ESI **The Erwin Schrödinger International Institute for Mathematical Physics**
Boltzmanngasse 9
A-1090 Wien, Austria

On the Boundary of the Group of Transformations Leaving a Measure Quasi–Invariant

Yury A. Neretin

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On the boundary of the group of transformations leaving a measure quasi-invariant

Yury A. Neretin[1]

Let A be a Lebesgue measure space. We interpret measures on $A \times A \times \mathbb{R}_+$ as 'maps' from A to A, which spread A along itself; their Radon-Nikodym derivatives also are spread. We discuss basic properties of the semigroup of such maps and the action of this semigroup in the spaces $L^p(A)$.

1 Purposes of the work

1.1. Groups $\mathrm{Ams}(A)$ and $\mathrm{Gms}(A)$ and their boundaries. Denote by $\mathbb{R}^\times$ the multiplicative group of positive real numbers. Let A be a space with a continuous probability measure α. Denote by $\mathrm{Ams}(A)$ the group of measurable transformations of A preserving α, by $\mathrm{Gms}(A)$ we denote the group of transformations leaving the measure α quasi-invariant.

The group $\mathrm{Ams}(A)$ has a well-known completion $\overline{\mathrm{Ams}(A)}$ (below we denote it by $\mathrm{Mar}(A, A)$), points of the completion are measures on $A \times A$ whose projections to both factors coincide with α. Elements of $\overline{\mathrm{Ams}(A)}$ can be regarded as 'maps' $A \to A$ spreading points along the set A. There is a well-defined composition of spreading maps.

Such objects are widely used in probability (since their definition is a rephrasing of Markov operators) and in ergodic theory (see, e.g., [7], [22], [21], [3]), they appear in mathematical hydrodynamics (see, e.g., [2]).

The group $\mathrm{Gms}(A)$ also has a natural completion $\overline{\mathrm{Gms}(A)}$ (below we denote it by $\mathrm{Pol}(A, A)$), whose points are measures on $A \times A \times \mathbb{R}^\times$, such measures can be regarded as spreading maps with spread Radon–Nikodym derivative; we call such 'maps' *polymorphisms*[2,3]. This object was introduced in [11], and initial motivation was the following theorem: *Any unitary representation of the group* $\mathrm{Gms}(A)$ *admits a unique continuous extension to the semigroup* $\overline{\mathrm{Gms}(A)}$.

1.2. Olshanski's problem on weak closure. Let ρ be a unitary representation of a group G in a Hilbert space H. Consider the set $\rho(G)$ of all operators $\rho(g)$, where g ranges in G. Consider its closure $\Gamma = \Gamma_\rho = \overline{\rho(G)}$ with respect to the weak operator topology. It can be readily shown that Γ is a compact semigroup. For a Lie group this object is not interesting (usually we get

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[2] May be it is better to say '$\mathbb{R}^\times$-polymorphisms'. Vershik [22] uses the term 'polymorphisms' for elements of Mar, more common is the term 'bistochastic kernels'. In [11], [12], [14] there were considered semigroups of measures on $A \times A \times G$, where G is an arbitrary group, they were called G-polymorphisms.

[3] These objects differ from 'substochastic kernels' [7].

a one-point compactification of G, see [4]). But for infinite-dimensional groups picture changes. The following 'experimental facts' hold:

— the Γ_ρ is essentially larger than G;

— $\Gamma = \Gamma_\rho$ admits a universalization (*mantle* of G) with respect to ρ;

— Γ admits an explicit description;

— Γ is an effective tool for investigation representations of G.

1.3. Action of a mantle on a measure space. Let an infinite dimensional group G act on a measure space by transformation leaving a measure quisiinvariant (a big zoo of such actions is known, see survey [13] and more recent constructions in [16], [6], [1]). In [13] there were proposed (partially precise, partially heuristic) arguments, which show that the mantle Γ acts on A by polymorphisms.

In [15] and [17] such actions were explicitly described in two simplest cases: for groups of natural symmetries of Gaussian measures and Poisson measures. It seems to me that formulas are unusual. There arises a problem to describe such actions in more complicated cases. The problem can be formulated in Olshanski's spirit: to describe the closure of G in $\overline{\mathrm{Gms}(A)}$.

1.4. Purposes of the paper. Basic facts about polymorphisms were formulated in [11], [15] without proofs. The present text is a step backward, we present these proofs and provide works [15], [17] and the problem formulated above by a necessary background. We discuss different versions of the definition of the product. Also for any polymorphism $\mathfrak{P} \in \mathrm{Pol}(A, B)$ we define the operator-valued function

$$u \mapsto T_u(\mathfrak{P}) : L^\infty(B) \to L^1(A),$$

where u ranges in the strip $0 \leqslant \operatorname{Re} u \leqslant 1$; on each line $\operatorname{Re} u = v$ the operators $T_u(\mathfrak{P})$ are bounded as operators $L^{1/v}(B) \to L^{1/v}(A)$. The product of polymorphisms corresponds to the point-wise product of operator-valued functions T_u. These functions provide us by a 'dual language' for work with polymorphisms (see [15], [17]), this require detailed description of the correspondence between polymorphisms and holomorphic operator-valued functions.

1.5. Structure of the paper. Sections 2 an 3 contain preliminaries on Lebesgue measure spaces and Markov operators. In Section 4 we discuss some simple properties of the semigroup of probability measures on $\mathbb{R}^\times$. Polymorphisms are defined in Section 5. In Section 6 we discuss the action of polymorphisms in spaces L^p.

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2 Preliminaries. Lebesgue spaces.

A fundamental work on Lebesgue measure spaces is Rohlin [20]. See an exposition in [10].

2.1. Lebesgue spaces. A *Lebesgue measure space* (A,α) is a space with a finite positive measure equivalent to a disjoint union of a segment $[p,q]\subset\mathbb{R}$ equipped with the Lebesgue measure and a finite or countable collection of points (atoms) having non-zero measures. We assume $\alpha(A)>0$.

We say that a measure is

— *probabilistic* if $\alpha(A)=1$;

— *continuous* if the set of atoms is empty;

— *discrete* if A is a union of atoms.

It is known that almost all spaces with *finite* measure, which appear in analysis, are Lebesgue.

We denote by $\alpha(M)$ the measure of a measurable subset $M\subset A$. By $\int f(a)\,d\alpha(a)$ we denote integral with respect to α.

Such a space (a union of a segment and a collection of atoms) has a natural Borel structure. Below the term '*measurable set*' (function) means measurable with respect to the Borel structure. Below a *measure* is a measure defined on the Borel σ-algebra.

2.2. Spaces L^p. For $1\leqslant p<\infty$ consider the space $L^p(A,\alpha)$ consisting of measurable functions (defined up to a.s) f satisfying

$$\|f\|_p:=\left(\int_A |f(a)|^p\,d\alpha(a)\right)^{1/p}<\infty.$$

In this way we get a separable Banach space with norm $\|f\|_p$. If $p>r$, then $L^p(A)\subset L^r(A)$.

For $p=\infty$, we set[4]

$$\|f\|_\infty:=\text{ess-sup}_{a\in A}|f(a)|.$$

In this way we get a nonseparable Banach space. To avoid the nonseparability, *we change the convergenve on L^∞*. We say that a sequence $f_j\in L^\infty$ converges to f if the sequence $\|f_j\|_\infty$ is bounded and for each $\varepsilon>0$ a measure the set $\bigl\{a\in A:\ |f_j(a)-f(a)|>\varepsilon\bigr\}$ tends to 0 as j tends to ∞. We say that a linear functional ℓ on $L^\infty(A)$ is continuous if convegence $f_j\to f$ implies the convergence $\ell(f_j)\to\ell(f)$.

Let $\frac{1}{p}+\frac{1}{q}=1$. Each continuous linear functional on $L^p(A,\alpha)$ has the form

$$\gamma(f)=\int_A f(a)g(a)\,d\alpha(a),\qquad \text{where } g\in L^q(A,\alpha)$$

(for $p=\infty$ this holds due the correction of convergence[5]). Moreover $\|\gamma\|=\|g\|_q$

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[4] Recall that the essential supremum of a subset $X\subset\mathbb{R}$ is the infimum of all x such that measure of set a $X\cap[x,\infty)$ is zero.

[5] We evaluate γ on identifier functions of measurable sets and get a countably additive charge on A. For a set of zero measure this charge is 0. Therefore this charge is determined by an integrable function.

2.3. Pushforward of a measure. Let (A, α) be a Lebesgue space, D be a space with the standard Borel structure. Let $\pi : A \to B$ be a measurable map. We define the measure β on B from the condition: $\beta(N) = \alpha\bigl(\pi^{-1}(N)\bigr)$. The space (B, β) becomes a Lebesgue measure space.

2.4. Conditional measures. A countable (or finite) partition X of a Lebesgue space (A, α) is a representation of A as a disjoint union of measurable subsets, $\mathsf{X} : A = \cup X_j$. The quotient space A/X is a discrete space, whose points a_j have measures $\alpha(X_j)$.

A continual *partition* $\mathsf{X} : A = \cup_{r \in R} X_r$, where r ranges in a continual space R and X_r are mutually disjoint, is *measurable*[6] if there exists a countable family of measurable subsets $U_j \subset A$ such that

— each U_j is a union $\cup_{r \in P} X_r$, where $P \subset R$ is a subset.

— the family U_j separates X_r, i.e., for two $X_r \ne X_q$ there exists U_i such that $X_r \subset U_i$, $X_q \not\subset U_i$.

We define a structure of a measure space on the quotient-space $A/\mathsf{X} \simeq R$: a subset $P \subset R$ is measurable iff $\cup_{r \in P} X_r$ is measurable and the measure of P is $\rho(P) := \alpha\bigl(\cup_{r \in P} X_r\bigr)$.

The space A/X is Lebesgue and the map $A \to A/\mathsf{X}$ is measurable.

Conversely, for a measurable map of Lebesgue spaces $g : A \to B$, the partition $A = \cup_{b \in B} g^{-1}(b)$ is measurable.

