

**Monodromy Representations of the Mapping Class Group
 B_n for the \mathfrak{su}_2 Knizhnik–Zamolodchikov Equation**

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MONODROMY REPRESENTATION OF THE MAPPING CLASS GROUP \mathcal{B}_n FOR THE su_2 KNIZHNIK-ZAMOLDCHIKOV EQUATION*

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1. INTRODUCTION

The study of a Fuchsian type differential equation in the complex domain has a substantial algebraic part: the characterization of the (finitely generated) monodromy group acting in the space of its solutions. The problem has its roots in a 19th century study of the monodromy of the hypergeometric equation. Understanding its algebraic nature, however, required a development of the mathematical physics of the 1980's: the dual concepts of a quantum group and a QUE algebra [Dri 85,86, 89] [Jim 85,86] [FRT 89]. The topic of the present lecture was first studied (under virtually the same heading) by Tsuchiya and Kanie [TK 87/88] and triggered a considerable activity which can be traced back from recent reviews - see, e.g. [FK 93]. In spite of vigorous efforts the subject continues to attract attention and we still face some intriguing open problems.

The present exposition is based on work of the authors (and of P. Furlan and L. Hadjiivanov) [FST 91], [STH 92,93], [TS 92] in which we construct and exploit an indecomposable representation of \mathcal{B}_n in a (finite dimensional) space \mathcal{L}_n of regular solutions of the KZ equation, [KZ 84] (that do not respect, in general, the physical fusion rules). We indicate the main steps of the argument showing that the monodromy of KZ amplitudes belongs to $GL(\mathcal{L}_n, \mathbb{Z}(q))$: the general linear group over the ring of *cyclotomic integers* that is, polynomials in q (where $q^h = -1$) with integer coefficients. This observation prepares the ground for an application to the Schwarz (finite monodromy) problem for the su_2 KZ equation [ST 94] reviewed in [Tod 95].

To understand the results covered in this paper one does not have to know much about *2-dimensional (2D) (rational) conformal field theory ((R)CFT)*. In order to accommodate such readers, we start in Sec. 2, essentially, with a study of the KZ equation which, supplemented by Möbius invariance and factorizability of solutions,

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reflects all CFT postulates. Here instead we shall summarize the quantum field theoretical (QFT) ingredients that yield the properties of conformal blocks listed in Sec. 2. (We assume some background knowledge in the subject - within the content of, say, the first 3 Secs. of [FST 89].)

In the axiomatic approach to 2D CFT, we are following here, one starts with (the vacuum representation of) an algebra generated by local currents (including the stress energy tensor as a composite field) and then construct its local (charged) field (positive energy) representations. Such an approach is manageable because of the simplicity of the *local (observable) algebra*: it splits into chiral parts, say \mathfrak{a} and $\bar{\mathfrak{a}}$, generated by current components depending on a single light-cone variable; as a consequence of locality left and right movers mutually commute. In the analytic compact (z) picture (the physical values, $z = e^{i(x^0 - x^1)}$, lying on the unit circle) the level k current algebra corresponding to a simple compact Lie group G is characterized by the local *commutation relations* CR

$$(1.1) \quad [J^1(z_1), J^2(z_2)] = [C_{12}, J^1(z_1)]\delta(z_{12}) - kC_{12}\delta'(z_{12}).$$

Here $z_{12} = z_1 - z_2$, the δ -function on the circle is defined to satisfy

$$\oint_{|z_2|=|z_1|} \delta(z_{12})f(z_2)\frac{dz_2}{2\pi i} = f(z_1);$$

if $\{t^a\}$ is a basis in the fundamental representation of \mathfrak{g} and $\{J^a\}$ is the corresponding basis of current then J is a matrix in the fundamental representation given by $J = \eta_{ab}J^at^b$ where

$$(1.2) \quad \eta^{ab} = \text{tr}(t^{1a}t^{2b}), \eta^{as}\eta_{sb} = \delta_b^a,$$

and we are using the abbreviate tensor product notation

$$(1.3) \quad J^1 = J \otimes \mathbf{1}, J^2 = \mathbf{1} \otimes J.$$

In general, $C_{12}(\Lambda_1, \Lambda_2)$ is the Casimir invariant in the tensor product of two irreducible representations (IRs) of weights Λ_1 and Λ_2 :

$$(1.4) \quad C_{12}(\Lambda_1, \Lambda_2) = \eta_{ab}t_{\Lambda_1}^{1a}t_{\Lambda_2}^{2b}.$$

In C_{12} appearing in (1.1) both Λ_1 and Λ_2 coincide with the fundamental representation.

The chiral ('right movers') component of the stress energy tensor T is given by the Sugawara formula*

$$(1.5) \quad T(z) = \frac{1}{2h} : \text{tr}(J(z))^2 :, \quad h = k + \check{g},$$

where the normal product of two currents can be defined in a standard fashion in terms of their mode expansion (each current component appearing as a free field)

*We use this, by now conventional, name without going into the intricate history of this relation which appears to go back to Kronig's energy operator in the neutrino theory of light [Kr 35]

- see [Tod 84,85], and h is the *height* or shifted level, \check{g} being the dual Coxeter number (see, e.g. [Kac 90]); $\check{g}(su_n) = n$. It is the height that characterizes (in a sense to be elaborated in these lectures) the resulting RCFT; we shall therefore denote the above chiral current algebra by $\mathfrak{a}_h(\mathfrak{g})$. In the quantum field theory, one is interested in positive energy Hilbert space representations. If we introduce the Fourier-Laurent expansion of basic local chiral conserved currents

$$(1.6a) \quad J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}, \quad T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

then we can formulate the following sharpened version of energy positivity. Each superselection sector of the chiral theory, that carries by definition a *factorial representation* (i.e. a finite multiple of an IR) of $\mathfrak{a}_h(\mathfrak{g})$, has a *minimal energy* subspace $\mathcal{H}_{\Lambda, \Delta}$, such that

$$(1.6b) \quad J_n \mathcal{H}_{\Lambda, \Delta} = L_n \mathcal{H}_{\Lambda, \Delta} = 0 \text{ for } n = 1, 2, \dots,$$

$\mathcal{H}_{\Lambda, \Delta}$ is a finite dimensional irreducible G module and is an eigensubspace of the chiral conformal energy operator L_0 so that

$$(1.6c) \quad \{\eta_{ab} J_0^a J_0^b - C_2(\Lambda)\} \mathcal{H}_{\Lambda, \Delta} = 0 = (L_0 - \Delta) \mathcal{H}_{\Lambda, \Delta}.$$

(1.5) then implies that the *conformal dimension* Δ is a function of the weight Λ of \mathfrak{g} :

$$(1.7) \quad \Delta = \Delta(\Lambda), \quad 2h\Delta(\Lambda) = C_2(\Lambda).$$

Fortunately, the unitary positive energy (called by mathematicians highest weight) representations of $\mathfrak{a}_h(\mathfrak{g})$ have been described by the time physicists felt the need to use them (for an up-to-date review by the chief contributor to the field, see [Kac 90]; a physicist's oriented version can be found in [GO 88]). They are given by the height h and the highest weight Λ . Remarkably, for a given h (which fixes the vacuum current algebra model) there is only a finite number of admissible Λ . For $\mathfrak{g} = su_2$ the weight $2I$ does not exceed the level, $2I \leq k (= 1, 2, \dots)$ (the dimension of the corresponding IR of SU_2 being smaller than the height, $2I + 1 < h = k + 2$). Note that the admissible set always contains the trivial *vacuum* representation for which $\Lambda = 0 = \Delta$ and \mathcal{H}_{00} is spanned by a single state $|0\rangle$ that is (\mathfrak{g} and) *Möbius invariant*: (1.6b) is to be supplemented by

$$(1.6d) \quad L_{-1}|0\rangle (= J_0|0\rangle = L_0|0\rangle) = 0.$$

The existence of only a *finite number of superselection sectors* (labeled in the above example by the isospin I) is a characteristic feature of a *rational* CFT.

