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1. Introduction

Let $p : \mathbb{C} \longrightarrow \mathbb{R}_+$ denote a $\mathcal{C}^1$–function and define $\Omega_p \subseteq \mathbb{C}^2$ by

$$\Omega_p = \{(z_1, z_2) \in \mathbb{C}^2 : \Im(z_2) > p(z_1)\}.$$

Weakly pseudoconvex domains of this kind were investigated by Nagel, Rosay, Stein and Wainger [NRSW1],[NRSW2] . For the case where $p(z) = |z|^k$, $k \in \mathbb{N}$, Greiner and Stein [GS] found an explicit expression for the Szegö kernel of Ω_p. If p is a subharmonic function, which depends only on the real or only on the imaginary part of z, then one can find analogous expressions and estimates in [N] (see also [Has1]). In [D] and in [K] properties of the Szegö projection for such domains are studied. The asymptotic behavior of the corresponding Szegö kernel was investigated in [Han] and [Has2].

Let $H^2(\partial\Omega_p)$ denote the subspace of $L^2(\partial\Omega_p)$ consisting of boundary values of holomorphic functions f on Ω_p such that

$$\sup_{y>0} \int_{\mathbb{C}} \int_{\mathbb{R}} |f(z, t + ip(z) + iy)|^2 \, d\lambda(z) dt < \infty,$$

where $d\lambda$ denotes the Lebesgue measure on $\mathbb{C}$. We identify $\partial\Omega_p$ with $\mathbb{C} \times \mathbb{R}$ and note that for each $f \in H^2(\partial\Omega_p)$ there exists a boundary function f_0 on $\partial\Omega_p$ such that $f_y(z,t) := f(z, t + ip(z) + iy)$ tends to $f_0(z,t)$ in $L^2(\mathbb{C} \times \mathbb{R})$ as y tends to 0, moreover we have

$$\int_{\mathbb{C}} \int_{\mathbb{R}} |f_0(z,t)|^2 \, dt d\lambda(z) = \sup_{y>0} \int_{\mathbb{C}} \int_{\mathbb{R}} |f(z, t + ip(z) + iy)|^2 \, dt d\lambda(z)$$

(see [M], [SW]).

We consider the tangential Cauchy–Riemann operator for $\partial\Omega_p$

$$L = \frac{\partial}{\partial \overline{z_1}} - 2i \frac{\partial p}{\partial \overline{z_1}}(z_1) \frac{\partial}{\partial \overline{z_2}}.$$

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After the identification of $\partial\Omega_p$ with $\mathbb{C}\times\mathbb{R}$ the tangential Cauchy–Riemann operator has the form

$$L = \frac{\partial}{\partial \overline{z}} - i\frac{\partial p}{\partial \overline{z}}\frac{\partial}{\partial t} \; .$$

Distributions ϕ satisfying $L(\phi) = 0$ are called CR-distributions (see [B]). The main result of this paper is to show that each function $f \in L^2(\partial\Omega_p)$ which is also a CR-distribution can be extended to a function holomorphic on Ω_p which belongs to $H^2(\partial\Omega_p)$.
In fact the space $H^2(\partial\Omega_p)$ can be identified with the space of all functions $f \in L^2(\partial\Omega_p)$ satisfying $L(f) = 0$ as distribution.

2. Weighted spaces of entire functions with parameters

Let E_p denote the space measurable functions F on $\mathbb{C}\times\mathbb{R}_+$ being entire with respect to the first variable and satisfying

$$\int_0^\infty \int_{\mathbb{C}} |F(z,t)|^2 \exp(-4\pi t p(z))\, d\lambda(z)dt < \infty .$$

The following lemma is a version of an important representation result for Hardy spaces (see [SW]).

Lemma 1. *Every function in $H^2(\partial\Omega_p)$ has the representation*

$$f(z,w) = \int_0^\infty F(z,t)e^{2\pi i t w}\, dt \; , \tag{1}$$

where $F \in E_p$.
In addition F can be gained back with the help of the boundary value of f by

$$F(z,\tau) = \int_{\mathbb{R}} f(z, t+ip(z))e^{-2\pi i\tau t}e^{2\pi\tau p(z)}\, dt \; . \tag{2}$$

Proof. For a function $g \in L^2(d\lambda(z)dt)$ let $\mathcal{F}$ denote the Fourier transform with respect to the variable $t \in \mathbb{R}$:

$$(\mathcal{F}g)(z,\tau) = \int_{\mathbb{R}} g(z,t)e^{-2\pi i t\tau}\, dt \; .$$

Then

$$\mathcal{F}L\mathcal{F}^{-1} = \frac{\partial}{\partial \overline{z}} + \tau\frac{\partial p}{\partial \overline{z}} \; . \tag{3}$$

$\mathcal{F}$ and $\mathcal{F}^{-1}$ are to be taken in the sense of the Plancherel theorem.

