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NORMAL BGG SOLUTIONS AND POLYNOMIALS

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Abstract. First BGG operators are a large class of overdetermined linear differential operators intrinsically associated to a parabolic geometry on a manifold. The corresponding equations include those controlling infinitesimal automorphisms, higher symmetries, and many other widely studied PDE of geometric origin. The machinery of BGG sequences also singles out a subclass of solutions called normal solutions. These correspond to parallel tractor fields and hence to (certain) holonomy reductions of the canonical normal Cartan connection. Using the normal Cartan connection, we define a special class of local frames for any natural vector bundle associated to a parabolic geometry. We then prove that the coefficient functions of any normal solution of a first BGG operator with respect to such a frame are polynomials in the normal coordinates of the parabolic geometry. A bound on the degree of these polynomials in terms of representation theory data is derived.

For geometries locally isomorphic to the homogeneous model of the geometry we explicitly compute the local frames mentioned above. Together with the fact that on such structures all solutions are normal, we obtain a complete description of all first BGG solutions in this case. Finally, we prove that in the general case the polynomial system coming from a normal solution is the pull-back of a polynomial system that solves the corresponding problem on the homogeneous model. Thus we can derive a complete list of potential normal solutions on curved geometries. Moreover, questions concerning the zero locus of solutions, as well as related finer geometric and smooth data, are reduced to a study of polynomial systems and real algebraic sets.

1. Introduction

It has long been known that certain natural overdetermined linear partial differential equations play a fundamental role in differential geometry. Archetypal examples in Riemannian geometry are the Killing equation and the conformal Killing equation on various types of tensor fields. In particular, solutions to these equations on vector fields are infinitesimal isometries and infinitesimal conformal isometries. The conformal Killing equation on differential forms (see e.g. [37, 34]) and the twistor equation, which is the analogous equation on spinors (see e.g. [19, 1]), have been intensively studied. More recently, it has been shown that solutions to the conformal Killing equation on symmetric tensor fields are equivalent to higher symmetries of the Laplacian, see [15], and generalisations to other operators have been obtained, see [25].

For all these equations, many questions arise concerning the nature and properties of solutions. For example the question of establishing the structure of the solution's zero locus has been taken up for specific equations in many places, see e.g. [2, 14, 28, 33]. Understanding the structure of solutions is also important because of a second line of applications of these operators. Aside from determining notions of symmetry, it is becoming

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evident that they can be used to define or characterise important geometric structures. For example the Poincaré-Einstein structures of [18], which form the basis of a geometric Poisson transform programme relating conformal and Riemannian geometry [26] as well as a related scattering programme [27], may be understood as a conformal manifold equipped with a solution to the overdetermined equation controlling the conformal-to-Einstein condition [20, 22]. In the study of this description in these the last references, the conformal tractor connection was used to classify the zero loci of solutions to the conformal to Einstein equation in a way which highlighted links with the homogeneous model. This motivated many of the developments in the current article.

All the equations alluded to above can be understood as special cases of first BGG equations on parabolic geometries, and this observation alone suggests powerful generalisations, see e.g. [8]. For a semisimple Lie group G with Lie algebra $\mathfrak{g}$ and a parabolic subgroup $P \subset G$, a parabolic geometry of type (G, P) is an n-manifold M equipped with a P–principal bundle $\mathcal{G} \to M$ endowed with a (suitably *normal*) *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ (see Definition 2.1). The homogeneous model of the geometry is the "Klein structure" consisting of G viewed as a P-principal bundle over the homogeneous space G/P. Parabolic geometries form a broad class of structures which includes conformal, CR, and projective geometries as special cases. For example in the case of signature (p, q) conformal geometry (with $p+q = n$), $G = SO(p+1, q+1)$ and (identifying G with its linear defining representation) P is the parabolic stabilising a nominated null ray.

On any parabolic geometry of type (G, P) the normal Cartan connection universally determines, for each irreducible representation $\mathbb{V}$ of G, a canonical sequence of differential operators with the property that on the model G/P this sequence is a finite resolution of $\mathbb{V}$ by linear differential operators [13]. These sequences are known as BGG sequences, due to the relations to the algebraic resolutions of Bernstein-Gelfand-Gelfand and others [3, 35]. The first operator in each such sequence is called a first BGG operator, its equation the first BGG equation, and ranging over the possible $(G, P, \mathbb{V})$ we obtain the class mentioned above.

First BGG equations have finite dimensional kernel and they have no non-trivial solutions in general. This is in fact part of their importance: the integrability conditions for the existence of non-trivial solutions are often extremely interesting non-linear systems intrinsic to the structure (such as the Einstein equation in certain cases). Normal (first) BGG solutions are a distinguished class of solutions; the solutions in the class are characterised by the fact that they correspond in a precise way to a holonomy reduction of the normal Cartan connection (see e.g. [34]). The correspondence is mediated by what is called the tractor connection; this linear connection is induced by the Cartan connection on the bundle associated to $\mathcal{G}$ via the relevant irreducible G-representation $\mathbb{V}$ (see Proposition 2.2). An important point is that on any homogeneous model G/P, for any G-irreducible $\mathbb{V}$ all solutions of the corresponding first BGG operator are normal and the vector space of such solutions is isomorphic to $\mathbb{V}$.

For the special case of projective structures, we have developed in [10] a new approach to the study of first BGG solutions. In particular, that article discusses applications to the construction and study of new compactifications, which emphasise the geodesic structure rather than conformal aspects. The developments in that article exhibited two main features, the polynomiality of normal solutions, and the comparison between curved geometries and the homogeneous model, which looks particularly simple on the level of tractor bundles. It turns out that the latter aspect can be understood in a conceptual way in the context of holonomy reductions of general Cartan geometries.

This is developed in [11], which may be viewed as a companion article to the current work, with parabolic geometries providing the most important examples. It turns out that holonomy reductions of parabolic geometries determined by normal solutions of first BGG operators govern a host of interesting constructions which relate apparently different geometries, such as [5, 17, 36, 31].