Recall the Rohlin Theorem. *For a measurable partition $\mathsf{X} : A = \cup_{r \in R} X_r$ there is a family of probability measures ξ_r defined on almost all (with respect to the measure on A/X) sets X_r such that for any measurable subset $M \subset A$ and for almost all r the sets $M \cap X_r \subset X_r$ are measurable in X_r and*

$$\alpha(M) = \int_{A/\mathsf{X}} \xi_r(M \cap X_r) d\rho(r).$$

Almost all spaces X_r are Lebesgue. For integrable functions on A we have

$$\int_A f(a)\, d\alpha(a) = \int_{A/\mathsf{X}} \int_{a \in X_r} f(a)\, d\xi_r(a)\, d\rho(r).$$

The measures ξ_r are called *conditional measures.*

2.5. Conditional expectation. Let $R = A/\mathsf{X}$, $\pi : A \to R$ be the projection, let ξ_r be the conditional measures. We define the operator of *conditional expectation*

$$J[A; \mathsf{X}] : L^1(A) \to L^1(R)$$

given by

$$J[A; \mathsf{X}] f(r) = \int_{X_r} f(a)\, d\xi_r(a).$$

On the other hand there is an isometric embedding

$$K[A; \mathsf{X}] : L^1(R) \to L^1(A)$$

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[6] See, [20], [10]. The partition of $\mathbb{R}$ with respect to the equivalence $x \sim y$ if $x - y \in \mathbb{Q}$ is an example of a non-measurable partition.

given by

$$K[A;\mathsf{X}]h(a) = h(\pi(a)).$$

We also define the operator of *conditional average*

$$I[A;\mathsf{X}] = K[A;\mathsf{X}]J[A;\mathsf{X}] : L^1(A) \to L^1(A).$$

It can be represented as

$$I[A;\mathsf{X}]f(a) = \int_{X_p \ni a} f(c)\, d\xi_p(c).$$

These operators satisfy properties

$$I^2 = I, \qquad IK = K, \qquad JI = J, \qquad JK = 1.$$

2.6. Groups $\mathrm{Ams}(A)$**.** Let (A,α) be a space with continuous Lebesgue measure. By $\mathrm{Ams}(A)$ we denote the group of all measure preserving bijective a.s. maps $A \to A$. Two elements g_1, g_2 of $\mathrm{Ams}(A)$ coincide if $g_1(a) = g_2(a)$ a.s.

The group $\mathrm{Ams}(A)$ acts in the space $L^p(A,\alpha)$ by the isometric operators

$$T(g)f(a) = f\bigl(g(a)\bigr).$$

The group $\mathrm{Ams}(A)$ is a separable topological group. The convergence is defined by the condition: $g_j \to g$ if for any measurable subsets M, $N \subset A$ we have

$$\lim_{j\to\infty} \alpha\bigl(g_j(M)\cap N\bigr) = \alpha\bigl(g(M)\cap N\bigr).$$

2.7. Groups $\mathrm{Gms}(A)$**.** Recall that the measure α is *quasi-invariant* with respect to a bijective a.s. map $A \to A$ if for any subset $M \subset A$ of zero measure, the sets $g(A)$ and $g^{-1}(A)$ have zero measure.

Equivalently there is a function $g'(a)$, which is called *Radon-Nikodym derivative*, such that for any measurable subset $M \subset A$

$$\mu(gM) = \int_M g'(a)\, d\alpha(a)$$

and $g'(a) \neq 0$ a.s. on A.

The Radon–Nikodym derivative satisfies the usual chain rule

$$(g\circ h)'(a) = g'\bigl(h(a)\bigr)\, h'(a).$$

By $\mathrm{Gms}(A)$ we denote the group of bijective a.s. maps $A \to A$ leaving the measure α quasi-invariant.

Fix p. For any $s \in \mathbb{R}$ the group $\mathrm{Gms}(A)$ acts in $L^p(A,\alpha)$ by isometric operators

$$T_{1/p+is}(g)f(a) = f\bigl(g(a)\bigr)g'(a)^{1/p+is}. \tag{2.1}$$

Due to the chain rule they satisfy

$$T_{1/p+is}(g_1)T_{1/p+is}(g_2) = T_{1/p+is}(g_1\circ g_2).$$

3 Markov category

Bistochastic kernels and Markov operators discussed below is a standard topic, see, e.g., [22], [7], [12], [3].

3.1. Markov category. The objects of the category Mar are Lebesgue spaces with probability measures. A morphism $\mathfrak{p} : (A, \alpha) \to (B, \beta)$ (a *bistochastic kernel*) is a measure $\mathfrak{p}$ on $A \times B$ such that

— the pushforward of $\mathfrak{p}$ under the projection $A \times B \to A$ is α;

— the pushforward of $\mathfrak{p}$ under the projection $A \times B \to B$ is β.

We denote the set of all morphisms $\mathfrak{p} : (A, \alpha) \to (B, \beta)$ by $\mathrm{Mar}(A, B)$.

Let $\mathfrak{p} : (A, \alpha) \to (B, \beta)$, $\mathfrak{q} : (B, \beta) \to (C, \gamma)$ be morphisms. We must define the product $\mathfrak{r} = \mathfrak{q} \circ \mathfrak{p} : (A, \alpha) \to (C, \beta)$. Let $M \subset A$, $K \subset C$. We restrict $\mathfrak{p}$ to $M \times A$ and take its pushforward $\mathfrak{p}_{M,b}$ under the projection $M \times B \to B$. Since $\mathfrak{p}_M(b)$ is dominated by $\beta(b)$, we have

$$\mathfrak{p}_M(b) = u_M(b)\, d\beta(b),$$

where $u_M(b)$ is a positive function $\leqslant 1$. Similarly, consider the restriction of $\mathfrak{q}$ to $B \times K$ and represent its pushforward $\mathfrak{q}_K(b)$ under the projection $B \times K \to B$ as

$$\mathfrak{q}_K(b) = v_K(b)\, d\beta(b).$$

Again, $0 \leqslant v_N(b) \leqslant 1$. We assign

$$\mathfrak{r}(M \times K) = \int_B u_M(b)\, v_K(b)\, d\beta(b).$$

Proposition 3.1 *The multiplication* $\mathrm{Mar}(A, B) \times \mathrm{Mar}(B, C) \to \mathrm{Mar}(A, C)$ *defined in this way is associative.*

3.2. Involution. The identity map $A \times B \to B \times A$ induces a map $\mathrm{Mar}(A, B) \to \mathrm{Mar}(B, A)$. We denote it by $\mathfrak{p} \mapsto \mathfrak{p}^\bigstar$. Obviously,

$$(\mathfrak{q} \circ \mathfrak{p})^\bigstar = \mathfrak{p}^\bigstar \circ \mathfrak{q}^\bigstar.$$

3.3. Sprcial case: spaces with discrete measures. Now let spaces A, B be countable. Let a_i (resp. b_j) be their points. Denote by α_i (resp. β_j) their measures. We can regard morphisms $\mathfrak{p} \in \mathrm{Mar}(A, B)$ as matrices $\mathfrak{P} = \mathfrak{p}_{ij}$ such that

$$\mathfrak{p}_{ij} \geqslant 0, \qquad \sum_i \mathfrak{p}_{ij} = \beta_j, \qquad \sum_j \mathfrak{p}_{ij} = \alpha_i.$$

If $\mathfrak{p} \in \mathrm{Mar}(A, B)$, $\mathfrak{q} \in \mathrm{Mar}(B, C)$, then the product is given by

$$\mathfrak{r}_{ik} = \sum_j \frac{\mathfrak{p}_{ij}\mathfrak{q}_{jk}}{\beta_j}$$

or

$$\mathfrak{R} = \mathfrak{Q}\Delta_\beta^{-1}\mathfrak{P}, \tag{3.1}$$

where Δ_β is the diagonal matrix with entries β_j.

3.4. Special case: absolutely continuous kernels. Let $p : A \times B \to \mathbb{R}$ be a nonnegative integrable function satisfying the conditions

$$\int_B p(a,b)\,d\beta(b) = 1 \qquad \int_A p(a,b)\,d\alpha(a) = 1 \qquad \text{a.s.}$$

Then we can define the bistochastic kernel $\mathfrak{p}$ on $A \times B$ by

$$\mathfrak{p}(M \times N) = \int_M \int_N p(a,b)\,d\beta(b)\,d\alpha(a)$$

If $p : A \times B \to \mathbb{R}$. $q : B \times C \to R$ are such functions. Then the product of bistochastic kernels corresponds to the function

$$r(a,c) := \int_B p(a,b)\,q(b,c)\,d\beta(b) \tag{3.2}$$

Lemma 3.2 *For almost all c for almost all a the integral converges.*

Proof. Fix c such that $q(b,c) \in L^1(B)$. The following integral converges

$$\int_B \int_A p(a,b)\,q(b,c)\,d\beta(b) = \int_B q(b,c)\,d\beta(b).$$

Applying the Fubini theorem we get that the integral 3.2 converges for almost all a. $\square$

3.5. Automorphisms. Let (A,α) be a space with continuous measure. Let $g \in \mathrm{Ams}(A)$. Consider the map $\iota_g : A \to A \times A$ given by $\iota(a) = \bigl(a, g(a)\bigr)$. Denote by $\xi[g]$ the pushforward of α under this map. Obviously, $\xi[g] \in \mathrm{Mar}(A,A)$. Moreover,

$$\xi[g_1 g_2] = \xi[g_1]\xi[g_2].$$

3.6. Convergence. A sequence $\mathfrak{p}_j \in \mathrm{Mar}(A,B)$ converges to $\mathfrak{p}_j \in \mathrm{Mar}(A,B)$ if for any subsets $M \subset A$, $N \subset B$,

$$\lim_{j\to\infty} \mathfrak{p}_j(M \times N) = \mathfrak{p}_j(M \times N).$$

Proposition 3.3 a) *Spaces* $\mathrm{Mar}(A,B)$ *are compact.*

b) *The product* $\mathrm{Mar}(A,B) \times \mathrm{Mar}(B,C) \to \mathrm{Mar}(A,C)$ *is separately continuous.*

Proposition 3.4 *Let a measure α be continuous. The group* $\mathrm{Ams}(A)$ *is dense in* $\mathrm{Mar}(A,A)$.

3.7. Another language and equivalent definition of the product. For $\mathfrak{p} \in \mathrm{Mar}(A,B)$ consider the map $A \times B \to B$. We have conditional probability measures $\mathfrak{p}_b(a)$ on almost all fibers, they satisfy the equation

$$\int_A \mathfrak{p}_a(b)\,d\alpha(a) = \beta(b)$$

or, more precisely, for any subset $N \subset B$

$$\int_A \mathfrak{p}_a(N)\, d\alpha(a) = \beta(N).$$

Informally, we can consider $\mathfrak{p}$ as a map $A \to B$, which sends each point $a \in A$ to a measure $\mathfrak{p}_a$ on B (or spread each a along B).

