho\}) = |\theta_z(\alpha_\varrho)|^2 \varrho(\{\alpha_\varrho\}), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Finally since θ_z is a real entire function, this proves the claim. □

Remark 10.9. *Given another singular Weyl–Titchmarsh function $\tilde{M}$ as in Remark 9.3, the corresponding spectral measures are related by*

$$\tilde{\rho} = e^{-2g}\rho,$$

where g is the real entire function appearing in Remark 9.3. Hence the measures are mutually absolutely continuous and the associated spectral transformations just differ by a simple rescaling with the positive function e^{-2g}. Also note that the spectral measure does not depend on the second solution.

11. Spectral transformation II

In this section let S be a self-adjoint restriction of $T_{\max}$ with separate boundary conditions as in the preceding section. We now want to look at the case where none of the endpoints satisfies the requirements of the previous section. In such a situation the spectral multiplicity of S could be two and hence we will need to work with a matrix-valued transformation.

In the following we will fix some point $x_0 \in (a,b)$ and consider the real entire fundamental system of solutions θ_z, ϕ_z with the initial conditions

$$\phi_z(x_0) = -\theta_z^{[1]}(x_0) = -\sin(\varphi_\alpha) \quad \text{and} \quad \phi_z^{[1]}(x_0) = \theta_z(x_0) = \cos(\varphi_\alpha), \quad z\in\mathbb{C}, \tag{11.1}$$

for some fixed $\alpha\in[0,\pi)$. The Weyl solutions are given by

$$\psi_{z,\pm}(x) = \theta_z(x) \pm m_\pm(z)\phi_z(x), \quad x\in(a,b),\ z\in\mathbb{C}\backslash\mathbb{R}, \tag{11.2}$$

such that ψ_- lies in $L^2((a,b);\varrho)$ near a and ψ_+ lies in $L^2((a,b);\varrho)$ near b. Here $m_\pm$ are the Weyl–Titchmarsh functions of the operators $S_\pm$ obtained by restricting S to (a,x_0) and (x_0,b) with a bundary condition $\cos(\varphi_\alpha)f(x_0)+\sin(\varphi_\alpha)f^{[1]}(x_0)=0$, respectively. According to Corollary 9.5 the functions $m_\pm$ are Herglotz functions. Now we introduce the 2×2 Weyl–Titchmarsh matrix

$$M(z) = \begin{pmatrix} -\frac{1}{m_-(z)+m_+(z)} & \frac{1}{2}\frac{m_-(z)-m_+(z)}{m_-(z)+m_+(z)} \\ \frac{1}{2}\frac{m_-(z)-m_+(z)}{m_-(z)+m_+(z)} & \frac{m_-(z)m_+(z)}{m_-(z)+m_+(z)} \end{pmatrix}, \quad z\in\mathbb{C}\backslash\mathbb{R}. \tag{11.3}$$

Note that $\det(M(z)) = -\frac{1}{4}$. The function M is a matrix Herglotz function with representation

$$M(z) = M_0 + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\Omega(\lambda), \quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{11.4}$$

where M_0 is a self-adjoint matrix and Ω is a symmetric matrix-valued measure given by the Stieltjes inversion formula

$$\Omega((\lambda_1,\lambda_2]) = \frac{1}{\pi}\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_{\lambda_1+\delta}^{\lambda_2+\delta} \mathrm{Im}(M(\lambda+\mathrm{i}\varepsilon))d\lambda, \quad \lambda_1,\lambda_2\in\mathbb{R},\ \lambda_1<\lambda_2. \tag{11.5}$$

Moreover, the trace $\Omega^{\mathrm{tr}} = \Omega_{1,1}+\Omega_{2,2}$ of Ω is a nonnegative measure and the components of Ω are absolutely continuous with respect to Ω^{tr}. The respective densities are denoted by $R_{i,j}$, $i,j\in\{0,1\}$, and are given by

$$R_{i,j}(\lambda) = \lim_{\varepsilon\downarrow 0}\frac{\mathrm{Im}(M_{i,j}(\lambda+\mathrm{i}\varepsilon))}{\mathrm{Im}(M_{1,1}(\lambda+\mathrm{i}\varepsilon)+M_{2,2}(\lambda+\mathrm{i}\varepsilon))}, \tag{11.6}$$

where the limit exists a.e. with respect to Ω^{tr}. Note that R is non-negative and has trace equal to one. In particular, all entries of R are bounded: $0\le R_{1,1},R_{2,2}\le 1$ and $|R_{1,2}| = |R_{2,1}| \le 1/2$.

The corresponding Hilbert space $L^2(\mathbb{R};\Omega)$ is associated with the inner product

$$\langle \hat f,\hat g\rangle_\Omega = \int_{\mathbb{R}} \hat f(\lambda)\hat g(\lambda)^* d\Omega(\lambda) = \int_{\mathbb{R}}\sum_{i,j=0}^{1}\hat f_i(\lambda)R_{i,j}(\lambda)\hat g_j(\lambda)^* d\Omega^{\mathrm{tr}}(\lambda).$$

For each $f \in L^2_c((a,b);\varrho)$ we define

$$\hat{f}(z) = \begin{pmatrix} \int_a^b \theta_z(x) f(x) d\varrho(x) \\ \int_a^b \phi_z(x) f(x) d\varrho(x) \end{pmatrix}, \quad z \in \mathbb{C}. \tag{11.7}$$

In the following lemma we will relate the 2×2 matrix-valued measure Ω to the operator-valued spectral measure E of S. If G is a measurable function on $\mathbb{R}$, we denote by M_G the maximally defined operator of multiplication by G in the Hilbert space $L^2(\mathbb{R};\Omega)$.

Lemma 11.1. *If $f, g \in L^2_c((a,b);\varrho)$, then we have*

$$\langle E((\lambda_1,\lambda_2])f, g\rangle = \langle M_{\mathbb{1}_{(\lambda_1,\lambda_2]}} \hat{f}, \hat{g}\rangle_\Omega,$$

for all λ_1, $\lambda_2 \in \mathbb{R}$ with $\lambda_1 < \lambda_2$.

Proof. This follows by evaluating Stones formula

$$\langle E((\lambda_1,\lambda_2])f, g\rangle = \frac{1}{\pi} \lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \int_{\lambda_1+\delta}^{\lambda_2+\delta} \operatorname{Im}\left(\langle R_{\lambda+\mathrm{i}\varepsilon} f, g\rangle\right) d\lambda$$

using our formula for the resolvent (8.3) together with Stieltjes inversion formula literally following the proof of [16, Thm. 2.12]. □

Lemma 11.1 shows that the transformation defined in (11.7) uniquely extends to an isometry $\mathcal{F}$ from $L^2((a,b);\varrho)$ into $L^2(\mathbb{R};\Omega)$.

Theorem 11.2. *The operator $\mathcal{F}$ is unitary with inverse given by*

$$(\mathcal{F}^{-1}\hat{f})(x) = (\mathcal{F}^*\hat{f})(x) = \lim_{N\to\infty} \int_{[-N,N)} \hat{f}(\lambda) \begin{pmatrix} \theta_\lambda(x) \\ \phi_\lambda(x) \end{pmatrix} d\Omega(\lambda), \tag{11.8}$$

where the limit exists in $L^2((a,b);\varrho)$. Moreover, $\mathcal{F}$ maps S onto M_{id}.

Proof. By our previous lemma it remains to show that $\mathcal{F}$ is onto. Since it is straightforward to verify that the integral operator on the right-hand side of (11.8) is the adjoint of $\mathcal{F}$, we can equivalently show $\ker(\mathcal{F}^*) = \{0\}$.