The main aim of the current article is to extend the polynomiality results from [10] to normal first BGG solutions on general parabolic geometries. This leads to remarkably strong results on the nature of these solutions. Given a point in the manifold, we first have to choose a point in the fibre of the Cartan bundle, so there is a freedom parametrised by the parabolic subgroup P. Having made this choice, we get local normal coordinates as well as a special class of local frames for all natural vector bundles associated to the parabolic geometry in question. We then prove that the coefficient functions of any normal solution with respect to such a frame are polynomials in the normal coordinates. The degree of these polynomials is known a priori in terms of data associated with the equation. These results form the main theorem, which is Theorem 2.10. The proof of that theorem is constructive: the polynomial is produced directly from the parallel tractor field via a preferred local trivialisation of the tractor bundle that we also describe.

As mentioned above, via the parallel tractor field corresponding to a normal solution of a first BGG operator, there is a connection to holonomy reductions of Cartan geometries as studied in [11]. One of the main features of such a reduction is an induced stratification of the underlying manifold into initial submanifolds, which reflects a certain orbit decomposition of the homogeneous model. The zero locus of the corresponding normal solution is encoded in this data (see Section 2.7 below); in fact, the full stratification reveals that considerable further information is available. Understanding this stratification from a more analytic point of view requires a description by functions and this is precisely the information captured by the polynomial systems we discover and describe here. Indeed, our results may be viewed as showing that the submanifolds arising can be understood as generalising, in a natural way, a class of projective algebraic sets.

In the case of the homogeneous model G/P we give an explicit description of the special frames on tensor bundles in Proposition 2.8, thus obtaining a completely explicit description of all solutions of the first BGG equations on such bundles. (Then the same result holds locally on any structure which is locally flat, i.e. locally isomorphic to the model). A second important, and surprising, consequence of the construction is that on any curved parabolic geometry of type (G, P) the polynomial system describing a normal solution is actually the pull-back, via a special diffeomorphism, of the polynomial system describing a normal solution (of the same first BGG equation) on G/P, see Proposition 2.13. This diffeomorphism comes from a comparison map that we construct in Definition 2.12. On the homogeneous space G/P the structures defined by the polynomial system can often be well understood via classical techniques (or are even well known), and our result here gives a precise statement to which extent these features must hold for a general normal solution.

Although the general theory is simple and universal, each particular case of a normal solution of a given first BGG equation on a specific parabolic geometry typically carries considerable information and detail. In Section 3 we illustrate how this can be obtained in a completely explicit way, and how it yields applications in familiar settings. Examples include conformal first BGG equations on densities, conformal Killing vector fields, conformal Killing r-forms, Killing vector fields, as well as several examples from projective geometry. In addition we show in that section how the generalised homogeneous projective coordinates, discovered in [10], arise using the very different perspective we develop here, see Proposition 3.1. This then enables the corresponding notion to be developed for conformal geometry, which is formalised in Definition 3.7.

2. Normal BGG solutions

We start by very briefly recalling the algebraic background and the definition of parabolic geometries, referring to Section 3.1 of [12] for details.

2.1. **$|k|$–graded Lie algebras and their representations.** The starting point for defining a parabolic geometry is a real or complex semisimple Lie algebra $\mathfrak{g}$ endowed with a so–called $|k|$–grading, i.e. a decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for some $k \geq 1$. We make the standard assumptions that none of the simple ideals of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$ and that the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$. In particular, this implies that $\mathfrak{g}_0 \subset \mathfrak{g}$ is a subalgebra which acts on each $\mathfrak{g}_i$ via the restriction of the adjoint representation.

Next, we consider the associated filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \cdots \supset \mathfrak{g}^k$ defined by $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$. This makes $\mathfrak{g}$ into a filtered Lie algebra in the sense that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$. In particular, this implies that $\mathfrak{p} := \mathfrak{g}^0$ is a subalgebra of $\mathfrak{g}$, which acts on each filtration component $\mathfrak{g}^i$ via the restriction of the adjoint action. In particular, $\mathfrak{p}_+ = \mathfrak{g}^1$ is an ideal in $\mathfrak{p}$, which is nilpotent by definition. It turns out that the subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is always a parabolic subalgebra in the sense of representation theory, and that any parabolic subalgebra can be realised via an appropriate $|k|$–grading. It further turns out that $\mathfrak{p}_+ \subset \mathfrak{p}$ is the nilradical of $\mathfrak{p}$ and $\mathfrak{g}_0$ is a reductive complement to $\mathfrak{p}_+$ in $\mathfrak{p}$, which is usually called a Levi–factor.

In this article, we will only consider finite dimensional representations. Any such representation $\mathbb{V}$ of $\mathfrak{g}$ inherits a grading, which is compatible with the grading on $\mathfrak{g}$. It can be shown that the grading of $\mathfrak{g}$ is the eigenspace decomposition with respect to the adjoint action of a uniquely determined element $E \in \mathfrak{g}$, called the *grading element*. This element then also acts diagonalisably on each irreducible representation $\mathbb{V}$ of $\mathfrak{g}$ and one can interpret the eigenspace decomposition as a decomposition $\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_N$ such that $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j}$. (In many cases it is more natural to write this decomposition in the form $\mathbb{V}_{-\ell} \oplus \cdots \oplus \mathbb{V}_\ell$, but one can always shift the degrees to obtain the form described above.)

2.2. **Parabolic geometries and BGG sequences.** Take a $|k|$–graded Lie algebra $\mathfrak{g}$ as in Section 2.1 and a Lie group G with Lie algebra $\mathfrak{g}$. Then it can be shown that the normaliser $N_G(\mathfrak{p})$ of $\mathfrak{p}$ in G has Lie algebra $\mathfrak{p}$, and one makes a choice of a *parabolic subgroup* $P \subset G$, i.e. a subgroup lying between $N_G(\mathfrak{p})$ and its connected component of the identity. Then the adjoint action of each element of P preserves each of the filtration components $\mathfrak{g}^i \subset \mathfrak{g}$, and one defines the *Levi–subgroup* $G_0 \subset P$ to consist of all elements whose adjoint action preserves the grading of $\mathfrak{g}$. One shows that $G_0 \subset P$ corresponds to the Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{p}$. Moreover, the exponential mapping defines a diffeomorphism from $\mathfrak{p}_+$ onto a closed normal subgroup $P_+ \subset P$ such that P is the semi–direct product of G_0 and P_+.

Definition 2.1. (1) A *parabolic geometry* of type (G, P) on a smooth manifold M is given by a principal P–bundle $p : \mathcal{G} \to M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. By definition, this means that for each $u \in \mathcal{G}$ the map $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism, that ω is equivariant with respect to the principal right action of P, and reproduces the generators of fundamental vector fields.

