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# Heat Kernel Estimates for the $\overline{\partial}$–Neumann Problem on G–manifolds

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# HEAT KERNEL ESTIMATES FOR THE $\bar{\partial}$-NEUMANN PROBLEM ON $G$-MANIFOLDS

JOE J. PEREZ AND PETER STOLLMANN

Abstract. We prove heat kernel estimates for the $\bar{\partial}$-Neumann Laplacian $\square$ acting in spaces of differential forms over noncompact manifolds with a Lie group symmetry and compact quotient. We also relate our results to those for an associated Laplace-Beltrami operator on functions.

*Dedicated to Barry Simon on his* $65^{\text{th}}$ *birthday*

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## 1. Introduction

1.1. **Description of the problem and principal results.** In this paper, we derive bounds on the heat kernel of the $\bar{\partial}$-Neumann Laplacian on manifolds with boundary possessing a Lie group symmetry. Heat kernel bounds are studied intensively and an attempt to describe only the most important works would go well beyond the scope of the

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present article. Instead we refer to [29] and point out the pecularities of the model we are dealing with before completely introducing the setup. The operator we deal with is the natural Laplacian coming from the PDE of several complex variables. It acts on complex-valued differential forms on a manifold with boundary and has non-coercive boundary conditions. Despite these differences from the usual situation treated in the theory of Dirichlet forms, some techniques from that discipline remain applicable in obtaining heat estimates. Due to the complications in our model, it comes as a nice surprise that these usual tools (particularly the intrinsic metric) remain useful. Apart from the results that will soon be described, this surprise is certainly a message we want to pass along. Since we would like to communicate our results to at least two communities, we will take our time to explain certain basics that might be familiar to some readers. We ask those to please bear with us.

Let $M$ be a complex manifold, $n = \dim_{\mathbb{C}} M$, and assume that $M$ has a smooth boundary $bM$ such that $\bar{M} = M \cup bM$. Assume further that $\bar{M}$ is contained in a slightly larger complex manifold $\widetilde{M}$ of the same dimension. The space of holomorphic functions on $M$ under various complex-geometric conditions on $bM \subset \widetilde{M}$ has been investigated from various standpoints, beginning with Hartogs and Levi [34, 43, 44] and, with Stein theory and sheaf-theoretic methods, culminating in the Oka-Grauert theorem, [27].

An approach to problems in several complex variables using partial differential equations was also developed by Morrey, Spencer, Andreotti-Vesentini, Kohn, Nirenberg, Hörmander, and others ([21, 62, 68]) and, for our purposes, bearing fruit in Kohn's solution to the $\bar{\partial}$-Neumann problem, [40, 41, 42]. This method heavily involves the analysis of a self-adjoint Laplace operator $\Box$ on differential forms in $\Lambda^{p,q}$, the subject of this article, which we describe here.

For any integers $p, q$ with $0 \leq p, q \leq n$ denote by $C^\infty(M, \Lambda^{p,q})$ the space of all $C^\infty$ forms of type $(p, q)$ on $M$. These are the differential forms which can be written in local complex coordinates $(z^1, z^2, \dots, z^n)$ as

$$\phi = \sum_{|I|=p,|J|=q} \phi_{I,J}\; dz^I \wedge d\bar{z}^J \tag{1}$$

where $dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, $dz^J = d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$, $I = (i_1, \dots, i_p)$, $J = (j_1, \dots, j_q)$, $i_1 < \cdots < i_p$, $j_1 < \cdots < j_q$, and the $\phi_{I,J}$ are smooth functions in local coordinates. For such a differential form $\phi$, the value of the antiholomorphic exterior derivative $\bar\partial\phi$ is

$$\bar\partial\phi = \sum_{|I|=p,|J|=q} \sum_{k=1}^{n} \frac{\partial\phi_{I,J}}{\partial\bar z^k}\, d\bar z^k \wedge dz^I \wedge d\bar z^J,$$

so $\bar\partial = \bar\partial_{p,q}$ defines a linear map $\bar\partial : C^\infty(M, \Lambda^{p,q}) \to C^\infty(M, \Lambda^{p,q+1})$.

With respect to a smooth measure on $M$ and a smoothly varying Hermitian structure in the fibers of the tangent bundle, define the spaces $L^2(M, \Lambda^{p,q})$. Let us extend the above $\bar\partial$ to the corresponding maximal operator in $L^2$ (and still call it $\bar\partial$) and let $\bar\partial^*$ be its adjoint operator (the differential forms in the domain of $\bar\partial^*$ will have to satisfy certain boundary conditions). Then, on

$$\text{(2)} \quad \mathrm{dom}(Q^{p,q}) := \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*),$$

$$\text{(3)} \quad Q^{p,q}(\phi,\psi) := \langle\bar\partial\phi, \bar\partial\psi\rangle_{L^2(M,\Lambda^{p,q+1})} + \langle\bar\partial^*\phi, \bar\partial^*\psi\rangle_{L^2(M,\Lambda^{p,q-1})}$$

defines a closed form $Q^{p,q}$ on $L^2(M, \Lambda^{p,q})$; we will frequently omit the superscripts indicating the type of forms and simply write $Q$ and $\mathrm{dom}(Q)$ instead. By standard theory (see details in Section 2.3 below) there is a unique self-adjoint operator $\Box = \Box_{p,q}$ corresponding to $Q = Q^{p,q}$ that we can write as

$$\Box = \Box_{p,q} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*.$$

This Laplacian $\Box$ is elliptic but its natural boundary conditions are not coercive, thus, in the interior of $M$, the operator gains two degrees in the Sobolev scale, as a second-order operator, while in neighborhoods of the boundary it gains less. The gain at the boundary depends on the geometry of the boundary, and the best such situation is that in which the boundary is *strongly pseudoconvex*, a condition already seen to be important in [34, 43, 44]; see [62]. In that case, the operator gains one degree on the Sobolev scale and so global estimates including both interior and boundary neighborhoods gain only one degree. More precisely, one obtains *a priori* (called Kohn-type) estimates of the form

$$\text{(4)} \qquad \|u\|_{H^{s+1}(M,\Lambda^{p,q})} \lesssim \|\Box u\|_{H^s(M,\Lambda^{p,q})} + \|u\|_{L^2(M,\Lambda^{p,q})},$$

uniformly for $u \in \mathrm{dom}(\Box) \cap C^\infty$ when the boundary is strongly pseudoconvex and $q > 0$. Such estimates are usually called *subelliptic* as the gain of the operator is less than its order. Geometric situations exist in which the gain is less than one as in the estimate (4); see particularly [10, 9].

Assuming for the moment that $\bar M$ is compact, under various well-investigated conditions on $bM$, the Laplacian satisfies a *pseudolocal estimate with gain* $\epsilon > 0$ *in* $L^2(M, \Lambda^{p,q})$. That is, if $U \subset \bar M$ is a neighborhood with compact closure, $\zeta, \zeta' \in C_c^\infty(U)$ for which $\zeta'|_{\mathrm{supp}(\zeta)} = 1$, and $\alpha|_U \in H^s(U, \Lambda^{p,q})$, then $\zeta(\Box + \mathbf{1})^{-1}\alpha \in H^{s+\epsilon}(\bar{M}, \Lambda^{p,q})$ and there exists a constant $C_{s,\zeta,\zeta'} > 0$ such that

$$(5)\quad \|\zeta(\Box + \mathbf{1})^{-1}\alpha\|_{H^{s+\epsilon}(M,\Lambda^{p,q})} \leq C_{s,\zeta,\zeta'}(\|\zeta'\alpha\|_{H^s(M,\Lambda^{p,q})} + \|\alpha\|_{L^2(M,\Lambda^{p,q})})$$

uniformly for all $\alpha$ satisfying the assumption. Still assuming that $\bar{M}$ is compact, Rellich's theorem provides that $(\Box + \mathbf{1})^{-1}$ is a compact operator and thus there exists an orthonormal basis of $L^2(M, \Lambda^{p,q})$ consisting of eigenforms of $\Box$, [21, Prop. 3.1.11]. With the eigenvalues and eigenforms of $\Box$, one can construct the heat operator and study it directly. In our case of noncompact $M$, this is not true and so we will take a different approach.

Still, to us, the most important result from the PDE of several complex variables remains the pseudolocal estimate (5), which holds even without assuming the compactness of $M$, as shown relatively recently in [18].

Finer methods have been developed with which to treat the $\bar{\partial}$-Neumann problem, originating in [22] and involving various pseudodifferential calculi, but these do not seem easily to alleviate the difficulty in going from the compact to the noncompact manifold case. Accordingly, one sees that the literature of the $\bar{\partial}$-Neumann problem rarely deals with noncompact manifolds.