The fact that space-time is not simply connected (the compactified light ray being a circle) is at the origin of the appearance of 'multivalued fields' which are conveniently described by *chiral vertex operators* (CVO) [TK 87/88]. In the case of the $\mathfrak{a}_h(su_2)$ chiral algebra, to which we specialize, a CVO $V_I(z)$ that intertwines the vacuum sector with a sector of isospin I , one needs, in general, two more labels

to specify the operator: the isospins I_i of the initial state (the domain) and I_f of the final state (the target). In particular, the monodromy of $V_{II_i}^{I_f}(z)$ is given by

$$(1.8) \quad V_{II_i}^{I_f}(e^{2\pi i}z) = \exp\{2\pi i(\Delta(I_f) - \Delta(I) - \Delta(I_i))\}V_{II_i}^{I_f}(z)$$

where $\Delta(I)$ is the *conformal dimension* which satisfies, according to (1.7),

$$(1.9) \quad h\Delta(I) = I(I+1) \quad (\text{as } C_2(2I) = 2I(I+1))$$

(the factor $e^{2\pi i}$ in the argument of V is a shorthand for an analytic continuation along a circle of radius $|z|$ in the positive direction). Single valued euclidean scalar fields $\varphi_I(z, \bar{z})$ are viewed as bilinear combinations of $V_I(z)$ and $\bar{V}_I(\bar{z})$ with matching additional labels. A (primary) CVO $V_I = V_{II_i}^{I_f}$ satisfies two local conditions (independent of the labels (I_f, I_i) which will be, therefore, omitted): CR with the stress energy tensor

$$(1.10) \quad [T(z_1), V_I(z_2)] = \delta(z_{12})\frac{\partial}{\partial z_2}V_I(z_2) - \Delta(I)\delta'(z_{12})V_I(z_2)$$

which reflects the property of T to generate local reparametrizations, and with the current,

$$(1.11) \quad [J^a(z_1), V_I(z_2)] = \delta(z_{12})V_I(z_2)t_I^a,$$

displaying the fact that J generates local gauge transformations. We shall make the internal symmetry properties of V_I more explicit by introducing a polynomial realization of the finite dimensional IRs of SU_2 . (The idea for such a realization, that goes back, at least to Herman Weyl, has been repeatedly revived in various contexts, see e.g. [BT 77], [ZF 86].) Regarding V_I as a polynomial of degree $2I$ in an internal space variable ζ (viewing it, in other words, as an SU_2 *coherent state operator*), we can substitute the matrices t_I^a in (1.11) by first order differential operators in ζ :

$$(1.12) \quad [J(z_1, \zeta_1), V_I(z_2, \zeta_2)] = -\delta(z_{12})\zeta_{12}(\zeta_{12}\partial_{\zeta_2} + 2I)V_I(z_2, \zeta_2)$$

where $\zeta_{12} = \zeta_1 - \zeta_2$,

$$(1.13) \quad J(z, \zeta) = J^-(z) + 2\zeta J^3(z) - \zeta^2 J^+(z) \quad (J^\pm = J^1 \pm iJ^2).$$

The current CR (1.1) can be obtained from (1.12) for $I = 1$ by adding the (central extension) Schwinger term:

$$(1.14) \quad [J(z_1, \zeta_1), J(z_2, \zeta_2)] = -\zeta_{12}(\zeta_{12}\partial_{\zeta_2} + 2)J(z_2, \zeta_2)\delta(z_{12}) - k\zeta_{12}^2\delta'(z_{12}).$$

Compatibility between (1.10), (1.12) and the quadratic expression (1.5) for T in terms of J implies the relation (1.9) for the conformal weights as well as the operator KZ equation

$$(1.15) \quad h\frac{\partial}{\partial z}V_I(z, \zeta) =: (IV_I(z, \zeta)\partial_\zeta J(z, \zeta) - J(z, \zeta)\partial_\zeta V_I(z, \zeta)) : .$$

(A derivation of (1.15) in these lines which also applies to the $U(1)$ Thirring model* was worked out in parallel with the KZ paper in [Tod 84,85].) The action of V_I on the vacuum state is specified without additional labels (like I_f) and extends the *field-vector correspondence* (known for local chiral operators):

$$(1.16) \quad V_I(z, \zeta)|0 \rangle = e^{(zL_{-1} + J_0^+ \zeta)} V_I(0, 0)|0 \rangle \quad (J_0^+ = J_0^1 + iJ_0^2)$$

$V_I(0, 0)|0 \rangle$ being a *lowest weight* vector of $\mathfrak{a}_h(su_2)$,

$$(1.17) \quad V_I(0, 0)|0 \rangle = |-I \rangle \quad (J_0^- |-I \rangle = (J_0^3 + I) |-I \rangle = 0).$$

The exponent in ζ is actually a polynomial of degree $2I$ since $(J_0^+)^{2I+1} |-I \rangle = 0$.

2. KZ EQUATION FOR n -POINT FUNCTIONS. REGULAR BASIS OF SOLUTIONS FOR $n = 4$

A significant part of the analysis of this section can be carried through for a current algebra model based on an arbitrary simple compact Lie group. Given, however, that the characteristics of the mapping class group we are aiming at, have only been understood in the case of su_2 , we decided to stick to this simple case from the very beginning. The advantage of such an approach is that we can keep the discussion elementary and concrete all along.

Let $\mathcal{V}_i = \mathcal{V}(I_i)$, $i = 1, \dots, n$, be irreducible $2I_i + 1$ dimensional SU_2 modules (of isospin I_i) such that the space of n -fold invariant tensors is non-empty

$$(2.1) \quad \mathcal{J}(I_1, \dots, I_n) = \text{Inv} \otimes_{i=1}^n \mathcal{V}_i, \quad d_{\mathcal{J}} = \dim \mathcal{J} > 0.$$

(This means that $\sum I_i \in \mathbb{N}$ and that none of the isospins exceeds the sum of the others.) For each ordered set of isospins (I_1, \dots, I_n) ($2I_i \in \mathbb{N}$) satisfying (2.1) we define a $d_{\mathcal{J}}$ dimensional vector space $\mathcal{L}_n = \mathcal{L}(I_1, \dots, I_n)$ of holomorphic functions $w_n = w(z_1, I_1; \dots; z_n, I_n)$ called (*conformal*) *n -point blocks* with values in $\mathcal{J}(I_1, \dots, I_n)$, (single valued) analytic in (a complex neighbourhood of) the convex cone

$$(2.2) \quad K_n : z_{i+1} = z_i - z_{i+1} > 0, i = 1, \dots, n-1,$$

and satisfying the following two conditions.

MI. Möbius invariance. The correlation function w_n is translation invariant, it only depends on the coordinate differences z_{ij} ; it is covariant under dilations

$$(2.3) \quad z_i \rightarrow \rho z_i, \rho > 0 : \rho^{\Delta_1 + \dots + \Delta_n} w(\rho z_1, \dots, \rho z_n) = w(z_1, \dots, z_n)$$

where $\Delta_i = \Delta(I_i)$ is given by (1.9) (and we have skipped the isospin arguments); it is covariant under infinitesimal special conformal transformations $z \rightarrow z(1 + \varepsilon z)^{-1}$, $\varepsilon \rightarrow 0$ implying the differential equation

$$(2.4) \quad \sum_{i=1}^n z_i \left(z_i \frac{\partial}{\partial z_i} + 2\Delta_i \right) w(z_1, I_1; \dots; z_n, I_n) = 0.$$

*As we are told by solid state physicists the first $2D$ CFT, the Thirring model [Th 58] also has a (non local) predecessor in Tomonaga's paper on Bloch's sound waves method [Tom 50], which has long remained unnoticed by field theorists

KZE. KZ equation for conformal blocks. The correlation functions satisfy the system of differential equations

$$(2.5) \quad \left(h \frac{\partial}{\partial z_i} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{C_{ij}}{z_{ij}} \right) w(z_1, I_1; \dots; z_n, I_n) = 0;$$

here $C_{ij} = C_{ij}(I_i, I_j)$ are the Casimir invariants (1.4) in the tensor product $\mathcal{V}_i \otimes \mathcal{V}_j$ (with Λ_i specialized to $2I_i$).