EXAMPLE. Let $\mathfrak{p} \in \mathrm{Mar}(A, B)$ be $\alpha \times \beta$. Then all $\mathfrak{p}_a(b) = \beta(b)$. The corresponding map uniformly "spreads" each point a along β. For any morphism $\mathfrak{q} \in \mathrm{Mar}(B, C)$, we have

$$\mathfrak{q} \circ (\alpha \times \beta) = \alpha \times \gamma.$$

For any $\mathfrak{o} \in \mathrm{Mar}(Z, A)$, we have

$$(\alpha \times \beta) \circ \mathfrak{o} = (\zeta \times \alpha). \qquad \square$$

The product of $\mathfrak{p} \in \mathrm{Mar}(A, B)$, $\mathfrak{q} \in \mathrm{Mar}(B, C)$ can be regarded as double spreading. Formally, let $\mathfrak{p}_a$, $\mathfrak{q}_b$ be the corresponding systems of conditional measures. Then the system $\mathfrak{r}_a(c)$ corresponding to $\mathfrak{r} = \mathfrak{q} \circ \mathfrak{p}$ is

$$\mathfrak{r}_a(c) = \int_B \mathfrak{q}_b(c)\, d\mathfrak{p}_a(b).$$

3.8. Markov operators. For a bistochastic kernel $\mathfrak{p} \in \mathrm{Mar}(A, B)$ we define the operator $T(\mathfrak{p})$ by

$$T(\mathfrak{p}) f(a) = \int_B f(b)\, d\mathfrak{p}_a(b).$$

Proposition 3.5 *For each $p \in [1, \infty]$ the operator $T(\mathfrak{p})$ is bounded as an operator $L^p(B) \to L^p(A)$. Moreover its norm is $\leqslant 1$ for each p*

Evidently.

$$T(\mathfrak{q} \circ \mathfrak{p}) = T(\mathfrak{p}) T(\mathfrak{q}).$$

3.9. Conditional expectations. Let $\mathsf{X} : A = \cup_{r \in R} X_r$ be a measurable partition of A, let (R, ρ) be the quotient space, $\pi : A \to R$ be the projection. Consider the map $\xi : A \to A \times (A/\mathsf{X})$ given by $a \mapsto (a, \xi(a))$. Denote by

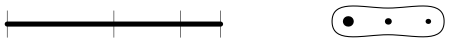
$$\mathfrak{m}[A; \mathsf{X}] \in \mathrm{Mar}(A, A/\mathsf{X})$$

the π-pushforward of the measure α. Denote

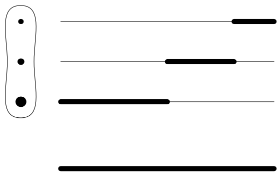
$$\mathfrak{l}[A; \mathsf{X}] := \mathfrak{m}[A; \mathsf{X}]^{\bigstar} : \in \mathrm{Mor}(A/\mathsf{X}, A).$$

Also define the morphism

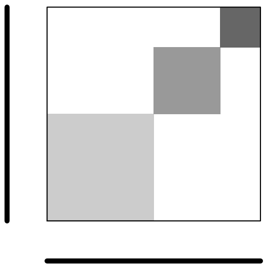
$$\mathfrak{t}[A; \mathsf{X}] = \mathfrak{l}[A; \mathsf{X}] \circ \mathfrak{m}[A; \mathsf{X}] : A \to A.$$



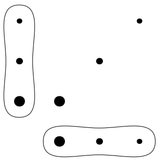
A segment $A=[0,1]$, its partition X into 3 pieces and the quotient space A/X.



The morphism $\mathfrak{l}[A;\mathsf{X}]$. On the picture the product $A/\mathsf{X}\times A$ is a union of 3 long horizontal segments. The measure $\mathfrak{l}[A;\mathsf{X}]$ is the uniform measure on the union of 3 thick horizontal subsegments.



The morphism $\mathfrak{t}[A;\mathsf{X}]$. We have a uniform measure on each sub-square $\subset[0,1]\times[0,1]$.



The unit morphism $A/\mathsf{X}\to A/\mathsf{X}$.

Figure 1: Reference to Subsection 3.9. Morphisms associated with a partition.

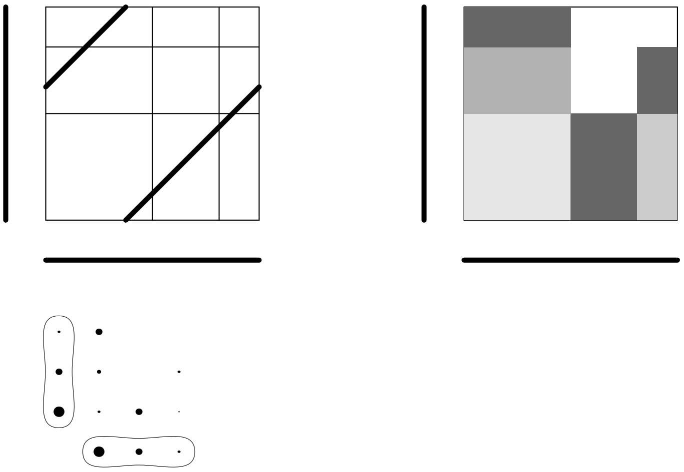


Figure 2: Reference to Subsection 3.9. The morphism $\mathfrak{t}[A/\mathsf{X}] \circ \mathfrak{p} \circ \mathfrak{t}[A/\mathsf{X}]$ is obtained from $\mathfrak{p}$ by uniform spreading measure $\mathfrak{p}$ along each rectangle.
The morphism $\mathfrak{m}[A/\mathsf{X}] \circ \mathfrak{p} \circ \mathfrak{l}[A/\mathsf{X}]$ is obtained from $\mathfrak{p}$ by concentration of measure $\mathfrak{p}$ of each rectangle.

Let us describe these measures more explicitly. The measure $\mathfrak{m}[A;\mathsf{X}]$ on $A \times A/\mathsf{X}$ is defined by

$$\int_{A\times R} F(a,r)\, d\mathfrak{m}[A;\mathsf{X}](a,r) = \int_R \left(\int_{X_r} F(a,r)\, d\xi_r(a) \right) d\rho(r).$$

The measure $\mathfrak{t}[A;\mathsf{X}]$ can be defined by

$$\int_{A\times A} F(a_1,a_2)\, d\mathfrak{t}[A;\mathsf{X}] = \int_R \left(\int_{X_r\times X_r} F(a_1,a_2)\, d\xi_r(a_1)\, d\xi_r(a_2) \right) d\rho(r).$$

In notation of Subsection 2.5,

$$I[A;\mathsf{X}] = T\bigl(\mathfrak{t}[A;\mathsf{X}]\bigr), \qquad J[A;\mathsf{X}] = T\bigl(\mathfrak{l}[A;\mathsf{X}]\bigr), \qquad K[A;\mathsf{X}] = T\bigl(\mathfrak{m}[A;\mathsf{X}]\bigr).$$

Now let $\mathfrak{p} \in \mathrm{Mar}(A,B)$, let $\mathsf{X} : A = \cup X_i$, $\mathsf{Y} : B = \cup Y_j$ be countable measurable partitions. First, consider the measure

$$\mathfrak{u} := \mathfrak{m}[B;\mathsf{Y}] \circ \mathfrak{p} \circ \mathfrak{l}[A;\mathsf{X}] \in \mathrm{Mar}(A/\mathsf{X}, B/\mathsf{Y}).$$

Both spaces A/X, B/Y are discrete. Therefore the measure $\mathfrak{u}$ is defined by a matrix with non-negative elements, it is given by

$$\mathfrak{u}_{ij} = \mathfrak{p}(X_i \times X_j). \tag{3.3}$$

Next, consider

$$\mathfrak{v} := \mathfrak{t}[B;\mathsf{Y}] \circ \mathfrak{p} \circ \mathfrak{t}[A;\mathsf{X}] \in \mathrm{Mar}(A,B).$$

This measure is given by

$$\mathfrak{v}(M\times N)=\sum_{i,j}\frac{\alpha(M\cap X_i)}{\alpha(M)}\frac{\beta(N\cap Y_j)}{\beta(N)}\mathfrak{p}(X_i\times Y_j),$$

where $M\subset A$, $N\subset B$ are measurable subsets of non-zero measure.

3.10. Definition of the product in the terms of approximations. Let $\mathsf{X}^{(1)}$, $\mathsf{X}^{(2)}$, … be a sequence of countable partitions of A. We say that it is *approximating* if for each p the partition $\mathsf{X}^{(p+1)}$ is a refinement of $\mathsf{X}^{(p)}$ and the sigma-algebra generated by all partitions coincides with the sigma-algebra of all measurable sets in A.

Now let A, B, C be spaces with probability measures and $\mathsf{X}^{(p)}$, $\mathsf{Y}^{(q)}$, $\mathsf{Z}^{(r)}$ be approximating sequences of partitions of A, B, C respectively.

Proposition 3.6 *The product of* $\mathfrak{p}\in\mathrm{Mar}(A,B)$*,* $\mathfrak{q}\in\mathrm{Mar}(B,C)$ *equals to*

$$\begin{aligned}&\mathfrak{q}\circ\mathfrak{p}=\\&\lim_{i\,j,\,k\to\infty}\mathfrak{t}[C;\mathsf{Z}_k]\circ\mathfrak{q}\circ\mathfrak{t}[B;\mathsf{Y}_l]\circ\mathfrak{p}\circ\mathfrak{t}[A;\mathsf{X}_i]=\\&\lim_{i\,j,\,k\to\infty}\mathfrak{l}[C;\mathsf{Z}_k]\circ\Bigl(\mathfrak{m}[C;\mathsf{Z}_k]\circ\mathfrak{q}\circ\mathfrak{l}[B;\mathsf{Y}_l]\Bigr)\circ\Bigl(\mathfrak{m}[B;\mathsf{Y}_j]\circ\mathfrak{p}\circ\mathfrak{l}[A;\mathsf{X}_i]\Bigr)\circ\mathfrak{m}[A;\mathsf{X}_i]\end{aligned}$$

Products inside brackets is nothing but writing of matrices as in (3.3). Product of two brackets is the product of matrices as in (3.1). In this way we get a measure on $A\times C$ and after this pass to the limit.

4 Semiring of measures on $\mathbb{R}^\times$

This section is a preparation to the definition of polymorphisms.

4.1. Semiring $\mathcal{M}^\triangledown$. Denote by $\mathcal{M}^\triangledown$ the set of all positive measures μ on $\mathbb{R}^\times$ such that

$$\int_{\mathbb{R}^\times}d\mu(t)<\infty,\qquad\int_{\mathbb{R}^\times}t\,d\mu(t)<\infty.$$

Evidently, if μ, $\nu\in\mathcal{M}^\triangledown$, then $\mu+\nu\in\mathcal{M}^\triangledown$ We also equip the set $\mathcal{M}^\triangledown$ by the convolution $(\mu,\nu)\mapsto\mu*\nu$ defined in the usual way,

$$\int_{\mathbb{R}^\times}f(t)\,d\mu*\nu(t)=\int\int_{\mathbb{R}^\times\times\mathbb{R}^\times}f(s_1s_2)\,d\mu(s_1)\,d\nu(s_2).$$

Evidently, $\mathcal{M}^\triangledown$ is closed with respect to the convolution. Indeed

$$\int\limits_{\mathbb{R}^\times}t^u d\mu*\nu(t)=\iint\limits_{\mathbb{R}^\times\times\mathbb{R}^\times}s_1^us_2^u\,d\mu(s_1)\,d\nu(s_2)=\int\limits_{\mathbb{R}^\times}s_1^ud\mu(s_1)\cdot\int\limits_{\mathbb{R}^\times}s_2^ud\nu(s_2)\qquad(4.1)$$

Substituting $u=0$ and $u=1$ we get $\mu*\nu\in\mathcal{M}^\triangledown$.

