ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Pasteurgasse 6/7
A-1090 Wien, Austria

Canonical Approach to the Quantum WZNW Model

Paolo Furlan
Ludmil K. Hadjiivanov
Ivan T. Todorov

Vienna, Preprint ESI 234 (1995) June 22, 1995

Supported by Federal Ministry of Science and Research, Austria
Available via http://www.esi.ac.at

IC/95/74

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

CANONICAL APPROACH TO THE QUANTUM WZNW MODEL

Paolo Furlan
Dipartimento di Fisica Teorica dell'Universitá di Trieste, Trieste, Italy
and
Istituto Nazionale di Fisica Nucleare (I.N.F.N.), Sezione di Trieste, Italy,

Ludmil K. Hadjiivanov
Istituto Nazionale di Fisica Nucleare (I.N.F.N.), Sezione di Trieste, Italy,
and
Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences,
BG–1784 Sofia, Bulgaria

and

Ivan T. Todorov[1]
International Centre for Theoretical Physics, Trieste, Italy
and
Erwin Schrödinger International Institute for Mathematical Physics,
A–1090 Vienna, Austria.

MIRAMARE – TRIESTE

May 1995

[1]On leave of absence from: Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, BG–1784 Sofia, Bulgaria.

ABSTRACT

The canonical approach to the chiral $SU(n)$ WZNW model with a monodromy independent r–matrix is reviewed. Taking the quantum group symmetry of the model (which reflects its classical Poisson–Lie symmetry) as a guiding principle, we derive a complete set of exchange relations in the enlarged chiral phase space that includes the Borel components $M_\pm$ of the monodromy matrix. Regarded as new dynamical variables the elements of M in the left and right sectors cannot be identified: their Poisson brackets have opposite signs. This is a technical reason why the canonical reduction of the pair of chiral models to a single 2–dimensional theory that involves the left and right movers' fields does not respect the quantum group symmetry. A simple modification of the Poisson brackets which does lead to an $SL_q(n)$ invariant model is proven unacceptable as a substitute for the 2D theory. As a way out we suggest a weak form of the monodromy and quantum group invariance of the extended 2D theory (involving mean values in a physical subspace of the tensor product of chiral state spaces).

1 Introduction

It took some five years after the multivalued action for the Wess–Zumino–Novikov–Witten (WZNW) model was written down [1] (during which its conformal current algebra treatment was developed [2, 3, 4]) before a canonical r–matrix approach to the theory was undertaken [5, 6, 7]. It aimed to understand the classical origin of (just discovered [8–12]) quantum group symmetries of rational conformal field theories (for a review and further references – see [13]). This development, reaching a peak in the early 1990ies [14–19], still continues to attract attention [20, 21].

The basic field of the theory is a map $g(= g(t,x))$ from the cylinder ($t \in \mathbb{R}$, $x \in \mathbb{R}/2\pi\mathbb{Z}$) to a simple compact Lie group G. To make the discussion concrete we shall work out in this paper the case $G = SU(n)$. It was demonstrated in [14] (with some details of the proof still unpublished, [18]) that the 2–dimensional (2D) symplectic form $\Omega^{(2)}$ of the classical WZNW theory can be written as a sum of two (symplectic) chiral parts, Ω and $\bar\Omega$, corresponding to the splitting[2]

$$g = u\bar{u} \ , \quad u = u(x-t) \ , \quad \bar{u} = \bar{u}(x+t) \ . \tag{1.1}$$

This is achieved by extending the phase space of u (and the one of $\bar u$) by the monodromy degrees of freedom, where the monodromy M is defined by the twisted periodicity condition

$$u(x+2\pi) = u(x)M, \quad \bar{u}(x+2\pi) = M^{-1}\bar{u}(x), \quad M \in G \ . \tag{1.2}$$

(We are working throughout the paper with equal time relations and will be skipping the time argument of u and $\bar u$.) In effect, Ω is a (closed) 2–form in the space of triples $(u(x), M_+, M_-)$ where $M_+ (M_-)$ is an upper (lower) triangular matrix in $SL(n,\mathbb{C})$ (for $G = SU(n)$) such that the diagonal elements of M_+ and M_-^{-1} coincide and

$$M = M_+ M_-^{-1} \ \ (diag\ M_+ \cdot diag\ M_- = \mathbb{1}) \ . \tag{1.3}$$

Furthermore, Ω (and $\bar\Omega$) can be chosen in such a way that the Poisson brackets (PB) of a pair of u's ($\bar u$'s) are monodromy independent. Such PB correspond (as anticipated in [6] and argued in [17]) to simple exchange relations for the quantized chiral fields that are manifestly quantum group symmetric. They are to be contrasted with the rather complicated exchange relations for u and $\bar u$ derived recently [21] from different principles.

[2]We are using throughout the space–like Lorentzian metric, $\eta_{11} = 1 = -\eta_{00}$, resulting, in particular, in interchanging the equations $(\partial_t + \partial_x)u = 0 = (\partial_t - \partial_x)\bar u$ for u and $\bar u$ as compared to [14].

It is usually presumed (and often stated) that the quantum group shifts in the left and right sectors have the form

$$u \to uT^{-1}, \quad M \to TMT^{-1}, \quad T \in SL_q(n) \tag{1.4}$$

and $\bar{u} \to T\bar{u}$ so that the 2D field (1.1) is $SL_q(n)$ invariant. This is, however, inconsistent with the canonical quantization scheme in which u and $\bar{u}$ commute[3]. The quantum groups in the two sectors turn out to be different; already the classical PB of the u and $\bar{u}$ monodromies have opposite signs. We indicate the flow of a simple minded attempt to cure the difficulty and suggest an alternative way out that mimics the indefinite metric space approach to a local gauge theory.

The paper is organized as follows.

We start in Section 2 with a concise review of classical PB for the chiral WZNW model making public some results of [18]. We point out on the way that the left and right sectors only decouple if we double the monodromy degrees of freedom regarding the monodromy matrices M and $\bar{M}$ in Ω and $\bar{\Omega}$ as independent. Section 3 is devoted to a derivation of the quantum exchange relations by expressing them in terms of the pair of (fundamental representations of) quantum group $R^{\pm}$ matrices. Respecting the symmetry of the classical theory reduces the ambiguity in quantization to a single deformation parameter q which is determined by the ratio of eigenvalues of the braid operator. In Section 4 we analyse the problem of combining the left and right sector in a 2D quantum theory.

2 Canonical formalism and chiral PB

The information contained in an action integral A in D space–time dimensions can be decoded from an associated closed $D+1$ form (for a general discussion – see [14], Sec.2). We shall review the realization of this approach in the WZNW model where the presence of a multivalued Wess–Zumino term in the 2D action A makes it particularly appropriate. The multivaluedness of A is translated into a cohomological property: the canonical 3–form

$$\omega \;=\; \frac{1}{4\pi}\,\mathrm{tr}\left\{ d\,[\,\frac{1}{2k}\,j_\alpha j^\alpha dt dx + i(j^1\,dt - j^0\,dx)\,g^{-1}\,dg\,] - \frac{k}{3}\,(g^{-1}dg)^3 \right\} \tag{2.1}$$

is closed (one readily checks that $d\omega = 0$) but not exact. The "coupling constant" k (to be identified with the level of the affine Kac–Moody algebra generated by the chiral components of the current j) has to be a positive integer in order to ensure the single valuedness of the field theoretic "partition function" e^{iA} [1].

[3]The contradiction was pointed out in a 1995 Schladming lecture by Anton Alekseev.

Rather than recalling the path from the WZNW action integral to the form ω we shall justify (2.1) by deriving from it the equations of motion. There are two ways to do that yielding the same answer. The canonical variational approach consists in contracting ω with a vertical vector field, $\partial/\partial j^\alpha$ or $\mathrm{tr}(gX\partial/\partial g), X \in \mathcal{G}$, and then pulling back the resulting 2–form to the base space setting $s^*(g^{-1}dg) = g^{-1}\partial_\alpha g\; dx^\alpha \equiv g^{-1}\partial_t g dt + g^{-1}\partial g\; dx$ etc. We thus obtain the equations of motion in the first order formalism:

$$\begin{aligned} s^*(\omega\lfloor\, \partial/\partial j^\alpha) &= \left(\frac{1}{k}\; j_\alpha + ig^{-1}\partial_\alpha g\right) dt\;\, dx = 0 \\ &\Rightarrow j_\alpha = -ikg^{-1}\partial_\alpha g \qquad\qquad (2.2a) \\ s^*(\omega\lfloor \mathrm{tr}(gX\;\partial/\partial g)) &= -\mathrm{tr}\left\{\left(k[g^{-1}\partial_0 g, g^{-1}\partial_1 g] + i\partial_\alpha j^\alpha\right) X\right\} dt\;\, dx = 0 \\ &\Rightarrow \partial_\alpha j^\alpha = \frac{1}{ik}\;[j_0, j_1]\;. \qquad\qquad (2.2b) \end{aligned}$$

(In deriving (2.2b) we have used the relation $g^{-1}dg\lfloor\mathrm{tr}(gX\;\partial/\partial g) = X$.) The second way requires regarding the forms dj^α and $g^{-1}dg$ as having both horizontal and vertical components so that, e.g., $-\frac{1}{3}\;\mathrm{tr}(g^{-1}dg)^3\lfloor\;\partial_0\wedge\partial_1 = \mathrm{tr}([g^{-1}\partial_0 g, g^{-1}\partial_1 g]g^{-1}dg)$, and setting the contraction of ω with the bivector field $\partial_0\wedge\partial_1$ equal to zero (then the coefficient to dj^α gives (2.2a), the coefficient to $g^{-1}dg$ reproduces (2.2b)). Taking the curl of (2.2a) and combining the result with (2.2b) we arrive at the left invariant chiral current

$$\partial_0 j_1 - \partial_1 j_0 (= \partial_0 j^1 + \partial_1 j^0) = \frac{1}{ik}\;[j_0, j_1] \Rightarrow (\partial_0 - \partial_1)\bar{j} = 0, \quad \bar{j} := \frac{1}{2}(j^0 - j^1) = ikg^{-1}\partial_+ g \qquad (2.2c)$$

($\partial_+ := \frac{1}{2}(\partial_0 + \partial_1)$). The right invariant counterpart j of $\bar{j}$ is

$$j = -ik(\partial_- g)g^{-1}, 2\partial_- = \partial_1 - \partial_0 \Rightarrow \partial_+ j = -g(\partial_-\bar{j})g^{-1} = 0\;. \qquad (2.2d)$$

These equations are in fact the starting point of [1] (suggested by the non–abelian boson–fermion correspondence).