If we use the polynomial realization of the IRs of SU_2 (with Ward identities for CVO derived from (1.12)) then C_{ij} appear as second order partial differential operators

$$(2.6) \quad C_{ij} = 2I_i I_j + 2\zeta_{ij}(I_i \partial_j - I_j \partial_i) - \zeta_{ij}^2 \partial_i \partial_j \quad (\partial_i = \partial_{\zeta_i}, \zeta_{ij} = \zeta_i - \zeta_j).$$

Both properties are implied by the field theoretical assumptions of Sec. 1. MI is a consequence of reparametrization covariance (1.10) and of the Möbius ($sl(2, \mathbb{R})$) invariance of the vacuum (1.6). KZE follows from (1.15) and (1.12). For someone who is not inclined to enter the intricacies of a 2D CFT the above formulation could as well serve as a starting point for the subsequent discussion.

We shall derive (in Secs. 3 and 4) the following implications of the definition of \mathcal{L}_n .

MR. Monodromy representations of the exchange algebra. The elements w_n of \mathcal{L}_n satisfying the above requirements admit a (path dependent, multivalued) analytic continuation to the product of complex planes minus the diagonal

$$(2.7) \quad \mathcal{Z}_n = \mathbb{C}^n \setminus \text{diag} = \{z_i \in \mathbb{C}, z_i \neq z_j \text{ for } i \neq j\}.$$

Thus \mathcal{L}_n can be reinterpreted as a section in a *holomorphic vector bundle* \mathcal{F}_n (see [FS 87]) with a base space \mathcal{Z}_n and a $d_{\mathcal{Z}}$ dimensional fibre spanned by analytic continuations of (su_2 invariant) conformal blocks w_n . It carries a representation of the *braid (or exchange) algebra* generated by the *exchange operators*

$$(2.8) \quad B_i = B_i^{I_1, \dots, I_n} : \mathcal{L}(I_1, \dots, I_i, I_{i+1}, \dots, I_n) \rightarrow \mathcal{L}(I_1, \dots, I_{i+1}, I_i, \dots, I_n), \\ i = 1, \dots, n-1,$$

which we proceed to define. Consider the analytic continuations of each $w_n \in \mathcal{L}_n$ along a pair of paths \mathcal{C}_i^{\pm} that exchange two neighbouring arguments z_i and z_{i+1} in positive/negative direction

$$(2.9) \quad \mathcal{C}_i^{\pm} : \begin{pmatrix} z_i \\ z_{i+1} \end{pmatrix} \rightarrow \frac{1}{2}(z_i + z_{i+1}) + \frac{1}{2} \begin{pmatrix} z_{ii+1} \\ -z_{ii+1} \end{pmatrix} e^{\pm i\pi t}, \quad 0 \leq t \leq 1.$$

They give rise to a pair of exchange operators B_i (2.8) and \bar{B}_i that intertwine the same two spaces and are inverse to each other:

$$(2.10) \quad \bar{B}_i^{I_1, \dots, I_{i+1}, I_i, \dots, I_n} B_i^{I_1, \dots, I_i, I_{i+1}, \dots, I_n} = 1.$$

For a basis $\{w_n^{\nu}\}$ of real analytic functions in the domain (2.2) the matrix \bar{B}_i is complex conjugate to B_i .

BIHF. Braid invariant hermitean form. If $\{w^\nu, \nu = 1, \dots, d_J\}$ is a basis in \mathcal{L}_n then there exists a sesquilinear form $Q = (Q_{\mu\nu}^{I_1 \dots I_n})$ such that

$$(2.11) \quad \begin{aligned} G_n &= G_n(z_1, \bar{z}_1, I_1; \dots; z_n, \bar{z}_n, I_n) \\ &= \bar{w}^\mu(\bar{z}_1, I_1; \dots, \bar{z}_n, I_n) Q_{\mu\nu} w^\nu(z_1, I_1; \dots; z_n, I_n) \end{aligned}$$

is invariant under the exchange of any pair of neighbouring arguments:

$$(2.12) \quad \begin{aligned} B_i^{*I_1 \dots I_{i+1} I_i \dots I_n} Q^{I_1 \dots I_{i+1} I_i \dots I_n} B_i^{I_1 \dots I_i I_{i+1} \dots I_n} \\ = Q^{I_1 \dots I_i I_{i+1} \dots I_n} \quad (i = 1, \dots, n-1). \end{aligned}$$

(In particular, for equal isospins, Q would be braid invariant in the usual sense.) Q can be chosen positive semidefinite and normalized in such a way that G_n would coincide with the euclidean $2D$ correlation function

$$(2.13) \quad G_n = \langle 0 | \varphi_1(z_1, \bar{z}_1) \dots \varphi_n(z_n, \bar{z}_n) | 0 \rangle .$$

The relative normalizations of G_n for different n 's (and different sets of isospins) are constrained by small distance factorization properties that can be summarized as follows.

OPE. 2D operator product expansions. For each pair of neighbouring arguments $ii+1$ there is a minimal isospin J ,

$$(2.14) \quad |I_i - I_{i+1}| \leq J = J(I_i, I_{i+1}) \leq I_i + I_{i+1},$$

for which both the 3-point function $\langle 0 | \varphi_i \varphi_{i+1} \varphi_J | 0 \rangle$ and the $(n-1)$ -point function

$$(2.15) \quad \begin{aligned} G_{n-1}^{(J)} = \langle 0 | \varphi_1(z_1, \bar{z}_1) \dots \varphi_{i-1}(z_{i-1}, \bar{z}_{i-1}) \\ \varphi_J(z_i, \bar{z}_i) \varphi_{i+2}(z_{i+2}, \bar{z}_{i+2}) \dots \varphi_n(z_n, \bar{z}_n) | 0 \rangle \end{aligned}$$

do not vanish; then the small z_{ii+1} behaviour of G_n is given by

$$(2.16) \quad \lim_{z_{i+1} \rightarrow z_i} z_{ii+1}^{\Delta_i + \Delta_{i+1} - \Delta(J)} G_n = C_{I_i I_{i+1}}^J G_{n-1}^{(J)}$$

where

$$(2.17a) \quad \begin{aligned} \langle 0 | \prod_{i=1}^3 \varphi_i(z_i, \bar{z}_i; \zeta_i, \bar{\zeta}_i) | 0 \rangle \\ = C_{I_1 I_2 I_3} \prod_{i < j} |z_{ij}|^{-2\Delta_{ij}} |\zeta_{ij}|^{2I^{(ij)}}, \end{aligned}$$

$$(2.17b) \quad \Delta_{ij} = \Delta_i + \Delta_j - \Delta_k, \quad I^{(ij)} = I_i + I_j - I_k,$$

$(i, j, k) = \text{Perm}(1, 2, 3)$ and

$$(2.18) \quad C_{I_1 I_2}^J C_{J J 0} = C_{I_1 I_2 J}.$$

Eq. (2.16) reflects the idea that the OPE $\varphi_i(z_i, \bar{z}_i) \varphi_{i+1}(z_{i+1}, \bar{z}_{i+1})$ splits into a finite sum of conformal families (labeled by isospins) the leading contribution to the J family being $C_{I_i I_{i+1}}^J z_{ii+1}^{\Delta(J) - \Delta_i - \Delta_{i+1}} \varphi_J(z_i, \bar{z}_i)$ - see [BPZ 84].