(2) The *homogeneous model* for parabolic geometries of type (G, P) is the homogeneous space G/P (which is a generalised flag manifold) with the left Maurer–Cartan form as the Cartan connection.

Parabolic geometries which satisfy the additional conditions of regularity and normality are equivalent encodings (in the categorical sense) of certain underlying structures, see Section 3.1 of [12] and Chapter 4 of this reference for many examples. These underlying structures include classical projective structures, conformal structures, almost quaternionic structures, hypersurface type CR structures, and several types of generic distributions. A congenial feature of the theory we develop below is that it does not depend at all on the details of the correspondence to underlying structures, nor does it need explicit details of the precise definitions of regularity and normality. Thus we shall simply take the parabolic geometry as an input and, for the general results (which form the main theorems), this results in an efficient simultaneous treatment of the entire class of parabolic geometries.

Forming associated bundles to the Cartan bundle, any representation of the Lie group P gives rise to a natural bundle on parabolic geometries of type (G, P). In particular, one can use the restrictions to P of (irreducible) representations of G, thus obtaining so–called *tractor bundles*. While these bundles may at first seem unusual as geometric objects, they have the critical advantage that (unlike general associated bundles) they inherit a canonical linear connection, called the *tractor connection* from the Cartan connection ω. The general theory of tractor bundles is developed in [9]. More conventional geometric objects like tensors and spinors are obtained by forming so–called *completely reducible* bundles, which are associated to completely reducible representations of P. It is well-known that these representations are obtained from completely reducible representations of G_0 via the projection $P \to G_0 = P/P_+$.

The tractor connections can be used to construct higher order differential operators acting between completely reducible bundles which are intrinsic to the geometric structure via the machinery of BGG sequences, which was introduced in [13] and improved in [6]. For our current purposes, we just need a small amount of information concerning the first operator in such a sequence. Given an irreducible representation $\mathbb{V}$ of $\mathfrak{g}$ consider the grading $\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_N$ as in Section 2.1 and the induced filtration $\{\mathbb{V}^i\}$ of $\mathbb{V}$ defined by $\mathbb{V}^i = \mathbb{V}_i \oplus \cdots \oplus \mathbb{V}_N$. Then each of the filtration components $\mathbb{V}^i$ is P–invariant and the induced action of P_+ on $\mathbb{V}^i/\mathbb{V}^{i+1}$ is trivial, so this is a completely reducible representation of P. In particular, this applies to the quotient $\mathbb{V}/\mathbb{V}^1 =: \mathbb{H}_0$, which is immediately seen to be even an irreducible representation of P. From the construction it is clear that $\mathbb{H}_0$ is obtained from the G_0–representation $\mathbb{V}_0$ via the quotient projection $P \to G_0$.

Passing to associated bundles, we write $\mathcal{V}M := \mathcal{G} \times_P \mathbb{V}$. The P–invariant filtration $\{\mathbb{V}^i\}$ of $\mathbb{V}$ corresponds to a filtration of $\mathcal{V}M$ by smooth subbundles $\mathcal{V}^iM$. We then have the irreducible quotient bundle $\mathcal{H}_0M := \mathcal{V}M/\mathcal{V}^1M$ and a natural projection $\Pi : \mathcal{V}M \to \mathcal{H}_0M$. The machinery of BGG sequences constructs a completely reducible subquotient $\mathcal{H}_1M$ of the bundle $T^*M \otimes \mathcal{V}M$ of $\mathcal{V}M$–valued one–forms and a natural differential operator $D^{\mathcal{V}} : \Gamma(\mathcal{H}_0M) \to \Gamma(\mathcal{H}_1M)$, called the *first BGG operator* associated to $\mathbb{V}$. Given $\mathbb{V}$, the representations $\mathbb{H}_0$ and $\mathbb{H}_1$ can be determined algorithmically, which also leads to a description of the order and the principal part of the operator $D^{\mathcal{V}}$. Simple algorithms for the formulae for the full operators $D^{\mathcal{V}}$ are available and explicit formulae in low order are explicitly available [7, 21]. Let us collect the main information we will need in the sequel. Proofs of the following facts can be found in [13].

Proposition 2.2. *Let us denote by $\Pi : \mathcal{V}M \to \mathcal{H}_0M$ the canonical projection from a tractor bundle to its canonical quotient as well as the induced operator on sections.*

(1) The kernel of $D^{\mathcal{V}}$ is always finite dimensional with dimension bounded by $\dim(\mathbb{V})$.

(2) If $s \in \Gamma(\mathcal{V}M)$ is parallel for the tractor connection on $\mathcal{V}M$, then $\Pi(s) \in \Gamma(\mathcal{H}_0)$ lies in the kernel of $D^{\mathcal{V}}$.

(3) The restriction of Π to the space of sections which are parallel for the tractor connection is injective.

(4) In the case of the homogeneous model G/P the tractor connection is flat with trivial holonomy and Π defines a linear isomorphism from the space of parallel sections of $\mathcal{V}(G/P)$ (which can be identified with $\mathbb{V}$) onto the kernel of $D^{\mathcal{V}}$.

Definition 2.3. (1) The *first BGG equation* determined by $\mathbb{V}$ is the natural differential equation $D^{\mathcal{V}}(\alpha) = 0$ on $\alpha \in \Gamma(\mathcal{H}_0M)$.

(2) A solution of the first BGG equation determined by $\mathbb{V}$ is called *normal* if it is of the form $\Pi(s)$ for a parallel section s of the tractor bundle $\mathcal{V}M$.

Note that part (4) of Proposition 2.2 says that on the homogeneous model (and hence also on geometries locally isomorphic to the homogeneous model) any solution of a first BGG equation is normal. It should be remarked at this point, that the name "normal" reflects that the solutions concerned correspond to parallel sections of the normal (in the sense of [9]) tractor connection, and this terminology is already established in the literature (cf. [34]). It is not directly related to normal coordinates and normal frames which we will discuss next.

2.3. **Normal coordinates and normal frames.** We now come to the basic construction needed to describe parallel sections of tractor bundles and hence normal solutions of first BGG operators.