To this end let $\hat{f} \in L^2(\mathbb{R},\Omega)$, $N \in \mathbb{N}$, and $z \in \rho(S)$, then

$$(S - z) \int_{-N}^{N} \frac{1}{\lambda - z} \hat{f}(\lambda) \begin{pmatrix} \theta_\lambda(x) \\ \phi_\lambda(x) \end{pmatrix} d\Omega(\lambda) = \int_{-N}^{N} \hat{f}(\lambda) \begin{pmatrix} \theta_\lambda(x) \\ \phi_\lambda(x) \end{pmatrix} d\Omega(\lambda)$$

since interchanging integration with the Radon–Nikodym derivatives can be justified using Fubini. Taking the limit $N \to \infty$ we conclude

$$\mathcal{F}^* \frac{1}{\cdot - z} \hat{f} = R_z \mathcal{F}^* \hat{f}, \qquad \hat{f} \in L^2(\mathbb{R},\Omega).$$

By Stone–Weierstraß we even conclude $\mathcal{F}^* M_F \hat{f} = F(S)\mathcal{F}^*\hat{f}$ for any continuous function F vanishing at ∞ and by a consequence of the spectral theorem (e.g, last part of [31, Thm. 3.1]) we can further extend this to characteristic functions of intervals I. Hence, for $\hat{f} \in \ker(\mathcal{F}^*)$ we conclude

$$\int_I \hat{f}(\lambda) \begin{pmatrix} \theta_\lambda(x) \\ \phi_\lambda(x) \end{pmatrix} d\Omega(\lambda) = 0$$

for any compact interval I. Moreover, after taking Radon–Nikodym derivatives we also have

$$\int_I \hat{f}(\lambda) \begin{pmatrix} \theta_\lambda^{[1]}(x) \\ \phi_\lambda^{[1]}(x) \end{pmatrix} d\Omega(\lambda) = 0.$$

Choosing $x = x_0$ we see

$$\int_I \hat{f}(\lambda) \begin{pmatrix} \cos(\varphi_\alpha) \\ -\sin(\varphi_\alpha) \end{pmatrix} d\Omega(\lambda) = \int_I \hat{f}(\lambda) \begin{pmatrix} \sin(\varphi_\alpha) \\ \cos(\varphi_\alpha) \end{pmatrix} d\Omega(\lambda) = 0$$

for any compact interval I and thus $\hat{f} = 0$ as required. □

Next, there is a measurable unitary matrix $U(\lambda)$ which diagonalizes $R(\lambda)$, that is,

$$R(\lambda) = U(\lambda)^* \begin{pmatrix} r_1(\lambda) & 0 \\ 0 & r_2(\lambda) \end{pmatrix} U(\lambda), \tag{11.9}$$

where $0 \le r_1(\lambda) \le r_2(\lambda) \le 1$ are the eigenvalues of $R(\lambda)$. Note $r_1(\lambda) + r_2(\lambda) = 1$ by $\mathrm{tr}(R(\lambda)) = 1$. The matrix $U(\lambda)$ provides a unitary operator $L^2(\mathbb{R};\Omega) \to L^2(\mathbb{R}; r_1 d\Omega^{\mathrm{tr}}) \oplus L^2(\mathbb{R}; r_2 d\Omega^{\mathrm{tr}})$ which leaves M_{id} invariant. From this observation we immediately obtain the analog of Corollary 10.4.

Corollary 11.3. *Introduce the Herglotz function*

$$M^{\mathrm{tr}}(z) = \mathrm{tr}(M(z)) = \frac{m_-(z)m_+(z) - 1}{m_-(z) + m_+(z)} \tag{11.10}$$

associated with the measure Ω^{tr}. Then the spectrum of S is given by

$$\sigma(S) = \mathrm{supp}(\Omega^{\mathrm{tr}}) = \overline{\{\lambda \in \mathbb{R} \,|\, 0 < \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M^{\mathrm{tr}}(\lambda + i\varepsilon))\}}.$$

Moreover,

$$\sigma_p(S) = \{\lambda \in \mathbb{R} \,|\, 0 < \lim_{\varepsilon\downarrow 0} \varepsilon \mathrm{Im}(M^{\mathrm{tr}}(\lambda + i\varepsilon))\},$$
$$\sigma_{ac}(S) = \overline{\{\lambda \in \mathbb{R} \,|\, 0 < \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M^{\mathrm{tr}}(\lambda + i\varepsilon)) < \infty\}}^{ess}.$$

and

$$\Sigma_s = \{\lambda \in \mathbb{R} \,|\, \limsup_{\varepsilon\downarrow 0} \mathrm{Im}(M^{\mathrm{tr}}(\lambda + i\varepsilon)) = \infty\}$$

is a minimal support for the singular spectrum (singular continuous plus pure point spectrum) of S.

Furthermore, this allows us to investigate the spectral multiplicity of S.

Lemma 11.4. *Define*

$$\Sigma_1 = \{\lambda \in \mathrm{supp}(\Omega^{tr}) | \det R(\lambda) = r_1(\lambda) r_2(\lambda) = 0\}, \tag{11.11}$$

$$\Sigma_2 = \{\lambda \in \mathrm{supp}(\Omega^{tr}) | \det R(\lambda) = r_1(\lambda) r_2(\lambda) > 0\}. \tag{11.12}$$

Then $M_{\mathrm{id}} = M_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_1}} \oplus M_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_2}}$ and the spectral multiplicity of $M_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_1}}$ is one and the spectral multiplicity of $M_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_2}}$ is two.

Proof. It is easy to see that $M_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_1}}$ is unitary equivalent to multiplication by λ in $L^2(\mathbb{R}; \mathbb{1}_{\Sigma_1} d\Omega^{\mathrm{tr}})$. Moreover, since $r_i \mathbb{1}_{\Sigma_2} d\Omega^{\mathrm{tr}}$ and $\mathbb{1}_{\Sigma_2} d\Omega^{\mathrm{tr}}$ are mutually absolutely continuous, $M_{\mathrm{id}\cdot\mathbb{1}_{\Sigma_2}}$ is unitary equivalent to M_{id} in the Hilbert space $L^2(\mathbb{R}; \mathbb{1}_2 \mathbb{1}_{\Sigma_1} d\Omega^{\mathrm{tr}})$. □

Combining (11.3) with (11.6) we see

$$\det R(\lambda) = \lim_{\varepsilon\downarrow 0} \frac{\mathrm{Im}(m_+(\lambda+\mathrm{i}\varepsilon))\mathrm{Im}(m_-(\lambda+\mathrm{i}\varepsilon))}{|m_+(\lambda+\mathrm{i}\varepsilon)+m_-(\lambda+\mathrm{i}\varepsilon)|^2}\frac{1}{\mathrm{Im}(M^{\mathrm{tr}}(\lambda+\mathrm{i}\varepsilon))^2}, \tag{11.13}$$

where the first factor is bounded by 1/4. Now Lemma 11.4 immediately gives

Lemma 11.5. *The singular spectrum of S has spectral multiplicity one. The absolutely continuous spectrum of S has multiplicity two on $\sigma_{ac}(S_+)\cap\sigma_{ac}(S_-)$ and multiplicity one on $\sigma_{ac}(S)\backslash(\sigma_{ac}(S_+)\cap\sigma_{ac}(S_-))$. Here $S_\pm$ are restrictions of S to (a,x_0) and (x_0,b), respectively.*

Proof. Using the fact that Σ_s is a minimal support for the singular part of S we obtain $S_s = S_{pp}\oplus S_{sc} = E(\Sigma_s)S$ and $S_{ac} = (1-E(\Sigma_s))S$. So we see that the singular part has multiplicity one by Lemma 11.4.

For the absolutely continuous part use that

$$\Sigma_{ac,\pm} = \{\lambda\in\mathbb{R} | 0 < \lim_{\varepsilon\downarrow 0}\mathrm{Im}(m_\pm(\lambda+\mathrm{i}\varepsilon)) < \infty\}$$

are minimal supports for the absolutely continuous spectrum of $S_\pm$. Again the remaining result follows from Lemma 11.4. □

Appendix A. Linear measure differential equations

In this appendix we collect some necessary facts from linear differential equations with measure coefficients. We refer to Bennewitz [5], Persson [24], Volkmer [36], Atkinson [4] or Schwabik, Tvrdý and Vejvoda [29] for further information. In order to make our presentation self-contained we have included proofs for all results.