**Principal results:** Though our results hold in somewhat greater generality, we will assume throughout this article that $M$ is a complex manifold with smooth, strongly pseudoconvex boundary $bM$. Assume also that $\bar{M} = M \cap bM$ is contained in the interior of a slightly larger complex manifold $\widetilde{M}$, of the same dimension, on which a Lie group $G$ acts freely and properly by holomorphic transformations. Finally, assume that restriction of the orbit space $\bar{X} = \bar{M}/G$ is compact.

The first of our principal results is a Nash-type inequality, *cf.* [50]:

**Theorem 1. (Nash inequality)** *Let $M$ be a strongly pseudoconvex $G$-manifold on which $G$ acts freely by holomorphic transformations with compact quotient $\bar{M}/G$. For integer $s > \dim_{\mathbb{C}} M$*

$$\|u\|^{2+\frac{1}{s}}_{L^2(M,\Lambda^{p,q})} \lesssim Q(u)\|u\|^{\frac{1}{s}}_{L^1(M,\Lambda^{p,q})}, \quad (u \in \mathrm{dom}(Q^{p,q}) \cap L^1(M, \Lambda^{p,q})).$$

Defining the heat semigroup by $P_t = e^{-t\Box}$, we obtain operator norm estimates in $L^p$ spaces as well as Sobolev spaces:

$$\|P_t\|_{L^2\to L^\infty}, \qquad \|P_t\|_{L^1\to L^\infty}, \qquad \|P_t\|_{H^r\to H^s},$$

valid for $t > 0$, $r, s \in \mathbb{R}$. This last property can be used to obtain that the Schwartz kernel of the heat operator is smooth for $t > 0$ and that $P_t$ is Fredholm in a generalized sense, which we will describe later.

We also obtain an off-diagonal estimate for the heat semigroup in terms of the intrinsic metric $d_\Box$ induced by $\bar\partial : C^\infty(M,\mathbb{R}) \to C^\infty(M,\Lambda^{0,1})$ and a $G$-invariant Hermitian structure on $\Lambda^{0,1}$. It turns out that $d_\Box$ is equivalent to the intrinsic metric $d_{LB}$ induced by the Laplace-Beltrami operator of a Riemannian metric simply related to the metric on $\Lambda^{0,1}$.

The off-diagonal estimate is

**Theorem 2. (Off-diagonal heat kernel estimate)** *Let $M$ be as above. For measurable subsets $A, B$ of $M$ it follows that the heat semigroup satisfies*

$$\|\,\mathbf{1}_B P_t\,\mathbf{1}_A\|_{L^2\to L^2} \le \exp\left[-\frac{d_\Box(A;B)^2}{4t}\right].$$

1.2. **Discussion of the assumptions.** We take the assumption that $\bar M$ possess a Lie group symmetry and compact quotient $\bar M/G$ partly because $G$-invariant metrics on such $M$ are all equivalent. A nice consequence is that, in Theorem 2, modulo a constant, we may replace $d_\Box$ with the ordinary Riemannian distance on $M$. More importantly, $M$ satisfying our assumptions possesses natural invariant Hilbert spaces and Sobolev structures and so our results involving metric properties have natural meaning. Additionally, on such $M$, there is a good generalized Fredholm theory for $\Box$ based on the harmonic analysis of $G$, which, together with generalized Paley-Wiener theorems, provides an effective framework for understanding the solvability of equations involving $\Box$. These are worked out in [51, 52, 53, 54, 14, 71]. In addition, we have recently established in [55] that the Laplacian in this setting is essentially self-adjoint and possesses generalized eigenforms with good properties. The treatment there depends heavily on our present treatment of the intrinsic metric.

For some of our results, the exact symmetries that we assume could be relaxed, say, to that of bounded geometry with an added assumption guaranteeing that the estimate (4) stay well-behaved at infinity. Geometric sufficient conditions for this latter property have not been worked out to our knowledge.

Our assumption that $M$ be strongly pseudoconvex implies that a pseudolocal estimate holds with gain $\epsilon = 1$ in $L^2(M,\Lambda^{p,q})$ for all $q > 0$. All the bundles constructed in [35] and which are treated in [33] are strongly pseudoconvex; we will briefly describe these later. In our results, one can revert to the more general setting, in which $0 < \epsilon < 1$, making inessential changes.

1.3. **Discussion of the results.** When $\bar M$ is compact, as we have suggested, the pseudolocal estimate (5) holds with the supports of the cutoff functions containing $\bar{M}$. Thus, by our method, for example, one could have demonstrated the validity of a Nash inequality with the machinery of [42] long before the appearance of [18]. To obtain Nash's inequality in the noncompact case, it seems that one would have to use substantially different methods. We are unaware of previous results like our off-diagonal estimate but long-distance asymptotics are not very meaningful in the compact case.

Our results contain the following peculiarity: as already pointed out, the $\bar{\partial}$-Neumann problem is not elliptic in the sense that inverse of $\Box$ does not gain two degrees in the Sobolev scale. This is due to the boundary conditions, which, even in the strong pseudoconvex case, give a gain of only one order of differentiability.

Our method of proof of the Nash inequality does not make use of the better estimates that are valid in the interior, where the gain is two as in [21, Thm. 2.2.9]. The resulting Sobolev estimates make our Theorem 1 somewhat weaker than what would be true for an elliptic operator with coercive (*e.g.* Dirichlet or Neumann) boundary conditions. In addition, the pseudolocal estimate that we use is given in terms of isotropic Sobolev norms while the problem is inherently anisotropic. In the compact case, finer anisotropic estimates have been worked out [22, 28] and it happens that the Laplacian actually does gain two orders of differentiability in all directions except for one "bad" direction in the boundary in which it gains one.

On the other hand, the off-diagonal bound, Theorem 2, is not affected at all by this. The intrinsic metric gives just the kind of decay that one would expect for an elliptic problem.

1.4. **Related work.** The pseudolocal and Kohn-type estimates that we use here were developed in the noncompact case in [18, 51] and applied in [52, 14] (with a group symmetry) to construct $L^2$ holomorphic functions in some cases, in a manner analogous to that of Kohn and Gromov, Henkin, Shubin, [40, 41, 21, 33]. This last reference contains other examples (regular covering spaces of compact, strongly pseudoconvex complex manifolds and two nonunimodular $G$-manifolds) to which our methods here apply.

The spectral theory of the $\bar{\partial}$-Neumann problem has been previously investigated in [48, 23, 24] in the compact situation and in [63, 66, 2, 3, 4], methods involving pseudodifferential operators are brought to bear on the problem, still in the compact case. In [57], a weighted $\bar{\partial}$-operator on $\mathbb{C}^n$ is treated and in [7, 8] the authors prove heat kernel estimates for the related but different $\Box_b$-operator.

Superficially, the work closest to ours is [64, 65], in that the manifolds studied are noncompact and there is a group acting by holomorphic transformations. In these papers, the $\bar\partial$-Neumann problem and its heat equation have been solved explicitly in regions in $\mathbb{C}^n$ called Siegel domains.

In [16], heat kernel asymptotics are developed for subelliptic operators on noncompact groups. In [46], an asymptotic expansion is developed for the heat kernel of a general elliptic operator with noncoercive boundary conditions.

Our work is also related to that of [15, 58], which treat the Hodge Laplacian on compact manifolds with boundary. The heat estimates there are derived for Dirichlet or Neumann boundary conditions on forms of all degrees independent of the degree, as are ours. But for the $\bar\partial$-Neumann problem, the antiholomorphic form degree (*i.e.* $q$) influences the boundary condition and the operator's character depends strongly on the degree of forms in which it is acting. See particularly the case of a $Z(q)$ boundary in [21, §3.2]. Still, in the strong pseudoconvex case, we show the estimates to be insensitive to the type of the form as long as $q > 0$, perhaps analogous to the setting in [15, 58].

The contents of the rest of this article are as follows. In Section 2 we will describe the basic constructions on $M$ and review the principal properties of the $\bar\partial$-Neumann problem relevant to our investigation. Also, we will draw the more directly accessible conclusions of these properties. In Section 3 we describe the intrinsic geometry carried by $M$ and derive the heat estimates for the $\bar\partial$-Neumann Laplacian. Section 4 provides examples on which our results hold.

## 2. The $\bar\partial$-Neumann problem

2.1. **Invariant structures.** We will need to describe smoothness of functions and differential forms using $G$-invariant Sobolev spaces which we describe here. We begin with an invariant Riemannian structure with respect to which all these objects will be given a scale.