Let us start with a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of some fixed type (G, P). Fix a point $u_0 \in \mathcal{G}$ and put $x_0 := p(u_0) \in M$. Consider the subalgebra $\mathfrak{g}_- \subset \mathfrak{g}$ which is complementary to $\mathfrak{p}$ as in 2.1. For any $X \in \mathfrak{g}_-$ we can consider the constant vector field $\tilde{X} \in \mathfrak{X}(\mathcal{G})$ which is characterised by $\omega(\tilde{X})(u) = X$ for all $u \in \mathcal{G}$. There is an open neighbourhood $V \subset \mathfrak{g}_-$ of zero such that the flow $\mathrm{Fl}^{\tilde{X}}_t(u_0)$ through u_0 is defined up to time $t = 1$ for all $X \in V$. Then $\Phi(X) := \mathrm{Fl}^{\tilde{X}}_1(u_0)$ defines a smooth map $\Phi : V \to \mathcal{G}$ and we define $\varphi := p \circ \Phi : V \to M$.

By construction, $\varphi(0) = x_0$ and the derivative $T_0\varphi : \mathfrak{g}_- \to T_{x_0}M$ at this point is given by $X \mapsto T_{u_0}p{\cdot}\omega_{u_0}^{-1}(X)$. Since $\mathfrak{g}_-$ is complementary to $\mathfrak{p}$, this is a linear isomorphism by the defining properties of a Cartan connection. Hence we can shrink V in such a way that φ defines a diffeomorphism from V onto an open neighbourhood U of x_0 in M.

Definition 2.4. (1) The *normal chart* determined by u_0 is the diffeomorphism $\varphi^{-1} : U \to V \subset \mathfrak{g}_-$. Choosing a basis in $\mathfrak{g}_-$, we get induced local coordinates on M called the *normal coordinates* determined by u_0.

(2) The *normal section* of $\mathcal{G}|_U$ determined by u_0 is the smooth map $\sigma : U \to \mathcal{G}$ characterised by $\sigma(p(\Phi(X)) = \Phi(X)$ for all $X \in V$.

Remark 2.5. One can slightly vary the construction of the normal section as follows. Since $\mathcal{G} \to M$ is a principal P–bundle, any closed subgroup of P acts freely on $\mathcal{G}$ via the principal right action. Applying this to the closed subgroup $P_+ \subset P$ from 2.1, one shows that $\mathcal{G}_0 := \mathcal{G}/P_+$ is a smooth manifold, $p : \mathcal{G} \to M$ descends to a map $p_0 : \mathcal{G}_0 \to M$ and this is a principal bundle with structure group $P/P_+ = G_0$. There is an obvious projection $q : \mathcal{G} \to \mathcal{G}_0$ which is actually a principal bundle with structure group P_+. As we have seen, $p_0 \circ q \circ \Phi = p \circ \Phi$ is a diffeomorphism, so $q \circ \Phi$ meets any fibre of $\mathcal{G}_0$ in at most one point. Hence the map sending $q(\Phi(X))$ to $\Phi(X)$ naturally extends to a G_0–equivariant smooth section of q over $(p_0)^{-1}(U)$. In the language of section 5.1.12 of [12], this is the *normal Weyl structure* centred at x_0 which is determined by $u_0 \in p^{-1}(x_0)$. In this way, one gets additional data, e.g. the so called Weyl connection on associated bundles, but we will not need those in the sequel.

Recall next, that the natural vector bundles for a parabolic geometry of type (G, P) are the associated vector bundles to the Cartan bundle, so they are equivalent to representations of the parabolic subgroup P. Likewise, natural bundle maps between such bundles are equivalent to P–equivariant maps between the inducing representations. Of course, a local section of a principal bundle gives rise to a local trivialisation and hence also to local trivialisations of all associated bundles.

Definition 2.6. (1) Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P), $u_0 \in \mathcal{G}$ a point and $\mathbb{W}$ a representation of P.

The *normal trivialisation* of the associated bundle $\mathcal{W}M = \mathcal{G} \times_P \mathbb{W}$ determined by u_0 is the trivialisation induced by the normal section determined by u_0.

(2) A *normal frame* for $\mathcal{W}M$ determined by u_0 is a frame obtained from a basis of $\mathbb{W}$ via a normal trivialisation.

Explicitly, given a representation $\mathbb{W}$ of P, the corresponding natural vector bundle $\mathcal{W}M$ is $\pi : \mathcal{G} \times_P \mathbb{W} \to M$, where $\mathcal{G} \times_P \mathbb{W}$ is the quotient of $\mathcal{G} \times \mathbb{W}$ by the action of P defined by $(u, w)\cdot b := (u\cdot b, b^{-1}\cdot w)$. Here we use the principal right action in the first component and the given representation of P on $\mathbb{W}$ in the second component. The trivialisation induced by σ is then given by the map $U \times \mathbb{W} \to \pi^{-1}(U)$ which maps (x, w) to the P–orbit of $(\sigma(x), w)$. Conversely, given $(u, w) \in \mathcal{G} \times \mathbb{W}$ with $x = p(u) \in U$, there is a unique element $b \in P$ such that $u = \sigma(x)\cdot b$, so the P–orbit of (u, w) corresponds to $(x, b\cdot w)$ in the trivialisation. By construction, the natural bundle map induced by a P–equivariant map $\alpha : \mathbb{W} \to \mathbb{W}'$ corresponds to $\mathrm{id}_U \times \alpha : U \times \mathbb{W} \to U \times \mathbb{W}'$ in this trivialisation.

It will also be very useful in the sequel to have a description of the trivialisations determined by σ in terms of smooth sections. Recall that smooth sections of the bundle $\mathcal{G} \times_P \mathbb{W} \to M$ over $U \subset M$ are in bijective correspondence with smooth maps $f : p^{-1}(U) \to \mathbb{W}$, which are P–equivariant in the sense that $f(u\cdot b) = b^{-1}\cdot f(u)$. In the local trivialisation determined by σ, this section is simply given by $x \mapsto (x, f(\sigma(x)))$, so it simply corresponds to the function $f \circ \sigma : U \to \mathbb{W}$.

Since natural bundle maps are induced by equivariant maps between the representations inducing the bundles, they are nicely compatible with normal trivialisations and with normal frames. Let us just formulate a particularly important case explicitly.