The symplectic form $\Omega^{(2)}$ of the theory is obtained by integrating (2.1) in x (over the circle) for constant time ($dt = 0$):

$$\Omega^{(2)} = \frac{1}{4\pi}\int_{-\pi}^{\pi}\mathrm{tr}\left\{i\;dj^0\;g^{-1}\;dg - (ij^0 + kg^{-1}\partial g)(g^{-1}dg)^2\right\} dx\;. \qquad (2.3)$$

The general solution g of (2.2) is given by the factorized expression (1.1) while the periodicity of g, $g(t, x+2\pi) = g(t,x)$ is reflected into the twisted periodicity condition (1.2) which introduces the monodromy degree of freedom. If we insert (1.1) into (2.3), using the first order equations for u and $\bar{u}$ to write

$$j^0(= -j_0) = ik\;g^{-1}\partial_t g = ik\;\bar{u}^{-1}\left((\partial\bar{u})\bar{u}^{-1} - u^{-1}\partial u\right)\bar{u}, \quad \partial = \frac{\partial}{\partial x}\;, \qquad (2.4)$$

we can express $\Omega^{(2)}$ as a sum of two chiral symplectic forms on the price of incorporating in the chiral phase spaces the boundary values $u_<$ and $\bar{u}_<$ of u and $\bar{u}$ at $x=-\pi$ and the monodromy (or, more precisely, its Gauss decomposition in the complexified group $G_{\mathbb{C}}$). The result can be written as

$$\Omega^{(2)} = \Omega(u, M) + \bar{\Omega}(\bar{u}, \bar{M}) \ , \tag{2.5}$$

with M and $\bar{M}$ treated as independent dynamical variables (to be identified after the reduction to $\Omega^{(2)}$) and

$$\Omega(= \Omega(u, M)) \ = \ \frac{k}{4\pi} \,\mathrm{tr} \left\{ \int_{-\pi}^{\pi} \partial(u^{-1}\ du) u^{-1}\ du\ dx + dMM^{-1}\ u_<^{-1}\ du_< \right\} + \frac{k}{4\pi}\,\rho(M) \ , \tag{2.6}$$

$$\bar{\Omega}(= \Omega(\bar{u}, \bar{M})) \ = \ \frac{k}{4\pi} \,\mathrm{tr}\ \left\{ \int_{-\pi}^{\pi} d\bar{u}\ \bar{u}^{-1}\ \partial(d\bar{u}\ \bar{u}^{-1}) dx + d\bar{M}\bar{M}^{-1}\ d\bar{u}_<\ \bar{u}_<^{-1} \right\} - \frac{k}{4\pi}\,\rho(\bar{M}) \ . \tag{2.$\bar{6}$}$$

The 2 form ρ that is added (subtracted) to Ω ($\bar{\Omega}$) is assumed to only depend on the monodromy and is designed to render the chiral forms closed:

$$\rho = \frac{1}{2}\,\mathrm{tr}(dMM^{-1}\ K\ dMM^{-1}), \quad K = K(M) \ ; \tag{2.7}$$

here K is a skew symmetric linear map of the complexified Lie algebra $\mathcal{G}_{\mathbb{C}}$ into itself, $\mathrm{tr}(XKY) = -\mathrm{tr}((KX)Y)$, restricted by

$$d\rho = \frac{1}{3}\,\mathrm{tr}(dMM^{-1})^3 \qquad (\Rightarrow d\Omega = 0 = d\bar{\Omega}) \ . \tag{2.8}$$

There is no global solution of (2.8) on the (compact) group manifold G. One can, however, construct such solutions on a dense open neighbourhood of the identity of the complexified group $G_{\mathbb{C}}$ using, in effect, the Gauss–like decomposition (1.3):

$$M = M_+ M_-^{-1}, \quad M_\pm \in B_\pm \subset G_{\mathbb{C}}, \quad M_+^{diag} M_-^{diag} = \mathbb{1} \ ; \tag{2.9}$$

here $B_+ (B_-)$ are Borel subgroups of $G_{\mathbb{C}}$, whose Lie algebras are generated (for the standard r–matrices (2.12) below) by the Cartan elements $h_i \in \mathcal{G}_{\mathbb{C}}$ and by the raising (respectively, lowering) operators $e_\alpha (e_{-\alpha})$ where α runs through the positive roots of $\mathcal{G}_{\mathbb{C}}$. (For $G = SU(n)$ $\quad B_\pm$ are the subgroups of $SL(n,\mathbb{C})$ of upper and lower triangular matrices.)

Theorem 2.1 ([14] Sec.3, [18]) *There is a one–to–one correspondence between the constant solutions $r^\pm$ of the classical Yang–Baxter equation (YBE)*

$$[r_{12}^\pm, r_{13}^\pm] + [r_{12}^\pm, r_{23}^\pm] + [r_{13}^\pm, r_{23}^\pm] = 0 \tag{2.10a}$$

of the form

$$r^{\pm}_{ij} = r_{ij} \pm C_{ij}, \quad r_{ij} = -r_{ji}, \quad C_{ij} = C_{ji}, \quad \mathrm{tr}_2 C_{12} \overset{2}{X} = X \ \ \textit{for any} \ \ X \in \mathcal{G}_{\mathbb{C}} \qquad (2.10b)$$

and the skew symmetric maps $K(M)$: $\mathcal{G}_{\mathbb{C}} \to \mathcal{G}_{\mathbb{C}}$, and hence the 2–forms ρ (2.7) satisfying (2.8), such that the resulting chiral symplectic form (2.6) yields PB involving the above (monodromy independent) $r^{\pm}$:

$$\left\{ \overset{1}{u}(x_1), \overset{2}{u}(x_2) \right\} = \frac{\pi}{k} \ \overset{1}{u}(x_1) \, \overset{2}{u}(x_2) (r^{+}_{12} \, \theta(x_{21}) + r^{-}_{12} \, \theta(x_{12})), \quad x_{ij} = x_i - x_j \ . \qquad (2.11)$$

Here θ is the step function satisfying $\theta(x) + \theta(-x) = 1$, $\theta(x) - \theta(-x) = sign \ x$ for $-2\pi < x < 2\pi$, and we are using the familiar Faddeev notation $\overset{1}{u} = u \otimes \mathbb{1}$, $\overset{2}{u} = \mathbb{1} \otimes u$.

Remark If $e_\alpha (e_{-\alpha})$ are the raising (lowering) operators in the defining representation of $\mathcal{G}_{\mathbb{C}}$ corresponding to positive roots α then the standard solution of (2.10) corresponds to

$$r_{12} = \sum_{\alpha > 0} \left(\overset{1}{e}_\alpha \overset{2}{e}_{-\alpha} - \overset{1}{e}_{-\alpha} \overset{2}{e}_\alpha \right) \ . \qquad (2.12)$$

The Casimir operator C_{12} appearing in (2.10b) is expressed in terms of the Cartan–Weyl basis $h_i, e_{\pm\alpha}$ of $\mathcal{G}_{\mathbb{C}}$ as

$$C_{12} = \sum_{\alpha > 0} \left(\overset{1}{e}_\alpha \overset{2}{e}_{-\alpha} + \overset{1}{e}_{-\alpha} \overset{2}{e}_\alpha \right) + (c^{-1})^{ij} \ \frac{\alpha_j^2}{\theta^2} \ \overset{1}{h}_i \overset{2}{h}_j \ . \qquad (2.13a)$$

Here $c_{ij} = 2 \ \frac{\alpha_i \alpha_j}{\alpha_j^2}$ is the Cartan matrix, c^{-1} is its inverse, α_i are the simple roots, θ is the highest root (the factor $\frac{\alpha_j^2}{\theta^2}$ is 1 for simply laced simple Lie algebras); in particular

$$C_{12} = P_{12} - \frac{1}{n} \ \mathbb{1}_{12} \quad \text{for} \quad G = SU(n) \qquad (2.13b)$$

(P_{12} is the permutation operator defined by $P_{12}(\overset{1}{\xi}\overset{2}{\eta}) = \overset{2}{\xi}\overset{1}{\eta}$ for any pair of vectors ξ, η in $\mathbb{C}^n$, so that $P_{12} \ \overset{1}{A}\overset{2}{B} = \overset{2}{A}\overset{1}{B} \ P_{12}$). Note that for $\xi^\alpha e_\alpha + \eta^i h_i + \xi^{-\alpha} e_{-\alpha} \in \mathcal{G}_{\mathbb{C}}$ the projections on the Lie algebras of B_+ and B_- corresponding to the splitting (2.9) of the group element M are $\xi^\alpha e_\alpha + \frac{1}{2} \ \eta^i h_i$ and $-\xi^{-\alpha} e_{-\alpha} - \frac{1}{2} \ \eta^i h_i$, respectively.

The form (2.10a) of $r^{\pm}$ with C_{12} given by (2.13) implies non–degeneracy of $r^{\pm}$.