**Lemma 2.1.** *There exists a $G$-invariant Riemannian metric $g$ on $M$ and any two such metrics are equivalent.*

*Proof.* Let $(O_k)_1^N$ be an open cover of $\bar X$ such that, for every $k$, the $G$-subbundle $G \to \pi^{-1}(O_k) \to O_k$ is trivial. Taking the direct product of a right-invariant metric on $G$ with any metric on $O_k$, we obtain a $G$-invariant metric on $G \times O_k$, hence on $\pi^{-1}(O_k)$. Let $(\phi_k)_1^N$ be a partition of unity on $X$ subordinate to the covering $(O_k)_k$ and lift the $\phi_k$ to obtain an invariant partition of unity $(\varphi_k)_k$ with $\varphi_k := \phi_k \circ \pi$. Now glue the metrics on the trivial bundles $\pi^{-1}(O_k)$ together with $(\varphi_k)_k$.

The equivalence follows from the fact that any $G$-invariant metric is uniquely determined by its restriction to the compact quotient. ♦

In $\Lambda^{p,q}$ we may thus introduce a $G$-invariant pointwise Hermitian structure $\langle\cdot,\cdot\rangle_{\Lambda^{p,q}}$. We denote by $C^\infty(M,\Lambda^{p,q})$ the space of smooth $(p,q)$-forms on $M$, by $C^\infty(\bar{M},\Lambda^{p,q})$ the subspace of those forms that can be smoothly extended to $\bar{M}$, and by $C_c^\infty(\bar{M},\Lambda^{p,q})$ the subspace of the latter consisting of those smooth forms with compact support. In terms of the $G$-invariant, pointwise Hermitian structure

$$C^\infty(\bar{M},\Lambda^{p,q}) \ni u, v \longmapsto \langle u(x), v(x)\rangle_{\Lambda^{p,q}_x} \in \mathbb{C}, \quad (x \in \bar{M}),$$

we define the $L^p$-spaces $L^p(M,\Lambda^{q,r})$ of differential forms as the completions of $C_c^\infty(\bar{M},\Lambda^{q,r})$ in the norms

$$\|u\|_{L^p(M,\Lambda^{q,r})} = \left[\int_M \langle u,u\rangle_{\Lambda^{q,r}}^{p/2}\right]^{1/p},$$

where the integral is taken with respect to an (invariant) Riemannian volume element. As in [30, 60] we may construct appropriate partitions of unity and, by differentiating componentwise with respect to local geodesic coordinates, assemble $G$-invariant integer Sobolev spaces $H^s(M,\Lambda^{p,q})$, for $s = 0,1,2,\ldots$. By Lemma 2.1, the spaces $H^s(M,\Lambda^{p,q})$ do not depend on the choices of an invariant metric on $M$ or of an invariant inner product on $\Lambda^{p,q}$. The usual duality relations for $L^p$ spaces hold (polarizing the above norm) as well as the Sobolev lemma, *etc.* Background on this is provided in [26].

2.2. **The complexified cotangent space.** We will introduce some complex-geometric concepts in this section, basically following [21]; see also [38, 39]. On a real, $2n$-dimensional $C^\infty$ manifold $M$, an *almost complex structure on* $M$ is a splitting of the complexification $TM \otimes_{\mathbb{R}} \mathbb{C}$ of the real tangent bundle $TM$,

$$TM \otimes_{\mathbb{R}} \mathbb{C} = T_{1,0}M \oplus T_{0,1}M,$$

with the following property; denoting the projections onto $T_{1,0}M$ and $T_{0,1}M$ by $\Pi_{1,0}$ and $\Pi_{0,1}$, respectively:

$$\Pi_{0,1}\zeta = \overline{\Pi_{1,0}\bar{\zeta}}, \tag{6}$$

where $^-$ denotes complex conjugation.

We can also describe an almost complex structure by a fibrewise linear mapping $J : TM \to TM$ with $J^2 = -\mathbf{1}$. These two descriptions are related via:

$$T_{1,0}M = \{X - iJX \mid X \in TM\} = \ker(J - i) \tag{7}$$

and

$$T_{0,1}M = \{X + iJX \mid X \in TM\} = \ker(J + i), \tag{8}$$

see [38, Chapter I, §7]. For a vector field $X \in TM$, a complex vector field in $TM \otimes_{\mathbb{R}} \mathbb{C}$ of the form $X - iJX \in T_{1,0}$ is called a holomorphic vector field while one of the form $X + iJX \in T_{0,1}$ is called antiholomorphic.

Dually, the projections $\Pi_{0,1}$, $\Pi_{1,0}$ induce a splitting of the exterior powers of the complexified cotangent bundle, $\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}$ into holomorphic and antiholomorphic parts so that $\Lambda^k = \bigoplus_{p+q=k} \Lambda^{p,q}$. The exterior derivative in $\Lambda^k T^*M$ can be combined with the splittings of the complexified cotangent bundle of $M$ to obtain holomorphic and antiholomorphic exterior derivatives $\partial$ and $\bar{\partial}$, respectively. The relations among these operators are given by

$$\bar{\partial} : C^\infty(\bar{M}, \Lambda^{p,q}) \to C^\infty(\bar{M}, \Lambda^{p,q+1}), \qquad \bar{\partial}\phi = \Pi_{p,q+1} d\phi$$

and

$$\partial : C^\infty(\bar{M}, \Lambda^{p,q}) \to C^\infty(\bar{M}, \Lambda^{p+1,q}), \qquad \partial\phi = \Pi_{p+1,q} d\phi$$

for $\phi \in C^\infty(\bar{M}, \Lambda^{p,q})$.

On a complex manifold, it is true that $d = \partial + \bar{\partial}$, see [21, Prop. 1.2.1] and that $\bar{\partial}^2 = 0$, which gives rise to the *$\bar{\partial}$-complex*,

$$0 \to C^\infty(\bar{M}, \Lambda^{p,0}) \xrightarrow{\bar{\partial}} C^\infty(\bar{M}, \Lambda^{p,1}) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} C^\infty(\bar{M}, \Lambda^{p,n}) \to 0$$

which is the starting point for various cohomology theories due to Dolbeault, Hodge-Kodaira, and unified by Spencer, *cf.* [42]. See also [51] for some results related to our current setting.

2.3. **Operators and forms.** As we said in the introduction, $\Box$ will be defined in terms of an associated quadratic form. Good references for background on the general concept of closed forms and their associated operators are [19, 36, 56], among others. Here we will give more details concerning the case at hand and also describe certain subsets of smooth forms that belong to the respective form and operator domains. We begin by collecting some information concerning the building blocks of $\Box$, the operators $\bar{\partial}$ and $\bar{\partial}^*$.

**Remark 2.2.** Let $M$ be as above.

(1) The maximal operator $\bar{\partial}$ in $L^2(M, \Lambda^{p,q})$ is given by: $\alpha \in \mathrm{dom}(\bar{\partial})$ whenever $\bar{\partial}\alpha \in L^2(M, \Lambda^{p,q+1})$ in the distributional sense. It acts from $L^2(M, \Lambda^{p,q})$ to $L^2(M, \Lambda^{p,q+1})$ and is a closed operator.

(2) The operator $\bar{\partial}^*$ in $L^2(M, \Lambda^{p,q})$ is the adjoint of $\bar{\partial}$ (in $L^2(M, \Lambda^{p,q-1})$); it is given by: $\alpha \in \mathrm{dom}(\bar{\partial}^*)$ whenever there exists $\beta \in L^2(M, \Lambda^{p,q-1})$ so that
$$\langle \bar{\partial}\gamma, \alpha\rangle_{L^2(M,\Lambda^{p,q})} = \langle \gamma, \beta\rangle_{L^2(M,\Lambda^{p,q-1})}$$
for all $\gamma \in L^2(M, \Lambda^{p,q-1})$ and $\bar{\partial}^*\alpha = \beta$.

(3) Since $\bar{\partial}$ is closed, the form
$$\mathrm{dom}(\bar{\partial}) \times \mathrm{dom}(\bar{\partial}) \ni (\alpha, \beta) \mapsto \langle \bar{\partial}\alpha, \bar{\partial}\beta\rangle_{L^2(M,\Lambda^{p,q+1})}$$
is a closed form in $L^2(M, \Lambda^{p,q})$; cf [19, 36].

(4) Since $\bar{\partial}^*$ is closed, the form
$$\mathrm{dom}(\bar{\partial}^*) \times \mathrm{dom}(\bar{\partial}^*) \ni (\alpha, \beta) \longmapsto \langle \bar{\partial}^*\alpha, \bar{\partial}^*\beta\rangle_{L^2(M,\Lambda^{p,q-1})}$$
is a closed form in $L^2(M, \Lambda^{p,q})$, provided, $q \geq 1$.

(5) $Q = Q^{p,q}$ is the sum of the closed forms defined in (3), (4) above and therefore a closed form as well for $q \geq 1$. $Q^{p,0}$ is the form defined in (3).