Next, we define the involution $\mu \mapsto \mu^\star$ in $\mathcal{M}^\triangledown$ by

$$\mu^\star(t) = t^{-1}\mu(t^{-1}),$$

i.e.,

$$\int_{\mathbb{R}^\times} f(t)\, d\mu^\star(t) = \int_{\mathbb{R}^\times} tf(t^{-1})\, d\mu(t). \tag{4.2}$$

For $\mu \in \mathcal{M}^\triangledown$, we have $\mu^\star \in \mathcal{M}^\triangledown$, also $(\mu * \nu)^\star = \mu^\star * \nu^\star$.

We say that a sequence $\mu_j \in \mathcal{M}^\triangledown$ converges to $\mu \in \mathcal{M}^\triangledown$ if for any bounded continuous function $f(t)$ on $\mathbb{R}^\times$ we have convergences

$$\int f(t)\, d\mu_j(t) \to \int f(t)\, d\mu(t), \qquad \int tf(t)\, d\mu_j(t) \to \int tf(t)\, d\mu(t).$$

In other words we require weak convergences (see. e.g. [9], Sect. 12.1) of two sequences of measures $\mu_j \to \mu$, $t\mu_j \to t\mu$.

4.2. Mellin transform. For a measure $\mu \in \mathcal{M}^\triangledown$, we define its *Mellin transform* by

$$\Phi_\mu(u) := \int_{\mathbb{R}^\times} t^u\, d\mu(t), \qquad \text{where } u = v + iw \in \mathbb{C}. \tag{4.3}$$

Remark. Pass to the variable $s := \ln t$. The measure $\nu(s) = \mu(\ln t)$ is a measure on $\mathbb{R}$, the conditions (4.2) transform to

$$\int_{\mathbb{R}} d\nu(s) < \infty, \qquad \int_{\mathbb{R}} e^s \nu(s) < \infty.$$

The function $\Phi(u)$ is the characteristic function (or Fourier transform) of the measure μ. This topic is quite standard (see, e.g., [8]), however we have not a convenient for our purpose reference. □

Proposition 4.1 a) *For any $\mu \in \mathcal{M}^\triangledown$, the function Φ_μ is uniformly continuous in the strip*

$$\Pi:\ 0 \leqslant v \leqslant 1 \qquad -\infty < w < \infty \tag{4.4}$$

and holomorphic in the open strip $0 < \operatorname{Re} u < 1$.

b) *The functions $\Phi_\mu(u)$ are positive definite, i.e. for any $u_1, \dots, u_n$ satisfying $0 \leqslant \operatorname{Re} u_j \leqslant 1/2$ and any $z_1, \dots, z_n \in \mathbb{C}$*

$$\sum_{l \leqslant n} \sum_{m \leqslant n} \Phi(u_l + \overline{u}_m) z_l \overline{z}_m \geqslant 0. \tag{4.5}$$

c) *Functions Φ_μ satisfy the following estimate*

$$|\Phi_\mu(v + iw)| \leqslant \Phi(0)^{1-v}\Phi(1)^v.$$

In particular, $\Phi_\mu(u)$ is bounded in the strip $0 \leqslant \operatorname{Re} u \leqslant 1$.

Proof. Convergence of integral (4.3) is obvious. Let prove uniform continuity:

$$|\Phi_\mu(u) - \Phi_\mu(u')| \leqslant \int_{\mathbb{R}_+} |t^u - t^{u'}|\, d\mu(t)$$

We split this integral as a sum of integrals over domains $t < 1/A$, $1/A \leqslant t \leqslant B$, $t > B$. We have

$$\int_{t>B} |t^u - t^{u'}|\, d\mu(t) \leqslant \int_{t>B} 2t\, d\mu(t).$$

For sufficiently large B this integral is as small as desired. In the same way we estimate the integral over $t < 1/A$:

$$\int_{t<1/A} |t^u - t^{u'}|\, d\mu(t) \leqslant \int_{t<1/A} 2\, d\mu(t).$$

Next, we fix A, B,

$$\int_{1/A\leqslant t\leqslant B} |t^u - t^{u'}|\, d\mu(t) = \int_{1/A\leqslant t\leqslant B} t^{\operatorname{Re} u}|t^{u'-u} - 1|\, d\mu(t) \leqslant$$
$$\leqslant \int_{1/A\leqslant t\leqslant 0} |t^{u'-u} - 1|\, d\mu(t) + \int_{0<t\leqslant B} t\,|t^{u'-u} - 1|\, d\mu(t)$$

If $|u' - u|$ is small, then $|t^{u'-u} - 1|$ is small on $[1/A, B]$. □

Proposition 4.2 *Let $\Phi(u)$ be a bounded continuous positive-definite function in the strip $0 \leqslant \operatorname{Re} u \leqslant 1$ holomorphic in the open strip. Then $\Phi(u)$ is a Mellin transform of a measure $\mu \in \mathcal{M}^\nabla$.*

Proof. By a Paley–Wiener theorem [5], Theorem 7.4.2, Φ is a Fourier transform of a tempered distribution $\nu(s)$ on $\mathbb{R}$. Applying the Bochner theorem (see, e.g. [9], Sect. 15.1) to the function $\Phi(iw)$ we get that $\nu(s)$ is a finite positive measure. Applying the Bochner theorem to $\Phi(1 + iw)$, we get that $e^s \cdot \nu(s)$ is a finite measure. Passing to the variable $t = e^s$ we get the desired statement. □

Proposition 4.3 a) $\Phi_{\mu*\nu}(u) = \Phi_\mu(u)\Phi_\nu(u)$.
b) $\Phi_{\mu^\star}(u) = \Phi_\mu(1 - u)$

Proof. a) is obvious, it was proved above in (4.1); b) also is obvious. □

4.3. Convergence of characteristic functions.

Proposition 4.4 *If μ_j converges to μ in $\mathcal{M}^\nabla$, then $\Phi_{\mu_j}(u)$ converges to $\Phi_\mu(u)$ uniformly on each rectangle $0 \leqslant v \leqslant 1$, $1/A \leqslant w \leqslant B$.*

Pointwise convergence is evident, proof of uniform convergence coincides with the standard proof, see [9], 13.2.C. □

Proposition 4.5 a) *Let μ_j, $\mu \in \mathcal{M}^\triangledown$. If*

$$\Phi_{\mu_j}(iw) \to \Phi_{\mu_j}(iw), \qquad \Phi_{\mu_j}(1+iw) \to \Phi_{\mu_j}(1+iw) \tag{4.6}$$

pointwise, then μ_j converges to μ.

b) *Let $\mu_j \in \mathcal{M}^\triangledown$. Assume that the sequence $\Phi_{\mu_j}(iw)$ converges pointwise to some function $\Psi(iw)$ and $\Phi_{\mu_j}(1+iw)$ converges pointwise to some function $\Theta(1+iw)$. If $\Psi(iw)$, $\Theta(1+iw)$ are continuous at $w=0$, then μ_j converges to some $\mu \in \mathcal{M}^\triangledown$ and $\Phi(iw) = \Psi(iw)$, $\Phi(1+iw) = \Theta(1+iw)$.*

Proof. Let us prove b). By the continuity theorem (see, e.g., [9], Theorem 15.2), the sequence μ_j weakly converges to a measure μ and $t \cdot \mu_j$ weakly converges to a measure ν. Let $f(t)$ be a continuous function with compact support. Then

$$\int_{\mathbb{R}^\times} f(t)\, d\nu(t) = \lim_{j\to\infty} \int_{\mathbb{R}^\times} f(t) t\, d\mu_j(t) =$$
$$= \lim_{j\to\infty} \int_{\mathbb{R}^\times} \bigl(t\, f(t)\bigr)\, d\mu_j(t) = \int_{\mathbb{R}^\times} \bigl(t\, f(t)\bigr)\, d\mu(t)$$

Therefore $\nu(t) = t\mu(t)$ and μ_j converges to μ in $\mathcal{M}^\triangledown$. $\square$

4.4. Exotics: semirings $\mathcal{M}^\triangledown_{a,b}$. For real $a<b$ we denote by $\mathcal{M}^\triangledown_{a,b}$ the set of positive measure on $\mathbb{R}^\times$ satisfying

$$\int_{\mathbb{R}^\times} t^a d\mu(t) < \infty, \qquad \int_{\mathbb{R}^\times} t^b \mu(t) < \infty.$$

All statements of this section can be extended automatically to this semiring. The only difference: the Mellin transform $\Phi(u)$ is defined in the strip

$$\Pi_{a,b} :\ a \leqslant \operatorname{Re} u \leqslant b$$

Notice also that for $\mu \in \mathcal{M}^\triangledown$, the measure $\nu(t) := t^{-a}\mu(t^{1/(b-a)})$ is contained in $\mathcal{M}^\triangledown_{a,b}$ and the map $\mu \to \nu$ is an isomorphism of semirings.

5 Polymorphisms. Basic definitions

5.1. Definition. Let (A,α), (B,β) be Lebesgue measure spaces. A *polymorphism $A \rightsquigarrow B$* is a measure $\mathfrak{P}$ on $A \times B \times \mathbb{R}^\times$ such that

1°. the pushforward of $\mathfrak{P}$ under the projection $A \times B \times \mathbb{R}^\times \to A$ is α;

2°. the pushforward of $t \cdot \mathfrak{P}$ under the projection $A \times B \times \mathbb{R}^\times \to B$ is β.

We denote the set of all polymorphism $A \rightsquigarrow B$ by $\operatorname{Pol}(A,B)$.

There is a well-defined associative product

$$\operatorname{Pol}(A,B) \times \operatorname{Pol}(B,C) \to \operatorname{Pol}(A,C).$$

Formal definition is given in Subsection 5.6. Before this we consider several simple cases.

5.2. Special case: category Mar. Now let A, B be spaces with probability measures. Any $\mathfrak{p} \in \mathrm{Mar}(A,B)$ can be regarded as an element of $\mathrm{Pol}(A,B)$, we simply consider the pushforward of the measure $\mathfrak{p}$ under the embedding

$$A \times B \to A \times B \times \mathbb{R}^\times$$

given by $(a,b) \mapsto (a,b,1)$.

5.3. Special case: $\mathcal{M}^\triangledown$. Consider single-point spaces A and B, denote by α and β their measures. Then a polymorphism $A \rightsquigarrow B$ is a measure on $\mathbb{R}^\times$ satisfying

$$\int_{\mathbb{R}^\times} d\mathfrak{P}(t) = \alpha, \qquad \int_{\mathbb{R}^\times} t d\mathfrak{P}(t) = \beta.$$

The product of $\mathfrak{P} : A \rightsquigarrow B$, $\mathfrak{Q} : B \rightsquigarrow C$ coincides with convolution of measures.