Proof of Theorem 2.1 Let $\hat{X}$ be an arbitrary vector field in the infinite dimensional phase space of chiral fields u with monodromy M treated as a dynamical variable. Its contractions with the basic (Lie algebra valued) 1–forms

$$u^{-1}(x) \ du(x) \lfloor \hat{X} = X(x), \qquad dM M^{-1} \lfloor \hat{X} = X_0 \qquad (2.14)$$

should satisfy, in view of (1.2), the *twisted periodicity* condition

$$X(x+2\pi) = M^{-1}(X(x)+X_0)M \ . \qquad (2.15)$$

If $\hat{X} = \hat{X}(|u(x))$ is the Hamiltonian vector field for $u(x)$ then its contraction with the symplectic form (2.6),

$$\begin{aligned} \Omega\lfloor\hat{X} &= \frac{k}{4\pi}\,\mathrm{tr}\Big\{dMM^{-1}(2X_< + X_0 + KX_0) - \\ &\quad -2\int_{-\pi}^{\pi} dx'(\partial' X(x')\ u^{-1}(x')\ du(x')\Big\} \ \left(\partial' = \frac{\partial}{\partial x'}\right)\ , \end{aligned} \qquad (2.16)$$

where $X_< = X(-\pi)$, should reduce to $du(x)$. This implies

$$2X_< + X_0 + KX_0 = 0 \qquad (2.17)$$

and

$$-k\ \mathrm{tr}\left\{\partial' X(x'|u(x))\ u^{-1}(x')\ du(x')\right\} = \delta(x-x')\ du(x) \qquad (2.18a)$$

δ being the periodic δ–function with respect to the line element $\frac{dx}{2\pi}$:

$$\delta(y) = \sum_{n\in\mathbb{Z}} e^{\pm iny} \qquad (2.18b)$$

(which is related to the ordinary δ–function, δ_0 on the interval $-2\pi < y < 2\pi$ by $\delta(y) = 2\pi\delta_0(y)$; note that for each x and x' belonging to the interval $(-\pi,\pi)$ their difference $x-x'$ runs through an interval of double length, $(-2\pi,2\pi)$). The general solution of (2.18) for $-\pi < x, x' < \pi$ is

$$X(x'|u(x)) = X_<(u(x)) - \theta(x'-x)\ Z(u(x))\ , \qquad (2.19)$$

where

$$\overset{1}{Z}\,(\overset{2}{u}) = \frac{2\pi}{k}\ \overset{2}{u}\ C_{12} \qquad (2.20a)$$

C_{12} being the Casimir operator (2.13) so that for $G = SU(n)$ we can write (in components):

$$Z^{\alpha'}_{\beta'}(u^\alpha_\beta) = \frac{2\pi}{k}\left(\delta^{\alpha'}_{\beta}\ u^{\alpha}_{\beta'} - \frac{1}{n}\ \delta^{\alpha'}_{\beta'}\ u^{\alpha}_{\beta}\right)\ . \qquad (2.20b)$$

The twisted periodicity (2.15) allows to express X_0 in terms of the remaining variables. Inserting the result, $X_0 = M(X_< - Z)M^{-1} - X_<$, in (2.17) we obtain a relation between $X_<$ and Z:

$$(1-K)X_< + (1+K)M(X_< - Z)M^{-1} = 0\ . \qquad (2.21)$$

The PB (2.11) can be expressed in terms of the Hamiltonian vector fields $\hat{X}(|u)$ as

$$\left\{\overset{1}{u}(x_1), \overset{2}{u}(x_2)\right\} = \Omega\lfloor\hat{X}(|\overset{1}{u}(x_1))\lfloor\hat{X}(|\overset{2}{u}(x_2)) =$$
$$= \overset{1}{u}(x_1)\left[\overset{1}{X}_<(\overset{2}{u}(x_2)) - \theta(x_{12})\,\overset{1}{Z}(\overset{2}{u}(x_2))\right] . \qquad (2.22)$$

Identifying (2.22) with (2.11) allows to determine the operator $K = K(M)$ in terms of its value K_1 at $M = 1\!\!1$. To begin with, assuming that the right–hand side of (2.22) is independent of M we deduce that so is $X_<$ which can then be computed from (2.21) for $M = 1\!\!1$:

$$X_<(u) = \frac{1}{2}(K_1 + 1)Z(u), \qquad K_1 := K(1\!\!1) . \qquad (2.23)$$

Inserting (2.23) into (2.22) and comparing with (2.11) we express K_1 in terms of r:

$$K_1 X = \mathrm{tr}_2(r_{12}\overset{2}{X}) \Leftrightarrow (K_1 \pm 1)X = \mathrm{tr}_2(r^{\pm}_{12}\overset{2}{X}) . \qquad (2.24)$$

(The last inference uses $\mathrm{tr}_2\, C_{12}\overset{2}{X} = X$ for any $X \in \mathcal{G}_{\mathbb{C}}$.) Substituting $X_<(u)$ (2.23) into (2.21) we obtain

$$\{(K-1)(1+K_1) + (K+1)Ad_M(1-K_1)\}Z = 0\ (Ad_M X := MXM^{-1}) . \qquad (2.25a)$$

Since $\overset{1}{Z}(\overset{2}{u}(x))$ spans $\mathcal{G}_{\mathbb{C}}$ when $\overset{2}{u}(x)$ varies, the operator in the braces should vanish:

$$(K-1)(K_1+1) = (K+1)Ad_M(K_1-1) . \qquad (2.25b)$$

We are interested in solutions $K(M)$ of (2.25) for which K_1 is given by (2.24) with $r^{\pm}$ satisfying the classical YBE (2.10). Eq.(2.10) can be translated into the following relation for K_1 (introduced and used in a similar context in [23]):

$$[K_1X, K_1Y] + [X,Y] = K_1([K_1X,Y] + [X,K_1Y]) \quad \forall\ \ X,Y \in \mathcal{G}_{\mathbb{C}} . \qquad (2.26)$$

The meaning of this "K_1–YBE" is revealed by the following important observation [23]. If K_1 satisfies (2.26) then the subspace $\tilde{\mathcal{G}}_{\mathbb{C}} \subset \mathcal{G}_{\mathbb{C}} \oplus \mathcal{G}_{\mathbb{C}}$ of pairs $((K_1+1)X, (K_1-1)X)$ with $X \in \mathcal{G}_{\mathbb{C}}$ is a (complex) Lie subalgebra of $\mathcal{G}_{\mathbb{C}} \oplus \mathcal{G}_{\mathbb{C}}$. In other words, the commutator of two such pairs is again a pair of this type; indeed, (2.26) implies

$$[(K_1 \pm 1)X, (K_1 \pm 1)Y] = (K_1 \pm 1)([K_1X,Y] + [X,K_1Y]) .$$

Let $\tilde{G}_{\mathbb{C}}$ be the Lie subgroup of $G_{\mathbb{C}} \times G_{\mathbb{C}}$ with Lie algebra $\tilde{\mathcal{G}}_{\mathbb{C}}$. It can be mapped into $G_{\mathbb{C}}$ by $(M_+, M_-) \to M_+M_-^{-1}$. One can use this map to pull $K(M)$ back to $\tilde{G}_{\mathbb{C}}$ [18].

Proposition 2.2 *The expression*

$$K(M) = Ad_{M_+}\ K_1\ Ad^{-1}_{M_+} \quad \left(\Rightarrow \rho = \mathrm{tr}(M_+^{-1}dM_+M_-^{-1}M_-)\right) \qquad (2.27)$$

satisfies (2.25) provided the K_1–YBE (2.26) holds (and the corresponding ρ satisfies (2.8)).

Proof. Inserting (2.27) in (2.25) and using $Ad_M = Ad_{M_+}\ Ad^{-1}_{M_-}$ we find

$$Ad_{M_+}(K_1+1)\left\{Ad^{-1}_{M_+}(K_1+1) - Ad^{-1}_{M_-}(K_1-1)\right\} = 2(K_1+1)\ . \tag{2.28}$$

But the pair $(Ad^{-1}_{M_+}(K_1+1)X, Ad^{-1}_{M_-}(K_1-1)X)$ belongs to $\tilde{\mathcal{G}}_{\mathbb{C}}$ for any $X \in \mathcal{G}_{\mathbb{C}}$ (since the adjoint action of $\tilde{\mathcal{G}}_{\mathbb{C}}$ maps $\tilde{\mathcal{G}}_{\mathbb{C}}$ into itself); hence

$$(Ad^{-1}_{M_+}(K_1+1)X,\quad Ad^{-1}_{M_-}(K_1-1)X) = ((K_1+1)Y, (K_1-1)Y)$$

where

$$2Y = Ad^{-1}_{M_+}(K_1+1)X - Ad^{-1}_{M_-}(K_1-1)X (\in \mathcal{G}_{\mathbb{C}})\ .$$

This proves the validity of (2.28) and hence of Proposition 2.2.

Remark It is instructive to see how Eq.(2.28) is realized for the standard solution (2.12) of the classical YBE; the map $K_1+1 : \mathcal{G}_{\mathbb{C}} \to \mathcal{G}_{\mathbb{C}}$ projects lower triangular matrices on their diagonal (Cartan) part and doubles the nondiagonal elements (corresponding to raising operators) in upper triangular matrices.

Conversely, for K given by (2.27) with K_1 (2.24) satisfying (2.26) we arrive at the monodromy independent PB (2.11) thus completing the proof of Theorem 2.1.

As a *corollary*, we shall deduce the PB of the monodromy M with u and with itself (now using the Hamiltonian vector field $\hat{X}(|M)$ of M). This computation again requires some work. A non–obvious step is the solution with respect to X of the equation

$$Y := 2X_< + (K+1)X_0 = 2X + Ad_{M_+}(K_1+1)(Ad^{-1}_{M_-} - Ad^{-1}_{M_+})X\ .$$

Here $X = X(x|M) = u^{-1}(x)du(x)\lfloor\hat{X}(|M)$ – cf.(2.14). Noting that $X(x|M)$ is actually independent of x, so that $X = X_<$, and using the twisted periodicity relation (2.15) we find $X_0 = MXM^{-1} - X$ and hence $4X = 2Y + (K_1+1)(Ad^{-1}_M - 1)Y$. Contracting Ω with the vector fields corresponding to u and to M or to $\overset{1}{M}$ and $\overset{2}{M}$ we obtain

$$\left\{\overset{1}{M}, \overset{2}{u}\right\} = \frac{\pi}{k}\ \left(\overset{1}{M}\overset{2}{u}\ r^-_{12} - \overset{2}{u}\ r^+_{12}\ \overset{1}{M}\right)\ , \tag{2.29}$$

$$\left\{\overset{1}{M}, \overset{2}{M}\right\} = \frac{\pi}{k}\left(r^{\pm}_{12}\ \overset{1}{M}\overset{2}{M} + \overset{1}{M}\overset{2}{M}\ r^{\mp}_{12} - \overset{1}{M}\ r^-_{12}\ \overset{2}{M} - \overset{2}{M}\ r^+_{12}\ \overset{1}{M}\right)\ . \tag{2.30}$$

The extended phase space of the chiral model contains as we have seen the pair of Borel subgroups (B_+, B_-) of $G_{\mathbb{C}}$. We shall therefore work out, following [18], the PB for

$M_\pm$. Using the relations $2dM_+ = M_+(K_1+1)Ad^{-1}_{M_+}(dMM^{-1}), (K_1 \pm 1)_{13}C_{32} = -r^{\mp}_{12}$, $\{(K_1+1)(Ad_{M_-} - Ad_{M_+})r^+\}_{12} = 2(Ad^{-1}_{\overset{1}{M}_+} - Ad_{\overset{2}{M}_+})r_{12}$ we find

$$\left\{\overset{1}{M}_\pm, \overset{2}{u}\right\} = -\frac{\pi}{k}\ \overset{2}{u}\ r^\pm_{12}\ \overset{1}{M}_\pm\ , \tag{2.31}$$

$$\left\{\overset{1}{M}_\pm, \overset{2}{M}_\pm\right\} = \frac{\pi}{k}\left[r_{12}, \overset{1}{M}_\pm\overset{2}{M}_\pm\right] = \frac{\pi}{k}\left[r^\varepsilon_{12}, \overset{1}{M}_\pm\overset{2}{M}_\pm\right], \quad \varepsilon = + \text{ or } -\ , \tag{2.32a}$$

$$\left\{\overset{1}{M}_\pm, \overset{2}{M}_\mp\right\} = \frac{\pi}{k}\left[r^\pm_{12}, \overset{1}{M}_\pm, \overset{2}{M}_\mp\right]\ . \tag{2.32b}$$

The PB (2.29, 30) for M (each containing twice as many terms as the corresponding PB for $M_\pm$) can be recovered from here by using (2.9) and the Leibniz rule for PB.