Recall that a closed operator is one whose graph is closed, while a form $Q$ is *closed* whenever its domain $\mathrm{dom}(Q)$ is a Hilbert space with respect to the *form inner product* $(\cdot \mid \cdot)_Q := Q(\cdot, \cdot) + \langle \cdot, \cdot\rangle$.

The stage is now set for the first form representation theorem, *cf.* [36], that asserts that for every semibounded closed form there is a unique self-adjoint operator associated with the form. In our case, there is a unique self-adjoint operator $\Box_{p,q}$ associated with $Q^{p,q}$, meaning that
$$\mathrm{dom}(\Box_{p,q}) \subset \mathrm{dom}(Q^{p,q}) \text{ and } Q(\alpha, \beta) = \langle \Box\alpha, \beta\rangle,$$
whenever $\alpha \in \mathrm{dom}(\Box_{p,q})$ and $\beta \in \mathrm{dom}(Q^{p,q})$. In fact, more is known:
$$\mathrm{dom}(\Box_{p,q}) = \{\alpha \mid \exists\gamma \in L^2(M, \Lambda^{p,q})\, \forall\beta \in \mathrm{dom}(Q^{p,q}) : Q^{p,q}(\alpha, \beta) = \langle\gamma, \beta\rangle\}$$
and, obviously, $\gamma = \Box_{p,q}\alpha$ is uniquely determined. Moreover, defining the square root $\Box_{p,q}^{\frac{1}{2}}$ by the functional calculus, we have that
$$\mathrm{dom}(Q^{p,q}) = \mathrm{dom}(\Box_{p,q}^{\frac{1}{2}}) \text{ and } Q(\alpha, \beta) = \langle \Box^{\frac{1}{2}}\alpha, \Box^{\frac{1}{2}}\beta\rangle.$$
We note that $\Box$ can be seen as the *form sum* of the operators $\bar{\partial}^*\bar{\partial}$ and $\bar{\partial}\bar{\partial}^*$. In fact, the former operator is the self-adjoint operator associated with the form in part (3) of the preceding remark and the latter is the self-adjoint operator associated with the form in part (4) of the preceding remark. In that sense, the formula
$$\Box = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$$
has now a precise meaning, interpreting the plus sign as the form sum, *cf.* [19, 36].

In principle, all domain questions are settled now and we have defined the forms and operators we will be dealing with. However, the results above give a rather implicit description so it is quite useful to have explicit subspaces of the operator and form domains given above.

We speak of a *core* of a form meaning a subspace of its domain that is dense in the domain with respect to the form norm. Similarly, a core of an operator is a subspace of its domain that is dense with respect to the graph norm.

The following lemma is from [33, Lemma 1.1] and [21, Lemma 2.3.2]. It serves to get our hands on the smooth elements of certain form and operator domains.

**Lemma 2.3.** *Let $M$ be as above, let $\vartheta$ be the formal adjoint operator to $\bar\partial$, and denote by $\sigma = \sigma(\vartheta, \cdot)$ its principal symbol.*

*(i) $\{u \in C_c^\infty(\bar M, \Lambda^\bullet) \mid \sigma(\vartheta, d\rho)u|_{bM} = 0\}$ is a core for $\bar\partial^*$ and on this space $\bar\partial^*$ agrees with $\vartheta$.*
*(ii) $\mathcal{D}^{p,q} := \{u \in C^\infty(\bar M, \Lambda^\bullet) \mid \sigma(\vartheta, d\rho)u|_{bM} = 0\}$ is a core for $Q^{p,q}$.*
*(iii) The domains of $\bar\partial^*$ and $Q$ are preserved by multiplication by cutoff functions.*

**Remark 2.4.** In [55] we go into much more detail about the domains of $Q$ and $\Box$, so we abbreviate the discussion here.

2.4. **Estimates for the Laplacian.** In this section we give our requirements on the boundary geometry and state the pseudolocal estimate in more precise language than in the introduction. As before, assume $M$ to be a complex manifold with nonempty smooth boundary $bM$, $\bar M = M \cup bM$, so that $M$ is the interior of $\bar M$, and $\dim_{\mathbb{C}}(M) = n$. Recall that we also assume that $\bar M$ is a closed subset in $\widetilde{M}$, a complex neighborhood of $\bar M$ so that the complex structure on $\widetilde{M}$ extends that of $M$, and every point of $\bar M$ is an interior point of $\widetilde{M}$. Let us choose a smooth function $\rho : \widetilde{M} \to \mathbb{R}$ so that

$$M = \{z \mid \rho(z) < 0\}, \quad bM = \{z \mid \rho(z) = 0\},$$

and for all $x \in bM$, we have $d\rho(x) \neq 0$. In local coordinates near any $x \in bM$ define the *holomorphic tangent plane* to the boundary at $x$ by

$$T_x^{\mathbb{C}}(bM) = \{w \in \mathbb{C}^n \mid \sum_{k=1}^n \left.\frac{\partial \rho}{\partial z^k}\right|_x w^k = 0\}$$

and define the Levi form $L_x$ by

$$L_x(w, \bar w) = \sum_{j,k=1}^n \left.\frac{\partial^2 \rho}{\partial z^j \partial \bar z^k}\right|_x w^j \bar w^k, \quad (w \in T_x^{\mathbb{C}}(bM)).$$

Then $M$ is said to be *strongly pseudoconvex* if for every $x \in bM$, the form $L_x$ is positive definite.

The following theorem will be our principal tool from the PDE of several complex variables.

**Theorem 2.5. (Pseudolocal estimate)** *Let $M$ be strongly pseudoconvex, $U$ an open subset of $\bar{M}$ with compact closure, and $\zeta, \zeta' \in C_c^\infty(U)$ for which $\zeta'|_{\mathrm{supp}\,(\zeta)} = 1$. If $q > 0$ and $\alpha|_U \in H^s(U, \Lambda^{p,q})$, then $\zeta(\square + 1)^{-1}\alpha \in H^{s+1}(\bar{M}, \Lambda^{p,q})$ and there exist constants $C_s > 0$ so that*

$$\|\zeta(\square + \mathbf{1})^{-1}\alpha\|_{H^{s+1}(M,\Lambda^{p,q})} \le C_s(\|\zeta'\alpha\|_{H^s(M,\Lambda^{p,q})} + \|\alpha\|_{L^2(M,\Lambda^{p,q})}). \tag{9}$$

*Proof.* This is [21, Prop. 3.1.1] extended to the noncompact case in [18]. ♦

**Remark 2.6.** Boundary geometries giving more general subelliptic estimates than does strong pseudoconvexity are harder to define, so we refer the interested reader to [21, §3.2], [10, 9] instead of pursuing this issue here. For completeness, we mention that the theorem holds when $M$ satisfies these weaker estimates, *mutatis mutandis* [18].

A word on notation: For two functions $A$ and $B$ on a set $S$, we write $A \lesssim B$ to mean that there exists a constant $C > 0$ such that $|A(\phi)| \le C|B(\phi)|$ for $\phi$ in $S$.

**Corollary 2.7.** *For $s \in \mathbb{N}$, $q > 0$, and $\zeta \in C_c^\infty(\bar{M})$,*

$$\|\zeta(\square + \mathbf{1})^{-s}\alpha\|_{H^s(M,\Lambda^{p,q})} \lesssim \|\alpha\|_{L^2(M,\Lambda^{p,q})}, \qquad (\alpha \in L^2(M, \Lambda^{p,q})). \tag{10}$$

*Proof.* By induction. Putting $s = 0$ in the theorem, we have

$$\|\zeta(\square + \mathbf{1})^{-1}\alpha\|_{H^1} \lesssim \|\zeta'\alpha\|_{L^2} + \|\alpha\|_{L^2} \lesssim \|\alpha\|_{L^2}, \qquad (\alpha \in L^2(M)).$$

Assuming the result for $s - 1$, it follows that $(\square + \mathbf{1})^{1-s}\alpha \in H^{s-1}_{\mathrm{loc}}(M)$ for all $\alpha \in L^2(M)$. Applying the theorem to this form, we have

$$\|\zeta(\square + \mathbf{1})^{-1}(\square + \mathbf{1})^{1-s}\alpha\|_{H^s} \lesssim \|\zeta'(\square + \mathbf{1})^{1-s}\alpha\|_{H^{s-1}} + \|(\square + \mathbf{1})^{1-s}\alpha\|_{L^2},$$

and $\|\zeta(\square + \mathbf{1})^{-s}\alpha\|_{H^s} \lesssim \|(\square + \mathbf{1})^{1-s}\alpha\|_{L^2} \lesssim \|\alpha\|_{L^2}$. ♦