5.4. Special case: Discrete spaces. Let spaces A, B be discrete, a_i, b_j be their points, α_i, β_j be measures of points. A measure $\mathfrak{P}$ on $A \times B \times \mathbb{R}^\times$ can be regarded as a matrix, whose matrix elements are positive measures $\mathfrak{p}_{ij} \in \mathcal{M}^\triangledown$, these measures satisfy additional conditions

$$\sum_i \int_{\mathbb{R}^\times} t\, d\mathfrak{p}_{ij} = \beta_j; \tag{5.1}$$

$$\sum_j \int_{\mathbb{R}^\times} d\mathfrak{p}_{ij} = \alpha_i. \tag{5.2}$$

For $\mathfrak{P} \in \mathrm{Pol}(A,B)$, $\mathfrak{Q} \in \mathrm{Pol}(B,C)$, their product $\mathfrak{R}$ is defined by

$$\mathfrak{r}_{ik} = \sum_j \frac{1}{\beta_j} \mathfrak{q}_{jk} * \mathfrak{p}_{ij}, \tag{5.3}$$

where $*$ denotes the convolution in $\mathcal{M}^\triangledown$. In fact, we multiply matrices whose elements are measures $\in \mathcal{M}^\triangledown$, see (3.1).

5.5. Special case. Absolutely continuous kernels. Let $p : A \times B \to \mathcal{M}^\triangledown$ be a measurable function. We define the measure $\mathfrak{P}$ on $A \times B \times \mathbb{R}^\times$ in the following way. For measurable subsets $M \subset A$, $N \subset B$, $K \subset \mathbb{R}$ we set

$$\mathfrak{P}(M \times N \times \mathbb{R}^\times) := \int_M \int_N p(K)\, d\beta(b)\, d\alpha(a).$$

If

$$\int_M \int_B \int_{\mathbb{R}^\times} dp(a,b)(t)\, d\beta(b)\, d\alpha(a) = \alpha(M),$$
$$\int_A \int_N \int_{\mathbb{R}^\times} tdp(a,b)(t)\, d\beta(b)\, d\alpha(a) = \beta(N),$$

then $\mathfrak{P}$ is a polymorphism. In this case, we say that $\mathfrak{P}$ is *absolutely continuous.*

Evidently, a polymorphism $\mathfrak{P}$ is absolutely continuous if the projection of $\mathfrak{P}$ to $A \times B$ is a measure absolutely continuous with respect to $\alpha \times \beta$.

REMARK. This includes the case discussed in the previous subsection. For a matrix $\mathfrak{p}_{ij}$, the function p is given by

$$p(a_i \times b_j) = \frac{\mathfrak{p}_{ij}}{\alpha_i \beta_j}. \qquad \square$$

Let $\mathfrak{P} : A \rightsquigarrow B$, $\mathfrak{Q} : B \rightsquigarrow C$ be absolutely continuous polymorphisms, p, q the corresponding $\mathcal{M}^\triangledown$-valued functions. We define the function $r : A \times C \to \mathcal{M}^\triangledown$ by

$$r(a,c) = \int_B p(a,b) * q(b,c)\, d\beta(b).$$

Lemma 5.1 a) $r(a,c) \in \mathcal{M}^\triangledown$ *a.s.*

b) r *determines a polymorphism* $A \rightsquigarrow C$.

PROOF. To prove a) we write the integral

$$\int_B \int_A \int_{\mathbb{R}^\times} t\, d(p(a,b) * q(b,c))\, d\alpha(a)\, d\beta(b) =$$
$$= \int_B \Bigl(\int_{\mathbb{R}^\times} t\, dq(b,c)(t)) \Bigr) \int_A \Bigl(\int_{\mathbb{R}^\times} t\, dp(a,b)(t) \Bigr)\, d\alpha(a)\, d\beta(b) =$$
$$= \int_B \Bigl(\int_{\mathbb{R}^\times} t\, dq(b,c)(t)) \Bigr)\, d\beta(b) = 1$$

and change the order of integration. By the Fubini theorem the integral

$$\int_B \int_{\mathbb{R}^\times} t\, d(p(a,b) * q(b,c))\, d\beta(b)$$

is convergent a.s. Next, we repate the same for the the integral

$$\int_B \int_C \int_{\mathbb{R}^\times} d(p(a,b) * q(b,c))\, d\gamma(c)\, d\beta(b).$$

b) is straightforward. $\square$

5.6. Definition of the product. Let $\mathfrak{P} \in \mathrm{Pol}(A,B)$. For any measurable subsets $M \subset A$, $N \subset B$ we have a measure $\mathfrak{p}[M \times N] \in \mathcal{M}^\triangledown$ defined as pushforward of $\mathfrak{P}$ under the projection

$$M \times N \times \mathbb{R}^\times \to \mathbb{R}^\times.$$

In this sense we can regard $\mathfrak{P}$ as a $\mathcal{M}^\triangledown$-valued measure $\mathfrak{p}(\cdot)$ on $A \times B$.

Lemma 5.2 a) *Let* $\mathfrak{P} \in \mathrm{Pol}(A, B)$. *For any measurable subset* $M \subset A$ *there is a system of measures* $\mathfrak{p}_{M,b}(t)$, *where* b *ranges in* B, *on* $\mathbb{R}^\times$ *defined for almost all* $b \in B$ *such that for any measurable* $N \subset N$ *we have*

$$\mathfrak{p}[M \times N] = \int_N \mathfrak{p}_{M,b}\, d\beta(b). \tag{5.4}$$

b) *Let* $\mathfrak{Q} \in \mathrm{Pol}(B, C)$. *For any measurable subset* $K \subset C$ *there is a system of measures* $\mathfrak{q}_{b,K}(t)$ *on* $\mathbb{R}^\times$ *such that for any measurable subset* $N \subset B$

$$\mathfrak{q}[N \times K] = \int_N \mathfrak{q}_{b,K}\, d\beta(b). \tag{5.5}$$

PROOF. a) Consider pushforwards of $t\mathfrak{P}$ under the projections

$$M \times B \times \mathbb{R} \xrightarrow{p} B \times \mathbb{R} \xrightarrow{q} B$$

The measure $q(p(t\mathfrak{P}))$ is dominated by β. Therefore there are well-defined conditional measures $\sigma_{N,b}(t)$ on the fibers of the projection $B \times \mathbb{R}^\times \to B$. The total measure $\sigma_{M,b}$ is $\leqslant 1$; We define the measures

$$\mathfrak{P}_{M,b} = t^{-1}\sigma_{M,b}(t)$$

b) We consider pushforwards of $\mathfrak{Q}$ under the maps

$$B \times K \times \mathbb{R}^\times \to B \times \mathbb{R}^\times \to B \qquad \square$$

Now we assign the element

$$\mathfrak{r}[M \times K] = \int_B \mathfrak{q}_{b,K} * \mathfrak{p}_{M,b}\, d\beta(b) \ \in \mathcal{M}^\triangledown \tag{5.6}$$

to the subset $M \times K \subset A \times C$ and come to $\mathcal{M}^\triangledown$-valued measure on $A \times C$.

Lemma 5.3 a) $\mathfrak{r}$ *is a countably additive* $\mathcal{M}^\triangledown$*-valued measure on* $A \times C$.

b) *The measure* $\mathfrak{r}$ *determines a polymorphism* $A \rightsquigarrow C$.

Lemma is proved in the next subsection.

Theorem 5.4 *The product* $\mathrm{Pol}(A, B) \times \mathrm{Pol}(B, C) \to \mathrm{Pol}(A, C)$ *defined in this way is associative, i.e. for any measure spaces* A, B, C, D *and any* $\mathfrak{P} \in \mathrm{Pol}(A, B)$, $\mathfrak{Q} \in \mathrm{Pol}(B, C)$, $\mathfrak{T} \in \mathrm{Pol}(C, D)$

$$(\mathfrak{T} \circ \mathfrak{Q}) \circ \mathfrak{P} = \mathfrak{T} \circ (\mathfrak{Q} \circ \mathfrak{P})$$

Proof is in Subsection 5.10

5.7. Proof of Lemma 5.3. First, we need in more detailed information about functions $\mathfrak{p}_{M,b}$ and $\mathfrak{q}_{b,K}$ defined in Lemma 5.2.

Lemma 5.5 a) $\mathfrak{p}_{M,b} \in \mathcal{M}^{\triangledown}$ *a.s. for* $b \in B$.

b)
$$\int_B \int_{\mathbb{R}^\times} d\mathfrak{p}_{M,b}(t)\, d\beta(b) = \alpha(M). \tag{5.7}$$

c)
$$\int_{\mathbb{R}^\times} t d\mathfrak{p}_{M,b}(t) \leqslant 1 \quad \text{for almost all } b \in B. \tag{5.8}$$

and

$$\int_{\mathbb{R}^\times} t d\mathfrak{p}_{A,b}(t) = 1 \tag{5.9}$$

d) *If* $\alpha(M_j)$ *tends to 0, then*

$$\int_B \int_{\mathbb{R}^\times} t \cdot d\mathfrak{p}_{M_j,b}(t)\, d\beta(b) \to 0. \tag{5.10}$$

Proof. Statements b), c) follow from the same reasoning as Lemma 5.2. By (5.7) measures $\mathfrak{p}_{M,b}$ are finite a.s. By (5.8), they are in $\mathcal{M}^{\triangledown}$.

The projection of the measure $t \cdot \mathfrak{P}$ to A is a probability measure absolutely continuous with respect to α. The statement d) is a reformulation of this fact. □

Next, we formulate the similar lemma for the measures $\mathfrak{q}_{b,K}$.

Lemma 5.6 a) $\mathfrak{q}_{b,K} \in \mathcal{M}^{\triangledown}$ *a.s. for* $b \in B$.

b)
$$\int_B \int_{\mathbb{R}^\times} t\, d\mathfrak{q}_{b,K}(t)\, d\beta(b) = \gamma(K) \tag{5.11}$$

c)
$$\int_{\mathbb{R}^\times} d\mathfrak{q}_{b,K}(t) \leqslant 1 \quad \text{for almost all } b \in B. \tag{5.12}$$

and

$$\int_{\mathbb{R}^\times} d\mathfrak{q}_{b,C}(t) = 1 \quad \text{for almost all } b \in B. \tag{5.13}$$

d) If $\gamma(K_j) \to 0$, *then*

$$\int_B \int_{\mathbb{R}^\times} d\mathfrak{q}_{b,K_j}(t)\, d\beta(b) \to 0$$

Proof is the same.

Proof of Lemma 5.3.a. If M_1, M_2 are disjoint, then $\mathfrak{p}_{M_1,b} + \mathfrak{p}_{M_2,b} = \mathfrak{p}_{M_1 \cup M_2,b}$. By (5.6), this implies finite additivity.