A similar computation for the $\bar{u}$–sector gives:

$$\left\{\overset{1}{\bar{u}}\,(x_1), \overset{2}{\bar{u}}\,(x_2)\right\} = -\frac{\pi}{k}\left(r^+_{12}\theta(x_{21}) + r^-_{12}\theta(x_{12})\right)\overset{1}{\bar{u}}\,(x_1)\ \overset{2}{\bar{u}}\,(x_2)\ , \tag{2.33}$$

$$\left\{\overset{1}{\bar{M}}, \overset{2}{\bar{u}}\right\} = \frac{\pi}{k}\left(\overset{1}{\bar{M}}\ r^-_{12}\ \overset{2}{\bar{u}} - r^+_{12}\ \overset{1}{\bar{M}}\overset{2}{\bar{u}}\right)\ , \tag{2.34}$$

$$\left\{\overset{1}{\bar{M}}, \overset{2}{\bar{M}}\right\} = -\frac{\pi}{k}\left(r^\pm_{12}\ \overset{1}{\bar{M}}\overset{2}{\bar{M}} + \overset{1}{\bar{M}}\overset{2}{\bar{M}}\ r^\mp_{12} - \overset{1}{\bar{M}}\ r^-_{12}\ \overset{2}{\bar{M}} - \overset{2}{\bar{M}}\ r^+_{12}\ \overset{1}{\bar{M}}\right)\ , \tag{2.35}$$

$$\left\{\overset{1}{\bar{M}}_\pm, \overset{2}{\bar{u}}\right\} = -\frac{\pi}{k}r^\pm_{12}\ \overset{1}{\bar{M}}_\pm\ \overset{2}{\bar{u}}\ , \tag{2.36}$$

$$\left\{\overset{1}{\bar{M}}_\pm, \overset{2}{\bar{M}}_\pm\right\} = -\frac{\pi}{k}\left[r_{12}, \overset{1}{\bar{M}}_\pm\overset{2}{\bar{M}}_\pm\right]\ , \tag{2.37a}$$

$$\left\{\overset{1}{\bar{M}}_\pm, \overset{2}{\bar{M}}_\mp\right\} = -\frac{\pi}{k}\left[r^\pm_{12}, \overset{1}{\bar{M}}_\pm, \overset{2}{\bar{M}}_\mp\right]\ . \tag{2.37b}$$

The sign difference between (2.30), (2.32) and (2.35), (2.37) indicates that we cannot identify M and $\bar{M}$ as dynamical variables.

The symplectic form (2.6) and the resulting chiral PB exhibit four types of symmetry which we choose to formulate for the u–sector.

(a) *Invariance under periodic left shifts* generated by the chiral current j:

$$\left\{\overset{1}{u}\,(x), \overset{2}{j}\,(y)\right\} = -iC_{12}\ \overset{1}{u}\,(x)\delta(x-y), \quad j(y) = -ik(\partial u(y))u^{-1}(y) = j(y+2\pi)\ . \tag{2.38}$$

(In deriving (2.38) we use the relation $2\pi\ \frac{\partial}{\partial x}\ \theta(x-y) = \delta(x-y)$ for δ given by (2.18b).)

(b) *Poisson–Lie symmetry* [22, 23] under constant right shifts (this is the classical counterpart of the quantum group symmetry of exchange relations to be displayed in Sec.3).

(c) *An infinite Poisson–Lie symmetry* under (infinitesimal) right shifts $u(x) \to u(x) - iu(x)\varepsilon(x)$ where $\varepsilon(x)$ satisfies the special boundary conditions [20])

$$\varepsilon(-\pi) = \varepsilon_+ := \varepsilon^i h_i + \sum_{\alpha>0} \varepsilon^\alpha e_\alpha, \quad \varepsilon(\pi) = \varepsilon_- := -\varepsilon^i h_i + \sum_{\alpha>0} \varepsilon^{-\alpha} e_{-\alpha} \; ; \tag{2.39}$$

(d) *An infinite conformal symmetry* $x \to f(x)$ where $f' > 0$ (f mapping the interval $(-\pi, \pi)$ into itself). Indeed, the step function appearing in the PB (2.11) can be characterized precisely by the property

$$\theta(f(x) - f(y)) = \theta(x - y) \quad \text{for} \quad f'(x) > 0 \; (-\pi < f(x) < \pi) \; . \tag{2.40}$$

This point is worth stressing since we started with the equal time symplectic form (2.3) that is not manifestly Lorentz invariant. In fact, we are dealing with the phase space of classical solutions of the equations of motion which exhibit the full symmetry of the theory (a point emphasized in [14, 17]).

3 Exchange relations for quantized chiral fields and their quantum group symmetry

We are now looking for *quantum exchange relations* for $u(x)$ and $M_\pm$ which respect all classical symmetries listed above.

Invariance under (local) left shifts suggests, in the first place, that the R matrix should multiply the u's from the right:

$$\overset{2}{u}(x_2)\, \overset{1}{u}(x_1) = \overset{1}{u}(x_1)\, \overset{2}{u}(x_2) R_{12}(x_{12}) \; . \tag{3.1}$$

Repeated application of this exchange relation should give the identity transformation:

$$R_{21}(x_{21})\; R_{12}(x_{12}) = \mathbb{1} \quad (R_{21}(x) = P\; R_{12}(x) P) \tag{3.2}$$

($P = P_{12}$ being the permutation operator that appears in (2.13b)). Lorentz invariance of R_{12} implies that it depends just on the sign of x_{12} (a fact emphasized in an early study of exchange relations in 1+1 dimension, [8]):

$$R_{12}(x_{12}) = R_{12}^+\; \theta(x_{21}) + R_{12}^-\; \theta(x_{12}) \quad (\theta(x) + \theta(-x) = 1) \; . \tag{3.3}$$

Combined with (3.2) Eq.(3.3) gives

$$R_{21}^-\; R_{12}^+ = \mathbb{1} = R_{21}^+\; R_{12}^- \; . \tag{3.2'}$$

Associativity of the exchange relations implies the *quantum* YBE:

$$R^{\varepsilon}_{12}\ R^{\pm}_{13}\ R^{\pm}_{23} = R^{\pm}_{23}\ R^{\pm}_{13}\ R^{\varepsilon}_{12} \quad (\varepsilon = + \ \text{ or } \ -)\ . \tag{3.4}$$

In order to actually implement the left shifts (2.35) we need the quantum version of the expression for the current j as a composite field. This involves a non–trivial ordering problem for the product of interacting fields ∂u and u^{-1}. An efficient way around this difficulty is to substitute the expression for j by the *operator Knizhnik–Zamolodchikov equation* [2]

$$-ih\ \frac{d}{dx}\ u(x) =: j(x)\ u(x) :, \quad h = k + n\ , \tag{3.5a}$$

while promoting the PB (2.35) to a commutation relation (CR):

$$\left[\overset{1}{u}(x), \overset{2}{j}(y)\right] = C_{12}\ \ \overset{1}{u}(x)\ \delta(x-y), \quad C_{12} = \eta^{ab}\ \ \overset{1}{t_a}\overset{2}{t_b}\ \ . \tag{3.5b}$$

Here $\{t_a\}$ is any basis in the fundamental representation of $\mathcal{G} = su(n)$; the expression for the Casimir invariant C_{12} is actually basis independent (provided $\eta_{ab} = \mathrm{tr}(t_a t_b)$, $\eta^{as}\eta_{sb} = \delta^a_b$) and hence coincides with (2.13). The normal product in (3.5a) is easy to define since the components of j (unlike those of u) behave like free fields. Setting

$$j(x) = \sum_{n\in\mathbb{Z}}\ J_n\ e^{inx}, \qquad J_n|0\rangle = 0, \quad n \geq 0 \tag{3.6a}$$

and $J_n = J^a_n t_a$ we can write [3]

$$: j(x)u(x) := j_{(+)}(x)u(x) + t_a u(x) j^a_{(-)}(x)\ , \quad j^a_{(-)}(x) = \sum_{n=0}^{\infty}\ J^a_n\ e^{inx} = j^a - j^a_{(+)}\ . \tag{3.6b}$$

Combining (3.1) and (3.5) actually requires substituting (3.5b) by a pair of operator Ward identities:

$$\left[\overset{1}{u}(x_1), \overset{2}{j}_{(+)}(x_2)\right] = i\left(2\sin\frac{x_{12}+i0}{2}\right)^{-1}\ \ e^{i\frac{x_{12}}{2}}\ C_{12}\ \ \overset{1}{u}(x_1)$$

$$\left[\overset{2}{j}_{(-)}(x_2), \overset{1}{u}(x_1)\right] = i\left(2\sin\frac{x_{12}-i0}{2}\right)^{-1}\ \ e^{i\frac{x_{12}}{2}}\ C_{12}\ \ \overset{2}{u}(x_2)\ . \tag{3.5c}$$

One way to derive these relations is to use the CR between u and the current modes:

$$\left[\overset{1}{u}(x), \overset{2}{J}_n\right] = e^{-inx}\ C_{12}\ \ \overset{1}{u}(x)$$

in conjunction with (3.6a). The latter agree with (3.5b) for δ given by (2.18b). In effect, Eq.(3.5c) suggests yet another limit expression for the periodic δ–function:

$$\frac{i}{2\sin\frac{x+i0}{2}} - \frac{i}{2\sin\frac{x-i0}{2}} = \delta(x)\ .$$

We point out that the proper definition of the product of the operator valued distribution $\overset{1}{u}(x_1)\,\overset{2}{u}(x_2)$ with a discontinuous (step) function, like in (3.1), requires care and will not be treated in the present paper.