**Corollary 2.8.** *Let $M$ be a strongly pseudoconvex $G$-manifold on which $G$ acts freely by holomorphic transformations with compact quotient $\bar{M}/G$. For integer $s > \dim_{\mathbb{C}} M$ and $q > 0$ we have the estimate*

$$\|(\square + \mathbf{1})^{-s}\alpha\|_{L^\infty(M,\Lambda^{p,q})} \lesssim \|\alpha\|_{L^2(M,\Lambda^{p,q})}, \qquad (\alpha \in L^2(M, \Lambda^{p,q})). \tag{11}$$

*Proof.* Choose $B \subset \bar{M}$ compact and sufficiently large so that $B \cdot G$ covers $\bar{M}$. This is possible since $\bar{X}$ is compact. Choose $\zeta \in C_c^\infty(\bar{M})$ such that $\mathrm{supp}\,\zeta \supset B$ in (10). Now, the Sobolev lemma provides that if

$s > k+m/2$, then $H^s(\mathbb{R}^m) \subset C^k(\mathbb{R}^m)$ and there is a constant $C = C_{s,k}$ such that

$$\sup_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^m} |\partial^\alpha u(x)| \leq C\|u\|_{H^s(\mathbb{R}^m)}, \tag{12}$$

thus, if we take $s > k + 1/2 \dim_{\mathbb{R}} M = k + \dim_{\mathbb{C}} M$, we have

$$\|(\Box + \mathbf{1})^{-s}\alpha\|_{C^k(\bar{M})} \lesssim \|\zeta(\Box + \mathbf{1})^{-s}\alpha\|_{H^s} \lesssim \|\alpha\|_{L^2}, \qquad (\alpha \in L^2(M))$$

by the $G$-invariance of $M$ and our choice of local geodesic coordinates. ♦

**Remark 2.9.** The exact invariances furnished by the group action assumed here are not essential and can be relaxed to assumptions on the uniformity of the estimates in (5), *etc.*

## 3. Heat kernel estimates and intrinsic geometry

**Definition 3.1.** Let $\Box = \int_0^\infty \lambda dE_\lambda$ be the spectral resolution of the Laplacian and for $t > 0$ put

$$P_t = \int_0^\infty e^{-t\lambda} dE_\lambda.$$

That is, $P_t = e^{-t\Box}$, and we would write $P_t^{p,q} = e^{-t\Box_{p,q}}$ to be completely explicit.

**Remark 3.2.** The semigroup $(e^{-tH}; t \geq 0)$ of a self-adjoint operator $H$ contains a wealth of information about its generator $H$ and satisfies the semigroup property $e^{-(t+s)H} = e^{-tH}e^{-sH}$; see [11, 25] for the general theory and [61] for the case of Schrödinger operators. In the case at hand, where $H \geq 0$, the semigroup consists of contractions, *i.e.*, $\|e^{-tH}\|_{L^2\to L^2} \leq 1$. The symbol $\|\cdot\|_{L^2\to L^2}$ denotes the operator norm of an operator from $L^2$ to $L^2$. Similar to what is known for the Laplacian, the semigroup of the $\bar{\partial}$-Neumann Laplacian $\Box$ is *ultracontractive*. That is, it maps $L^2$ into $L^\infty$ continuously. This is equivalent to the validity of a Nash-type inequality and will be discussed below.

### 3.1. Ultracontractivity and Nash inequalities.

The heat operator's ultracontractivity (*i.e.* boundedness from $L^2 \to L^\infty$) follows immediately from the Sobolev estimate in Cor. 2.8 above. The proof is formally very similar to that from Davies [12]. The difference between the two cases is that our basic spaces consist of vector-valued functions and so certain concepts and manipulations are not available. For example, we cannot identify nonnegative elements or take the absolute value in a naive way.

**Proposition 3.3.** *Let $M$ be a strongly pseudoconvex $G$-manifold on which $G$ acts freely by holomorphic transformations with compact quotient $\bar{M}/G$. For integer $s > \dim_{\mathbb{C}} M$ and $q > 0$, we have*

$$\|P_t\alpha\|_{L^\infty(M,\Lambda^{p,q})} \lesssim \max(1, t^{-s})\|\alpha\|_{L^2(M,\Lambda^{p,q})}, \quad (\alpha \in L^2(M, \Lambda^{p,q})). \tag{13}$$

*Proof.* We plug $(\Box + \mathbf{1})^s P_t\alpha$ into the inequality (11) and obtain:

$$\begin{aligned} \|P_t\alpha\|_{L^\infty} &= \|(\Box + \mathbf{1})^{-s}(\Box + \mathbf{1})^s P_t\alpha\|_{L^\infty} \\ &\lesssim \|(\Box + \mathbf{1})^s P_t\alpha\|_{L^2} \\ &\lesssim t^{-s}\|\alpha\|_{L^2} \end{aligned}$$

for any $0 < t \leq 1$, by functional calculus, since the maximum of the function $\lambda \mapsto (\lambda + 1)^s e^{-\lambda t}$ goes like $t^{-s}$ for $t > 0$. This gives the result for arbitrary $t \geq 0$, as the semigroup is a contraction on $L^2$. ♦

Recall that the usual duality properties of the $L^p$ spaces hold in our setting, Sect. 2.1.

**Corollary 3.4.** *Let $M$ be as in the previous proposition. Then, for integer $s > \dim_{\mathbb{C}} M$ and $q > 0$ we have*

$$\|P_t\alpha\|_{L^\infty(M,\Lambda^{p,q})} \lesssim \max(1, t^{-2s})\|\alpha\|_{L^1(M,\Lambda^{p,q})}, \tag{14}$$

*uniformly for $\alpha \in L^1 \cap L^2(M, \Lambda^{p,q})$.*

*Proof.* Since $P_t$ is symmetric, $\|P_t\|_{L^2\to L^\infty} = \|P_t\|_{L^1\to L^2}$ by duality, and

$$\|P_t\|_{L^2\to L^\infty} \lesssim \max(1, t^{-s}),$$

from the previous statement, we have

$$\|P_t\|_{L^1\to L^\infty} \leq \|P_t\|_{L^2\to L^\infty}\|P_t\|_{L^1\to L^2} \leq \|P_{t/2}\|^2_{L^2\to L^\infty} \lesssim \frac{1}{t^{2s}}$$

by the semigroup property. ♦

**Remark 3.5.** The basic tool in the estimates to come is the fundamental theorem of calculus applied to the function $t \mapsto \|P_t u\|^2_{L^2}$ or variants thereof. This rests on the following immediate consequence of functional calculus: For any $u \in \mathrm{dom}(\Box)$,

$$P_t u \in \mathrm{dom}(\Box) \text{ and } \frac{d}{dt}[P_t u] = -\Box P_t u.$$

**Proposition 3.6.** *Let $M$ be as in the previous proposition. For any real-valued function $w \in C^\infty(\bar{M}) \cap L^\infty(M)$ for which $\langle \bar\partial w, \bar\partial w\rangle_{\Lambda^{0,1}}$ is bounded in $M$ and $u \in L^2(M, \Lambda^{p,q})$,*

$$\frac{d}{dt}\|e^w P_t u\|^2_{L^2(M,\Lambda^{p,q})} = -2\mathfrak{Re}\, Q(P_t u, e^{2w} P_t u).$$

*In particular, for $w = 0$ we get:*

$$\frac{d}{dt}\|P_t u\|^2_{L^2(M,\Lambda^{p,q})} = -2Q(P_t u).$$

*Proof.* For any $t > 0$ we have

$$\begin{aligned}
\frac{d}{dt}\|e^w P_t u\|^2_{L^2} &= \lim_{h\to 0}\frac{1}{h}\left[\langle P_{t+h}u, e^{2w}P_{t+h}u\rangle - \langle P_t u, e^{2w}P_t u\rangle\right] \\
&= \lim_{h\to 0}\left[\langle \frac{1}{h}(P_{t+h}u - P_t u), e^{2w}P_{t+h}u\rangle + \langle e^{2w}P_t u, \frac{1}{h}(P_{t+h}u - P_t u)\rangle\right] \\
&= \langle -\Box P_t u, e^{2w}P_t u\rangle + \langle e^{2w}P_t u, -\Box P_t u\rangle \\
&= -Q(P_t u, e^{2w}P_t u) - Q(e^{2w}P_t u, P_t u),
\end{aligned}$$

where, in the last step we used that $e^{2w}u$ is in the domain of $Q$, by part $(iii)$ of Lemma 2.3. ♦

***Proof of Theorem 1.*** From Prop. 3.3 and duality we get

$$t^{-2s}\|u\|^2_{L^1} \geq \langle P_t u, P_t u\rangle_{L^2} = \|P_t u\|^2_{L^2}.$$