To prove countable additivity consider a chain $M_1 \supset M_2 \supset \dots$ in A such that $\alpha(M_j) \to 0$:

$$\int_{\mathbb{R}^\times} d\mathfrak{r}[M_j \times K](t) = \int_B \Bigl(\int_{\mathbb{R}^\times} d\mathfrak{p}_{M_j,b}(t)\Bigr) \cdot \Bigl(\int_{\mathbb{R}^\times} d\mathfrak{q}_{b,K}(t)\Bigr)\, d\beta(b) =$$
$$= \alpha(M_j) \int_B \int_{\mathbb{R}^\times} d\mathfrak{q}_{b,K}(t)\, d\beta(b) \to 0$$

here we applied (5.7), (5.12); since the function 1 is positive, we can change the order of integration. Next,

$$\int_{\mathbb{R}^\times} t\cdot d\mathfrak{r}[M_j\times K](t)=\int_B\Bigl(\int_{\mathbb{R}^\times} t\cdot d\mathfrak{p}_{M_j,b}(t)\Bigr)\cdot\Bigl(\int_{\mathbb{R}^\times} t\cdot d\mathfrak{q}_{b,K}(t)\Bigr)\,d\beta(b)= \\ =\gamma(K)\int_B\int_{\mathbb{R}^\times} t\cdot d\mathfrak{p}_{M_j,b}(t)\,d\beta(b)\to 0,$$

here we applied (5.11), (5.10).

In the same way we show that $\mathfrak{r}[M\times K_j]\to 0$ if $\gamma(K_j)\to\infty$. □

PROOF OF LEMMA 5.3.B.

$$\int_{\mathbb{R}^\times} d\mathfrak{r}[M\times C](t)=\int_B\Bigl(\int_{\mathbb{R}^\times} d\mathfrak{p}_{M,b}(t)\Bigr)\cdot\Bigl(\int_{\mathbb{R}^\times} d\mathfrak{q}_{b,C}(t)\Bigr)\,d\beta(b)= \\ =\int_B\int_{\mathbb{R}^\times} d\mathfrak{p}_{M,b}(t)\,d\beta(b)=\alpha(M),$$

we applied (5.13) and (5.7). Next,

$$\int_{\mathbb{R}^\times} t\cdot d\mathfrak{r}[M\times C](t)=\int_B\Bigl(\int_{\mathbb{R}^\times} t\cdot d\mathfrak{p}_{A,b}(t)\Bigr)\cdot\Bigl(\int_{\mathbb{R}^\times} t\cdot d\mathfrak{q}_{b,K}(t)\Bigr)\,d\beta(b)= \\ =\int_B\int_{\mathbb{R}^\times} t\cdot d\mathfrak{q}_{b,K}(t)\,d\beta(b)=\gamma(K),$$

we applied (5.9) and (5.11). □

5.8. Involution. Let $\mathfrak{P}\in\mathrm{Pol}(A,B)$. We define $\mathfrak{P}^\star\in\mathrm{Pol}(B,A)$ as the measure $t^{-1}\mathfrak{P}(a,b,t^{-1})$ regarded as a measure on $B\times A\times\mathbb{R}^\times$.

Lemma 5.7 *For* $\mathfrak{P}\in\mathrm{Pol}(A,B)$*,* $\mathfrak{Q}\in\mathrm{Pol}(B,C)$*, we have*

$$(\mathfrak{Q}\circ\mathfrak{P})^\star=\mathfrak{P}^\star\circ\mathfrak{Q}^\star.$$

PROOF. Multiplying $\mathfrak{P}^\star\circ\mathfrak{Q}^\star$, we get in (5.6) the expression

$$\int_B (t^{-1}\mathfrak{q}_{b,K})*(t^{-1}\mathfrak{p}_{M,b})\,d\beta(b)=t^{-1}\mathfrak{r}[M\times K](t^{-1}).$$

5.9. Convergence. Let us regard polymorphisms $A\rightsquigarrow B$ as $\mathcal{M}^\triangledown$-valued measures on $A\times B$ as above. Let $\mathfrak{P}_j$, $\mathfrak{P}\in\mathrm{Pol}(A,B)$. We say that the sequence $\mathfrak{P}_j$ converges to $\mathfrak{P}$ if for any measurable $M\subset A$, $N\subset B$ we have convergence

$$\mathfrak{p}_j(M\times N)\to\mathfrak{p}(M\times N)$$

in the sense of $\mathcal{M}^\triangledown$.

Theorem 5.8 *The* $\circ$*-product is separately continuous.*

PROOF. We keep the notation of Subsection 5.6. Since we have an involution, it suffices to prove the one-side continuity. Let $\mathfrak{P}^{(j)} \to \mathfrak{P}$. The the functions $\mathfrak{p}_{M,b}$ satisfy the condition

$$\int_N \mathfrak{p}^{(j)}_{M,b}\, d\beta(b) \quad \text{converges to} \quad \int_N \mathfrak{p}_{M,b}\, d\beta(b) \quad \text{in } \mathcal{M}^\triangledown. \tag{5.14}$$

Also

$$\int_B \mathfrak{p}^{(j)}_{M,b}\, d\beta(b) \leqslant \mathfrak{p}[A \times B].$$

We wish to show that

$$\mathfrak{r}^{(j)}[M \times K] \to \mathfrak{r}[M \times K] \quad \text{in } \mathcal{M}^\triangledown.$$

It suffices to verify pointwise convergence of Mellin transforms on lines $u = iw$, $u = 1 + iw$. We have

$$\int_{\mathbb{R}^\times} t^{iw} d\bigl(\mathfrak{r}^{(j)}[M \times K] - \mathfrak{r}[M \times K]\bigr)\, d\beta(b) =$$
$$= \int_B \Bigl[\int_{\mathbb{R}^\times} t^{iw} d\mathfrak{p}^{(j)}_{M,b}(t) - \int_{\mathbb{R}^\times} t^{iw} d\mathfrak{p}_{M,b}(t)\Bigr] \cdot \Bigl\{\int_{\mathbb{R}^\times} t^{iw} d\mathfrak{q}_{b,K}(t)\Bigr\}\, d\beta(b). \tag{5.15}$$

The factor $\{F(b)\}$ in the curly brackets is a bounded function, see (5.12). The factor $[G^{(j)}(b) - G(b)]$ in the square brackets is contained in $L^1(B)$ and its L^1-norm is uniformly bounded by $2\alpha(M)$ (by (5.7)). The convergence (5.14) implies weak convergence $G^{(j)} \to G$ (see criterion in [19]), thus the sequence (5.15) converges to 0.

$$\int_{\mathbb{R}^\times} t^{1+iw} d\bigl(\mathfrak{r}^{(j)}[M \times K] - \mathfrak{r}[M \times K]\bigr)\, d\beta(b) =$$
$$= \int_B \Bigl[\int_{\mathbb{R}^\times} t^{1+iw} d\mathfrak{p}^{(j)}_{M,b}(t) - \int_{\mathbb{R}^\times} t^{1+iw} d\mathfrak{p}_{M,b}(t)\Bigr] \cdot \Bigl\{\int_{\mathbb{R}^\times} t^{1+iw} d\mathfrak{q}_{b,K}(t)\Bigr\}\, d\beta(b). \tag{5.16}$$

Now the factor in the curly brackets is contained in $L^1(B)$ by (5.11), the factor in the square brackets is $\leqslant 2$ by (5.8), i.e., it is contained in a ball in $L^\infty(B)$. Also the sequence in curly brackets converges to 0 weakly in L^∞. Therefore, we get the desired convergence of (5.16) to 0. □

5.10. Proof of Theorem 5.4. Associativity of the product. The set of absolutely continuous polymorphisms $A :\rightsquigarrow B$ is dense in $\mathrm{Pol}(A, B)$. Evidently, the product of absolutely continuous kernels is associative. On the other hand the product of polymorphisms is separately continuous.

5.11. Definition of the product in terms of finite approximations. Return to the notation of Subsection 3.9. For a countable partition X of A we can define the morphisms $\mathfrak{l}[A; \mathsf{X}] : A/\mathsf{X} \rightsquigarrow A$, $\mathfrak{m}[A; \mathsf{X}] : A \rightsquigarrow A/\mathsf{X}$, $\mathfrak{t}[A; \mathsf{X}] : A \to A$ as above (since we have canonical embeddings $\mathrm{Mar}(A, B) \to \mathrm{Pol}(A, B)$).

Let $\mathsf{X} : A = \cup X_i$, $\mathsf{Y} : B = \cup Y_j$ be countable partitions.

Lemma 5.9 a) *For $\mathfrak{P} : A \rightsquigarrow B$ the morphism*

$$\mathfrak{m}[A;\mathsf{Y}] \circ \mathfrak{P} \circ \mathfrak{l}[A;\mathsf{X}] : \ A/\mathsf{X} \rightsquigarrow B/\mathsf{Y}$$

is determined by the $\mathcal{M}^{\triangledown}$-valued matrix $\mathfrak{p}_{ij} = \mathfrak{p}[X_i \times Y_j]$.

b) *The measure*

$$\mathfrak{t}[A;\mathsf{Y}] \circ \mathfrak{P} \circ \mathfrak{t}[A;\mathsf{X}] : \ A \rightsquigarrow B$$

is determined by the following rule: its restriction to $X_i \times Y_j \times \mathbb{R}^\times \subset A \times B \times \mathbb{R}^\times$ is

$$\frac{1}{\alpha(X_i)\beta(Y_j)} \cdot \alpha \times \beta \times \mathfrak{p}_{ij}$$

This can be verified in a straightforward way. In any case the statement follows from Theorem 6.7 proved below.

For measure spaces A, B, C consider approximating sequences of countable partitions $\mathsf{X}^{(i)}$, $\mathsf{Y}^{(j)}$, $\mathsf{Z}^{(k)}$.

Proposition 5.10 *For any $\mathfrak{P} : A \rightsquigarrow B$, $\mathfrak{Q} : B \rightsquigarrow C$ their product is given by*

$$\mathfrak{Q} \circ \mathfrak{P} = \lim_{i,\,j,\,k\to\infty} \mathfrak{t}[C;\mathsf{Z}^{(k)}] \circ \mathfrak{Q} \circ \mathfrak{t}[B;\mathsf{Y}^{(l)}] \circ \mathfrak{P} \circ \mathfrak{t}[A;\mathsf{X}^{(i)}] =$$
$$= \lim_{i,\,j,\,k\to\infty} \mathfrak{l}[C;\mathsf{Z}^{(k)}] \circ \Bigl(\mathfrak{m}[C;\mathsf{Z}^{(k)}] \circ \mathfrak{Q} \circ \mathfrak{l}[B;\mathsf{Y}^{(l)}]\Bigr) \circ$$
$$\circ \Bigl(\mathfrak{m}[B;\mathsf{Y}^{(l)}] \circ \mathfrak{P} \circ \mathfrak{l}[A;\mathsf{X}^{(i)}]\Bigr) \circ \mathfrak{m}[A;\mathsf{X}^{(i)}]. \quad (5.17)$$

In big brackets we have polymorphisms of countable sets and their products can be evaluated as above (5.3). Now we get a sequence of measures weakly convergent to $\mathfrak{Q} \circ \mathfrak{P}$.