Let (a,b) be a finite or infinite interval and ω a positive Borel measures on (a,b). Furthermore, let M be a $\mathbb{C}^{n\times n}$ valued measurable function on (a,b) and F a $\mathbb{C}^n$ valued measurable function on (a,b), such that $\|M(\cdot)\|$, $\|F(\cdot)\| \in L^1_{loc}((a,b);\omega)$. Here $\|\cdot\|$ denotes some norm on $\mathbb{C}^n$ as well as the corresponding operator norm on $\mathbb{C}^{n\times n}$.

For $c\in(a,b)$ and $Y_c\in\mathbb{C}^n$, some $\mathbb{C}^n$ valued function Y on (a,b) is a solution of the initial value problem

$$\frac{dY}{d\omega} = MY + F, \qquad Y(c) = Y_c, \tag{A.1}$$

if the components of Y are locally absolutely continuous with respect to ω, their Radon–Nikodym derivatives satisfy (A.1) almost everywhere with respect to ω and $Y(c) = Y_c$. An integration shows that some function Y is a solution of the initial value problem (A.1) if and only if it solves the vector integral equation

$$Y(x) = Y_c + \int_c^x (M(t)Y(t)+F(t))\,d\omega(t), \quad x\in(a,b). \tag{A.2}$$

Before we prove existence and uniqueness for solutions of the initial value problem, we need a Gronwall lemma. The proof follows [5, Lemma 1.2 and Lemma 1.3] (see also Atkinson [4, p. 455]).

Lemma A.1. *Let $c\in(a,b)$ and $v\in L^1_{loc}((a,b);\omega)$ be real-valued such that*

$$0\le v(x)\le K + \int_c^x v(t)d\omega(t), \quad x\in[c,b),$$

for some constant $K \geq 0$, then v can be estimated by

$$v(x) \leq K \mathrm{e}^{\int_c^x d\omega}, \quad x \in [c,b).$$

Proof. First of all note that the function $F(x) = \int_c^x d\omega$, $x \in [c,b)$, satisfies

$$F(x)^{n+1} \geq (n+1) \int_c^x F(t)^n d\omega(t), \quad x \in [c,b), \tag{A.3}$$

by a variant of the substituion rule for Lebesgue–Stieltjes integrals [32, Cor. 5.3]. Now we will prove that

$$v(x) \leq K \sum_{k=0}^n \frac{F(x)^k}{k!} + \frac{F(x)^n}{n!} \int_c^x v(t) d\omega(t), \quad x \in [c,b),$$

for each $n \in \mathbb{N}_0$. For $n = 0$ this is just our assumption. If $n > 0$ we get inductively

$$\begin{aligned} v(x) &\leq K + \int_c^x v(t) d\omega(t) \\ &\leq K + \int_c^x \left(K \sum_{k=0}^n \frac{F(t)^k}{k!} + \frac{F(t)^n}{n!} \int_c^t v d\omega \right) d\omega(t) \\ &\leq K \left(1 + \sum_{k=0}^n \int_c^x \frac{F(t)^k}{k!} d\omega(t) \right) + \int_c^x \frac{F(t)^n}{n!} d\omega(t) \int_c^x v d\omega \\ &\leq K \sum_{k=0}^{n+1} \frac{F(x)^k}{k!} + \frac{F(x)^{n+1}}{(n+1)!} \int_c^x v d\omega, \quad x \in [c,b), \end{aligned}$$

where we used (A.3) twice in the last step. Now taking the limit $n \to \infty$ yields the claim. □

Because of the definition of our integral the assertion of this lemma is only true to the right of the point c. Indeed, a simple reflection proves that the estimate

$$0 \leq v(x) \leq K + \int_{x+}^{c+} v(t) d\omega(t), \quad x \in (a,c],$$

for some constant $K \geq 0$, implies

$$v(x) \leq K e^{\int_{x+}^{c+} d\omega}, \quad x \in (a,c]. \tag{A.4}$$

We are now ready to prove the basic existence and uniqueness result.

Theorem A.2. *The initial value problem* (A.1) *has a unique solution for each $c \in (a,b)$ and $Y_c \in \mathbb{C}^n$ if and only if the matrix*

$$I + \omega(\{x\}) M(x) \quad \textit{is regular} \tag{A.5}$$

for all $x \in (a,b)$. In this case solutions are real if M, F, and Y_c are real.

Proof. First assume that the initial value problem (A.1) has a unique solution for each $c \in (a,b)$ and $Y_c \in \mathbb{C}^n$. Now if the matrix (A.5) were not regular for some $x_0 \in (a,b)$, we had two solutions Y_1, Y_2 such that $Y_1(x_0) \neq Y_2(x_0)$ but

$Y_1(x_0+) = Y_2(x_0+)$. Indeed one only has to take solutions with different initial conditions at x_0 such that

$$\begin{aligned} Y_1(x_0+) + \omega(\{x_0\})F(x_0) &= (I + \omega(\{x_0\})M(x_0))\, Y_1(x_0) \\ &= (I + \omega(\{x_0\})M(x_0))\, Y_2(x_0) \\ &= Y_2(x_0+) + \omega(\{x_0\})F(x_0). \end{aligned}$$

But then one had

$$\begin{aligned} \|Y_1(x) - Y_2(x)\| &= \left\| \int_{x_0+}^{x} M(t)\,(Y_1(t) - Y_2(t))\, d\omega(t) \right\| \\ &\leq \int_{x_0+}^{x} \|M(t)\|\, \|Y_1(t) - Y_2(t)\|\, d\omega(t), \quad x \in (x_0, b), \end{aligned}$$

and hence by Lemma A.1, $Y_1(x) = Y_2(x)$ for all $x \in (x_0, b)$. But this is a contradiction since now Y_1 and Y_2 are two different solutions of the initial value problem with $Y_c = Y_1(c)$ for some $c \in (x_0, b)$.

Now assume (A.5) holds for all $x \in (a,b)$ and let α, $\beta \in (a,b)$ with $\alpha < c < \beta$. It suffices to prove that there is a unique solution of (A.2) on (α, β). In order to prove uniqueness, take a solution Y of the homogenous system, i.e. $Y_c = 0$ and $F = 0$. We get

$$\|Y(x)\| \leq \int_c^x \|M(t)\|\, \|Y(t)\|\, d\omega(t), \quad x \in [c, \beta),$$

and hence $Y(x) = 0$, $x \in [c, \beta)$, by Lemma A.1. To the left-hand side we have

$$\begin{aligned} Y(x) &= -\int_x^c M(t)Y(t)d\omega(t) = -\int_{x+}^c M(t)Y(t)d\omega(t) - \omega(\{x\})M(x)Y(x) \\ &= -\left(I + \omega(\{x\})M(x)\right)^{-1} \int_{x+}^c M(t)Y(t)d\omega(t), \quad x \in (\alpha, c), \end{aligned}$$

and hence

$$\|Y(x)\| \leq \left\| (I + \omega(\{x\})M(x))^{-1} \right\| \int_{x+}^{c+} \|M(t)\|\, \|Y(t)\|\, d\omega(t), \quad x \in (\beta, c].$$

Now the function in front of the integral is bounded. Indeed, since M is locally integrable, we have $\omega(\{x\})\|M(x)\| < \frac{1}{2}$ for all but finitely many $x \in [\beta, c]$. Moreover, for those x we have

$$\left\| (I + \omega(\{x\})M(x))^{-1} \right\| = \left\| \sum_{n=0}^{\infty} \left(-\omega(\{x\})M(x)\right)^n \right\| \leq 2.$$

Therefore estimate (A.4) applies and yields $Y(x) = 0$, $x \in (\beta, c]$.