We use the fundamental theorem of calculus and the above Prop. 3.6 in

$$\begin{aligned}
\ldots &= \|u\|^2_{L^2} - 2\int_0^t Q(P_s u)ds \\
&\geq \|u\|^2_{L^2} - 2tQ(u) \qquad (15)
\end{aligned}$$

where, in the last inequality we use the following straightforward consequence of functional calculus:

$$Q(P_s u) = \|\Box^{\frac{1}{2}}e^{-s\Box}u\|^2_{L^2} \leq \|\Box^{\frac{1}{2}}u\|^2_{L^2}.$$

Putting $t = Q(u)^{-\frac{1}{2s+1}}\|u\|^{\frac{2}{2s+1}}_{L^1(M,\Lambda^{p,q})}$ in (15) gives the assertion. ♦

3.2. **The intrinsic metric.** We will measure the bounds on off-diagonal terms in the heat kernel with respect to the metric given by

**Definition 3.7.** We define the $G$-invariant pseudo-metric $d_\Box$ on $M$ by

$$d_\Box(x,y) = \sup\{w(y) - w(x) \mid w \in L^\infty \cap C^\infty(\bar{M},\mathbb{R}), \langle\bar\partial w, \bar\partial w\rangle_{\Lambda^{0,1}} \leq 1\}.$$

The distance between sets is given by

$$d_\Box(A;B) := \sup\{\inf_B w - \sup_A w \mid w \in L^\infty \cap C^\infty(\bar{M},\mathbb{R}), \langle\bar\partial w, \bar\partial w\rangle_{\Lambda^{0,1}} \leq 1\}$$

for arbitrary $A, B \subset \bar{M}$.

The definition above is geared to the intrinsic metric of Dirichlet forms, as used in slightly different versions, *e.g.* in [6, 13, 17, 69, 70, 67] as well as the metrics considered in [20, 37, 49] and see [31, 32] as well. Note however, that our application of this concept is somewhat nonstandard. We use this metric, defined on functions, to estimate the heat kernels acting on forms! We now show that the metric above is equivalent to an associated Riemannian distance. To this end, let us describe the metric structure of $M$ in more detail, in the notation of Sect. 2.2 above.

On the tangent bundle $TM$ of the $2n$-dimensional real $G$-manifold underlying $M$, we have a $G$-invariant almost complex structure $J : TM \to TM$, induced by the complex structure on $M$. Assume that we also have a $G$-invariant Riemannian metric $g$ on $TM$ so that $J$ is an isometry with respect to $g$; $g(X,Y) = g(JX,JY)$. Such a metric exists because a metric obtained from Lemma 2.1 can be averaged over the action of $J$. Note that with respect to any such metric, $X \perp JX$. Indeed,

$$g(X,JX) = g(JX,-X) = -g(JX,X) = -g(X,JX) = 0.$$

We may extend any Riemannian structure for which $J$ is an isometry by complex sesquilinearity (linear in the first slot, conjugate-linear in the second slot) to obtain Hermitian inner products which we say are *associated to* $g$ in $T_{1,0}, T_{0,1} \subset TM \otimes_{\mathbb{R}} \mathbb{C}$:

$$\begin{aligned} \langle X - iJX, Y - iJY\rangle_{T_{1,0}} &:= g(X,Y) + ig(X,JY), \\ \langle X + iJX, Y + iJY\rangle_{T_{0,1}} &:= g(X,Y) + ig(JX,Y). \end{aligned}$$

By duality, these structures extend naturally to $\Lambda^{1,0}$ and $\Lambda^{0,1}$ and by tensoriality to each of the spaces $\Lambda^{p,q}$. We will also metrize the bundle of complex $k$-forms as an orthogonal sum

$$\Lambda^k = \bigoplus_{p+q=k} \Lambda^{p,q}, \qquad (k = 0, 1, \dots, n). \tag{16}$$

Let us describe the $(0,1)$-forms in terms of $J$ analogously to our vector fields in (7), (8). Since $\Lambda^{0,1}$ is the dual of $T_{0,1}$ in the Hermitian metric above, we have $\xi_X \in \Lambda^{0,1}$, the dual of $X + iJX \in T_{0,1}$, naturally of the form

$$\begin{aligned} \xi_X(Y + iJY) =& \langle Y + iJY, X + iJX\rangle_{T_{0,1}} \\ =& g(Y,X) + ig(JY,X) = g(X,Y) - ig(JX,Y). \end{aligned} \tag{17}$$

We compute the last term in coordinates. Since by assumption we have $g(X,Y) = g(JX,JY)$, it is true that

$$g_{kl} J_i^k J_j^l = g_{ij},$$

with the convention that repeated indices be summed over. Multiplying this identity by $J$ and using $J^l_j J^j_k = -\delta^l_k$, the Kronecker $\delta$, we get

$$g_{kj} J^j_i = -g_{ij} J^j_k,$$

from which it follows that $g(JX, \cdot) = -Jg(X, \cdot)$ since

$$g(JX, \cdot) = g_{ij} J^j_k X^k dx^i \quad \text{and} \quad Jg(X, \cdot) = J^j_i g_{jk} X^k dx^i.$$

Going back to (17) and writing $Jg(X, \cdot)|_Y$ too simply "$Jg(X, Y)$," we see that

$$\xi_X(Y + iJY) = g(X, Y) + iJg(X, Y)$$

thus $\Lambda^{0,1} \ni \xi_X = \phi_X + iJ\phi_X$ for the real 1-form $\phi_X = g(X, \cdot)$. Similarly, a form $\Lambda^{1,0} \ni \xi_X = \phi_X - iJ\phi_X$ again for the real 1-form $\phi_X = g(X, \cdot)$

Now we return to the description of the intrinsic metric. For $w \in C^\infty(\bar{M}, \mathbb{R})$, consider the following computation:

$$\langle dw, dw \rangle_{\Lambda^1} = \langle (\bar{\partial} + \partial)w, (\bar{\partial} + \partial)w \rangle_{\Lambda^1} = \langle \bar{\partial} w, \bar{\partial} w \rangle_{\Lambda^{0,1}} + \langle \partial w, \partial w \rangle_{\Lambda^{1,0}}$$

since $\bar{\partial} w \in \Lambda^{0,1}$ and $\partial w \in \Lambda^{1,0}$ are orthogonal by the decomposition (16).

Now, $w$ is real so $\bar{\partial} w$ is the complex conjugate of $\partial w$ by (6), thus there is a *single* real 1-form $\phi$ such that $\bar{\partial} w = \phi + iJ\phi$ and $\partial w = \phi - iJ\phi$. In fact, $\phi = \frac{1}{2} dw$ since $d = \partial + \bar{\partial}$. Computing the inner products,

$$\langle \bar{\partial} w, \bar{\partial} w \rangle_{\Lambda^{0,1}} = \langle \partial w, \partial w \rangle_{\Lambda^{1,0}} = 2g(\phi, \phi)$$

since $g(\phi, J\phi) = 0$. Thus $\langle dw, dw \rangle_{\Lambda^1} = 2\langle \bar{\partial} w, \bar{\partial} w \rangle_{\Lambda^{0,1}} = 4g(\phi, \phi)$ in our metric.

Since the Laplace-Beltrami operator on functions is induced by the quadratic form $w \mapsto \int \langle dw, dw \rangle_{\Lambda^1}$, *cf.* [59, 70], we have shown

**Proposition 3.8.** *For a $J$-invariant Riemannian structure $g$, let $\Delta_{LB}$ be the corresponding Laplace-Beltrami operator. Given the Hermitian structure on $\Lambda^{0,1}$ associated to $g$, the intrinsic metric $d_\Box$ is equivalent to the one induced by the intrinsic metric of $-\Delta_{LB}$ on functions.*

**Remark 3.9.** (1) At least in the case of complete manifolds without boundary it is well-known, *cf.* [70] that the intrinsic metric $d_{LB}$ of the Laplace-Beltrami operator coincides with the Riemannian distance, *i.e.*,

$$d_{LB}(x, y) = \inf\{L(\gamma) \mid \gamma : I \to M \text{ a curve joining } x, y \in M\}.$$

In view of [1, 67], the presence of a boundary should not change this picture and Lemma 2.1 together with the intrinsic metric's manifest $G$-invariance provide the equivalence of all these structures.

(2) For Kähler manifolds, $\Box = \frac{1}{2}\Delta$, *cf.* [38, Chap. III, §2], acting componentwise on forms, therefore it is clear in this case that we recover the intrinsic metric of the Laplacian up to a factor of $\sqrt{2}$.

(3) These properties of $d_\Box$ are important for the method of [55].

3.3. **Off-diagonal heat kernel estimates.** Here, we basically use the proof from [17], pointing out once more that our setup is substantially different as our spaces are spaces of differential forms rather than functions. Let us also remind the reader that multiplication by functions preserves the domain of $Q$ and this is crucial to our treatment.