Proof of the proposition is given in Subsection 6.12.

REMARK. Notice that a reference to a separate continuity of the product allows to claim that $\mathfrak{Q} \circ \mathfrak{P}$ coincides with the iterated limit

$$\mathfrak{Q} \circ \mathfrak{P} = \lim_{i\to\infty} \lim_{j\to\infty} \lim_{k\to\infty} \bigl(\dots\bigr).$$

But we have a triple limit in (5.17). □

5.12. Group Gms(A)**.** Let A be a space with continuous measure. For $g \in \mathrm{Gms}(A)$ consider the map $\mathfrak{I}_g : A \to A \times A \times \mathbb{R}^\times$ given by

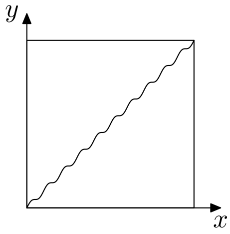
$$a \mapsto \bigl(a, g(a), g'(a)\bigr).$$

Denote by $\mathfrak{I}[g]$ the pushforfard of the measure α under this map.

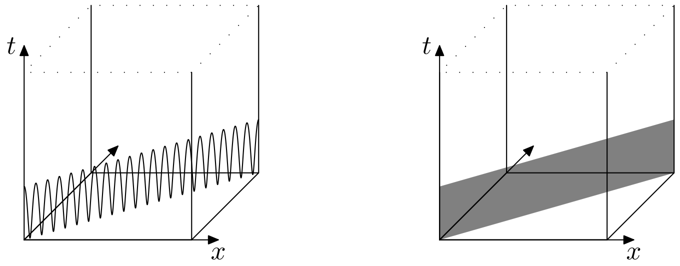
Proposition 5.11 a) $\mathfrak{I}[g] \in \mathrm{Pol}(A, A)$

b) *The map $g \mapsto \mathfrak{I}(g)$ is a homomorphism.*

This is obvious.



A map $y = x + \frac{1}{n}\sin nx$ of the segment $[0, 2\pi]$ to itself.



The image of the measure in $[0, 2\pi] \times [0, 2\pi] \times \mathbb{R}^\times$ is supported by a oblate helical line. The limit as $n \to \infty$ is a (non-uniform) measure supported by the rectangle $x = y$, $0 < t \leqslant 2$.

Figure 3: Reference to Subsection 5.12

Theorem 5.12 *Let A be a space with continuous measure. Then the group* $\mathrm{Gms}(A)$ *is dense in* $\mathrm{Pol}(A,A)$.

Proof. Fix $\mathfrak{P}\in\mathrm{Pol}(A,A)$. Consider a finite partition $\mathsf{X}: A=\cup X_j$ of the space A. Denote $\mathfrak{p}_{ij}=\mathfrak{p}(X_i\times X_j)$. Consider a subpartition $X_i=\cup Y_{ij}$ of each A_i such that

$$\alpha(Y_{ij})=\int_{\mathbb{R}^\times} d\mathfrak{p}_{ij}(t)$$

Consider another subpartition of each $X_i=\cup Z_{ij}$ such that

$$\alpha(Z_{ij})=\int_{\mathbb{R}^\times} t\,d\mathfrak{p}_{ij}(t).$$

For each pair (i,j) consider a map $Y_{ij}\to Z_{ij}$, whose Radon–Nikodym derivative is distributed as $\mathfrak{p}_{ij}$. Joining maps $X_{ij}\to Y_{ij}$ we get a (non-canonical) element $g[\mathsf{X}]$ of $\mathrm{Gms}(A)$.

Now we consider an approximating sequence of partitions $\mathsf{X}^{[p]}$ and come to a sequence $g[\mathsf{X}^{[p]}]\in\mathrm{Gms}(A)$, which converges to $\mathfrak{P}$. □

5.13. Dilatations. Fix a space E with continuous probability measure. Any Lebesgue space A with probability measure can be represented as a quotient space $A=E/\mathsf{U}$, we simply consider the product $A\times[0,1]$ and identify it with E.

Theorem 5.13 *Let A, B be spaces with probability measures, $A=E/\mathsf{U}$, $B=E/\mathsf{V}$, where E is a space with continuous measure. For any $\mathfrak{P}\in\mathrm{Pol}(A,B)$ there is $g\in\mathrm{Gms}(A)$ such that*

$$\mathfrak{P}=\mathfrak{m}[E;\mathsf{U}]\circ g\circ\mathfrak{l}[E;\mathsf{V}].$$

Proof. First, without loss of generality we can assume that the measures on A and B are continuous. Otherwise consider spaces $A'=A\times[0,1]$, $B'=B\times[0,1]$ and

$$\mathfrak{P}'=\mathfrak{P}\times[0,1]\times[0,1]\ \subset\ (A\times[0,1])\times(B\times[0,1])\times\mathbb{R}^\star.$$

Consider the map $A\times B\times\mathbb{R}^\times\to A$, which can be regarded as polymorphism $\mathfrak{m}[...]:\mathfrak{P}\rightsquigarrow A$ (the product $\times B\times\mathbb{R}^\times$ is equipped with the measure $\mathfrak{P}$). Consider also the space $A\times B\times\mathbb{R}^\times$ equipped with the measure the map $t\cdot\mathfrak{P}\to B$, which can be regarded as a polymorphism $B\rightsquigarrow\mathfrak{P}$.

Thus we have an identical map $A\times B\times\mathbb{R}^\times\ \to\ A\times B\times\mathbb{R}^\times$ and $\mathfrak{P}:A\rightsquigarrow B$ is represented as $\mathfrak{m}[\dots]\circ 1\circ\mathfrak{l}[\dots]$. □

6 Markov–Mellin transform

6.1. Markov-Mellin transform. Let $u=v+iw$ range in the strip

$$\Pi:\ 0\leqslant v\leqslant 1 \qquad -\infty<w<\infty \tag{6.1}$$

Denote $p = 1/(1-v)$, $q = 1/v$. For a polymorphism $\mathfrak{P} \in \operatorname{Pol}(A, B)$ consider the following bilinear form on $L^{1/(1-v)}(A) \times L^{1/v}(B)$

$$S_u(\mathfrak{P}; f, g) = S_{v+iw}(\mathfrak{P}; f, g) = \iiint_{A\times B\times\mathbb{R}^\times} f(a) g(b) t^{v+iw}\, d\mathfrak{P}(a, b, t) \qquad (6.2)$$

Lemma 6.1

$$|S_{v+iw}(\mathfrak{P}; f, g)| \leqslant \|f\|_{1/(1-v)} \cdot \|g\|_{1/v}.$$

Lemma 6.2 *For fixed* f, $g \in L^\infty$ *the function* $u \mapsto S_u(\mathfrak{P}; f, g)$ *is continuous and positive definite in the strip* Π *and holomorphic in the open strip.*

Lemma 6.3 $S_u(\mathfrak{P}^\bigstar; f, g) = S_{1-u}(\mathfrak{P}; f, g)$.

The last statement is obtained by a substitution $t \mapsto t^{-1}$ to (6.2).

As an immediate corollary of Lemma 6.1, we get the following theorem.

Theorem 6.4 a) *For each* $u = v + iw \in \Pi$ *there exists a linear operator*

$$T_u(\mathfrak{P}) : L^p(B) \to L^p(A), \qquad \textit{where } p = \frac{1}{1-v}$$

defined by

$$\int_A \bigl(T_u(\mathfrak{P}) g\bigr)(a)\, f(a)\, d\alpha(a) = S_u(\mathfrak{P}; f, g). \qquad (6.3)$$

b)
$$\|T_u(\mathfrak{P})\|_{L^p} \leqslant 1. \qquad (6.4)$$

We say that the map $u \mapsto T_u(\mathfrak{P})$ is the *Markov–Mellin transform* of $\mathfrak{P}$, it is a holomorphic operator-valued function in the strip Π.

6.2. Direct definition of Markov–Mellin transform. First, we reformulate the definition of polymorphism. Fix a polymorphism $\mathfrak{P} : A \rightsquigarrow B$. Consider the map $A \times B \times \mathbb{R} \to A$. For each $a \in A$ we consider the conditional (probability) measure $\mathfrak{P}_a(b, t)$ on $B \times \mathbb{R}^\times$. Next, consider the map $B \times \mathbb{R}^\times \to B$. Denote the pushforward by $\mathfrak{P}_a(b)$. By $\mathfrak{P}_{a,b}(t)$ we denote the conditional measures on the fibers. We get

$$\iiint\limits_{A\times B\times\mathbb{R}^\times} F(a, b, t)\, d\mathfrak{P}(a, b, t) = \int\limits_A \biggl(\int\limits_B \biggl(\int\limits_{\mathbb{R}^\times} F(a, b, t)\, d\mathfrak{P}_{a,b}(t) \biggr) d\mathfrak{P}_a(b) \biggr) d\alpha(a). \qquad (6.5)$$

Thus we can define a polymorphism in the terms of two systems of measures $\mathfrak{P}_a(b)$, $\mathfrak{P}_{a,b}(t)$. These measures are probabilistic and satisfy the integral identity corresponding to the condition 2° for polymorphisms (see Subsection 5.1):

$$\int\limits_A \biggl(\int\limits_B \biggl(\int\limits_{\mathbb{R}^\times} t\, g(b)\, d\mathfrak{P}_{a,b}(t) \biggr) d\mathfrak{P}_a(b) \biggr) d\alpha(a) = \int_B g(b)\, d\beta(b). \qquad (6.6)$$

This holds for all $g \in L^1(B)$. The identity also can be written as

$$\int\limits_A \Bigl[\Bigl(\int\limits_{\mathbb{R}^\times} t\, d\mathfrak{P}_{a,b}(t)\Bigr) \cdot \mathfrak{P}_a(b)\Bigr]\, d\alpha(a) = \beta(b). \tag{6.7}$$

In square brackets we have a product of an integrable function and a measure.

Theorem 6.5 *For $\mathfrak{P} : A \rightsquigarrow B$ and $g \in L^1(B)$ we have*

$$T_u(\mathfrak{P})g(a) = \int_B \int_{\mathbb{R}^\times} t^u g(b)\, d\mathfrak{P}_{a,b}(t)\, d\mathfrak{P}_a(b).$$

For absolutely continuous kernels the formula is more transparent. Let $p : A \times B : \mathcal{M}^\nabla$ be the same function as in Subsection 5.5.