Since our basic dynamical variables, $u(x)$ and $M_\pm$, are dimensionless, and the "coupling constant" k in (1.1) should be a (positive) integer [1], there is no room for introducing the Planck constant. Instead, the *(quasi) classical limit* is defined by the large k behaviour of R_{12}:

$$\overset{2}{u}(x_2)\,\overset{1}{u}(x_1) \approx \overset{1}{u}(x_1)\,\overset{2}{u}(x_2) - i\left\{\overset{1}{u}(x_1), \overset{2}{u}(x_2)\right\} \quad \text{or} \quad R^\pm_{12} \approx \mathbb{1} - \frac{i\pi}{k}\, r^\pm_{12} \quad \text{for} \quad k \gg 1\ . \tag{3.7}$$

Conditions (3.1–7) still leave considerable freedom in the choice of R_{12}. For instance, if u and R_{12} satisfy (3.1–6) and S is a (constant) element of $G_{\mathbb{C}}$ then uS^{-1} and $\overset{1}{S}\overset{2}{S}\,R_{12}\,\overset{1}{S}{}^{-1}\,\overset{2}{S}{}^{-1}$ also satisfy these relations. If in addition S differs from the unit matrix by a term of order $1/k$ then (3.7) is satisfied as well. The $G_{\mathbb{C}}$ orbit space of solutions of the quantum YBE has been shown [24] to be rather singular even in the simplest case of $G = SU(2)$ (in which R_{12} is a 4×4 matrix). Happily, the non–degeneracy condition (implied at the classical level by (2.10b)) strongly reduces the freedom.

The key to selecting an appropriate R matrix is given by the requirement that the exchange relations (3.1) admit a *quantum group symmetry under right shifts*, $u \to uT^{-1}$, that is the quantum version of the Poisson–Lie symmetry (b). The point is that quantum groups provide a standard tool, the universal R–matrix [25], for constructing exchange algebras (– see [26]). Thus, in the case of $G = SU(r+1)$, one starts with the following pair of $U_q(s\ell_{r+1})$ R–matrices (written, one at a time, in [27, 28]):

$$R^+_{12} \;=\; \bar{q}^{H_{12}}\,\Theta^+_{12}, \quad H_{12} = (c^{-1})^{ij}\;\overset{1}{H}_i\overset{2}{H}_j, \quad \Theta^+ = \overrightarrow{\prod_{\alpha>0}}\;\Theta^+(\alpha) \tag{3.8a}$$

$$R^-_{12} \;=\; \Theta^-_{12}\,q^{H_{12}}, \quad \Theta^- = \overleftarrow{\prod_{\alpha>0}}\;\Theta^-(\alpha), \quad (\bar{q} = q^{-1})\ . \tag{3.8b}$$

Here c^{-1} is the inverse (A_r-) Cartan matrix, the arrow $\to$ indicates an ordering, say $(\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\ldots+\alpha_r, \alpha_2, \ldots, \alpha_2+\ldots+\alpha_r, \ldots, \alpha_r)$ of positive roots, $\leftarrow$ standing for the opposite ordering. The intertwiner Θ^+ exchanges the co–product Δ_q with $\Delta_{\bar{q}}$, $\Theta^+\Delta_q(X) = \Delta_{\bar{q}}(X)\;\Theta^+$, Θ^- intertwines the corresponding permuted products Δ' : $\Theta^-\Delta'_{\bar{q}}(X) = \Delta'_q(X)\Theta^-$. (The operator Θ^- has been subsequently used by Lusztig under the name "quasi–R–matrix" – see [29] Chapter 4; note that the uniqueness theorem proven there uses a specific completion of the tensor product $U_q(s\ell_{r+1}) \otimes U_q(s\ell_{r+1})$; Θ^- and Θ^+ belong, in fact, to different completions in the sense of Lusztig.) The factors in (3.8) are given by q–exponentials:

$$\Theta^+(\alpha) \;=\; e_+(\rho\;\overset{1}{E}_\alpha\overset{2}{F}_\alpha), \quad \Theta^-(\alpha) = e_-(\bar{\rho}\;\overset{1}{F}_\alpha\overset{2}{E}_\alpha), \quad \rho = \bar{q} - q = -\bar{\rho} \tag{3.9}$$

$$e_\pm(x) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{(n)_\pm!}, \quad (n)_\pm! = \prod_{\ell=1}^{n} (\ell)_\pm, \quad (\ell)_\pm = \frac{q^{\pm 2\ell}-1}{q^{\pm 2}-1} \; ; \tag{3.10}$$

$\alpha_i, \; i = 1, \ldots, r$ label the simple roots of $s\ell_{r+1}$, E_α are raising, F_α are lowering operators:

$$E_{ij+1} = E_{ij}E_{j+1} - qE_{j+1}E_{ij}, \quad F_{ij+1} = F_{j+1}F_{ij} - \bar{q}F_{ij}F_{j+1}, \quad i \le j \le r-1$$

$(E_{ii} \equiv E_i, F_{ii} \equiv F_i)$. The Chevalley–Cartan generators E_i, F_i, H_i satisfy the basic CR

$$q^{H_i}\; E_j \;=\; E_j\; q^{H_i + c_{ij}}, \quad q^{H_i}\; F_j = F_j\; q^{H_i - c_{ij}} \tag{3.11a}$$

$$[E_i, F_j] \;=\; \delta_{ij}[H_i], \quad [H] = \frac{q^H - \bar{q}^H}{q - \bar{q}} \quad (i, j = 1, \ldots, r) \; , \tag{3.11b}$$

and the Serre relations

$$E_i\; E_{i\pm 1}\; E_i = E_i^{(2)}\; E_{i\pm 1} + E_{i\pm 1}\; E_i^{(2)}, \quad [E_i, E_j] = 0 \quad \text{for} \quad |i-j| \ge 2 \; , \tag{3.12}$$

where $[2]\; E_i^{(2)} = E_i^2$ (and similar relations for F_i).

In the $(r+1)$–dimensional defining representation of $U_q(s\ell_{r+1})$ E_α, F_α and H_i are given in terms of the (undeformed) Weyl matrices:

$$E_i = e_{i\; i+1}, \quad F_i = e_{i+1\; i}, \quad H_i = e_{ii} - e_{i+1\; i+1}, \quad i = 1, \ldots, r, \tag{3.13a}$$

where e_{ij} are characterized by the multiplication law

$$e_{ij}\; e_{k\ell} = \delta_{jk}\; e_{i\ell}\; (\Rightarrow E_{ij} = e_{ij+1}, F_{ji} = e_{j+1\; i}) \; . \tag{3.13b}$$

Proposition 3.1 *The deformation parameter q can be chosen in such a way that for the defining representation* (3.13) *of $U_q(s\ell_{r+1})$ the R–matrices* (3.8) *satisfy the necessary conditions* (3.4,5,7) *for the quantum exchange relations* (3.1).

Proof The YBE (3.4) is established in general for the universal R–matrix by Drinfeld [25]. Eq.(3.2) is also general and follows from the property $e_+(x)e_-(-x) = 1$ of deformed exponents. The quasi classical limit is obtained for $\rho = \bar{q} - q \approx \frac{2\pi}{ik}$ (for $k \gg 1$) using that in the fundamental $n = r+1$ dimensional representation of $U_q(s\ell_n)$ the q–exponents (3.9) are given by the first two terms of their expansions:

$$\Theta^\pm(\alpha) = 1 \pm \rho \;\; \overset{1}{e}_{\pm\alpha}\; \overset{2}{e}_{\mp\alpha} \;\; . \tag{3.14}$$

With these choices our quantum theory depends on a single parameter q. The quasi-classical asymptotics only gives the first two terms in its $1/k$ expansion: $q \approx 1 + \frac{i\pi}{k} + O(\frac{1}{k^2})$. In order to fix q we shall compute the eigenvalues of the braid matrix

$$\check{R} = P\; R_{12}^+ = q^{\frac{1}{n}} \left\{ \sum_{i,j=1}^{n} \bar{q}^{\delta_{ij}}\;\; \overset{1}{e}_{ij}\; \overset{2}{e}_{ji} + \rho \sum_{j<\ell} \overset{1}{e}_{\ell\ell} \overset{2}{e}_{jj} \right\} \tag{3.15}$$

where $n = r + 1$, and shall compare them with the exponents of conformal dimensions computed with the aid of the Sugawara stress energy tensor of the current algebra theory.

Proposition 3.2 *The $n^2 \times n^2$ matrix $\check{R}$ (3.15) has two eigenvalues: a $\binom{n+1}{2}$ degenerate one, $\bar{q}^{\frac{n-1}{n}}$, corresponding to the eigensubspace of q–symmetric tensors and a $\binom{n}{2}$ degenerate one, $-q^{\frac{n+1}{n}}$, with an eigensubspace of q–skewsymmetric tensors.*

Proof We realize (following [30]) a q–symmetric tensor in terms of products of "q–bosonic" variables b^α satisfying

$$b^\alpha b^\beta = q b^\beta b^\alpha \quad \text{for} \quad \alpha < \beta, \quad \alpha, \beta = 1, \ldots, n \; . \tag{3.16}$$

Noting that

$$\sum_{ij} \bar{q}^{\delta_{ij}} (e_{ij})^\alpha_\gamma \; (e_{ji})^\beta_\delta \; b^\gamma b^\delta = \bar{q}^{\delta_{\alpha\beta}} \; b^\beta b^\alpha$$

and using (3.16) we find

$$\left(\check{R} - \bar{q}^{\frac{n-1}{n}}\right) \overset{1}{b}\overset{2}{b} = 0 \; . \tag{3.17}$$

Similarly, introducing q–fermions f^α satisfying

$$(f^\alpha)^2 = 0 = f^\alpha f^\beta + \bar{q} f^\beta f^\alpha \quad \text{for} \quad \alpha < \beta \; , \tag{3.18a}$$

we obtain

$$\left(\check{R} + q^{\frac{n+1}{n}}\right) \overset{1}{f}\overset{2}{f} = 0 \; . \tag{3.18b}$$

Corollary It follows from (3.2′) that the matrix $\check{R}^- = P R^-_{12}$ satisfies similar relations with $q \leftrightarrow \bar{q}$. They are both *Hecke*, e.g.