Geometric interpretation. We can interpret the operator in the left hand side of the KZ equation

$$(2.19) \quad \nabla^i = \frac{\partial}{\partial z_i} + A^i, A^i = \sum_{j \neq i} r_{ij}(z_{ij}), r_{ij}(u) = \frac{C_{ij}}{u},$$

as a covariant derivative which defines a *flat connection* in the Friedan-Shenker bundle \mathcal{F}_n . Indeed, the corresponding connection 1-form has *zero curvature*:

$$(2.20) \quad A = A^i dz_i \Rightarrow dA + A \wedge A = 0.$$

Actually, both dA and $A \wedge A$ vanish: the first, due to the symmetry of C_{ij} the second, as a consequence of the *classical Yang-Baxter equation*.

$$[r_{12}(u), r_{23}(v)] + [r_{12}(u), r_{13}(u+v)] + [r_{13}(u+v), r_{23}(v)] = 0$$

for r_{ij} defined in (2.19). In verifying (2.21) we use (1.4) to prove

$$(2.22) \quad [C_{12}, C_{23}] = [C_{13}, C_{12}] = [C_{23}, C_{13}] = f_{abc} t_{\Lambda_1}^{1a} t_{\Lambda_2}^{2b} t_{\Lambda_3}^{3c}$$

f_{abc} being the structure constants of \mathfrak{g} ($[t_{\Lambda}^a, t_{\Lambda}^b] = f_c^{ab} t_{\Lambda}^c$), $f_{abc} = \eta_{aa'} \eta_{bb'} f_c^{a'b'}$.

The interest in this remark stems from the fact that a flat connection on a holomorphic vector bundle defines a (holonomy) representation of the fundamental group of the base space.

Möbius invariant 4-point amplitudes. We now specialize to $n = 4$, $\mathfrak{g} = su_2$, and to pairwise equal isospins setting

$$(2.23a) \quad I_4 = \min_{1 \leq i \leq 4} I_i = I, \quad I_1 + I_2 + I_3 - I_4 = 2I', \\ |I_{ij}| = I_{k4} \text{ for } (i, j, k) = Perm(1, 2, 3), \quad I_{ij} = I_i - I_j.$$

We shall single out the configuration

$$(2.23b) \quad I_1 = I_4 = I(\leq) I_2 = I_3 = I'$$

which simplifies formulae and is appropriate for imposing positivity on 4-point blocks. Möbius invariance allows us to reduce Eq. (2.5) to a system of ordinary differential equations. As a first step we use SU_2 invariance and Eq. (2.4) to write

$$w(z_1, \zeta_1, I; z_2, \zeta_2, I'; z_3, \zeta_3, I'; z_4, \zeta_4, I) = p_{II'}(z_{ij}, \zeta_{ij}) F(\eta; \xi_1, \xi_2)$$

where $p_{II'}$ is a homogeneous prefactor

$$(2.25) \quad p_{II'} = \frac{\zeta_{23}^{2I_-}}{z_{14}^{2\Delta} z_{23}^{2\Delta} \eta^{2\Delta(I, I')}}, \quad I_- = I' - I \\ h\Delta - I(I+1), \quad h\Delta' = I'(I'+1), \quad h\Delta(I, I') = I(I'+1),$$

F is a homogeneous polynomial of degree $2I$ in the pair (ξ_1, ξ_2) :

$$(2.26) \quad F(\eta; \xi_1, \xi_2) = (\xi_1 + \xi_2)^{2I} f(\eta, \xi),$$

$$\xi_1 = \zeta_{12}\zeta_{34}, \quad \xi_2 = \zeta_{14}\zeta_{23}, \quad \xi = \frac{\xi_1}{\xi_1 + \xi_2}$$

η and ξ being the Möbius invariant cross ratios

$$(2.27) \quad \eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \left(= 1 - \frac{z_{14}z_{23}}{z_{13}z_{24}} \right), \quad \xi = \frac{\zeta_{12}\zeta_{34}}{\zeta_{13}\zeta_{24}}.$$

In fixing the η dependence of $p_{II'}$, we have required that F has a finite (non-zero) limit for $\eta \rightarrow 0$ and $\eta \rightarrow 1$. Inserting (2.24) in the KZ equation (2.5) (say, for $i = 2$, using $\{C_{12} + C_{23} + C_{24} + 2I_2(I_2 + 1)\}w_4 = 0$) we obtain

$$(2.28) \quad \left(h \frac{\partial}{\partial \eta} - \frac{\tilde{C}_{12}}{\eta} + \frac{\tilde{C}_{23}}{1 - \eta} \right) F(\eta; \xi_1, \xi_2) = 0$$

where \tilde{C}_{ij} differ C_{ij} (2.6) by constant shifts:

$$(2.29a) \quad \begin{aligned} \tilde{C}_{ij} &= C_{ij} + I_i(I_i + 1) + I_j(I_j + 1) - I_{k4}(I_{k4} + 1) \\ &= C_{ij} + (I_i + I_j - |I_{ij}|)(I' + 1), \quad (i, j, k) = \text{Perm}(1, 2, 3) \end{aligned}$$

or for the configuration (2.23b)

$$(2.29b) \quad \tilde{C}_{12} = C_{12} + 2I(I' + 1), \quad \tilde{C}_{23} = C_{23} + 2I'(I' + 1).$$

Taking the homogeneity of F (2.26) into account we can express \tilde{C}_{12} as a differential operator in ξ_2 and \tilde{C}_{23} as a differential operator in ξ_1 :

$$(2.30a) \quad \tilde{C}_{12} = (\xi_1 + \xi_2)\xi_2 \frac{\partial^2}{\partial \xi_2^2} + [(2I_- + 1)\xi_1 + 2(I_- + 1)\xi_2] \frac{\partial}{\partial \xi_2},$$

$$(2.30b) \quad \tilde{C}_{23} = (\xi_1 + \xi_2)\xi_1 \frac{\partial^2}{\partial \xi_1^2} + (2\xi_1 + \xi_2) \frac{\partial}{\partial \xi_1}.$$

The space $\mathcal{J}_4 = \mathcal{J}(I_1, \dots, I_4)$ of 4-point SU_2 invariants has dimension $d_{\mathcal{J}} = 2I + 1$. A natural basis in it in our analytic picture is given by

$$(2.31a) \quad \begin{aligned} J_l &= \zeta_{12}^{I_{14}+I_{23}} \zeta_{23}^{I_{34}-I_{12}} \zeta_{13}^{I_{12}+I_{34}} \tilde{J}_l \\ & (= \zeta_{23}^{2I-} \tilde{J}_l \text{ for } I_{34} = I_{21} = I_-, I_{14} = I_{23} = 0), \end{aligned}$$

$$(2.31b) \quad \tilde{J}_l \equiv \tilde{J}_l^{(2I)}(\xi_1, \xi_2) = \xi_1^{2I-l} \xi_2^l, \quad l = 0, \dots, 2I.$$

Clearly, \tilde{C}_{12} does not increase l while \tilde{C}_{23} does not decrease it:

$$(2.32) \quad \begin{aligned} \tilde{C}_{12}\tilde{J}_l &= l\{(2I_- + l + 1)\tilde{J}_l + (2I_- + l)\tilde{J}_{l-1}\}, \\ \tilde{C}_{23}\tilde{J}_l &= l(2I - l)\{(2I - l + 1)\tilde{J}_l + (2I - l)\tilde{J}_{l+1}\}. \end{aligned}$$