Lemma 2.7. *Let $\mathbb{W}$ and $\mathbb{W}''$ be representations of P and let $\mathbb{W}' \subset \mathbb{W}$ be a P–invariant subspace such that $\mathbb{W}/\mathbb{W}' \cong \mathbb{W}''$, and let n' and n'' be the dimensions of $\mathbb{W}'$ and $\mathbb{W}''$, respectively. Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P), and let $\tau : \mathcal{W}M/\mathcal{W}'M \to \mathcal{W}''M$ be the induced isomorphism of associated bundles.*

Then for any point $u_0 \in \mathcal{G}$ there is a normal frame of $\mathcal{W}M$ over the corresponding subset U which has the form $\{s_1, \dots, s_{n'}, s_{n'+1}, \dots, s_{n'+n''}\}$ where $\{s_1, \dots, s_{n'}\}$ is a normal frame for $\mathcal{W}'M$ over U and $\{\tau(s_{n'+1}), \dots, \tau(s_{n'+n''})\}$ is a normal frame for $\mathcal{W}''M$ over U.

Proof. We just have to choose a basis of $\mathbb{W}'$ and extend it to a basis of $\mathbb{W}$. The additional elements will then descend to a basis of $\mathbb{W}/\mathbb{W}'$, so there is a corresponding basis of $\mathbb{W}''$. Then the normal frames of the three induced bundles determined by these three bases will be related in the way claimed in the lemma. □

2.4. **The case of the homogeneous model.** The homogeneous model of parabolic geometries of type (G, P) is the principal P–bundle $p : G \to G/P$ together with the (left) Maurer Cartan form as a Cartan connection. For this example, we can describe normal sections and normal coordinates in a completely explicit way. We also obtain an explicit description of the normal frames of the tangent bundle. By naturality of normal frames, this also provides normal frames for all tensor bundles. For more complicated examples of parabolic geometries, which involve a non–trivial filtration of the tangent bundle, we can obtain normal frames adapted to the filtration in this way. Via Lemma 2.7, these can be used to obtain normal frames of the quotients of subsequent filtration components which provide the main constituents for natural bundles for such geometries.

The results we are going to prove in the sequel all describe the component functions obtained by expanding certain sections in terms of a normal frame. Together with the description of normal frames we will prove now, we get a complete understanding of first BGG solutions in the case of the homogeneous model. To obtain similar results for general geometries, one needs an explicit description of a normal frame, which essentially amounts to getting an explicit description of the canonical Cartan connection.

By homogeneity, we can take the distinguished point in the total space of the bundle $G \to G/P$ to be the identity element e. By definition, the constant vector field $\tilde{X} \in \mathfrak{X}(G)$ for the Maurer Cartan form generated by $X \in \mathfrak{g}$ is simply the left invariant vector field L_X. Hence $\mathrm{Fl}_t^{\tilde{X}}(e) = \exp(tX)$, $\Phi(X) = \exp(X)$ and $\varphi(X) = \exp(X)P$ for $X \in \mathfrak{g}_-$. It is well known that in this case φ is defined on all of $\mathfrak{g}_-$ and it defines a diffeomorphism onto an open subset of G/P. The normal Weyl structure on this open subset of G/P obtained in this way is the *very flat Weyl structure* as described in Example 5.1.12 of [12].

To formulate the description of normal frames for the tangent bundle, recall that there is a unique connected and simply connected Lie group G_- with Lie algebra $\mathfrak{g}_-$. This is often called the *Carnot–group* associated to $\mathfrak{g}_-$. It is well known that the exponential map for this group defines a global diffeomorphism $\exp_{G_-} : \mathfrak{g}_- \to G_-$. Now any vector $X \in \mathfrak{g}_-$ generates a left invariant vector field on G_-. In particular, choosing a basis for $\mathfrak{g}_-$ we obtain a global left invariant frame for the tangent bundle TG_-.

Proposition 2.8. *In the normal chart determined by $e \in G$, the normal frames for the tangent bundle $T(G/P)$ are exactly the pullbacks along the diffeomorphism $\exp_{G_-} : \mathfrak{g}_- \to G_-$ of the left invariant frames of the tangent bundle TG_-.*

In particular, if $\mathfrak{g}_-$ is abelian, then these are exactly the holonomic frames of coordinate vector fields determined by a choice of basis of $\mathfrak{g}_-$.

Proof. Given a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) an element $X \in \mathfrak{g}$ and a point $u \in \mathcal{G}$, one immediately sees that $T_u p \cdot \tilde{X}(u) \in T_{p(u)}M$ depends only on $X + \mathfrak{p} \in \mathfrak{g}/\mathfrak{p}$. This easily implies that the tangent bundle TM can be identified with the associated bundle $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$. To obtain a normal frame for $T(G/P)$, we thus have to choose a basis of $\mathfrak{g}/\mathfrak{p}$, choose preimages of the basis elements in $\mathfrak{g}$, and then project the corresponding left invariant vector fields to G/P. Since $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$, there are unique preimages contained in $\mathfrak{g}_-$ and conversely, any basis of $\mathfrak{g}_-$ projects onto a basis of $\mathfrak{g}/\mathfrak{p}$. Thus we may as well start from a basis of $\mathfrak{g}_-$.

Now the inclusion of $\mathfrak{g}_-$ into $\mathfrak{g}$ uniquely lifts to an injective homomorphism $i : G_- \to G$. In particular, this implies that the tangent map Ti maps left invariant vector fields to left invariant vector fields. Since $\Phi = i \circ \exp_{G_-}$, we obtain the first result.

If $\mathfrak{g}_-$ is abelian, then $G_- = \mathfrak{g}_-$ (with the group structure given by addition) and $\exp_{G_-} = \mathrm{id}$. Moreover, the left invariant vector fields for this group are just the constant vector fields for the obvious trivialisation of $T\mathfrak{g}_-$, i.e. the coordinate vector fields. □

2.5. **The basic polynomiality result.** We are now ready to prove the basic fact that the coefficients of any normal solution to a first BGG operator in a normal frame are polynomials in the corresponding normal coordinates; we also obtain a bound on the degree of these polynomials. First, we need a lemma:

Lemma 2.9. *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of some fixed type (G, P) and consider the tractor bundle $\mathcal{V}M \to M$ corresponding to a representation $\mathbb{V}$ of G. Fix a point $u_0 \in \mathcal{G}$, write $x_0 = p(u_0) \in M$ and consider the normal section $\sigma : U \to \mathcal{G}$ centred at x_0 which is determined by u_0.*

If $s \in \Gamma(\mathcal{V}M)$ is parallel for the canonical tractor connection, then the function $f : U \to \mathbb{V}$, which describes s in the given normal trivialisation, is given by $f(\varphi(X)) = \exp(-X)\cdot f(x_0)$.