**Lemma 3.10.** *For $w \in L^\infty \cap C^1(\bar{M}, \mathbb{R})$, we have*

$$Q(u,u) = Q(e^{-\epsilon w}u, e^{\epsilon w}u) - 2i\epsilon\, \mathfrak{Im}\, \left\{\langle \bar\partial u, \bar\partial w \wedge u\rangle_{L^2} + \langle \star(\partial w \wedge \star u), \bar\partial^* u\rangle_{L^2}\right\} + \epsilon^2 \left\{\|\bar\partial w \wedge u\|^2_{L^2} + \|\partial w \wedge \star u\|^2_{L^2}\right\}$$

*for all $u \in \mathrm{dom}(Q)$.*

*Proof.* By definition,

$$Q(e^{-\epsilon w}u, e^{\epsilon w}u) = \langle \bar\partial e^{-\epsilon w}u, \bar\partial e^{\epsilon w}u\rangle + \langle \bar\partial^* e^{-\epsilon w}u, \bar\partial^* e^{\epsilon w}u\rangle.$$

The first term simplifies as follows

$$\langle \bar\partial e^{-\epsilon w}u, \bar\partial e^{\epsilon w}u\rangle = \langle \bar\partial u, \bar\partial u\rangle + 2i\epsilon\, \mathfrak{Im}\langle \bar\partial u, \bar\partial w \wedge u\rangle - \epsilon^2\langle \bar\partial w \wedge u, \bar\partial w \wedge u\rangle.$$

For the second term, note that $\bar\partial^* = -\star\partial\star$ where $\star$ is the Hodge operator and $\partial = d - \bar\partial$, (*cf.* Prop. 5.1.1, [21]). Thus

$$\begin{aligned}\bar\partial^* e^{-w}u &= -\star\partial\star(e^{-w}u) = -\star[\partial e^{-w}(\star u)] = -\star[\partial e^{-w} \wedge \star u + e^{-w}\partial\star u] \\ &= -\star[\partial e^{-w} \wedge \star u] + e^{-w}\bar\partial^* u = e^{-w}\star[\partial w \wedge \star u] + e^{-w}\bar\partial^* u.\end{aligned}$$

With the corresponding expression

$$\bar\partial^* e^{w}u = -e^{w}\star[\partial w \wedge \star u] + e^{w}\bar\partial^* u,$$

we obtain

$$\begin{aligned}\langle \bar\partial^* e^{-w}u, \bar\partial^* e^{w}u\rangle =& \langle \bar\partial^* u, \bar\partial^* u\rangle + 2i\, \mathfrak{Im}\, \langle \star(\partial w \wedge \star u), \bar\partial^* u\rangle \\ &- \langle (\partial w \wedge \star u), (\partial w \wedge \star u)\rangle,\end{aligned}$$

where we have used the fact that the Hodge $\star$ is an isometry. ♦

**Corollary 3.11.** *Assuming $\langle \bar\partial w, \bar\partial w\rangle_{\Lambda^{0,1}} \leq 1$, we have*

$$-\mathfrak{Re}\, Q(e^{-w}u, e^{w}u) \leq 2\|u\|^2_{L^2(M,\Lambda^{p,q})}.$$

*Proof.* The previous assertion implies

$$-\mathfrak{Re}\, Q(e^{-\epsilon w}u, e^{\epsilon w}u) = \epsilon^2\left\{\|\bar\partial w \wedge u\|^2 + \|\partial w \wedge \star u\|^2\right\} - Q(u,u)$$

and since we have assumed $\langle \partial w, \partial w\rangle_{\Lambda^{1,0}} = \langle \bar\partial w, \bar\partial w\rangle_{\Lambda^{0,1}} \leq 1$, (see Sect. 3.2) we have the result by Cauchy-Schwarz and again the fact that the Hodge $\star$ is an isometry. ♦

***Proof of Theorem 2***. For arbitrary $f \in \operatorname{dom}(Q)$, the computation in Prop. 3.6 gives

$$\begin{aligned}\|e^w P_t f\|_{L^2}^2 - \|e^w f\|_{L^2}^2 &= \int_0^t \frac{d}{ds}\|e^w P_s f\|_{L^2}^2 ds \\ &= -2\mathfrak{Re} \int_0^t ds\ Q(P_s f, e^{2w} P_s f). \qquad (18)\end{aligned}$$

Writing

$$Q(P_s f, e^{2w} P_s f) = Q(e^{-w} e^w P_s f, e^w e^w P_s f)$$

and applying Cor. 3.11, the integrand in (18) satisfies

$$-\mathfrak{Re}\, Q(P_s f, e^{2w} P_s f) \leq \|e^w P_s f\|_{L^2}^2, \qquad (19)$$

as usual, assuming that $\langle \bar\partial w, \bar\partial w\rangle_{\Lambda^{0,1}} \leq 1$. It follows that

$$\|e^w P_t f\|_{L^2}^2 - \|e^w f\|_{L^2}^2 \leq 2\int_0^t ds\ \|e^w P_s f\|_{L^2}^2.$$

Gronwall's inequality implies that

$$\|e^w P_t f\|_{L^2}^2 \leq e^{2t}\|e^w f\|_{L^2}^2$$

and replacing $w$ by $\delta w$ we obtain $\|e^{\delta w} P_t f\|_{L^2} \leq e^{\delta^2 t}\|e^{\delta w} f\|_{L^2}$ by inspection in (19). This implies that

$$\|e^{\delta w} P_t e^{-\delta w}\|_{L^2 \to L^2} \leq e^{\delta^2 t}$$

since $f$ was arbitrary in the domain.

Now, for arbitrary $\alpha, \beta \in L^2$

$$\begin{aligned}|\langle \mathbf{1}_B P_t \mathbf{1}_A \alpha, \beta\rangle| &= \left|\langle e^{\delta w} P_t e^{-\delta w} e^{\delta w} \mathbf{1}_A \alpha, e^{-\delta w} \mathbf{1}_B \beta\rangle\right| \\ &\leq \|e^{\delta w} P_t e^{-\delta w} e^{\delta w} \mathbf{1}_A \alpha\|_{L^2(M)} \|e^{-\delta w} \mathbf{1}_B \beta\|_{L^2(M)} \\ &\leq \|e^{\delta w} P_t e^{-\delta w}\|_{L^2 \to L^2} \|e^{\delta w} \mathbf{1}_A \alpha\|_{L^2(M)} \|e^{-\delta w} \mathbf{1}_B \beta\|_{L^2(M)}. \\ &\leq e^{\delta^2 t}\|e^{\delta w} \mathbf{1}_A \alpha\|_{L^2(M)} \|e^{-\delta w} \mathbf{1}_B \beta\|_{L^2(M)}.\end{aligned}$$

For $\varepsilon > 0$ choose a weight function $w$ as in the definition of $d_\square(A;B)$ above, with $\langle \bar\partial w, \bar\partial w\rangle_{\Lambda^{0,1}} \leq 1$ and so that

$$d_\square(A;B) - \varepsilon \leq \inf_B w - \sup_A w \quad \text{and} \quad \sup_A w = 0$$

(we can achieve the latter by adding a suitable constant). This gives

$$\inf_B w \geq d_\square(A;B) - \varepsilon.$$

Inserting gives

$$|\langle \mathbf{1}_B P_t \mathbf{1}_A \alpha, \beta\rangle| \leq e^{\delta^2 t} e^{-\delta(d_\square(A;B) - \varepsilon)}\|\alpha\|\|\beta\|$$

so that (since $\varepsilon$ is arbitrary)

$$\| \mathbf{1}_B P_t \mathbf{1}_A \| \leq e^{\delta^2 t} e^{-\delta d_\Box (A;B)}.$$

For $d_\Box(A;B) < \infty$, choose $\delta = d_\Box(A;B)/(2t)$. ♦

**Remark 3.12.** In light of Prop. 3.8, we may replace $d_\Box$ with $d_{LB}$, making the necessary changes.

3.4. **Sobolev estimates for the heat operator.** Here we extend some $L^p$ results from the preceding treatment to Sobolev spaces. First note that for $t > 0$ and $k \in \mathbb{N}$ arbitrary, we have $P_t : L^2 \to \mathrm{dom}(\Box^k)$.