$$\check{R}^2 - \rho q^{1/n} \; \check{R} - q^{2/n} = 0 \; . \tag{3.19}$$

We now turn to writing down the above eigenvalues of $\check{R}$ in terms of the conformal dimensions $\Delta_h(\Lambda)$ of primary chiral vertex operators of highest weight $\Lambda = (\lambda_1, \ldots, \lambda_r)$ in a height $h = k + n$ (level k) $su(n)$ current algebra theory. $\Delta_h(\Lambda)$ is given by

$$2h\Delta_h(\Lambda) = C_2(\Lambda) = \lambda_i (c^{-1})^{ij} (\lambda_j + 2) \quad (1 \leq i, j \leq r = n - 1), \quad h = k + n \; . \tag{3.20}$$

In particular, for the "quark representation" $\Lambda_1 = (1, 0, \ldots, 0)$ and symmetric and skewsymmetric tensor representations $2\Lambda_1$ and $\Lambda_2 = (0, 1, 0, \ldots, 0)$ we have $C_2(\Lambda_1) = \frac{n^2-1}{n}$, $C_2(2\Lambda_1) = 2\frac{(n+2)(n-1)}{n}$, $C_2(\Lambda_2) = 2\frac{(n-2)(n+1)}{n}$, so that the eigenvalues of the conformal braid matrix for the product of two "quark fields" are

$$e^{i\pi(2\Delta_h(\Lambda_1) - \Delta_h(2\Lambda_1))}, \quad -e^{i\pi(2\Delta_h(\Lambda_1) - \Delta_h(\Lambda_2))} \tag{3.21}$$

(the minus sign in the second term comes from the permutation of $SU(n)$ indices in a skewsymmetric tensor (of weight Λ_2)). They agree with the eigenvalues computed in Proposition 3.2 for

$$q = e^{i\frac{\pi}{h}}, \quad h = k + n \quad (\text{for } G = SU(n)) \ . \tag{3.22}$$

The ratio of the two exponentials (3.21) $e^{i\pi(\Delta_h(2\Lambda_1)-\Delta_h(\Lambda_2))}$ is just q^2.

We remark that the calculation of the eigenvalues of the simplest braid matrix not only allows to fix the parameter q, but also provides a consistency check of the theory since two numbers are matched by a single parameter (with a given large k behaviour).

We now proceed to computing the exchange relations for the (Borel components of the) monodromy matrix demanding as a test that $u(x)$ and $u(x+2\pi) = u(x)M$ obey the same exchange relations (3.1).

We start with the Borel components $M_\pm$ of M whose PB are given by (2.31, 32). Assuming that their exchange relations only involve the above matrices $R^\pm_{12}$ we find the following (unique) expressions that fit the quasi classical limit:

$$\overset{1}{M}_\pm \overset{2}{u} = \overset{2}{u} \ R^\pm_{12} \ \overset{1}{M}_\pm \ , \tag{3.23}$$

$$R^\varepsilon_{12} \ \overset{1}{M}_\pm \overset{2}{M}_\pm = \overset{2}{M}_\pm \ \overset{1}{M}_\pm \ R^\varepsilon_{12}, \quad \varepsilon = \pm \ , \tag{3.24}$$

$$R^+_{12} \ \overset{1}{M}_+ \overset{2}{M}_- = \overset{2}{M}_- \overset{1}{M}_+ \ R^+_{12} \ \left(\Leftrightarrow R^-_{12} \ \overset{1}{M}_- \overset{2}{M}_+ = \overset{2}{M}_+ \overset{1}{M}_- \ R^-_{12}\right) \ . \tag{3.25}$$

These relations imply the following "reflection equations" for the monodromy (2.9):

$$\overset{2}{u} \ R^+_{12} \ \overset{1}{M} = \overset{1}{M} \overset{2}{u} \ R^-_{12} \ , \tag{3.26}$$

$$\overset{2}{M} \ R^+_{12} \ \overset{1}{M} \ R^+_{21} = R^+_{12} \ \overset{1}{M} \ R^+_{21} \ \overset{2}{M} \Leftrightarrow R^-_{21} \ \overset{2}{M} \ R^+_{12} \ \overset{1}{M} = \overset{1}{M} \ R^+_{21} \ \overset{2}{M} \ R^-_{12} \ . \tag{3.27}$$

The consistency check of the exchange relations for $u(x)M$ needs some care: we should make sure that after the shift (1.2) the argument $\pm x_{12}$ of θ in (3.3) is still in the interval $(-2\pi, 2\pi)$. (The alternative, to extend $\theta(x)$ beyond this interval setting $\theta(x+2\pi n) = \theta(x) + n$ for $n \in \mathbb{Z}$, would require appropriate modification of the above exchange relations depending on the interval to which belong the arguments of u.) Let, for instance,

$$\begin{aligned} 0 < x_{21} < 2\pi \quad \Rightarrow \quad & \overset{2}{u} (x_2) \ \overset{1}{u} (x_1) = \overset{1}{u} (x_1) \ \overset{2}{u} (x_2) R^+_{12}, \quad \overset{2}{u} (x_2) \ \overset{1}{u} (x_1 + 2\pi) = \\ = \ & \overset{1}{u} (x_1 + 2\pi) \ \overset{2}{u} (x_2) R^-_{12} \ ; \end{aligned} \tag{3.28}$$

inserting $\overset{1}{u} (x_1 + 2\pi) = \overset{1}{u} (x_1) \ \overset{1}{M}$ we see that (3.26) agrees with (3.28).

We shall now display the quantum group symmetry of the exchange relations. We start by recalling a basic property of the algebra $SL_q(n)$ (a Hopf algebra dual to the quantum universal enveloping algebra $U_q(s\ell_n)$, – see [26] or [31] Chapter 7). $SL_q(n)$ is defined as the polynomial algebra generated by the matrix elements of an $n \times n$ matrix T satisfying the following two equivalent sets of defining relations (for the two signs of ε in R^{ε}_{12} – cf. (3.24))

$$R^{\pm}_{12} \overset{1}{T}\overset{2}{T} = \overset{2}{T}\overset{1}{T} R^{\pm}_{12}; \qquad \det_q T = 1 \ . \tag{3.29}$$

Here the q–determinant $\det_q T$ is the central element of the algebra which appears in the relation

$$T^1_{\alpha_1} f^{\alpha_1} \dots T^n_{\alpha_n} f^{\alpha_n} = (\det_q T) f^1 \dots f^n \ , \tag{3.30}$$

where f^{α} are the q–fermionic variables (3.18).

Example [30] Let $n = 2$,

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ .$$

We leave it to the reader to verify that each of the two sets of quadratic relations (3.29) is equivalent to the commutation relations

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb, \quad ad - da = (q - \bar{q})bc$$

while $\det_q T = ad - qbc = da - \bar{q}bc$.

The following statement is an adaption of well known results [26, 30].

Proposition 3.3 *If $T \in SL_q(n)$ and T^{α}_{β} commute with u (and M) then the substitutions*

$$u \to uT^{-1}, \qquad M \to TMT^{-1} \tag{3.31}$$

respect all quantum exchange relations.

(In verifying (3.1,3) we need the first equation (3.29) for both R^{+}_{12} and R^{-}_{12}. In verifying T invariance of (3.27) we use the implication $\overset{1}{T} R^{\pm}_{12} \overset{2}{T}{}^{-1} = \overset{2}{T}{}^{-1} R^{\pm}_{12} \overset{1}{T}$ of (3.29).)

The monodromy extended chiral phase space with a manifest quantum group (Hopf algebra) symmetry matches the "axiomatic" description of the quantum group extended chiral theory of [11, 12, 32]. The price for the manifest $SL_q(n)$ (or $U_q(s\ell_n)$) symmetry is a relaxation of the unitarity assumption for the chiral conformal field theory. (One only has a *semidefinite braid invariant hermitian form* [13, 32].) Insisting on strict (Wightman positivity and) unitarity is only compatible with a weak quasi Hopf quantum symmetry [33]. In contrast to Ref. [21] we do not require unitarity of the R matrix; we just assume that the eigenvalues of $\check{R}$ (3.15) coincide with those of a conformal current algebra braid matrix ($\check{R}$ is then unitarizable whenever it is diagonalizable).

Quantized right invariant sector. The exchange relations between two $\bar{u}$'s that fit the PB (2.33) and are consistent with a symmetry under (undeformed) right $SU(n)$ shifts are

$$\overset{2}{\bar{u}}(x_2)\,\overset{1}{\bar{u}}(x_1) = R_{21}(x_{21})\,\overset{1}{\bar{u}}(x_1)\,\overset{2}{\bar{u}}(x_2)\ ; \tag{3.32}$$

here we have used

$$r^{\pm}_{21} = -r^{\mp}_{12}, \quad R_{21}(x_{21})R_{12}(x_{12}) = 1\!\!1\ . \tag{3.33}$$

Preparing for the discussion of the 2D theory (in Sec.4 below) we also list the CR involving $\bar{M}_\pm$:

$$\overset{2}{\bar{u}}\overset{1}{\bar{M}}_\pm = R^{\mp}_{21}\ \overset{1}{\bar{M}}_\pm\overset{2}{\bar{u}} \Rightarrow R^{+}_{12}\ \overset{2}{\bar{u}}\overset{1}{\bar{M}} = \overset{1}{\bar{M}}\ R^{-}_{12}\ \overset{2}{\bar{u}} \tag{3.34}$$

$$R^{\varepsilon}_{21}\ \overset{1}{\bar{M}}_\pm\overset{2}{\bar{M}}_\pm = \overset{2}{\bar{M}}_\pm\overset{1}{\bar{M}}_\pm\ R^{\varepsilon}_{21}\quad ,\quad R^{-}_{21}\ \overset{1}{\bar{M}}_+\overset{2}{\bar{M}}_- = \overset{2}{\bar{M}}_-\overset{1}{\bar{M}}_+\ R^{-}_{21} \tag{3.35a}$$

$$R^{-}_{21}\ \overset{1}{\bar{M}}\ R^{-}_{12}\ \overset{2}{\bar{M}} = \overset{2}{\bar{M}}\ R^{-}_{21}\ \overset{1}{\bar{M}}\ R^{-}_{12}\ . \tag{3.35b}$$

(in accord with (2.34) – (3.37)). The relation

$$\bar{R}^{+}_{ij}\ =\ R^{-}_{ji} \qquad (R^{-}_{ji}R^{+}_{ij} = 1) \tag{3.36}$$

(where the bar on R stands for complex conjugation), valid for

$$R^{+}_{12} = q^{\frac{1}{n}}\left\{\sum_{i,j=1}^{n}\ \bar{q}^{\delta_{ij}}\ \overset{1}{e}_{ii}\ \overset{2}{e}_{jj} + \rho\sum_{i<j}\ \overset{1}{e}_{ij}\overset{2}{e}_{ji}\right\}\ , \tag{3.37a}$$

$$R^{-}_{12} = \bar{q}^{\frac{1}{n}}\left\{\sum_{i,j=1}^{n}\ q^{\delta_{ij}}\ \overset{1}{e}_{ii}\ \overset{2}{e}_{jj} - \rho\sum_{i>j}\ \overset{1}{e}_{ij}\overset{2}{e}_{ji}\right\}\ , \tag{3.37b}$$

shows that the CR (3.35a) for $\bar{M}_\pm$ are related to the CR (3.25) for $M_\pm$ by complex conjugation. This remark can be extended to the CR involving u if we view $\bar{u}$ as complex conjugate to u^{-1}.