Proof. Let $\tilde{f} : \mathcal{G} \to \mathbb{V}$ be the P–equivariant function corresponding to s. Denoting the tractor connection by ∇^V, the function $\mathcal{G} \to L(\mathfrak{g}/\mathfrak{p}, \mathbb{V})$ corresponding to $\nabla^V s$ is given by $u \mapsto ((X + \mathfrak{p}) \mapsto (\omega_u^{-1}(X)\cdot\tilde{f})(u) + X\cdot(\tilde{f}(u)))$. Now consider the smooth curve $c(t) := \mathrm{Fl}_t^{\tilde{X}}(u_0)$, where $\tilde{X}$ is the constant vector field determined by $X \in \mathfrak{g}_-$. Then $c'(t) = \omega_{c(t)}^{-1}(X)$ for all t, so $(\tilde{f} \circ c)'(t) = \omega_{c(t)}^{-1}(X)\cdot\tilde{f}$. If the initial section s is parallel, we conclude that $\tilde{f} \circ c$ satisfies the differential equation $(\tilde{f} \circ c)'(t) = -X\cdot(\tilde{f} \circ c)(t)$. Since the unique solution of this is $(\tilde{f} \circ c)(t) = \exp(-tX)\cdot\tilde{f}(c(0))$, $c(0) = \sigma(x_0)$ and for X close enough to zero we have $c(1) = \sigma(\varphi(X))$ the result follows immediately from $f = \tilde{f} \circ \sigma$. □

Now recall, from Section 2.2, that given an irreducible G–representation $\mathbb{V}$ we have the filtration $\{\mathbb{V}^i\}$ and the P–irreducible quotient $\mathbb{H}_0$. For a given parabolic geometry $(p : \mathcal{G} \to M, \omega)$ we can then consider normal solutions of the first BGG operator determined by $\mathbb{V}$. By definition, these are the images of those sections of $\mathcal{V}M = \mathcal{G} \times_P \mathbb{V}$ which are parallel, for the tractor connection, under the natural bundle map $\Pi : \mathcal{V}M \to \mathcal{H}_0 M$ induced by the quotient projection. Now we are ready to formulate our basic polynomiality result.

Theorem 2.10. *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) and let $\mathbb{V}$ be a representation of G with natural grading $\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_N$, and suppose that $\alpha \in \Gamma(\mathcal{H}_0 M)$ is a normal solution to the first BGG operator determined by $\mathbb{V}$.*

Then for any normal section σ, the coefficients of α in a normal frame are polynomials of degree at most N in the normal coordinates determined by σ.

Proof. Having given the normal section σ, we use normal frames as in Lemma 2.7 by choosing an appropriate basis of $\mathbb{V}$. Having chosen this basis, we obtain an identification of $L(\mathbb{V},\mathbb{V})$ with the space of real matrices of appropriate size. Likewise, we have to choose a basis of $\mathfrak{g}_-$ in order to obtain normal coordinates, and we extend this to a basis of $\mathfrak{g}$.

The representation ρ of $\mathfrak{g}$ on $\mathbb{V}$ is a linear map, so for $X \in \mathfrak{g}_-$ and $v \in \mathbb{V}$ the coordinates of $X \cdot v = \rho(X)v$, with respect to the chosen basis of $\mathbb{V}$, are linear expressions in the coordinates of X in the chosen basis of $\mathfrak{g}_-$. Now the grading $\{\mathbb{V}_j\}$ of $\mathbb{V}$ has the property that $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j}$. In particular, for any element $X \in \mathfrak{g}_i$ with $i < 0$, the corresponding linear map $\rho(X) : \mathbb{V} \to \mathbb{V}$ has the property that $\rho(X)^{N+1} = 0$.

Now let $s \in \Gamma(VM)$ be a section which is parallel for the tractor connection induced by ω and let $f : U \to \mathbb{V}$ be the function describing s in the normal trivialisation induced by σ. Then from Lemma 2.9 we know that

$$f(\varphi(X)) = \exp(-X) \cdot v_0 = \sum_{k=0}^{N} \frac{(-1)^k}{k!} \rho(X)^k v_0 \tag{2.1}$$

for an appropriate element $v_0 \in \mathbb{V}$. But this implies that the components of s in the normal frame determined by σ and the chosen basis of $\mathbb{V}$ are polynomials of degree at most N in the normal coordinates. By construction, the bundle map $\Pi : \mathcal{V}M \to \mathcal{H}_0 M$ maps some elements of the normal frame of $\mathcal{V}M$ to the normal frame of $\mathcal{H}_0 M$ and the remaining elements to zero, so since $\alpha = \Pi \circ s$, the result follows. □

Remark 2.11. Note that in the proof above we obtain more than is claimed in the theorem. The right-hand-side of expression (2.1) *is* the polynomial system describing the parallel tractor s in the normal trivialisation.

Obtaining the explicit presentation of the polynomial system giving α now simply requires the description of Π arising from the choice of adapted basis for $\mathbb{V}$. This requires an application of elementary representation theory, the details of which depend on G, P, and $\mathbb{V}$. For a given parabolic geometry (so with G, and P fixed) a practical solution to this, which we shall take up in section 3 below, is to exploit the fact that representations of G can be built up via tensorial constructions from some basic representations and correspondingly tractor bundles can be built up from some basic bundles.

Before we discuss examples in detail we continue with a line of argument that does not depend on knowing the particular parabolic geometry or details of the representation $\mathbb{V}$, namely the comparison to the homogeneous model, which is the main tool in the study of projective structures in [10] and of holonomy reductions of general Cartan geometries in [11].

2.6. **Comparison to the homogeneous model.** Recall the description of the normal section and normal chart determined by $e \in G$ for the homogeneous model $G \to G/P$ from section 2.4. From now on, to distinguish this setting from the general case, we shall write $\underline{\Phi}$ and $\underline{\varphi}$ for these maps. Recall that $\underline{\varphi}$ is defined on all of $\mathfrak{g}_-$ and gives a diffeomorphism onto an open subset of G/P. Now we can combine the construction of section 2.3 for a general geometry of type (G, P) with the same construction for the homogeneous model.