Now we will construct the solution by successive approximation. To this end we define

$$Y_0(x) = Y_c + \int_c^x F(t)d\omega(t), \quad x \in [c, \beta), \tag{A.6}$$

and inductively for each $n \in \mathbb{N}$

$$Y_n(x) = \int_c^x M(t)Y_{n-1}(t)d\omega(t), \quad x \in [c, \beta). \tag{A.7}$$

These functions are bounded by

$$\|Y_n(x)\| \leq \sup_{t\in[c,x)} \|Y_0(t)\| \frac{\left(\int_c^x \|M(t)\|\, d\omega(t)\right)^n}{n!}, \quad x \in [c,\beta). \tag{A.8}$$

Indeed for $n = 0$ this is obvious, for $n > 0$ we get inductively, using (A.3),

$$\begin{aligned} \|Y_n(x)\| &\leq \int_c^x \|M(t)\|\, \|Y_{n-1}(t)\|\, d\omega(t) \\ &\leq \sup_{t\in[c,x)} \|Y_0(t)\| \int_c^x \|M(t)\| \frac{\left(\int_c^t \|M(s)\|\, d\omega(s)\right)^n}{n!} d\omega(t) \\ &\leq \sup_{t\in[c,x)} \|Y_0(t)\| \frac{\left(\int_c^x \|M(t)\|\, d\omega(t)\right)^{n+1}}{(n+1)!}. \end{aligned}$$

Hence the sum $Y(x) = \sum_{n=0}^\infty Y_n(x)$, $x \in [c,\beta)$ converges absolutely and uniformly. Moreover, we have

$$\begin{aligned} Y(x) &= Y_0(x) + \sum_{n=1}^\infty \int_c^x M(t) Y_{n-1}(t) d\omega(t) \\ &= Y_c + \int_c^x M(t)Y(t) + F(t) d\omega(t), \quad x \in [c,\beta). \end{aligned}$$

In order to extend the solution to the left of c, take points $x_k \in [\alpha, c]$, $k = -N, \dots, 0$ with

$$\alpha = x_{-N} < x_{-N+1} < \cdots < x_0 = c$$

such that

$$\int_{(x_k,x_{k+1})} \|M(t)\| d\omega(t) < \frac{1}{2}, \quad -N \leq k < 0, \tag{A.9}$$

which is possible since M is locally integrable. First of all, take all points $x \in (\alpha, c)$ with $\omega(\{x\})\|M(x)\| \geq \frac{1}{2}$ (these are at most finitely many because $\|M(\cdot)\|$ is locally integrable). Then divide the remaining subintervals such that (A.9) is valid. Now let $-N < k \leq 0$ and assume Y is a solution on $[x_k, \beta)$. We will show that Y can be extended to a solution on $[x_{k-1}, \beta)$. To this end we define

$$Z_0(x) = Y(x_k) + \int_{x_k}^x F(t) d\omega(t), \quad x \in (x_{k-1}, x_k], \tag{A.10}$$

and inductively for each $n \in \mathbb{N}$

$$Z_n(x) = \int_{x_k}^x M(t) Z_{n-1}(t) d\omega(t), \quad x \in (x_{k-1}, x_k]. \tag{A.11}$$

Using (A.9) it is easy to prove inductively that these functions are bounded by

(A.12)

$$\|Z_n(x)\| \leq \left(\|Y(x_k)\| + \int_{[x_{k-1},x_k]} \|F(t)\|\, d\omega(t) \right) \frac{1}{2^n}, \quad x \in (x_{k-1}, x_k],\ n \in \mathbb{N}.$$

Hence we may extend Y onto (x_{k-1}, x_k) by

$$Y(x) = \sum_{n=0}^{\infty} Z_n(x), \quad x \in (x_{k-1}, x_k),$$

where the sum converges absolutely and uniformly. As above one shows that Y is a solution of (A.2) on (x_{k-1}, β). Now if we extend Y further by (note that the right-hand limit exists because of (A.2))

(A.13)

$$Y(x_{k-1}) = (I - \omega(\{x_{k-1}\})M(x_{k-1}))^{-1} \left(Y(x_{k-1}+) + \omega(\{x_{k-1}\})F(x_{k-1})\right),$$

then it is an easy calculation to prove that Y satisfies (A.2) for all $x \in [x_{k-1}, \beta)$. After finitely many steps one arrives at a solution Y, satisfying (A.2) for all $x \in (\alpha, \beta)$.

If the data M, F, and Y_c are real, one easily sees that all quantities in the construction stay real. □

The proof of Theorem A.2 shows that condition (A.5) is only needed for all points x to the left of the initial point c. Indeed it is always possible to extend solutions uniquely to the right of the initial point but not to the left. For a counterexample take $n = 1$, the interval $(-2, 2)$, $y_0 \in \mathbb{C}$ and $\omega = -\delta_{-1} - \delta_1$. Then one easily checks that the integral equation

$$y(x) = y_0 + \int_0^x y(t)d\omega(t), \quad x \in (-2, 2)$$

has the solutions

$$y_d(x) = \begin{cases} d, & \text{if } -2 < x \leq -1, \\ y_0, & \text{if } -1 < x \leq 1, \\ 0, & \text{if } 1 < x < 2, \end{cases}$$

for each $d \in \mathbb{C}$. Hence the solutions are not unique to the left of the initial point $c = 0$.

Corollary A.3. *Assume* (A.5) *holds for each* $x \in (a, b)$. *Then for each* $c \in (a, b)$ *and* $Y_c \in \mathbb{C}^n$, *the initial value problem*

$$\frac{dY}{d\omega} = MY + F \quad \textit{with} \quad Y(c+) = Y_c,$$

has a unique solution. If M, F, *and* Y_c *are real then the solution is real.*

Proof. Some function Y is a solution of this initial value problem if and only if it is a solution of

$$\frac{dY}{d\omega} = MY + F \quad \text{with} \quad Y(c) = (I + \omega(\{c\})M(c))^{-1} \left(Y_c - \omega(\{c\})F(c)\right).$$

□

Theorem A.4. *Assume* $\|M(\cdot)\|$ *and* $\|F(\cdot)\|$ *are integrable near* a *and* Y *is a solution of the initial value problem* (A.1). *Then the limit*

$$Y(a) := \lim_{x \downarrow a} Y(x)$$

exists and is finite. A similar result holds for the endpoint b.

Proof. By assumption there is a $c \in (a,b)$ such that

$$\int_a^c \|M(t)\| \, d\omega(t) \leq \frac{1}{2}.$$

We first prove that $\|Y(\cdot)\|$ is bounded near a. Indeed if it were not, we had a monotone sequence $x_n \in (a,c)$ with $x_n \downarrow a$ such that $\|Y(x_n)\| \geq \|Y(x)\|$, $x \in [x_n, c]$. From the integral equation which Y satisfies we get

$$\begin{aligned} \|Y(x_n)\| &\leq \|Y(c)\| + \int_{x_n}^c \|M(t)\| \|Y(t)\| d\omega(t) + \int_{x_n}^c \|F(t)\| d\omega(t) \\ &\leq \|Y(c)\| + \|Y(x_n)\| \int_{x_n}^c \|M(t)\| d\omega(t) + \int_a^c \|F(t)\| d\omega(t) \\ &\leq \|Y(c)\| + \int_a^c \|F(t)\| d\omega(t) + \frac{1}{2} \|Y(x_n)\|. \end{aligned}$$

Hence $\|Y(\cdot)\|$ is bounded near a by some constant K. Now the first claim follows because we have

$$\begin{aligned} \|Y(x_1) - Y(x_2)\| &= \left\| \int_{x_2}^{x_1} M(t) Y(t) d\omega(t) \right\| \\ &\leq K \int_{x_1}^{x_2} \|M(t)\| \, d\omega(t), \end{aligned}$$

for each x_1, $x_2 \in (a,c)$, $x_1 < x_2$ i.e. $Y(x)$ is a Cauchy-sequence as $x \downarrow a$. □