4 2D observables and chiral fields

The 2D observables are the field g (1.1) and the left and right currents which can be written at the classical level as

$$j(x_-) = -ik(\partial_- g)g^{-1} = -ik(\partial_x u)u^{-1}, \quad \bar{j}(x_+) = -ikg^{-1}\partial_+ g = -ik\bar{u}^{-1}\partial_x\bar{u}\ , \tag{4.1}$$

where $x_\pm = x \pm t$ and $\partial_\varepsilon x_{\varepsilon'} = \delta_{\varepsilon\varepsilon'}$, for $\varepsilon, \varepsilon' = \pm$. The quantum CR among these fields can be obtained from the exchange relations involving u and $\bar{u}$ in the chiral models, provided they are supplemented by the assumption that u and $\bar{u}$ commute,

$$\left[\overset{1}{u}(x), \overset{2}{\bar{u}}(y)\right] = 0 = \left[\overset{1}{j}(x), \overset{2}{\bar{u}}(y)\right] = \left[\overset{1}{u}(x), \overset{2}{\bar{j}}(y)\right] . \tag{4.2}$$

To begin with, using (3.1), (3.32), (3.33) and (4.2) we find

$$\left[\overset{1}{g}(t,x), \overset{2}{g}(t,y)\right] = 0 . \tag{4.3}$$

It follows further from (3.5b) and from its counterpart for $\bar{u}$ that

$$\left[\overset{1}{g}(t,x), \overset{2}{j}(y_-)\right] = C_{12} \overset{1}{g}(t,x)\delta(x_- - y_-) \tag{4.4a}$$

$$\left[\overset{1}{g}(t,x), \overset{2}{\bar{j}}(y_+)\right] = -\overset{1}{g}(t,x)C_{12}\,\delta(x_+ - y_+) . \tag{4.4b}$$

(Clearly, for equal times $x_\pm - y_\pm = x - y$.) In view of (4.2) the KZ equation (3.5a) can be written in terms of the 2D field g:

$$-ih\partial_- g(t,x) = :j(x_-)\, g(t,x): \tag{4.5a}$$

$$-ih\partial_+ g(t,x) = :g(t,x)\, \bar{j}(x_+): \quad (2\partial_\pm = \partial_x \pm \partial_t) . \tag{4.5b}$$

These *physical CR* give rise in the classical limit to PB that can be derived directly from the symplectic form (2.3) on the *reduced (physical) phase space* spanned by g and j^0 (the space of maps from the 2D cylinder to the cotangent bundle T^*G).

The exchange relations (3.1), (3.32) together with the CR (4.2) do *not* admit a quantum group symmetry $y \to uT^{-1}$, $\bar{u} \to T\bar{u}$ ($T \in SL_q(n)$, T^α_β non–commuting – cf. (3.29)). Indeed, if the matrix elements of u and $\bar{u}$ commute, those of $u^{-1}T$ and $T\bar{u}$ won't. One reason why already the classical Poisson Lie symmetry of the left and the right chiral models does not carry through to the 2D theory is that the left and right symmetries are not the same. Classically, this is reflected in the overall minus sign in the right–hand side of (2.33a) and in the already noted sign difference between the PB of M (2.30) and $\bar{M}$ (2.34). In the quantum picture the $\bar{u}$–sector symmetry under left shifts

$$\bar{u} \to \bar{T}\bar{u} \quad (\bar{M} \to \bar{T}\bar{M}\bar{T}^{-1}) \tag{4.6}$$

involves a matrix $\bar{T}$ satisfying (due to (3.32))

$$R^\pm_{21}\ \overset{1}{\bar{T}}\overset{2}{\bar{T}} = \overset{2}{\bar{T}}\overset{1}{\bar{T}}\ R^\pm_{21} , \tag{4.7}$$

instead of (3.29). In view of (3.33) this implies

$$\bar{T} \in SL_{\bar{q}}(n), \qquad \bar{q} = q^{-1} . \tag{4.8}$$

The question arises: cannot we modify the Poisson–Lie structure of the chiral phase spaces in such a way that the combined 2D system does exhibit a quantum symmetry without affecting the physical CR (4.3), (4.4)? The following unsuccessful attempt, presented here to serve a pedagogical purpose – as a Coleman style "false theorem", demonstrates that a simple minded approach, that keeps the u–sector unchanged, fails.

The fact that each of the forms Ω and $\bar{\Omega}$ is closed by itself and that the discrepancy between the PB of the u and $\bar{u}$ sectors is just a sign difference suggests a "Columbus egg solution" to the puzzle: postulate that the Poisson structure of the 2D model is determined by the difference $\Omega - \bar{\Omega}$ rather than by the sum of symplectic forms. At first sight, it works beautifully: the u–sector CR do not change, the monodromy in the two sectors has identical exchange relations, so that we can set $M = \bar{M}$, the CR involving $\bar{u}$ become

$$\overset{2}{\bar{u}}(x_2)\,\overset{1}{\bar{u}}(x_1) \;=\; R_{12}(x_{12})\,\overset{1}{\bar{u}}(x_1)\,\overset{2}{\bar{u}}(x_2)\;, \tag{4.9}$$

$$\overset{2}{M}_{\pm}\overset{1}{\bar{u}}= R_{12}^{\mp}\,\overset{1}{\bar{u}}\overset{2}{M}_{\pm} \;\Rightarrow\; R_{12}^{-}\,\overset{1}{\bar{u}}\overset{2}{M}=\overset{2}{M}\,R_{12}^{+}\,\overset{1}{\bar{u}}\;; \tag{4.10}$$

u and $\bar{u}$ no longer commute,

$$\overset{1}{\bar{u}}(x_1)\,\overset{2}{\bar{u}}(x_2) =\overset{2}{\bar{u}}(x_2)R_{12}(x_{12})\,\overset{1}{\bar{u}}(x_1) \Leftrightarrow \overset{2}{\bar{u}}\overset{1}{u}=\overset{1}{u}\,R_{21}\,\overset{2}{\bar{u}}\;, \tag{4.11}$$

but the locality property (4.3) for the 2D field (1.1) is preserved. The 2D model does admit a quantum group symmetry (1.4), (4.6) with $T = \bar{T}$ satisfying (3.29). Assuming only part of these properties one finds that this solution is unique ... What goes wrong is the fact that the above CR for $\bar{u}$ (and the CR for u and M derived in Sec.3) imply that g (1.1) is not just local, it is central: it commutes with all dynamical variables,

$$\left[\overset{1}{g}(t,x), \overset{2}{u}(y)\right] = 0 = \left[\overset{1}{g}(t,x), \overset{2}{\bar{u}}(y)\right] = \left[\overset{1}{g}(t,x), \overset{2}{M}_{\pm}\right]\;.$$

Hence the CR (4.4) between g and the chiral currents cannot possibly hold. It turns out that the CR (4.9–11) of $\bar{u}$ are exactly the same as the CR of u^{-1} that follow from the exchange relations of Sec.3. Hence, the CR of g are consistent with the assumption that $g = \mathbb{1}$.

This wrong start can be elevated to a no–go result since an exchange relation of type (4.11) is, in fact, necessary in order to have a consistent quantum group symmetry $(u, \bar{u}) \to (uT^{-1}, T\bar{u}), T \in SL_q(n)$.

Is not this result vindicating the point of view expressed in [19]: the quantum group symmetry is an artefact of cutting the WZNW model into chiral parts (and changing each), there is no trace of it in the 2D theory? We shall argue why such a negative conclusion seems to be premature.

Physical subspace of $\mathcal{H}_L \otimes \mathcal{H}_R$ corresponding to a diagonal theory The extended WZNW model defined in the tensor product of left and right chiral spaces exhibits some familiar features of a gauge theory. On top of the physical 2D field g and the chiral currents it involves multivalued chiral fields which require arbitrary choices. The inner product in each chiral state space is not positive definite (as discussed at the end of Sec.3). A crucial ingredient in a "covariant gauge" setup consists of selecting the physical subspace of the theory. We are looking for a subspace $\mathcal{H}$ of the (extended) tensor product space $\mathcal{H}_L \otimes \mathcal{H}_R$ of the chiral models which contains the vacuum $|0>=|0>_L \otimes |0>_R$ and is such that the matrix elements of $g(t,x)$ in $\mathcal{H}$ are periodic in x:

$$< \Phi | g(t, x+2\pi) - g(t,x) | \Psi >= 0 \quad \text{for} \quad \Phi, \Psi \in \mathcal{H} \tag{4.12}$$

(whenever the corresponding matrix elements of g are well defined). We assume further that the operators which commute with g and hence with all observables are multiples of the unit operator in $\mathcal{H}$ (i.e., have $\mathcal{H}$ as their common eigensubspace). Such are the products $M_\pm^{-1} \bar{M}_\pm$; as it follows from (3.23), (3.34),

$$\overset{1}{M}{}_\pm^{-1} \overset{1}{\bar{M}}_\pm \overset{2}{u} \overset{2}{\bar{u}} = \overset{1}{M}{}_\pm^{-1} \overset{2}{u} R_{12}^\pm \overset{2}{\bar{u}} \overset{1}{\bar{M}}_\pm = \overset{2}{u} \overset{2}{\bar{u}} \overset{1}{M}{}_\pm^{-1} \overset{1}{\bar{M}}_\pm \, . \tag{4.13}$$

Extending the defining property of $M_\pm$ to have inverse eigenvalues to the quantum case we shall write

$$(M_+^{-1} \bar{M}_+ - \bar{M}_-^{-1} M_-) \mathcal{H} = 0 \tag{4.14}$$

($M_\pm^{-1} \bar{M}_\pm |0>$ being multiples of the unit matrix).