We expand F in terms of \tilde{J}_l setting

$$F(\eta; \xi_1, \xi_2) = \sum_{l=0}^{2I} (\xi_2 \eta)^l [\xi_1 (1 - \eta)]^{2I-l} f_l(\eta)$$

and insert in (2.28). The resulting system of $2I + 1$ ordinary differential equations for f_l has $2I + 1$ linearly independent solutions f_l^λ , $\lambda = 0, 1, \dots, 2I$ which are given for $0 < \eta < 1$ by the integral representation [STH 92,93]

$$(2.34) \quad f_l^\lambda(\eta) = N_\lambda \int_0^\eta dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{\lambda-1}} dt_\lambda \int_\eta^1 dt_{\lambda+1} \int_{t_{\lambda+1}}^1 dt_{\lambda+2} \cdots \int_{t_{2I-1}}^1 dt_{2I} P_l(t_i, \eta)$$

where

$$(2.35) \quad \begin{aligned} P_l &= \prod_{i=1}^{2I} t_i^{\frac{1}{h}(1+2I_-)} (1 - t_i)^{\frac{1}{h}} |\eta - t_i|^{\frac{1}{h}-1} \prod_{i < j} |t_{ij}|^{\frac{2}{h}} \\ &\sum_{\sigma} \frac{1}{l!(2I-l)!} \prod_{s=1}^l t_{i_s}^{-1} \prod_{r=l+1}^{2I} (1 - t_{i_r})^{-1} \end{aligned}$$

the sum extending over all permutations $\sigma: (1, \dots, 2I) \rightarrow (i_1, \dots, i_{2I})$ (the numerical factor cancels the number of permutations that do not change the integrand); the constant $N_\lambda = N_\lambda(I, I')$ is independent of l .

3. BRAID GROUP STATISTICS AND QUANTUM SYMMETRY. EXCHANGE MATRICES FOR 4-POINT BLOCKS

We now come to a remarkable property of chiral CFT: multivaluedness of conformal blocks yields braid group statistics and quantum group symmetry. Let us consider this assertion in the context of general QFT. Multivalued correlation functions in complexified space-time are certainly not newcomers in field theory. It is, however, a theorem for any local QFT, say in $D = 4$ space-time dimensions that Wightman functions are single valued analytic in a real domain of space like separated arguments (see, e.g. [BLOT 90]). This is a reflection of basic principles like microcausality and the fact that space like separated points form a simply connected manifold in the product of Minkowski spaces for $\mathcal{D} \geq 3$. It follows that for 3 and higher space time dimensions fields should satisfy permutation group statistics and the superselection structure is associated to a compact gauge symmetry group [DoR 91]. By contrast, primary CVO in a $2D$ CFT have nontrivial monodromy and obey braid group exchange relations for causally disjoint arguments. As a consequence their symmetry can only be described by a ‘quantum group’ - in general, by a weak quasi Hopf algebra [MaS 92]. If one gives up Wightman positivity for chiral n -point blocks (and associated unitarity condition for exchange operators) then the quantum symmetry can be described [STH 92,93] [TS 92] by a standard (Hopf) QUE algebra. As noted in Section 2, the existence of a positive semidefinite braid invariant hermitean form (BIHF) allows to recover positivity for $2D$ physical correlators.

Exchange algebra. We shall display the matrices B_1 and B_2 in the basis (2.24) (2.33) (2.34) of 4-point blocks where we are using the abbreviate notation

$$(3.1) \quad B_i^{I_1 I_2 I_3 I_4} = B_i \text{ for } (I_1 I_2 I_3 I_4) = (II'I'I),$$

and will demonstrate that B_3 is expressed in terms of B_1 (for equal isospins we have $B_1 = B_3$).

Proposition 3.1. *If we write $B_i : w^\lambda \rightarrow B_{i\mu}^\lambda w^\mu$ then the exchange matrix B_1 is upper triangular*

$$(3.2a) \quad B_i^\lambda \mu = \frac{N_\lambda}{N_\mu} (-1)^{2I-\mu} q^{I-(I_-+2\lambda+1)+\mu(\lambda+1)-I(I+1)-I'(I'+1)} \begin{bmatrix} \mu \\ \lambda \end{bmatrix}, q = e^{i\frac{\pi}{h}}$$

while B_2 is lower triangular

$$(3.2b) \quad B_2^\lambda \mu = \frac{N_\lambda}{N_\mu} (-1)^\mu q^{2I-\mu(2I-\lambda+1)-2I'(I'+1)} \begin{bmatrix} 2I-\mu \\ 2I-\lambda \end{bmatrix};$$

B_2 is similar to $B_1^{I'I'II}$:

$$(3.3) \quad B_2^{I'I'I'I} = F^{I'I'II} B_1^{I'I'II} F^{I'I'I'I} \quad (F^{I'I'II} F^{II'I'I} = 1)$$

where F^{I_1, \dots, I_4} is a multiple of the operator exchanging the arguments 1 and 3:

$$(3.4) \quad \begin{aligned} & B_1^{I_2 I_3 I_1 I_4} B_2^{I_2 I_1 I_3 I_4} B_1^{I_1 I_2 I_3 I_4} \\ &= B_2^{I_3 I_1 I_2 I_4} B_1^{I_1 I_3 I_2 I_4} B_2^{I_1 I_2 I_3 I_4} \\ &= (-1)^{I_1+I_2+I_3-I_4} q^{I_4(I_4+1)-I_1(I_1+1)-I_2(I_2+1)-I_3(I_3+1)} F^{I_1 I_2 I_3 I_4}. \end{aligned}$$

If the ratio $\frac{N_\lambda}{N_\mu}$ of normalization constants is assumed to be independent of the order of isospins then so are the matrix elements of the involutive matrix F which is antidiagonal:

$$(3.5) \quad (F^{I_1 \dots I_4})^\lambda \mu \equiv F^\lambda \mu = \frac{N_\lambda}{N_\mu} \delta_{\lambda+\mu}^{2I} (\lambda, \mu = 0, \dots, 2I = 2 \min(I_1, \dots, I_4)).$$

We have used in (3.2) the q -binomial coefficients

$$(3.6) \quad \begin{aligned} \begin{bmatrix} n \\ m \end{bmatrix} &= \frac{[n]}{[m][n-m]!}, [n]! = [n][n-1]!, \\ [0]! &= 1, [n] = \frac{q^n - \bar{q}^n}{q - \bar{q}} (\bar{q} = q^{-1} = q^{2h-1}) \end{aligned}$$

The proof ([STH 92,93] [TS 92]) uses analytic continuation from the real domain (2.2) as well as the transformation law for the SU_2 invariants (2.31) under permutation:

$$(3.7a) \quad (I_1, \zeta_1) \leftrightarrow (I_2, \zeta_2) \Rightarrow J_l \rightarrow (-1)^{2I-l} \sum_{\nu=0}^l \binom{l}{\nu} J_l'$$

$$(3.7b) \quad (I_2, \zeta_2) \leftrightarrow (I_3, \zeta_3) \Rightarrow J_l \rightarrow (-1)^{2I-l} \sum_{l'=l}^{2I} \binom{2I-l}{2I-l'} J_{l'}.$$

(Note that in deriving (3.7) one needs the general expression (2.31a) for arbitrary isospins.)

The family of normalizations for which the N -dependent factor in $F^{\lambda\mu}$ (3.5) disappears, $N_\lambda = N_{2I} - \lambda$, contains the value

$$(3.8) \quad N_\lambda = N(I_1, \dots, I_4) \left[\begin{matrix} 2I \\ \lambda \end{matrix} \right]^{-1} \leftrightarrow \frac{N_\lambda}{N_\mu} = \frac{[\lambda]![2I-\lambda]!}{[\mu]![2I-\mu]!} \quad (I = \min_i I_i)$$

for which

$$(3.9) \quad (B_1^{I_1 I_2 I_3 I_4})^\lambda \mu = (-1)^{I_{14} + I_{23} + 2I - \mu} q^{(I_{34} + \mu)(I_{34} + \lambda + 1) + I_{12}(\mu - \lambda) - I_1(I_1 + 1) - I_2(I_2 + 1)} \left[\begin{matrix} 2I - \lambda \\ \mu - \lambda \end{matrix} \right].$$

The q binomial coefficient vanishes outside the range of indices (e.g., for $\mu > 2I$). The choice $N_\lambda/N_\mu = 1$ (of the same family) was made in [STH 92,93]. Physically relevant quantities (such as the eigenvalues of the braid matrices) are independent of N_λ .