Under the assumption of Theorem A.4 one can show that there is always a unique solution of the initial value problem

$$\frac{dY}{d\omega} = MY + F \quad \text{with} \quad Y(a) = Y_a$$

with essentially the same proof as for Theorem A.2. If $\|M(\cdot)\|$ is integrable near b one furthermore has to assume that (A.5) holds for all $x \in (a,b)$ in order to get unique solutions of the initial value problem

$$\frac{dY}{d\omega} = MY + F \quad \text{with} \quad Y(b) = Y_b.$$

In the following let M_1, M_2 be $\mathbb{C}^{n \times n}$ valued functions on (a,b) such that M_1, $M_2 \in L^1_{loc}((a,b);\omega)$. We are interested in the analytic dependence on $z \in \mathbb{C}$ of solutions to the initial value problems

$$\frac{dY}{d\omega} = (M_1 + zM_2)\, Y + F \quad \text{with} \quad Y(c) = Y_c. \tag{A.14}$$

Theorem A.5. *Assume* (A.5) *holds for each* $x \in (a,b)$. *If for each* $z \in \mathbb{C}$, Y_z *is the unique solution of* (A.14), *then* $Y_z(x)$ *is analytic for each* $x \in (a,b)$.

Proof. We show that the construction in the proof of Theorem A.2 yields analytic solutions. Indeed let α, $\beta \in (a,b)$ with $\alpha < c < \beta$ as in the proof of Theorem A.2. Then the now z dependent functions $Y_{z,n}(x)$, $n \in \mathbb{N}_0$ (defined as in (A.6) and (A.7)) are polynomials in z for each fixed $x \in (c,\beta)$. Furthermore, the sum $\sum_{n=0}^\infty Y_{z,n}(x)$ converges locally uniformly in z by (A.8) which proves that $Y_z(x)$ is analytic. Now in order to prove analyticity to the left of c fix some $R \in \mathbb{R}^+$. Then there are points $x_k \in [\alpha, c]$, $k = -N, \ldots, 0$ as in the proof of Theorem A.2 such that (A.9) holds for all $M = M_1 + zM_2$, $|z| < R$. It suffices to prove that if $Y_z(x_k)$ is analytic for some $-N < k \leq 0$ then $Y_z(x)$ is analytic for all $x \in [x_{k-1}, x_k)$. Indeed for each $n \in \mathbb{N}_0$ and $x \in (x_{k-1}, x_k)$ the functions $Z_{z,n}(x)$ (defined as in (A.10) and (A.11)) are analytic and locally bounded in $|z| < R$. From the bound (A.12) one sees that $\sum_{n=0}^{\infty} Z_{z,n}(x)$ converges locally uniformly in $|z| < R$. Hence $Y_z(x)$ is analytic in $\mathbb{C}$. Furthermore, $Y_z(x_{k-1})$ is analytic by (A.13) (note that $Y_z(x_{k-1}+)$ is also analytic since $Y_z(x)$ is bounded locally uniformly in z to the right of x_{k-1}). □

Under the assumptions of the last theorem we even see that the right-hand limit $Y_z(x+)$ is analytic for each $x \in (a, b)$. Indeed this follows since

$$Y_z(x+) = \lim_{\xi \downarrow x} Y_z(\xi), \quad z \in \mathbb{C}$$

and $Y_z(x)$ is bounded locally uniformly in x and z. Furthermore, one can show (see the proof of Corollary A.3) that if for each $z \in \mathbb{C}$, Y_z is the solution of the initial value problem

$$\frac{dY}{d\omega} = (M_1 + zM_2)\, Y + F \quad \text{with} \quad Y(c+) = Y_c,$$

$Y_z(x)$ as well as $Y_z(x+)$ are analytic in $z \in \mathbb{C}$ for each $x \in (a, b)$.

Appendix B. Linear relations in Hilbert spaces

Let X and Y be linear spaces over $\mathbb{C}$. A linear relation of X into Y is a linear subspace of $X \times Y$. The space of all linear relations of X into Y is denoted by $\mathrm{LR}(X, Y)$. Linear relations generalize the notion of linear operators. Indeed, if D is a linear subspace of X and $T : D \to Y$ is a linear operator, we may identify T with its graph, which is a linear relation of X into Y. In this way any operator can be regarded as a linear relation. Motivated by this point of view, we define the domain, range, kernel and multi-valued part of some linear relation $T \in \mathrm{LR}(X, Y)$ as

$$\begin{aligned} \mathrm{dom}\,(T) &= \{x \in X \,|\, \exists y \in Y : (x, y) \in T\}\,, \\ \mathrm{ran}\,(T) &= \{y \in Y \,|\, \exists x \in X : (x, y) \in T\}\,, \\ \ker\,(T) &= \{x \in X \,|\, (x, 0) \in T\}\,, \\ \mathrm{mul}\,(T) &= \{y \in Y \,|\, (0, y) \in T\}\,. \end{aligned}$$

Note that some relation T is (the graph of) an operator if and only if $\mathrm{mul}\,(T) = \{0\}$. In this case these definitions are consistent with the usual definitions for operators.

Again motivated by an operator theoretic viewpoint, we define the following algebraic operations. For T, $S \in \mathrm{LR}(X, Y)$ and $\lambda \in \mathbb{C}$ we set

$$T + S = \{(x, y) \in X \times Y \,|\, \exists y_1, y_2 \in Y : (x, y_1) \in T, (x, y_2) \in S, y = y_1 + y_2\}\,,$$

and

$$\lambda T = \{(x, y) \in X \times Y \,|\, \exists y_0 \in Y : (x, y_0) \in T, y = \lambda y_0\}\,.$$

It is simple to check that both, $T + S$ and λT are linear relations of X into Y. Moreover, we can define the composition of two linear relations. If $T \in \mathrm{LR}(X, Y)$ and $S \in \mathrm{LR}(Y, Z)$ for some linear space Z, then

$$ST = \{(x, z) \in X \times Z \,|\, \exists y \in Y : (x, y) \in T, (y, z) \in S\}\,,$$

is a linear relation of X into Z. One may even define an inverse of a linear relation $T \in \mathrm{LR}(X,Y)$ by

$$T^{-1} = \{(y,x) \in Y \times X \,|\, (x,y) \in T\},$$

as a linear relation of Y into X. For further properties of these algebraic operations of linear relations see [3, 2.1 Theorem], [11, Chapter 1] or [17, Appendix A].

From now on assume X and Y are Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$. The adjoint of a linear relation $T \in \mathrm{LR}(X,Y)$, given by

$$T^* = \{(y,x) \in Y \times X \,|\, \forall (u,v) \in T : \langle u, x\rangle_X = \langle v, y\rangle_Y\},$$

is a linear relation of Y into X. The adjoint of a linear relation is always closed, i.e. T^* is closed with respect to the product topology on $Y \times X$. Moreover, one has

$$T^{**} = \overline{T}, \quad \ker(T^*) = \mathrm{ran}\,(T)^{\perp} \quad \text{and} \quad \mathrm{mul}\,(T^*) = \mathrm{dom}\,(T)^{\perp}. \tag{B.1}$$

If $S \in \mathrm{LR}(X,Y)$ is another linear relation we have

$$T \subseteq S \quad \Rightarrow \quad T^* \supseteq S^*. \tag{B.2}$$

All these properties of adjoints may be found in [3, Section 3] or in [17, Proposition C.2.1].

Now let T be a closed linear relation of X into X. The resolvent set $\rho(T)$ of T consists of all numbers $z \in \mathbb{C}$ such that $R_z = (T - z)^{-1}$ is an everywhere defined operator. Here $T - z$ is short-hand for the relation $T - zI$, where I is the identity operator on X. The mapping $z \mapsto R_z$ on $\rho(T)$, called the resolvent of T, has the following properties (see [11, Section VI.1] or [17, Proposition A.2.3]).