**Proposition 3.13.** *For $t > 0$ and $q > 0$ we have*

$$P_t : L^2(M, \Lambda^{p,q}) \to C^\infty(\bar{M}, \Lambda^{p,q}).$$

*Proof.* We will proceed by induction and use the Sobolev lemma, (12) above. Fix $t > 0$. For any $\alpha \in L^2$, since $\mathrm{im}(P_t) \subset \mathrm{dom}(\Box)$, and $(\Box + \mathbf{1})^{-1} : L^2 \to \mathrm{dom}(\Box)$ is onto, we may apply Thm. 2.5 to the form $\alpha = (\Box + \mathbf{1})P_t\beta$, $\beta \in \mathrm{dom}(\Box)$, to obtain

$$\|\zeta P_t \beta\|_{H^1} \lesssim \|\zeta'(\Box + \mathbf{1})P_t\beta\|_{L^2} + \|(\Box + \mathbf{1})P_t\beta\|_{L^2} \lesssim \|\beta\|_{L^2},$$

and conclude that $\mathrm{im}(P_t) \in H^1_{\mathrm{loc}}$. Furthermore, since $P_t$ is a function of $\Box$, they commute and we also have

$$(\Box + \mathbf{1})P_t\beta = P_t(\Box + \mathbf{1})\beta \in H^1_{\mathrm{loc}} \qquad (\alpha \in L^2).$$

Assuming now that $(\Box + \mathbf{1})P_t\beta \in H^{s-1}_{\mathrm{loc}}$, the same theorem provides

$$\|\zeta P_t\beta\|_{H^s} \lesssim \|\zeta'(\Box + \mathbf{1})P_t\beta\|_{H^{s-1}} + \|(\Box + \mathbf{1})P_t\beta\|_{L^2},$$

so $P_t\beta \in H^s_{\mathrm{loc}}$. ♦

We will need the following *a priori* estimate for $\Box$, proven in our setting by a small variation on the methods of [42, 21], in [51, Thm. 4.5].

**Lemma 3.14. (Kohn inequality)** *If $M$ is a strongly pseudoconvex $G$-manifold on which $G$ acts freely by holomorphic transformations with compact quotient $\bar{M}/G$ and $q > 0$, then for every integer $s \geq 0$ there exists a positive constant $C_s$ so that*

$$\|u\|_{H^{s+1}} \leq C_s(\|\Box u\|_{H^s} + \|u\|_{L^2}), \quad (u \in \mathrm{dom}(\Box) \cap C^\infty(\bar{M}, \Lambda^{p,q}))$$

*uniformly.*

**Corollary 3.15.** *For $t > 0$ and $q > 0$ we have* $\mathrm{im}(P_t) \subset H^\infty(M, \Lambda^{p,q})$.

*Proof.* Combining the results of Prop. 3.13 and Lemma 3.14, we have

$$\|u\|_{H^{s+1}} \leq C_s(\|\Box u\|_{H^s} + \|u\|_{L^2}) \qquad (u \in \operatorname{im}(P_t))$$

but $\operatorname{im}(P_t) \subset \operatorname{dom}(\Box^k)$ for all powers of the Laplacian, so this estimate can be iterated. Thus the estimates

$$\|\Box^{k-s}u\|_{H^{s+1}} \lesssim \|\Box^{k-s+1}u\|_{H^s} + \|\Box^{k-s}u\|_{L^2}, \quad (s = 1, 2, \dots, k) \tag{20}$$

hold for $u \in \operatorname{im}(P_t)$ and these imply the result. ♦

**Proposition 3.16.** *If $M$ is as above, $t > 0$, and $q > 0$, then the heat operator $P_t$ is bounded from $H^{-s}(\bar{M}, \Lambda^{p,q}) \to H^s(M, \Lambda^{p,q})$ for any positive integer $s$.*

*Proof.* First recall the following fact about Sobolev spaces on manifolds with boundary from Remark 12.5 of [45]. For $s > 0$, the dual space of $H^s(M)$, denoted $H^{-s}(\bar{M})$, consists of elements of $H^{-s}(\widetilde{M})$ whose support is in $\bar{M}$. Now, from Cor. 3.15 we have that for all $s > 0$, $P_t : L^2 \to H^s(M)$ continuously. Since $P_t$ is self-adjoint, its domain can be extended to the dual of $H^s(M)$ so that $P_t : H^{-s}(\bar{M}) \to L^2(M)$. The semigroup law $P_t^2 = P_{2t}$ holds on $C_c^\infty(\bar{M}) \subset L^2(M)$, a dense subspace of all the $H^s(\bar{M})$, $(s \in \mathbb{R})$ so we may conclude that $P_t : H^{-s}(\bar{M}) \to H^s(M)$ for all $s > 0$. ♦

**Remark 3.17.** These results have three easy consequences.

1) For an operator norm estimate, we can put $u = P_t\alpha$ in the estimates (20) and telescope them to find that for $s \in \mathbb{N}$,

$$\|P_t\alpha\|_{H^s} \lesssim \sum_{k=0}^{s} \|\Box^k P_t\alpha\|_{L^2} \lesssim \sum_{k=0}^{s} t^{-k}\|\alpha\|_{L^2},$$

which yields an estimate analogous to that in Prop. 3.3.

2) Combining Cor. 3.15 with Gagliardo-Nirenberg-Sobolev embeddings, *e.g.*

$$H^s(\mathbb{R}^n) \subset L^p(\mathbb{R}^n), \qquad p = \frac{2n}{n-2s}, \quad 0 \leq s < \frac{n}{2},$$

[5], one obtains results overlapping those of the previous sections in $L^p$ spaces. With other such embeddings can obtain results for $L^p$-Sobolev spaces.

3) One can continue the treatment in Sect. 6 of [51] to obtain that, for $t > 0$, the heat operator's Schwartz kernel $K_t \in C^\infty(\bar{M} \times \bar{M})$ and

$$\int_{\frac{M \times M}{G}} |K_t|^2 < \infty, \qquad (t > 0), \tag{21}$$

noting that $\Box$ and thus $K_t$ are $G$-invariant. When $G$ is unimodular, (21) means that von Neumann's $G$-trace of $P_{2t}$ is finite.

## 4. Examples

Let us describe some classes of complex manifolds to which our results apply. As in [33], let $X$ be a strongly pseudoconvex, complex manifold with compact closure $\bar{X} = X \cup bX$. Assume also that the fundamental group $\pi_1(X)$ is infinite. It follows that $\pi_1(X)$ acts properly discontinuously on the universal cover $\widetilde{X} = M$ of $X$ by deck transformations, and estimates involving the boundary are uniform as they are determined on the compact $\bar{X}$. Covers of $X$ corresponding to subgroups of $\pi_1$ will share this uniformity property.

For Lie group symmetries, in [35] a large class of manifolds was constructed which also satisfy our assumptions: Suppose that a Lie group $G$ acts freely and properly by $C^\omega$ transformations on a $C^\omega$ manifold $Y$, for example the underlying manifold of $G$ itself. It turns out that the action of $G$ on $Y$ can be extended to a complexification $Y^{\mathbb{C}}$ of $Y$ in such a way that the action of $G$ on $Y^{\mathbb{C}}$ is by holomorphic transformations. In addition, there exists a strictly plurisubharmonic function $\varphi$ in a neighborhood of $Y$ in $Y^{\mathbb{C}}$ such that $\varphi$ is constant on the orbits of $G$. It follows that for $\epsilon > 0$ sufficiently small, the tube $M = \{\varphi < \epsilon\}$ is a strongly pseudoconvex complex $G$-manifold and if $Y/G$ is compact, then $\bar{M}/G$ is too.

Whenever the group in the setting of [35] contains a cocompact lattice, of course the present situation reduces to (roughly) that of [33]. However, even for the restricted class of unimodular Lie groups, it is generically not the case that a Lie group $G$ possess such a subgroup, [47]. We should note that the methods of [35] are predominantly Stein-theoretic and their results extend to proper actions.

A concrete example of a tube of a matrix group can be found in [14], constructed explicitly by the abstract technique of [35]. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, define the three-dimensional Heisenberg group

$$\mathbb{H}_3(\mathbb{K}) = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_k \in \mathbb{K} \right\}.$$

The function $\varphi : \mathbb{H}_3(\mathbb{C}) \to \mathbb{R}$ given by

$$\varphi(Z) = (\mathfrak{Im} z_1)^2 + (\mathfrak{Im} z_2)^2 + (\mathfrak{Im} z_3 - \mathfrak{Re} z_2\, \mathfrak{Im} z_1)^2$$

is invariant under right multiplication by matrices in $\mathbb{H}_3(\mathbb{R})$. An easy calculation shows that $M_\epsilon = \{\varphi < \epsilon\} \subset \mathbb{C}^3$ is strongly pseudoconvex as long as $\epsilon < 1$, and it is true that $M_1$ satisfies a pseudolocal estimate though it is not strongly pseudoconvex. Since $\mathbb{H}_3(\mathbb{R})$ contains lattices, the manifolds $M_\epsilon$ are examples of the setting of the discrete structure group as well as that of a bundle.

Finally, [33, §3] contains a remarkable example of a $G$ manifold ($G$ is a nonunimodular matrix group here) in $\mathbb{C}^2$ which is not a tube but satisfies all of our requirements. This manifold has a trivial Bergman space though the $\bar\partial$-Neumann problem is somewhat tractable, as shown in [53]. Our present treatment is valid there as well.

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