The first equation (3.4) has the form of a (parameter free) *quantum Yang-Baxter equation*. For equal isospins it reduces to the basic third degree relation for braid group generators. The exchange matrices obey additional relations (like the involutivity of F in (3.3)) that give rise to a projective representation of the *mapping class group* \mathcal{B}_4 of the 2-sphere with 4 punctures. In order to write them down we introduce the maps

$$(3.10a) \quad B(I_1 | I_2 I_3 I_4) = B_3^{I_2 I_3 I_1 I_4} B_2^{I_2 I_1 I_3 I_4} B_1^{I_1 I_2 I_3 I_4} : \mathcal{J}^{I_1 I_2 I_3 I_4} \rightarrow \mathcal{J}^{I_2 I_3 I_4 I_1}$$

$$(3.10b) \quad B(I_2 I_3 I_4 | I_1) = B_1^{I_2 I_1 I_3 I_4} B_2^{I_2 I_3 I_1 I_4} B_3^{I_4 I_1 I_2 I_3} : \mathcal{J}^{I_4 I_1 I_2 I_3} \rightarrow \mathcal{J}^{I_1 I_2 I_3 I_4}$$

The additional relations characterizing \mathcal{B}_4 have the form

$$(3.11) \quad \begin{aligned} B(I_2 I_3 I_4 | I_1) B(I_1 | I_2 I_3 I_4) &= \bar{q}^{4I_1(I_1+1)}, \\ B(I_4 | I_1 I_2 I_3) B(I_1 I_2 I_3 | I_4) &= \bar{q}^{4I_4(I_4+1)} \end{aligned}$$

$$(3.12) \quad \begin{aligned} B(I_4 | I_1 I_2 I_3) B(I_3 | I_4 I_1 I_2) B(I_2 | I_3 I_4 I_1) B(I_1 | I_2 I_3 I_4) \\ = \bar{q}^2 \{I_1(I_1+1) + I_2(I_2+1) + I_3(I_3+1) + I_4(I_4+1)\}. \end{aligned}$$

Eq. (3.11) reflects the fact that the monodromy of a CVO of dimension Δ around the point at infinity is given by $e^{-4\pi i \Delta}$. (A more general statement is proven in Appendix A, to [TS 92].)

Corollary.

Define, in general, the *monodromy operators*

$$(3.13) \quad m_i = B_i^{I_1 \dots I_{i+1} I_i \dots I_n} B_i^{I_1 \dots I_i I_{i+1} \dots I_n} : \mathcal{J}^{I_1 \dots I_i I_{i+1} \dots I_n} \rightarrow \mathcal{J}^{I_1 \dots I_i I_{i+1} \dots I_n}$$

Eqs. (3.4) (3.11) and (3.12) imply

$$(3.14) \quad m_1(= B_1^{I_2 I_1 I_3 I_4} B_1^{I_1 I_2 I_3 I_4}) \\ = q^2 \{ I_3(I_3+1) + I_4(I_4+1) - I_1(I_1+1) - I_2(I_2+1) \} m_3.$$

In order to relate B_1 and B_3 we need the following observation. Jost version of the *TCP theorem* (see, e.g. [StW 64] and reference therein) hold for chiral conformal blocks:

$$(3.15) \quad w(z_1, \zeta_1, I_1; z_2, \zeta_2, I_2; \dots; z_n, \zeta_n, I_n) \\ = w(-z_n, -\zeta_n, I_n; \dots; -z_2, -\zeta_2, I_2; -z_1, -\zeta_1, I_1).$$

Hence there is an involution operator Θ in the direct sum $\mathcal{L}(I_1, I_2, \dots, I_n) \oplus \mathcal{L}(I_n, \dots, I_2, I_1)$ which interchanges the two terms. For $n = 4$ it intertwines B_1 and B_3 :

$$(3.16) \quad B_1^{I_4 I_3 I_2 I_1} \Theta = \Theta B_3^{I_1 I_2 I_3 I_4}.$$

The subgroup $\mathcal{B}(I, I', I', I)$ of exchange operators that leave the order (I, I', I', I) unchanged is generated by m_1 and B_2 . Note that for this order Eq. (3.14) becomes simply $m_1 = m_3$.

The algebra of (4-point) exchange operators so constructed has two remarkable properties. Most importantly, B_i are well defined for the commonly used tree like bases (see, e.g. [MS 90]) which pretend to diagonalize non-diagonalizable matrices. Our second observation is a number theoretical one: there is a family of normalizations N_λ - including ours, (3.8) (as well as the choice N_λ made in [STH 92]) - such that the entries of the exchange matrices are *cyclotomic integers*:

$$(3.17) \quad B_i^{I_1 I_2 I_3 I_4} \in GL(2I + 1, \mathbb{Z}(q^{1/2})).$$

where $\mathbb{Z}(q^{1/2})$ is the ring of polynomials of integer coefficients in the cyclotomic unit $q^{1/2}$ (such that $(q^{1/2})^{2h} = -1$). It follows that the determinant $\det B_i$ is a power of $q^{1/2}$. This is also seen directly from our triangular matrix realization; for B_1 (3.2a) we have

$$(3.18) \quad \det B_1^{II' I' I} = (-1)^{I(2I+1)} \bar{q}^{\frac{2}{3}I(I+1)(2I+1)}$$

Violation of unitarity. There is a price for the regularity and the nice arithmetic properties of the above representation of the exchange algebra: it is, in general, not unitarizable. Whenever $I_1 + I_2 + I_3 + I_4 \geq h - 1$ the representation is actually indecomposable (hence, a fortiori, nonunitarizable). Postponing the general proof of this statement to Sec. 4 we shall verify it on the simplest example of four isospins 1/2 fields in a level 1 theory. For equal isospins the upper indices of B_i are superfluous and we have

$$(3.19) \quad B_1 = \bar{q}^{-1/2} \begin{bmatrix} -\bar{q} & 1 \\ 0 & q \end{bmatrix}, \quad B_2 = \bar{q}^{-1/2} \begin{bmatrix} q & 0 \\ 1 & -\bar{q} \end{bmatrix} \quad (I_i = I = 1/2, q + \bar{q} = 1).$$

We leave it to the reader to verify that the vectors proportional to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ form a 1-dimensional invariant subspace of \mathcal{B}_4 for $[2] = 1$ with no invariant complement.