Theorem B.1. *The resolvent set $\rho(T)$ is open and the resolvent identity*

$$R_z - R_w = (z-w) R_z R_w, \quad z,\ w \in \rho(T),$$

holds. Moreover, the resolvent is analytic as a mapping into the space of everywhere defined, bounded linear operators on X, equipped with the operator norm.

The spectrum $\sigma(T)$ of a closed linear relation T is the complement of the resolvent set. One may divide the spectrum into three disjoint parts.

$$\begin{aligned}
\sigma_p(T) &= \{\lambda \in \sigma(T) \,|\, \ker(T-\lambda) \neq \{0\}\} \\
\sigma_c(T) &= \{\lambda \in \sigma(T) \,|\, \ker(T-\lambda) = \{0\},\, \mathrm{ran}\,(T-\lambda) \neq X,\, \overline{\mathrm{ran}\,(T-\lambda)} = X\}, \\
\sigma_r(T) &= \{\lambda \in \sigma(T) \,|\, \ker(T-\lambda) = \{0\},\, \overline{\mathrm{ran}\,(T-\lambda)} \neq X\}.
\end{aligned}$$

The set $\sigma_p(T)$ is called the point spectrum, $\sigma_c(T)$ is the continuous spectrum and $\sigma_r(T)$ is the residual spectrum of T. Elements of the point spectrum are called eigenvalues. The spaces $\ker(T-\lambda)$ corresponding to some eigenvalue λ are called eigenspaces, the non zero elements of the eigenspaces are called eigenvectors.

We need a variant of the spectral mapping theorem for the resolvent (see [11, Section VI.4] or [17, Proposition A.3.1]).

Theorem B.2. *For each $z \in \rho(T)$ we have*

$$\sigma(R_z) \backslash \{0\} = \left\{ \frac{1}{\lambda - z} \,\middle|\, \lambda \in \sigma(T) \right\}.$$

Some linear relation T is said to be symmetric provided that $T \subseteq T^*$. If T is a closed symmetric linear relation, we have $\rho(T) \subseteq \mathrm{r}(T)$ and $\mathbb{C}\backslash\mathbb{R} \subseteq \mathrm{r}(T)$, where

$$\mathrm{r}(T) = \left\{ z \in \mathbb{C} \,|\, (T-z)^{-1} \text{ is a bounded operator} \right\},$$

denotes the points of regular type of T. Moreover, some linear relation S is said to be self-adjoint, if $S = S^*$. In this case S is closed, the spectrum of S is real and from (B.1) one sees that

$$\text{(B.3)} \qquad \operatorname{mul}(S) = \operatorname{dom}(S)^{\perp} \quad \text{and} \quad \ker(S) = \operatorname{ran}(S)^{\perp}.$$

In particular S is an operator if and only if its domain is dense. Furthermore,

$$\text{(B.4)} \qquad S_{\mathfrak{D}} = S \cap (\mathfrak{D} \times \mathfrak{D})$$

is a self-adjoint operator in the Hilbert space $\mathfrak{D}$, where $\mathfrak{D}$ is the closure of the domain of S. These properties of symmetric and self-adjoint linear relations may be found in [12, Theorem 3.13, Theorem 3.20 and Theorem 3.23]. Moreover, the following result shows that S and $S_{\mathfrak{D}}$ (as an operator in the Hilbert space $\mathfrak{D}$) have many spectral properties in common.

Lemma B.3. *S and $S_{\mathfrak{D}}$ have the same spectrum and*

$$R_z f = (S_{\mathfrak{D}} - z)^{-1} P f, \quad f \in X,\ z \in \rho(S),$$

where P is the orthogonal projection onto $\mathfrak{D}$. Moreover, the eigenvalues as well as the corresponding eigenspaces coincide.

Proof. From the equalities

$$\operatorname{ran}(S_{\mathfrak{D}} - z) = \operatorname{ran}(S - z) \cap \mathfrak{D} \quad \text{and} \quad \ker(S_{\mathfrak{D}} - z) = \ker(S - z), \quad z \in \mathbb{C},$$

one sees that S and $S_{\mathfrak{D}}$ have the same spectrum as well as the same point spectrum and corresponding eigenspaces. Now let $z \in \rho(S)$, $f \in X$ and set $g = (S-z)^{-1} f$, i.e. $(g, f) \in S - z$. If $f \in \mathfrak{D}$, then since $g \in \mathfrak{D}$, we have $(g, f) \in S_{\mathfrak{D}} - z$, i.e. $(S_{\mathfrak{D}} - z)^{-1} f = g$. Finally note that if $f \in \mathfrak{D}^{\perp}$, then $g = 0$ since we have $\operatorname{mul}(S - z) = \operatorname{mul}(S) = \operatorname{dom}(S)^{\perp}$. □

Applying a variant of the spectral theorem to $S_{\mathfrak{D}}$, we get the following result for the self-adjoint relation S.

Lemma B.4. *For each f, $g \in X$ there is a unique complex Borel measure $E_{f,g}$ on $\mathbb{R}$ such that*

$$\text{(B.5)} \qquad \langle R_z f, g \rangle_X = \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{f,g}(\lambda), \quad z \in \rho(S).$$

Moreover,

$$\langle P f, g \rangle_X = \int_{\mathbb{R}} dE_{f,g}, \quad f,\ g \in X,$$

and for each $f \in X$, $E_{f,f}$ is a positive measure with

$$P f \in \operatorname{dom}(S) \quad \Leftrightarrow \quad \int_{\mathbb{R}} |\lambda|^2 dE_{f,f}(\lambda) < \infty.$$

In this case

$$\langle f_S, P g \rangle_X = \int_{\mathbb{R}} \lambda \, dE_{f,g}(\lambda),$$

whenever $(P f, f_S) \in S$.

Proof. Since $S_{\mathfrak{D}}$ is a self-adjoint operator in $\mathfrak{D}$, there is an operator-valued spectral measure E such that for all $f,\, g \in \mathfrak{D}$

$$\langle (S_{\mathfrak{D}} - z)^{-1} f, g\rangle_X = \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{f,g}(\lambda), \quad z \in \rho(S_{\mathfrak{D}}),$$

where $E_{f,g}$ is the complex measure given by $E_{f,g}(B) = \langle E(B)f, g\rangle_X$, for each Borel set B. For arbitrary $f,\, g \in X$ we set $E_{f,g} = E_{Pf,Pg}$. Because of Lemma B.3 we get for each $z \in \rho(S)$ the claimed equality

$$\begin{aligned}\langle R_z f, g\rangle_X = \langle R_z Pf, Pg\rangle_X = \langle (S_{\mathfrak{D}} - z)^{-1} Pf, Pg\rangle_X &= \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{Pf,Pg}(\lambda) \\ &= \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{f,g}(\lambda),\end{aligned}$$

where we used $R_z = PR_zP$ (see (8.1)). Uniqueness of these measures follows from the Stieltjes inversion formula. The remaining claims follow from the fact that E is the spectral measure of $S_{\mathfrak{D}}$. □

We are interested in self-adjoint extensions of symmetric relations in X. Therefore let T be a closed symmetric linear relation in $X \times X$. The linear relations

$$M_{\pm}(T) = \{(x, y) \in T^* \,|\, y = \pm \mathrm{i} x\},$$

are called deficiency spaces of T. Note that $M_{\pm}(T)$ are operators with

$$\mathrm{dom}\,(M_{\pm}(T)) = \mathrm{ran}\,(T \mp \mathrm{i})^{\perp} = \ker\,(T^* \pm \mathrm{i}).$$

Furthermore, one gets an analog of the first von Neumann formula (see [12, Theorem 6.1])

$$T^* = T \oplus M_+(T) \oplus M_-(T), \tag{B.6}$$

where the sums are orthogonal with respect to the usual inner product in $X \times X$. Now the existence of self-adjoint extensions of T is determined by these subspaces (see [10, Theorem 15] or [12, Corollary 6.4]).