Definition 2.12. Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P). The *normal comparison map* determined by a point $u_0 \in \mathcal{G}$ is the composition $\psi := \underline{\varphi} \circ \varphi^{-1}$, where φ^{-1} denotes the normal chart determined by u_0.

Evidently, the normal comparison map defines a diffeomorphism from the domain U of the normal chart onto an open neighbourhood $\underline{U}$ of $o = eP$ in G/P.

Proposition 2.13. *Let $\mathbb{V}$ be an irreducible representation of G with P–irreducible quotient $\mathbb{H}_0$. Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) and for a point $u_0 \in \mathcal{G}$ consider the normal comparison map $\psi : U \to \underline{U}$.*

If $\alpha \in \Gamma(\mathcal{H}_0 M)$ is a normal solution of the first BGG operator determined by $\mathbb{V}$, then the coordinate functions of α with respect to a normal frame of $\mathcal{H}_0 M$ are the pullbacks along ψ of the coordinate functions of a unique corresponding solution of the first BGG operator determined by $\mathbb{V}$ in the corresponding normal frame of $\mathcal{H}_0(G/P)$.

Proof. It clearly suffices to prove this for one choice of normal frames, so we may use normal frames as in Lemma 2.7. Thus the normal frames for the $\mathcal{H}_0$–bundles are just the projections of some elements of normal frames of corresponding the tractor bundles. By definition, there is a parallel section $s \in \Gamma(VM)$ such that $\alpha = \Pi(s)$. From Lemma 2.9 we know that the function $f : U \to \mathbb{V}$ corresponding to s is given by $f(\varphi(X)) = \exp(-X)\cdot f(x_0)$. Now on the homogeneous model, any tractor bundle is canonically trivial and the tractor connection is the flat connection induced by this trivialisation. (Thus on the homogeneous model all first BGG solutions are normal.) In particular, there is a unique parallel section $\underline{s} \in \Gamma(\mathcal{V}(G/P))$ such that the corresponding function $\underline{U} \to \mathbb{V}$ maps o to $f(x_0)$. Again by Lemma 2.9, this function must map $\underline{\varphi}(X)$ to $\exp(-X)\cdot f(x_0)$. This means that the component functions of s and $\underline{s}$ in any compatible pair of normal frames are intertwined by ψ, and since $\underline{s}$ projects to a solution of the first BGG operator determined by $\mathbb{V}$, and conversely this solution determines $\underline{s}$, the result follows. □

Of course, this result implies that the comparison map $\psi : U \to \underline{U}$ restricts to a bijection between the zero sets of the two normal solutions in question. In particular, the zero sets of normal solutions on curved geometries locally look like zero sets of solutions on the model. The latter can often be nicely analysed using algebraic geometry (or even linear algebra), c.f. [10]; Proposition 2.13 shows that the *same* polynomial systems apply in the curved setting.

To get stronger consequences from this line of argument, one has to analyse the properties of the comparison maps for individual structures in more detail. Of course, the comparison map cannot be a morphism of parabolic geometries unless $(p : \mathcal{G} \to M, \omega)$ is locally flat. (It is a local isomorphism in the locally flat case.) However, by construction it is compatible with the projections of flow lines of constant vector fields through x_0 respectively through o. Each of these flow lines is a distinguished curve of the geometry in the sense of Section 5.3 of [12], so one gets compatibility with some canonical curves through x_0. However, not all canonical curves through x_0 are obtained in that way (since the point u_0 remains fixed). For most geometries, at least some of the canonical curves (e.g. null geodesics in pseudo–Riemannian conformal structures and chains in CR structures) are uniquely determined by their initial direction up to parametrisation so one can get nice information on those. This works particularly well in the situation of projective structures as discussed in [10]. There all canonical curves are uniquely determined by their initial direction up to parametrisation, so one simply gets compatibility of ψ with the unparametrised canonical curves through x_0, which is heavily used in [10].

2.7. **Remark on G–types and P–types.** We only touch on this topic briefly, because it is discussed in detail in our article [11] in the more general context of holonomy reductions of parabolic geometries and in [10] for projective structures. Consider a representation $\mathbb{V}$ of G, the corresponding tractor bundle $\mathcal{V}M$ for a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) and a section $s \in \Gamma(\mathcal{V}M)$. Then for a given point $x \in M$, the image of the fibre $\mathcal{G}_x$ under the equivariant function corresponding to s is a P–orbit in $\mathbb{V}$ that we term the *P–type of s at x*. For a given normal solution α, it is obvious from Proposition 2.2 that any point in the zero locus of α lies in a different P-type to a point where α is non-zero. In fact much more information is available and in both the articles mentioned examples related to normal solutions are discussed.

It is shown in [10, 11] that in the case of a parallel section s and a connected base M, the P–types of all points are contained in a single G–orbit $\mathcal{O} \subset \mathbb{V}$, which is called the G–type of s. If $\mathcal{O} = \cup_i \mathcal{O}_i$ is the decomposition of the G–orbit into P–orbits, there is an induced decomposition $M = \cup_i M_i$ according to P–type. In [11] it is proved that each of the M_i is an initial submanifold in M. For the homogeneous model, this decomposition is the decomposition of G/P into orbits under the action of the isotropy group H of some chosen element of $\mathcal{O}$. Thus, the general decomposition is called a *curved orbit decomposition*. If α is the normal solution of the first BGG operator determined by s, then the zero–set of α is stratified into a union of P–types.

Finally, the H–orbits in G/P are all homogeneous spaces and thus homogeneous models of Cartan geometries. In the curved case, one obtains Cartan geometries of the same types over the individual curved orbits. The curvatures of these induced Cartan geometries are related in a precise way (that we describe) to the curvature of the original geometry of type (G, P).

3. Examples and applications

In this section, we show how to apply the general principles and results obtained above to specific structures. We mainly exploit the fact that the representations of a classical Lie group G (and hence the corresponding tractor bundles on parabolic geometries of type (G, P) for any parabolic subgroup P of G) can be built from the standard representation (and spin representations in the orthogonal case). This can be used to explicitly describe potential parallel sections of tractor bundles as well as their projections to the canonical quotients. The details of this of course strongly depend on the choices of G and P, but the method is quite universal.