Quantum group symmetry. The most natural way to come to the notion of quantum group symmetry starts with the Wess-Zumino-Novikov-Witten (WZNW) multivalued action [Wit 84] or, equivalently, with the closed 3-form [Gaw 91],

(3.20)

$$\omega = \frac{1}{4\pi} \text{tr} \left\{ \frac{1}{k} j^\alpha dj^\alpha dx^0 dx^1 + i(dj^0 g^{-1} dg dx^1 + dj^1 dx^0 g^{-1} dg) - j^0 (g^{-1} dg)^2 dx^1 + j^1 dx^0 (g^{-1} dg)^2 - \frac{k}{3} (g^{-1} dg)^3 \right\}.$$

The equations of motion (obtained by contracting ω with the vertical vector fields $\frac{\partial}{\partial j^\alpha}$ and $gX \frac{\partial}{\partial g}$, $X \in su_2$, and then *pulling back*, - i.e. setting $dj^\alpha = (\partial_\beta j^\alpha) dx^\beta$ etc.),

$$(3.21) \quad j_\alpha = -ikg^{-1} \partial_a g, \quad \partial_+ ((\partial_- g)g^{-1}) = 0 = \partial_-(g^{-1} \partial_+ g) \quad \partial_\pm = 1/2(\partial_1 \pm \partial_0),$$

yield a factorized solution

$$(3.22) \quad g(x) = g_-(x_-)g_+(x_+), \quad x_\pm = x^1 \pm x^0 \quad (g_\pm \in SU(2)).$$

Here g is a group valued field (in the case at hand $g(x) \in SU(2)$) defined on the cylindric space-time ($x^0 = t \in \mathbb{R}$, $x^1 = x \in \mathbb{R}/2\pi\mathbb{Z}$), - i.e. periodic in the space coordinate x which only implies a twisted periodicity for g_\pm :

$$(3.23) \quad \begin{aligned} g(t, x + 2\pi) &= g(t, x) \Rightarrow g_-(x + 2\pi) = g_-(x)M, \\ g_+(x + 2\pi) &= M^{-1}g_+(x), \quad M \in SU(2). \end{aligned}$$

The symplectic form $\int_{x=-\pi}^{\pi} \omega(dt=0)$ splits into a sum of a left and a right invariant chiral symplectic form [Gaw 91] [FG 93]. Upon quantization [FG 93] [FHT 95] the g -form yields the operator exchange relations

$$(3.24) \quad g_-^2(x_2)g_-^1(x_1) = g_-^1(x_1)g_-^2(x_2)R_{12}(x_{12}), \quad x_{12} = x_1 - x_2$$

where R_{12} depends only on the sign of the coordinate difference ($R_{12}(x_{12}) = R_{12}^- \theta(x_{12}) + R_{12}^+ \theta(x_{21})$). In order to make contact with our axiomatic results we introduce the analytic picture chiral field $v(z)$ related to $g_-(x)$ by

$$(3.25) \quad g_-(x) = e^{-i\Delta x} v(e^{-ix}), \quad \Delta = \frac{3}{4h} \quad (h = k + 2)$$

The exchange relation (3.24) can be recast as follows: for z_1 and z_2 exchanged along the path $C_1^+(2.9)$ we have

$$(3.26) \quad v^2(z_2)v^1(z_1) = v^1(z_1)v^2(z_2)R_{12}^- \quad \text{or} \quad Pv^1(z_2)v^2(z_1) = v^1(z_1)v^2(z_2)\check{R}_{12}$$

where P is the permutation operator \check{R}_{12} is the *braid operator*:

$$(3.27) \quad (Pv^1(z_2)v^2(z_1))^{\alpha_1 \alpha_2} \beta_1 \beta_2 = v_{\beta_1}^{\alpha_2}(z_2)v_{\beta_2}^{\alpha_1}(z_1), \quad \check{R}_{12} = R_{12}^- P.$$

The exchange relation (3.26) admits a quantum group symmetry $v \rightarrow vT^{-1}$ where $T \in SL_q(2)$ - i.e., it is a 2×2 matrix with non-commuting entries satisfying

$$(3.28) \quad R_{12}^\pm T^1 T^2 = T^2 T^1 R_{12}^\pm \text{ or } [\check{R}_{12}, T^1 T^2] = 0 (\Rightarrow T_1^\alpha T_2^\alpha = q T_2^\alpha T_1^\alpha \text{ etc. } .)$$

$$(3.29) \quad \det_q T := T_1^1 T_2^2 - q T_2^1 T_1^2 = T_2^2 T_1^1 - \bar{q} T_2^1 T_1^2 = 1.$$

The 4-point function of $v(z)$ can be expanded in terms of the basis w^λ (2.24) (2.33) (2.34) with coefficients \mathcal{J}_λ that are quantum group invariant rank 4 tensors:

$$(3.30) \quad \langle 0 | v(z_1, \zeta_1) v(z_2, \zeta_2) v(z_3, \zeta_3) v(z_4, \zeta_4) | 0 \rangle = w^0 \mathcal{J}_0 + w^1 \mathcal{J}_1.$$

The relation between B_1 (3.9) and \check{R}_{12} (3.27) is then given by

$$(3.31) \quad \mathcal{J}_\lambda B_{1\mu}^\lambda = \mathcal{J}_\mu \check{R}_{12}, \quad B_1 = \bar{q}^{1/2} \begin{pmatrix} -\bar{q} & 1 \\ 0 & q \end{pmatrix} (I_i = 1/2, i = 1, \dots, 4).$$

4. A POSITIVE SEMIDEFINITE BRAID INVARIANT HERMITEAN FORM

The construction of an (SU_2 and) quantum group covariant chiral field can be extended as follows. There exists for any I (not just for $I = 1/2$) a *chiral field* V_I which transforms as an isospin I SU_2 tensor under left shifts and as a q -spin I $U_q(sl_2)$ tensor under right shifts. It thus combines in a single entity all CVO V_I of a given I . Its correlation functions can be written as a finite sum of type (3.30) of solutions of the KZ equation multiplied by appropriate (n point) quantum group invariants so that the exchange relations (3.26) are valid (for vacuum expectation values). There is a polynomial (coherent state) realization of $U_q(sl_2)$ representations [GP 89] used in [STH 92,93] in which the quantum group invariants appear as a q -deformation of the SU_2 invariant monomials

$$(4.1) \quad \prod_{i < j} \zeta_{ij}^{\mu_{ij}}, \quad \zeta_{ij} = \zeta_i - \zeta_j, \quad \sum_{j \neq i} \mu_{ij} = 2I_i, \quad \mu_{ij} \mu_{kl} = 0 \text{ for } i < k < j < l.$$

For the general isospin configuration (2.23a) we have the following basis of 4-point $U_q(sl_2)$ invariants ([FST 91] [TS 92])

$$(4.2) \quad \mathcal{J}_\lambda(u_1 I_1, u_2 I_2, u_3 I_3, u_4 I_4) = a_\lambda \mathfrak{p}_{12} \mathfrak{p}_{23} \mathfrak{p}_{34} \mathfrak{p}_{14} \mathfrak{p}_{13}, \quad a_\lambda = \begin{bmatrix} 2I \\ \lambda \end{bmatrix},$$

$$(4.2b) \quad \mathfrak{p}_{ij} \equiv \mathfrak{p}(u_i, u_j; \mu_{ij}, \rho_{ij}) := \prod_{n=0}^{\mu_{ij}-1} (\bar{q}^{\rho_{ij}} u_i - q^{\mu_{ij} + \rho_{ij} - 2n} u_j)$$

where $2\rho_{ij}$ (and μ_{ij}) are (non-negative) integers given by

$$(4.3a) \quad \begin{aligned} 0 \leq \mu_{14} \equiv \lambda \leq 2I, \quad \mu_{23} &= I_{34} - I_{12} + \lambda, \quad \mu_{12} = I_{14} + I_{23} + 2I - \lambda, \\ \mu_{34} &= 2I - \lambda, \mu_{13} = I_{12} + I_{34}, \quad 2\rho_{12} = \mu_{23}, \quad 2\rho_{13} = -I_{14} - I_{23}, \\ 2\rho_{34} &= \lambda, \quad 2\rho_{23} = I_{14} - I_{23} + 2I - \lambda, \quad 2\rho_{14} = \lambda - 2I; \end{aligned}$$

$$(4.3b) \quad \begin{aligned} \mu_{13} = 0, \quad \mu_{12} = \mu_{34} = 2\rho_{23} = 2\rho_{14} = 2I - \lambda \\ \text{for } I_{34} = I_{21} = I_- (\equiv I' - I), \quad I_{23} = I_{14} = 0. \end{aligned}$$

This basis is conjugate to the basis w^λ of solutions of the KZ equation constructed in Sec. 2 in the sense that the 4-point correlation function has the form

$$\langle 0 | \prod_{i=1}^4 V_{I_i}(z_i, \zeta_i, u_i) | 0 \rangle = w^\lambda(z_1, \zeta_1, I_1; \dots; z_4, \zeta_4, I_4) \mathcal{J}_\lambda(u_1 I_1, \dots, u_4 I_4).$$

(Note that the product of the normalization constants N_λ (3.8) and a_λ (4.2a) is independent of λ .) The 4-point function (4.4) is characterized by the following property: the monodromy representation of the exchange generators B_i constructed in Sec. 3 reduces to the action of the quantum group braid operators $\tilde{R}_{i,i+1}$ on the $U_q(sl_2)$ invariants: an extension of Eq. (3.31) is valid for arbitrary isospins.