The requirement (4.12) implies that the helicities $\Delta_h(\Lambda) - \Delta_h(\bar{\Lambda})$ have integer values on physical state vectors: the *univalence operator* $e^{2\pi i (L_0 - \bar{L}_0)}$ should coincide with the identity on $\mathcal{H}$,

$$(e^{2\pi i (L_0 - \bar{L}_0)} - 1) \mathcal{H} = 0 \, . \tag{4.15}$$

Then Eq.(4.12) follows since

$$g(t, x+2\pi) = e^{2\pi i (\bar{L}_0 - L_0)} g(t,x) e^{2\pi i (L_0 - \bar{L}_0)} \, . \tag{4.16}$$

This is achieved, in particular, in a *diagonal theory* in which the physical state space is a direct sum of positive energy (highest weight) current algebra modules of equal left and right weights:

$$\mathcal{H} = \bigoplus_\Lambda \mathcal{H}_L(\Lambda) \otimes \mathcal{H}_R(\Lambda) \, . \tag{4.17}$$

Let Π be the orthogonal projector on $\mathcal{H}$ in $\mathcal{H}_L \otimes \mathcal{H}_R$ $\quad (\Pi^2 = \Pi = \Pi^* \; , \; \Pi \mathcal{H}_L \otimes \mathcal{H}_R = \mathcal{H})$. We shall display the observable (the *conditional expectation* in the terminology of [34])

$\Pi g(t,x)\Pi$ for $G = SU(r+1)$ (setting $\Lambda = (\lambda_1, \ldots, \lambda_r)$ as in (3.20)). There are exactly $r+1$ *chiral vertex operators*

$$u(x;\mu_1,\ldots,\mu_r): \ \mathcal{H}_L(\lambda_1,\ldots,\lambda_r) \longrightarrow \mathcal{H}_L(\lambda_1+\mu_1,\ldots,\lambda_r+\mu_r) \tag{4.18}$$

where the admissible $(\mu_1, \ldots, \mu_r)$ are

$$(1,0,\ldots,0,0)\ ,\ (-1,1,0,\ldots,0)\ ,\ldots,\ (0,0,\ldots,-1,1)\ ,\ (0,0,\ldots,0,-1) \tag{4.19}$$

and the space in the right hand side of (4.18) consists of the 0–vector whenever one of the entries $\lambda_i + \mu_i$ is negative. (In the case of $SU(2)$ the CVOs $u(x;\pm 1)$ are nothing but the creation and annihilation operators which increase, respectively decrease the isospin of the state vector by 1/2 .) With these notations the observable projection of g is just the diagonal in $\{\mu_i\}$ part

$$\Pi g(t,x)\Pi = \sum_{\{\mu_i\}} u(x-t;\mu_1,\ldots,\mu_r)\bar{u}(x+t;\mu_1,\ldots,\mu_r) \tag{4.20}$$

where the sum runs through the $r+1$ sequences (4.19). Indeed, the orthogonality of subspaces of different highest weight implies the vanishing of the conditional expectation of a product $u(x-t;\{\mu_i\})\bar{u}(x+t;\{\bar{\mu}_i\})$ for $\{\mu_i\} \neq \{\bar{\mu}_i\}$. The single valuedness (i.e., the periodicity in x) of the conditional expectation (4.20) is an immediate consequence of (4.16) and (4.15).

Verifying the consistency of the above analysis with the properties of the quantum monodromy matrices requires some work. To begin with there is an ordering problem in the quantum version of (1.2):

$$u(x+2\pi) \ = \ \vdots u(x)M \vdots \quad , \quad \bar{u}(x+2\pi) = \vdots M^{-1}\bar{u}(x)\vdots \ ; \tag{4.21}$$

the meaning of the triple dot product (already restricted by the consistency condition (3.28)) should be disclosed in a systematic matching of the operator formalism with the known braiding properties of conformal current algebra blocks (for $G = SU(2)$ - see [32]). A preliminary study indicates that the two pictures are indeed consistent with one another. This suggests that the conditional expectation (4.20) is also quantum group invariant (in spite of the fact that the operator equation $T^{-1}\bar{T} = 1$ does not hold in $\mathcal{H}_L \otimes \mathcal{H}_R$). We hope to return to these questions shortly.

Acknowledgments

The authors are greatly indebted to Krzysztof Gawędzki for acquainting them with his unpublished work, for a number of enlightening discussions, and for a critical reading of an earlier version of the manuscript. We also thank A. Alekseev and P. Kulish for stimulating discussions. L.H. thanks I.N.F.N., Sezione di Trieste, for support. I.T. thanks IHES, Bures–sur–Yvette, SISSA, Trieste, and ESI, Vienna for hospitality and support during the course of this work. This work is supported in part by the Bulgarian Foundation for Scientific Research under contract F-404.

References

[1] E. Witten, Commun. Math. Phys. **92** (1984) 455.

[2] V.G. Knizhnik, A.B. Zamolodchikov, Nucl. Phys. **247** (1984) 83.

[3] I.T. Todorov, *International Conference on Differential Geometric Methods in Physics*, Shumen 1984, eds. H.–D. Döbner, T. Palev (World Scientific, Singapore) pp.297–347; Phys. Lett. **B153** (1985) 77.

[4] A. Tsuchia and Y. Kanie, Lett. Math. Phys. **13** (1987) 303.

[5] O. Babelon, Phys. Lett. **B215** (1988);
B. Blok, Phys. Lett. **B233** (1989) 359.

[6] L.D. Faddeev, Commun. Math. Phys. **132** (1990) 131.

[7] A. Alekseev and S. Shatashvili, Commun. Math. Phys. **133** (1990) 353.

[8] J. Fröhlich, in: *Non–perturbative Quantum Field Theory*, Cargèse 1987, G.'t Hooft et al. (eds.) (Plenum, New York, 1988) pp.71–100.

[9] V. Pasquier, Commun. Math. Phys. **118** (1988) 355.

[10] L. Alvarez–Gaumé, C. Gomez and G. Sierra, Nucl. Phys. **B319** (1989) 155;
A. Ganchev and V. Petkova, Phys. Lett. **B233** (1989) 374;
V. Pasquier and H. Saleur, Nucl. Phys. **B330** (1990) 523;
P. Furlan, A. Ganchev and V. Petkova, Nucl. Phys. **B343** (1990) 205;
J.–L. Gervais, Commun. Math. Phys. **130** (1990) 257 and **138** (1991) 301.

[11] I.T. Todorov, in: *Quantum Groups*, Clausthal 1989, H.–D. Doeber, J.–D. Hennig (eds.) Lecture Notes in Physics **370** (Springer, Berlin, 1990) pp.231–277;
L.K. Hadjiivanov, R.R. Paunov and I.T. Todorov, Nucl. Phys. **B356** (1991) 387.

[12] K. Gawędzki, Nucl. Phys. B (Proc. Suppl.) **18B** (1990) 78.

[13] J. Fröhlich and T. Kerler, *Quantum Groups, Quantum Categories and Quantum Field Theory*, Lecture Notes in Mathematics **1542** (Spinger, Berlin, 1993).

[14] K. Gawędzki, Commun. Math. Phys. **139** (1991) 201.

[15] A.Yu. Alekseev, L.D. Faddeev, and M.A. Semenov–Tian–Shansky, Commun. Math. Phys. **149** (1992) 335.

[16] M. Chu, P. Goddard, I. Halliday, D. Olive and A. Schwimmer, Phys. Lett. **B266** (1991) 71.

[17] F. Falceto and K. Gawędzki, Proceedings of the *XXth International Conference on Differential Geometric Methods in Physics*, NY, eds. S. Catto and A. Rocha, (Scientific, Singapore, 1991) pp.972–985; J. Geom. Phys. **11** (1993) 251.

[18] F. Falceto and K. Gawędzki, Quantum group symmetries in WZW models, unpublished notes (I.H.E.S., Bures–sur–Yvette, 1991).

[19] G. Papadopoulos and B. Spence, Phys. Lett. **B295** (1992) 44.

[20] A.Yu. Alekseev and I.T. Todorov, Nucl. Phys. **B421** (1994) 413.

[21] M. Chu and P. Goddard, Quantization of the $SU(N)$ WZW model at level k, Univ. of Cambirdge preprint, hep–th/9407108.

[22] V.G. Drinfel'd, Sov. Math. Doklady **27** (1983) 68.

[23] M.A. Semenov–Tian–Shanksy, Publ. Res. Inst. Math. Sci. **21** (1985) 1237.

[24] J. Hietarinta, J. Math. Phys. **34** (1993) 1725.

[25] V.G. Drinfeld, Sov. Math. Doklady **32** (1985) 254;
Proc. Int. Congr. Math. Berkeley 1986, Vol.1 (AMS, 1987) pp.798–820.

[26] N.Yu. Reshetikhin, L.A. Takhtajan and L.D. Faddeev, Leningrad Math. J. **1** (1990) 193.

[27] S.M. Khoroshkin and V.N. Tolstoy, Commun. Math. Phys. **141** (1991) 599.

[28] M.S. Iotov and I.T. Todorov, Lett. Math. Phys. **23** (1991) 241.

[29] G. Lusztig, *Introduction to Quantum Groups* (Birkhäuser, Boston, 1993).

[30] Yu.I. Manin, *Quantum Groups and Non–commutative Geometry* (Centre de Recherches Mathématiques, Université de Montréal, 1988).

[31] V. Chari and A. Pressley, *A Guide to Quantum Groups* (Cambirdge Univ. Press, 1994).

[32] Ya.S. Stanev, I.T. Todorov and L.K. Hadjiivanov, Phys. Lett. **B276** (1992) 87; and in: *Quantum Symmetries*, H.–D. Döbner and V.K. Dobrev (eds.) (World Scientific, Singapore, 1993) pp.24–40.

[33] G. Mack and V. Schomerus, Phys. Lett. **B267** (1991) 207; Nucl. Phys. **B370** (1992) 185.

[34] K.-H.Rehren, Y.S.Stanev, I.T.Todorov, *Characterizing invariants for local extensions of current algebras*, Hamburg and Vienna preprint DESY 94-164, ESI 132 (1994), hep-th/9409165, Commun.Math.Phys. (to be published).