Now let M denote the multiplication operator

$$M \ : \ L^2(d\lambda(z)dt) \longrightarrow L^2(e^{-4\pi t p(z)} d\lambda(z)dt)$$

defined by

$$(Mg)(z,\tau) = e^{2\pi\tau p(z)} g(z,\tau) \ ,$$

for $g \in L^2(d\lambda(z)dt)$. Then from (3) we get

$$\mathcal{F} L \mathcal{F}^{-1} = M^{-1} \frac{\partial}{\partial \overline{z}} M \ . \tag{4}$$

If $f \in H^2(\partial\Omega_p)$, then its boundary function f_0 satisfies $L(f_0) = 0$ in the sense of distributions (the functions f_y are holomorphic in a neighborhood of $\partial\Omega_p$, they satisfy the equation $L(f_y) = 0$ (see [Ra]) and they converge to the boundary function f_0 in L^2). Now let F be as in (2). Then, using Plancherel's theorem, we get

$$\mathcal{F}^{-1} M^{-1} F = f_0$$

and from (4)

$$0 = L(f_0) = L\mathcal{F}^{-1} M^{-1} F = \mathcal{F}^{-1} M^{-1} \frac{\partial}{\partial \overline{z}} F,$$

which implies that $\frac{\partial}{\partial \overline{z}} F = 0$.
Again by Plancherel's theorem we obtain

$$\int_0^\infty |F(z,\tau)|^2 e^{-4\pi\tau p(z)} \, d\tau = \int_{\mathbb{R}} |f_0(z,t)|^2 \, dt = \int_{\mathbb{R}} |f(z, t + ip(z))|^2 \, dt,$$

hence

$$\int_{\mathbb{C}} \int_0^\infty |F(z,\tau)|^2 e^{-4\pi\tau p(z)} \, d\tau d\lambda(z) = \int_{\mathbb{C}} \int_{\mathbb{R}} |f(z, t + ip(z))|^2 \, dt d\lambda(z).$$

Since

$$\int_{\mathbb{C}} \int_{\mathbb{R}} |f(z, t + ip(z))|^2 \, dt d\lambda(z) = \sup_{y>0} \int_{\mathbb{C}} \int_{\mathbb{R}} |f(z, t + ip(z) + iy)|^2 \, dt d\lambda(z) < \infty,$$

we get $F \in E_p$.

For $(z,w) \in \Omega_p$ we set

$$h(z,w) = \int_0^\infty F(z,\tau) e^{2\pi i \tau w} \, d\tau,$$

we can write $w = t + i(p(z) + \rho)$, $t \in \mathbb{R}$, $\rho > 0$ and from $F \in E_p$ we derive that h is holomorphic in Ω_p (see for instance [Ru], pg. 404). A further application of Plancherel's theorem implies that $h = f$. $\square$

2. Extension of CR-distributions

Proposition 1. *Let* $f \in L^2(\partial\Omega_p)$ *and suppose that* $L(f) = 0$ *in the sense of distributions. Then* f *can be extended to a function holomorphic in* Ω_p*, in fact belonging to* $H^2(\partial\Omega_p)$*, whose boundary value coincides with the original function.*

Proof. We define a function F on $\mathbb{C} \times \mathbb{R}_+$ by formula (2)

$$F(z,\tau) = \int_{\mathbb{R}} f(z, t + ip(z))e^{-2\pi i\tau t}e^{2\pi\tau p(z)}\,dt,$$

and conclude again from (4) that F is entire with respect to z and belongs to E_p and

$$h(z,w) = \int_0^\infty F(z,\tau)e^{2\pi i\tau w}\,d\tau,$$

for $(z,w) \in \Omega_p$ yields the desired extension. □

Remarks:
If f belongs to $H^2(\partial\Omega_p)$, then its boundary value f_0 satifies $L(f_0)$ in the sense of distributions. Hence $H^2(\partial\Omega_p)$ can be identified with the space of all functions $f \in L^2(\partial\Omega_p)$ satisfying $L(f) = 0$ in the sense of distributions.
The Szegö projection

$$S : L^2(\partial\Omega_p) \longrightarrow H^2(\partial\Omega_p)$$

can therefore be viewed as the projection onto the kernel of the tangential Cauchy-Riemann operator.

If Ω_p is a domain in $\mathbb{C}^{n+1}$ of the form

$$\Omega_p = \{(z,w) \ : \ z \in \mathbb{C}^n \ , \ w \in \mathbb{C} \ , \ \Im w > p(z)\},$$

then the tangential Cauchy-Riemann operator has the form

$$\overline{\partial}_b(f) = \sum_{j=1}^n L_j(f)\,d\overline{z}_j,$$

where

$$L_j = \frac{\partial}{\partial \overline{z}_j} - i\frac{\partial p}{\partial \overline{z}_j}\frac{\partial}{\partial t} \ .$$

The whole reasoning can be carried over to this case dealing with the operators L_j separately for $j = 1, \ldots, n$.

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