The space of all solutions of the KZ equation needed to define correlation functions with such a property is bigger than the space on n -point blocks obeying the *physical fusion rules* if the sum of isospins exceeds $h - 1$. According to [Gaw 90] [GK 91] the *physical n -point blocks* are singled out by the constraint

$$(4.5) \quad \frac{\partial^{l_1 + \dots + l_n}}{\partial \zeta_1^{l_1} \partial \zeta_n^{l_n}} w_n |_{\{\zeta_i = z_i\}} = 0 \text{ for } \sum_{i=1}^n (I_i - l_i) \geq h - 1$$

The violation of this condition is the source of the breaking of full reducibility (and hence of unitarity) of the exchange algebra illustrated by (3.19). We shall now write down a braid invariant hermitean form in the space of 4-point blocks which contains the unwanted pieces in its kernel. It corresponds to the following recipe for the inner product of monomials belonging to the same irreducible representation of q -spin I [TS 92] [STH 93]:

$$(4.6) \quad \left[\begin{matrix} 2I \\ n \end{matrix} \right] (\bar{u}^{\bar{n}}, u^n) = (-1)^n q^{(I-n)(2I-1)} \delta_{n+\bar{n}}^{2I}, \quad n = 0, 1, \dots, 2I$$

and is continued in a factorizable way to tensor products of representations. The overall normalization in (4.6) is chosen in such a way that the norm square of the 2-point function

$$\mathfrak{p}(u_1, u_2; 2I, 0) = \sum_{m=0}^{2I} (-q)^m \left[\begin{matrix} 2I \\ m \end{matrix} \right] u_1^{2I-m} u_2^m,$$

obtained from (4.2) for $I_1 = I_2 = I, I_3 = I_4 = 0$, is given by the *quantum dimension*

$$(\mathfrak{p}(\bar{u}_1, \bar{u}_2; 2I, 0), \mathfrak{p}(u_1, u_2; 2I, 0)) = \sum_{n=0}^{2I} q^{2n-2I} = [2I + 1].$$

The hermitean form $Q = (Q_{\lambda\mu}^{I_1 \dots I_4})$ is now given by the inner products of the quantum group invariants (4.2)

$$(4.9) \quad Q_{\lambda\mu}^{I_1 \dots I_4} = (\mathcal{J}_\lambda(\bar{u}_1 I_1, \dots, \bar{u}_4 I_4), \mathcal{J}_\mu(u_1 I_1, \dots, u_4 I_4)) = S_{\nu\lambda} D_\nu S_{\nu\mu}.$$

Here S , the matrix that diagonalizes Q , can be written as

(4.10a)

$$S_{\lambda\mu} = (-1)^{\mu-\lambda} \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \frac{[\mu - I_{12} + I_{34}]![2I_{34} + 2\lambda + 1]![2I - \lambda]![\lambda]!}{[\lambda - I_{12} + I_{34}]![2I_{34} + \lambda + \mu + 1]![\mu]![2I - \mu]!} \text{ for } 0 \leq \lambda \leq m$$

(4.10b) $S_{\lambda\mu} = \delta_{\lambda\mu}$ for $m + 1 \leq \lambda \leq 2I$ (if $2I + 2I' > k$)

(4.11) $m = m(k; I, I') = \min(2I, k - 2I') = 1/2(k - 2I_- - |2I + 2I' - k|)$.

The diagonal matrix $D = (D_\lambda \delta_{\lambda\mu})$ is degenerate if $2I + 2I' > k$ and has $2I + 2I' - k$ zero eigenvalues. First of all, if $4I > k$ then $D_\lambda = 0$ for $\lambda > m(k; I, I) = \min(2I, k - 2I)$. In the complementary region we find the expression

(4.12)

$$D_\lambda^{II'I'I} = \left(\frac{[2I' + \lambda + 1]![2I_- + \lambda]!}{[2I']![2I_- + 2\lambda + 1]!} \right)^2 [2I_- + 2\lambda + 1] \text{ for } 0 \leq \lambda \leq \min(2I, k - 2I)$$

which is verified to vanish for $2I_-$ more values of λ (assuming $I_- > 0$):

(4.13) $D_\lambda^{II'I'I} = 0$ for $k + 1 - 2I' \leq \lambda \leq \min(2I, k - 2I)$.

The kernel $\ker Q$ of $Q^{II'I'I}$ defines a braid invariant subspace of $\mathcal{L}_4 = \mathcal{L}(II'I'I)$ with no invariant complement. The factor space $\mathcal{L}_4/\ker Q$ carries a unitarizable representation of the stabilizer $\mathcal{B}(I, I', I', I)$ of the configuration (2.23b) in the exchange algebra, provided the intermediate quantum dimensions are positive:

(4.14) $[2I_- + 2\lambda + 1] > 0$ for $0 \leq \lambda \leq m$.

That is indeed true for q satisfying

(4.15) $[2] = q + \bar{q} = 2 \cos \frac{\pi}{h}$.

Note that the positivity requirement (4.14) is the first case in which the value of the complex number q among the primitive roots of the equation $q^h = 1$ does matter (cf. [Tod 95]). The braid invariance of the form (4.9-12) is verified by a direct computation ([TS 92] [STH 93]).

The factor representation of $\mathcal{B}(I, I', I', I)$ in the 'physical' quotient space

(4.16) $\mathcal{P}(I, I', I', I) = \mathcal{L}(I, I', I', I)/\ker Q$

may still be reducible (then the braid invariant hermitean form constructed here is not unique.) Classifying its irreducible components should yield the ADE classification of $A^{(1)}$ current algebra CFTs [CIZ 87]. (For the time being only the opposite path has been followed: using the classification of modular invariant $A^{(1)}$ partition functions to detect the cases of reducibility of the unitary monodromy representations of the mapping class group - see [ST 94] [Tod 95].)

5. CONCLUDING REMARKS: OPEN PROBLEMS

We have reviewed in these lectures one of the best understood facets of RCFT. Even within this limited scope (leaving aside current efforts to classify higher rank current algebra theories [Gan 94] [Sta 95a,b] [CG 94] [FSS 95]) the subject is far from being closed. We shall just mention two open problems directly related to the content of this paper.

Quantum group symmetry is only understood for diagonal (A -series) models. Its precise identification for the two infinite D series (even and odd) is still a challenge. Simple algebraic (even arithmetic) properties of ratios of structure constants of non-diagonal versus diagonal theories uncovered recently ([PZ 95] [RST 94]) also await a better understanding. We should note in this connection that the algebraic QFT approach [RST 94] to the study of local extensions of conformal current algebras [MST 92] (for an essay on the subject addressed to a wider audience - see [Tod 94]) has also remained out of the scope of the present paper.

The second problem concerns the existence of a consistent (continuum limit) operator formalism for chiral conformal fields satisfying exchange relations of the type (3.24). The difficulty stems from the fact that one multiplies a (bilocal) operator valued distribution by a discontinuous step function. So far, it has only been treated in the framework of a lattice WZNW theory [FG 93].

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