Theorem B.5. *The relation T has a self-adjoint extension if and only if the dimensions of the deficiency subspaces are equal. In this case all self-adjoint extensions S of T are of the form*

$$S = T \oplus (I - V) M_+(T), \tag{B.7}$$

where V is an isometry from $M_+(T)$ onto $M_-(T)$. Conversely, for each such isometry V the linear relation S given by (B.7) *is self-adjoint.*

In particular, we are interested in the case when the deficiency subspaces are finite-dimensional, i.e.

$$n_{\pm}(T) = \dim M_{\pm}(T) < \infty.$$

The numbers $n_{\pm}(T)$ are called deficiency indices of T.

Corollary B.6. *Suppose T has equal and finite deficiency indices, i.e.*

$$n_-(T) = n_+(T) = n \in \mathbb{N},$$

then the self-adjoint extensions of T are precisely the n-dimensional symmetric extensions of T.

Proof. By Theorem B.5 each self-adjoint extension is an n-dimensional symmetric extension of T, since $\dim(I - V)M_+(T) = n$. Conversely assume S is an n-dimensional symmetric extension of T, i.e. $S = T\dot{+}M$ for some n-dimensional symmetric subspace M. Then since $\dim \operatorname{ran}(M \mp \mathrm{i}) = n = \dim X/\operatorname{ran}(T \mp \mathrm{i})$ (note that $(M \mp \mathrm{i})^{-1}$ is an operator) we get

$$\operatorname{ran}(S \mp \mathrm{i}) = \operatorname{ran}(T \mp \mathrm{i}) \dot{+} \operatorname{ran}(M \mp \mathrm{i}) = X.$$

Hence we get $\dim M_\pm(S) = 0$ and therefore $S^* = S$ in view of (B.6). □

The essential spectrum $\sigma_e(S)$ of a self-adjoint linear relation S consists of all eigenvalues of infinite multiplicity and all accumulation points of the spectrum. Moreover, the discrete spectrum $\sigma_d(S)$ of S consists of all eigenvalues of S with finite multiplicity which are isolated points of the spectrum of S. From Lemma B.3 one immediately sees that

$$\sigma_e(S) = \sigma_e(S_{\mathfrak{D}}) \quad \text{and} \quad \sigma_d(S) = \sigma_d(S_{\mathfrak{D}}).$$

Using this equality, we get the following two theorems on the stability of the essential spectrum.

Theorem B.7. *Let T be a symmetric relation with equal and finite deficiency indices n and S_1, S_2 be self-adjoint extensions of T. Then the operators*

$$(S_1 \pm \mathrm{i})^{-1} - (S_2 \pm \mathrm{i})^{-1}$$

are at most n-dimensional. In particular we have

$$\sigma_e(S_1) = \sigma_e(S_2).$$

Proof. Because of $\dim \operatorname{ran}(T \pm \mathrm{i})^\perp = n$ and

$$(S_1 \pm \mathrm{i})^{-1} f = (T \pm \mathrm{i})^{-1} f = (S_2 \pm \mathrm{i})^{-1} f, \quad f \in \overline{\operatorname{ran}(T \pm \mathrm{i})},$$

the difference of the resolvents is at most n-dimensional. Now the remaining claim follows from Lemma B.3 and [31, Theorem 6.19]. □

Theorem B.8. *Let X_1, X_2 be closed, linear subspaces of X such that $X = X_1 \oplus X_2$. If S_1 is a self-adjoint linear relation in X_1 and S_2 is a self-adjoint linear relation in X_2, then $S_1 \oplus S_2$ is a self-adjoint linear relation in x with*

$$\sigma_e(S_1 \oplus S_2) = \sigma_e(S_1) \cup \sigma_e(S_2).$$

Proof. A simple calculation shows that $(S_1 \oplus S_2)^* = S_1^* \oplus S_2^* = S_1 \oplus S_2$. Since

$$\mathfrak{D} = \overline{\operatorname{dom}(S_1 \oplus S_2)} = \overline{\operatorname{dom}(S_1)} \oplus \overline{\operatorname{dom}(S_2)} = \mathfrak{D}_1 \oplus \mathfrak{D}_2,$$

and

$$(S_1 \oplus S_2)_{\mathfrak{D}} = S_{1\mathfrak{D}_1} \oplus S_{2\mathfrak{D}_2},$$

the claim follows from the corresponding result for operators. □

Appendix C. One dimensional Sturm–Liouville problems

For the sake of completeness we consider in this section the case when ϱ is not necessarily supported on more than one point, i.e. we only assume (i) to (v) of Hypothesis 3.7. Because of the lack of the identification of Proposition 3.9 in this case, we make the following definition. Some linear subspace $S \subseteq \mathfrak{D}_\tau$ is said to give rise to a self-adjoint relation if the map

$$\begin{array}{rcc} S & \rightarrow & L^2((a,b);\varrho) \times L^2((a,b);\varrho) \\ f & \mapsto & (f, \tau f) \end{array} \tag{C.1}$$

is well-defined, injective and its range is a self-adjoint relation of $L^2((a,b);\varrho)$ into $L^2((a,b);\varrho)$. By the identification of Proposition 3.9 one sees that we already determined all linear subspaces of $\mathfrak{D}_\tau$ which give rise to a self-adjoint relation if $|\operatorname{supp}(\varrho)| > 1$. Hence we need only consider the case when $|\operatorname{supp}(\varrho)| = 1$. Indeed we will do this by proving a version of Theorem 7.6 (note that τ is in the l.c. case). Therefore assume in the following $\varrho = \varrho_0 \delta_{x_0}$ for some $\varrho_0 \in \mathbb{R}^+$ and $x_0 \in (a,b)$. In this case each function $f \in \mathfrak{D}_\tau$ is of the form

$$f(x) = \begin{cases} u_a(x), & \text{if } x \in (a, x_0], \\ u_b(x), & \text{if } x \in (x_0, b), \end{cases}$$

where u_a and u_b are solutions of $\tau u = 0$ with $u_a(x_0-) = u_b(x_0+)$, i.e. f is continuous in x_0 but in general the quasi-derivative $f^{[1]}$ is not. In this case τf is given by

$$\tau f(x_0) = \frac{1}{\varrho_0}\left(-f^{[1]}(x_0+) + f^{[1]}(x_0-) + f(x_0)\chi(\{x_0\})\right). \tag{C.2}$$

Furthermore, for two functions f, $g \in \mathfrak{D}_\tau$, the limits

$$W(f,g)(a) := \lim_{x\downarrow a} W(f,g)(x) \quad \text{and} \quad W(f,g)(b) := \lim_{x\uparrow b} W(f,g)(x)$$

exist and are finite. Indeed the Wronskian is constant away from x_0. Now as in Section 7 let w_1, $w_2 \in \mathfrak{D}_\tau$ with

$$\begin{aligned} W(w_1, w_2^*)(a) &= 1 \qquad \text{and} \qquad W(w_1, w_1^*)(a) = W(w_2, w_2^*)(a) = 0, \\ W(w_1, w_2^*)(b) &= 1 \qquad \text{and} \qquad W(w_1, w_1^*)(b) = W(w_2, w_2^*)(b) = 0, \end{aligned}$$

and define the linear functionals BC_a^1, BC_a^2, BC_b^1 and BC_b^2 on $\mathfrak{D}_\tau$ by

$$\begin{aligned} BC_a^1(f) &= W(f, w_2^*)(a) \qquad \text{and} \qquad BC_a^2(f) = W(w_1^*, f)(a) \qquad \text{for } f \in \mathfrak{D}_\tau, \\ BC_b^1(f) &= W(f, w_2^*)(b) \qquad \text{and} \qquad BC_b^2(f) = W(w_1^*, f)(b) \qquad \text{for } f \in \mathfrak{D}_\tau. \end{aligned}$$

Again we may choose special functions w_1, w_2 as in Proposition 7.2.