Proposition 6.6

$$T_u g(a) = \int_B \int_{\mathbb{R}^\times} t^u g(b)\, dp(a,b)(t)\, d\beta(b).$$

6.3. Some properties of Markov–Mellin transform.

Theorem 6.7 a) *For any $\mathfrak{P} \in \mathrm{Pol}(A,B)$, $\mathfrak{Q} \in \mathrm{Pol}(B,C)$, we have*

$$T_u(\mathfrak{P})T_u(\mathfrak{Q}) = T_u(\mathfrak{Q} \circ \mathfrak{P}). \tag{6.8}$$

b) *The operator $T_u(\mathfrak{P}^\bigstar)$ is dual to $T_{1-u}(\mathfrak{P})$.*

The following statement is obvious.

Proposition 6.8 a) *For $\mathfrak{P} \in \mathrm{Mar}(A,B)$ the operators $T_u(\mathfrak{P})$ coincide with the Markov operators defined in Subsection* 3.8.

b) *For $g \in \mathrm{Gms}(A)$ these operators coincide with operators $T_u(g)$ defined by* (2.1).

c) *For single-point spaces T_u coincides with the characteristic function $\Phi(u)$ discussed in Section* 4.

6.4. Characterization of the image and the inverse construction. For a measurable subset $M \subset A$ denote by I_M the identifier function, $I_M(a) = 1$ if $a \in M$, $I_M(a) = 0$ if $a \notin M$.

Theorem 6.9 a) *Let $u \mapsto T_u$ be a function on the strip Π taking values in the space of bounded operators*[7]*, $L^\infty(A) \to L^1(B)$ such that*

i) *for fixed $f \in L^\infty(A)$, $g \in L^\infty(B)$ the matrix elements*

$$u \mapsto \varphi_{f,g}(u) = \int_B T_u f(b)\, g(b)\, d\beta(b)$$

---

[7] Recall that we use a non-standard convergenve in L^∞.

are continuous and bounded for $u \in \Pi$ and holomorphic in the open strip;

ii) *for nonnegative f, g the functions $\varphi_{f,g}(u)$ are positive definite in Π;*

iii) $\varphi_0(1,1) = 1$, $\varphi_1(1,1) = 1$.

Then $T_u = T_u(\mathfrak{P})$ for a unique $\mathfrak{P} \in \mathrm{Pol}(A, B)$.

b) *The polymorphism $\mathfrak{P}$ is determined by the condition:*

$$\int_{\mathbb{R}^\times} t^u \, d\mathfrak{p}[M \times N](a, b, t) = \varphi_{I_M, I_N}(u)$$

for all measurable $M \subset A$, $B \subset B$.

6.5. Convergence.

Theorem 6.10 a) *If $\mathfrak{P}_j$ converges to $\mathfrak{P}$, then for each $u \in \Pi$ the operators $T_u(\mathfrak{P}_j)$ weakly converge*[8] *to $T_u(\mathfrak{P})$.*

b) *Let $\mathfrak{P}_j$, $\mathfrak{P} \in \mathrm{Pol}(A, B)$. Let $T_u(\mathfrak{P}_j)$ converges weakly to $T_u(\mathfrak{P})$ for each $u \in \Pi$. Then $\mathfrak{P}_j$ converges to $\mathfrak{P}$. It suffices to require the weak convergence on the lines $u = iw$ and $u = 1 + iw$.*

6.6. Proof of Lemma 6.1 (inequality for bilinear form). To obtain the assertion a) we apply the Hölder inequality (see [9], Sect. 9.3) and the definition of polymorphisms

$$\begin{aligned}|S_{v+iw}(f,g)| \leqslant \Bigl(\iiint_{A\times B\times\mathbb{R}^\times} |f(a)|^{1/(1-v)} \, d\mathfrak{P}(a,b,t)\Bigr)^{1-v} \times \\ \times \Bigl(\iiint_{A\times B\times\mathbb{R}^\times} |g(b)t^{v+iw}|^{1/v} \, d\mathfrak{P}(a,b,t)\Bigr)^{v} = \\ = \Bigl(\int_A |f(a)|^{1/(1-v)} \, d\alpha(a)\Bigr)^{1-v} \cdot \Bigl(\int_B |g(b)|^{1/v} \, d\beta(b)\Bigr)^{v}\end{aligned}$$

6.7. Proof of Lemma 6.2 (properties of matrix elements). First, we formulate it in a more precise form.

Lemma 6.11 a) *For fixed $f \in L^p(A)$, $g \in L^r(B)$, the function $S_u(\mathfrak{P}, f, g)$ is continuous in the strip*

$$\frac{1}{r} \leqslant \mathrm{Re}\, u \leqslant 1 - \frac{1}{p} \qquad (6.9)$$

and holomorphic in the corresponding open strip.

b) *If $f \in L^p(A)$, $g \in L^r(B)$ are non-negative, then the function $\Psi(u) = S_u(\mathfrak{P}, f, g)$ is positive definite in the strip* (6.9).

---

[8] For a definition and discussion of weak and strong operator convergences, see, e.g., [19], Section VI.1

c) *If* $\mathfrak{P}_j \in \operatorname{Pol}(A, B)$ *converges to* $\mathfrak{P}$*, then for any* $f \in L^p(A)$*,* $g \in L^r(B)$*, and* u *being in the strip* (6.9)*, we have*

$$S_u(\mathfrak{P}_j; f, g) \to S_u(\mathfrak{P}; f, g).$$

For fixed f*,* g *the convergence is uniform in each rectangle* $0 \leqslant v \leqslant 1$*,* $-A \leqslant w \leqslant A$*.*

Proof. a) follows from Lemma 6.1. Positive definiteness is clear. Now we apply Proposition 4.2 and get the following corollary:

If $f \in L^p(A)$*,* $g \in L^q(B)$ *are non-negative, then the pushforward of the measure* $f(a)g(b)\mathfrak{P}(a, b, t)$ *under the map* $A \times B \times \mathbb{R}^\times \to \mathbb{R}^\times$ *is contained in the semiring* $\mathcal{M}^\triangledown_{1/p,1-1/q}$*.*

Next, we apply Proposition 4.4 and get c) for positive f and g. But any function is a linear combination of positive functions. ☐

6.8. Proof of Theorem 6.5 (integral formula for operators).

$$\int_A T_u(\mathfrak{P})g(a)\, f(a)\, \alpha(a) = \int_A \int_B \int_{\mathbb{R}^\times} t^u f(a)\, g(b)\, d\mathfrak{P}_{a,b}(t)\, d\mathfrak{P}_a(b)\, \alpha(a).$$

By the definition of our conditional measures (see (6.5)) we have

$$d\mathfrak{P}_{a,b}(t)\, d\mathfrak{P}_a(b)\, \alpha(a) = d\mathfrak{P}(a, b, t)$$

and we get $S_u(f, g)$. ☐

6.9. Proof of Theorem 6.9 (inversion). Let $M \subset A$, $N \subset B$ be measurable sets. The function $\varphi_{I_M, I_N}(u)$ is positive definite and bounded in the strip Π and therefore it is a characteristic function of a certain measure $\mathfrak{p} := \mathfrak{p}[M \times N]$.

Obviously, for disjoint sets M_1, M_2, we have

$$\mathfrak{p}\big[(M_1 \cup M_2) \times N\big] = \mathfrak{p}[M_1 \times N] + \mathfrak{p}[M_2 \times N].$$

Next, let measurable subsets $M_1, M_2, \dots \subset A$ be pairwise disjoint. Then $I_{\cup M_j} = \sum I_{M_j}$ in the topology of L^∞. Therefore the sequence $\varphi_{I_{M_1 \cup \dots \cup M_j}, I_N}(u)$ converges to $\varphi_{I_{\cup_j M_j}, I_N}(u)$ pointwise. By Proposition 4.5 we get

$$\sum_j \mathfrak{p}[M_j \times N] = \mathfrak{p}\big[(\cup M_j) \times N\big].$$

Thus our $\mathcal{M}^\triangledown$-valued measure on $A \times B$ is countably additive.

By the condition iii) this measure is a polymorphism.

6.10. Proof of Theorem 6.10 (convergence). The statement a) is contained in Lemma 6.11.c.

To prove b) we continue considerations of the previous subsection. For a polymorphism $\mathfrak{P}$ the function $u \mapsto S_u(\mathfrak{P}; I_M, I_N)$ is the Mellin transform of the measure $\mathfrak{p}[M \times N] \in \mathcal{M}^\triangledown$. Pointwise convergence of Mellin transforms implies convergence of measures (see Proposition 4.4), i.e. convergence $S_u(\mathfrak{P}_j; I_M, I_N)$ to $S_u(\mathfrak{P}; I_M, I_N)$ implies convergence $\mathfrak{p}_j[M \times N] \to \mathfrak{p}[M \times N]$. $\square$

6.11. Proof of Theorem 6.7. The statement is obvious for absolutely continuous kernels. By separate continuity of the product of polymorphisms and separate weak continuity of the product of operators the statement is valid always.

6.12. Proof of Proposition 5.10. It is easy to show that

$$T_u(\mathfrak{t}[A, \mathsf{X}]) = I[A; \mathsf{X}].$$

If a sequence $\mathsf{X}^{(i)}$ is approximating then $I[A; \mathsf{X}^{(i)}]$ strongly converges to 1. A product of strongly convergent sequences strongly converges. Therefore the sequence

$$\begin{aligned} T_u\Big(\mathfrak{t}[C; \mathsf{Z}^{(k)}] \circ \mathfrak{Q} \circ \mathfrak{t}[B; \mathsf{Y}^{(l)}] \circ \mathfrak{P} \circ \mathfrak{t}[A; \mathsf{X}^{(i)}]\Big) = \\ = I[C; \mathsf{Z}^{(k)}]\; T_u(\mathfrak{Q})\; I[B; \mathsf{Y}^{(l)}]\; T_u(\mathfrak{P})\; I[A; \mathsf{X}^{(i)}] \end{aligned}$$

converges to

$$T_u(\mathfrak{Q}) T_u(\mathfrak{P}) = T_u(\mathfrak{Q} \circ \mathfrak{P})$$

as i, j, $k \to \infty$. Therefore $\mathfrak{t}[C; \mathsf{Z}^{(k)}] \circ \mathfrak{Q} \circ \mathfrak{t}[B; \mathsf{Y}^{(l)}] \circ \mathfrak{P} \circ \mathfrak{t}[A; \mathsf{X}^{(i)}]$ converges to $\mathfrak{Q} \circ \mathfrak{P}$. $\square$

For general case, we calculate the product using finite approximations. We start from a partition X of A containing a part M and partition Y of C containing a part K. Then for all the approximations we get the same result for $\mathfrak{r}[M \times K]$.

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Math.Dept., University of Vienna,
Nordbergstrasse, 15, Vienna, Austria
&
Institute for Theoretical and Experimental Physics,
Bolshaya Cheremushkinskaya, 25, Moscow 117259, Russia
&
MechMath. Dept., Moscow State University,
Vorob'evy Gory, Moscow
e-mail: neretin(at) mccme.ru
URL:www.mat.univie.ac.at/∼neretin
wwwth.itep.ru/∼neretin