Theorem C.1. *Let $S \subseteq \mathfrak{D}_\tau$ be a linear subspace of the form*

$$S = \left\{ f \in \mathfrak{D}_\tau \;\middle|\; \begin{array}{c} BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0 \\ BC_b^1(f)\cos\varphi_\beta - BC_b^2(f)\sin\varphi_\beta = 0 \end{array} \right\} \tag{C.3}$$

for some φ_α, $\varphi_\beta \in [0,\pi)$. Then S gives rise to a self-adjoint relation if and only if one of the following inequalities

$$w_2(x_0-)\cos\varphi_\alpha + w_1(x_0-)\sin\varphi_\alpha \neq 0, \tag{C.4a}$$

$$w_2(x_0+)\cos\varphi_\beta + w_1(x_0+)\sin\varphi_\beta \neq 0, \tag{C.4b}$$

holds. This relation is an operator if and only if (C.4a) *and* (C.4b) *hold.*

Proof. The boundary conditions can by written as

$$W(f, w_2^* \cos\varphi_\alpha + w_1^* \sin\varphi_\alpha)(x_0-) = 0,$$
$$W(f, w_2^* \cos\varphi_\beta + w_1^* \sin\varphi_\beta)(x_0+) = 0.$$

From this one sees that the mapping (C.1) is injective if and only if one of the inequalities (C.4a) or (C.4b) holds. Hence for the first part it remains to show that in this case the range of the mapping (C.1) is a self-adjoint relation. First consider the case when both inequalities hold. Then we get from the boundary conditions

$$f^{[1]}(x_0-) = f(x_0)\frac{\cos\varphi_\alpha w_2^{[1]}(x_0-)^* + \sin\varphi_\alpha w_1^{[1]}(x_0-)*}{\cos\varphi_\alpha w_2(x_0-)^* + \sin\varphi_\alpha w_1(x_0-)*}, \quad f \in S,$$

and similar for the right-hand limit. A simple calculation shows that the imaginary part of this fraction as well as the imaginary part of the corresponding fraction for the right-hand limit vanish. Hence from (C.2) we infer that the range of the mapping (C.1) is a self-adjoint operator (multiplication with a real scalar). Now in the case when one inequality, say (C.4a) does not hold, we get $f(x_0) = 0$ for each $f \in S$ from the boundary condition at a. Hence it suffices to prove that $\tau f(x_0)$ takes each value in $\mathbb{C}$ if f runs through S, i.e. S corresponds to the self-adjoint, multi-valued relation $\{0\} \times L^2((a,b);\varrho)$. But this follows since all functions of the form

$$f(x) = \begin{cases} u_a(x), & \text{if } x \in (a, x_0], \\ 0, & \text{if } x \in (x_0, b), \end{cases}$$

where u_a is a solution of $\tau u = 0$ with $u_a(x_0) = 0$, lie in S. □

The preceding theorem corresponds to separate boundary conditions. Next we discuss the case of coupled boundary conditions.

Theorem C.2. *Let $S \subseteq \mathfrak{D}_\tau$ be a linear subspace of the form*

$$S = \left\{ f \in \mathfrak{D}_\tau \,\middle|\, \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = e^{\mathrm{i}\varphi} R \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} \right\} \tag{C.5}$$

for some $\varphi \in [0,\pi)$ and $R \in \mathbb{R}^{2\times 2}$ with $\det R = 1$ and set

$$\tilde{R} = \begin{pmatrix} w_2^{[1]}(x_0+)^* & -w_2(x_0+)^* \\ -w_1^{[1]}(x_0+)^* & w_1(x_0+)^* \end{pmatrix}^{-1} R \begin{pmatrix} w_2^{[1]}(x_0-)^* & -w_2(x_0-)^* \\ -w_1^{[1]}(x_0-)^* & w_1(x_0-)^* \end{pmatrix}.$$

Then S gives rise to a self-adjoint relation if and only if

$$\tilde{R}_{12} \neq 0 \qquad \text{or} \qquad e^{\mathrm{i}\varphi}\tilde{R}_{11} \neq 1 \neq e^{\mathrm{i}\varphi}\tilde{R}_{22}.$$

This relation is an operator if and only if $\tilde{R}_{12} \neq 0$.

Proof. The boundary conditions can be written as

$$\begin{pmatrix} f(x_0+) \\ f^{[1]}(x_0+) \end{pmatrix} = e^{\mathrm{i}\varphi}\tilde{R}\begin{pmatrix} f(x_0-) \\ f^{[1]}(x_0-) \end{pmatrix},$$

First of all note that $\tilde{R}$ is a real matrix. Indeed since for each $i = 1,2$, w_i and w_i^* are solutions of $\tau u = 0$ on (a, x_0) we see that they must be linearly dependent, hence we get $w_i(x) = w_i(x)^*$, $x \in (a, x_0)$. Of course the same holds to the right of x_0 and since R is real also $\tilde{R}$ is real. If $\tilde{R}_{12} \neq 0$, then the boundary conditions show that the mapping (C.1) is injective. Furthermore, using (C.2) one gets

$$\tau f(x_0)\varrho_0 = f(x_0)\frac{1 - \mathrm{e}^{\mathrm{i}\varphi}\left(\tilde{R}_{11} + \tilde{R}_{22}\right) + \mathrm{e}^{2i\varphi}\det\tilde{R}}{\mathrm{e}^{\mathrm{i}\varphi}\tilde{R}_{12}} + f(x_0)\chi(\{x_0\}) \quad f \in S.$$

A simple calculation shows that $\det\tilde{R} = \det R = 1$ and that the fraction is real. Hence we see that S gives rise to a self-adjoint, single-valued relation.

Now assume $\tilde{R}_{12} = 0$ and $\mathrm{e}^{\mathrm{i}\varphi}\tilde{R}_{11} \neq 1 \neq \mathrm{e}^{\mathrm{i}\varphi}\tilde{R}_{22}$, then again the boundary conditions show that the mapping (C.1) is injective. Furthermore, they show that each function $f \in S$ satisfies $f(x_0) = 0$. Hence it suffices to show that $\tau f(x_0)$ takes every value as f runs through S. But this is true since all functions

$$f_c(x) = \begin{cases} cu_a(x), & \text{if } x \in (a, x_0], \\ c\mathrm{e}^{\mathrm{i}\varphi}\tilde{R}_{22}u_b(x), & \text{if } x \in (x_0, b) \end{cases} \tag{C.6}$$

with $c \in \mathbb{C}$ and u_a, u_b are solutions of $\tau u = 0$ with $u_a(x_0-) = u_b(x_0+) = 0$ and $u_a^{[1]}(x_0-) = u_b^{[1]}(x_0+) = 1$ lie in S. If $\tilde{R}_{12} = 0$ but $\mathrm{e}^{\mathrm{i}\varphi}\tilde{R}_{22} = 1$, then the mapping (C.1) is not injective. Indeed all functions of the form (C.6) are mapped onto zero. Finally if $\tilde{R}_{12} = 0$ and $\mathrm{e}^{\mathrm{i}\varphi}\tilde{R}_{11} = 1 \neq \mathrm{e}^{\mathrm{i}\varphi}\tilde{R}_{22}$, then since S is two-dimensional it does not give rise to a self-adjoint relation. □

Note that if we choose for BC_a^1, BC_a^2, BC_b^1 and BC_b^2 the functionals from Proposition 7.2, then we get $\tilde{R} = R$.

The resolvents of the self-adjoint relations given in Theorem C.1 and Theorem C.2 can be written as in Section 8. In fact, Theorem 8.1 and Corollary 8.2 are obviously valid since the resolvents are simply multiplication by some scalar. Moreover, Theorem 8.3 and Corollary 8.4 for self-adjoint relations as in Theorem C.1 may be proven along the same lines as in the general case. The remaining theorems of Section 8 are void of meaning here, since all self-adjoint relations have purely discrete spectrum. Finally the results of Section 9 and Section 10 are also valid for self-adjoint relations as in Theorem C.1 since all proofs in these sections also apply in this simple case.

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