3.1. **Normal frames and generalised homogeneous coordinates for oriented projective structures.** Here we put $G = SL(n+1, \mathbb{R})$, identified with its defining representation, and $P \subset G$ the stabiliser of the ray spanned by the first vector in the standard basis, which corresponds to oriented projective structures. Let us start by showing how to recover the generalised homogeneous coordinates from [10] in our setting.

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$ consists of matrices of the form $\begin{pmatrix} -\operatorname{tr}(A) & Z \\ X & A \end{pmatrix}$ with $X \in \mathbb{R}^n$, $Z \in \mathbb{R}^{n*}$ and $A \in \mathfrak{gl}(n, \mathbb{R})$, and X, A, and Z represent the grading components $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$, and $\mathfrak{g}_1$, respectively. We start the construction of normal frames with the *standard tractor bundle* $\mathcal{T}M$, which corresponds to the standard representation $\mathbb{R}^{n+1}$ of G. Let us denote the standard basis of $\mathbb{R}^{n+1}$ by $e_0, \dots, e_n$, so P stabilises the line $\mathbb{R}{\cdot}e_0$, and the quotient $\mathbb{R}^{n+1}/\mathbb{R}{\cdot}e_0$ is an irreducible representation of P. In the standard notation for projective structures the bundle induced by $\mathbb{R}{\cdot}e_0$ is a density bundle usually denoted by $\mathcal{E}(-1)$, while $\mathbb{R}^{n+1}/\mathbb{R}{\cdot}e_0$ induces the bundle $TM \otimes \mathcal{E}(-1)$. We will denote the latter bundle by $TM(-1)$ and in general use the convention that adding "(w)" to the name of a bundle indicates a tensor product with the density bundle $\mathcal{E}(w)$.

Given a projective structure, the corresponding parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P), and a normal chart U, we consider the corresponding normal frames. The normal frame for the density bundle $\mathcal{E}(-1)$ is a specific non–vanishing section determined by the basis vector e_0. It will be better, however, to take the corresponding non–vanishing section X^0 of $\mathcal{E}(1) = \mathcal{E}(-1)^*$ as the basic object, so then e_0 corresponds to the density $(X^0)^{-1}$. The basis vectors $\{e_1, \dots, e_n\}$ then can be used to complete this to the adapted normal frame of the standard tractor bundle $\mathcal{T}M$, and projecting them to the quotient, one obtains the normal frame for $TM(-1)$, which we denote by $\{\xi_1, \dots, \xi_n\}$.

Let us use the obvious identification of $\mathfrak{g}_{-1}$ with $\mathbb{R}^n$ to define coordinates and denote the corresponding normal coordinates on M by $x_1, \dots, x_n \in C^\infty(M, \mathbb{R})$. Then for $i = 1, \dots, n$ we define $X^i \in \Gamma(\mathcal{E}(1))$ by $X^i := x_i X^0$.

Proposition 3.1. *The densities $X^0, \dots, X^n \in \Gamma(\mathcal{E}(1))$ are exactly the (local) generalised homogeneous coordinates on M introduced in* [10].

Proof. By its construction, it is clear that the comparison map to the homogeneous model is compatible with the normal coordinates x_i and thus also with the densities X^i. Comparing with the construction in [10], it thus suffices to show that on the homogeneous model the X^i are obtained from the standard coordinates on $\mathbb{R}^{n+1} \setminus \{0\}$. The normal chart in this case simply is given by $X \mapsto \exp(X)(e_0)$, and from the presentation of $\mathfrak{g}$ above it is obvious that this is given by $e_0 + \sum_{i=1}^n x_i e_i$, where the x_i are the components of X. This immediately implies the claim. □

3.2. **Normal solutions for projective structures.** To proceed further, we need some information on the first BGG operators on projective structures, which is taken from [4]. An irreducible representation $\mathbb{V}$ of G and hence the corresponding tractor bundle can be determined by its irreducible quotient $\mathbb{H}_0$ (viewed as a representation of the semisimple part of G_0) and a single non–negative integer k. In this case, we will say that $\mathbb{V}$ corresponds to $(\mathbb{H}_0, k)$. The first BGG operator is then defined on the bundle $\mathcal{H}_0 M(w)$ (induced by $\mathbb{H}_0$), for an appropriate value of w, and has order k. The range of the operator lies in the bundle induced by the P–representation $S^k\mathbb{R}^{n*} \odot \mathbb{H}_0(w')$ for appropriate w'. Here $\odot$ denotes the *Cartan product*, i.e. the irreducible component of maximal highest weight in $S^k\mathbb{R}^{n*} \otimes \mathbb{H}_0$. Moreover, if $\mathbb{V}$ is the representation of G corresponding to $(\mathbb{H}_0, 1)$, then the G–representation corresponding to $(\mathbb{H}_0, k)$ is $S^{k-1}\mathbb{R}^{(n+1)*} \odot \mathbb{V}$ for any $k \geq 2$.

The basic building blocks of representations of G are the fundamental representations $\Lambda^r\mathbb{R}^{n+1}$ for $r = 1, \dots, n$ (of course $\Lambda^n\mathbb{R}^{n+1} \cong \mathbb{R}^{(n+1)*}$). The description of the composition series of $\Lambda^r\mathbb{R}^{n+1}$ follows readily from the one of $\mathbb{R}^{n+1}$. There is a P–invariant subspace isomorphic to $\Lambda^{r-1}\mathbb{R}^n$ (spanned by the wedge products of basis elements which involve e_0) and the quotient by this is the irreducible representation $\Lambda^r\mathbb{R}^n$. In the above description of irreducible representations of G, $\Lambda^r\mathbb{R}^{n+1}$ corresponds to $(\Lambda^r\mathbb{R}^n, 1)$ for $r < n$ and $(\mathbb{R}, 2)$ for $r = n$. Hence the corresponding first BGG operator has order one for $r < n$ and order two for $r = n$. On the level of associated bundles, the invariant subspace corresponds to $\Lambda^{r-1}TM(-r)$, while the quotient corresponds to $\Lambda^r TM(-r)$ and $\Lambda^n TM(-n) \cong \mathcal{E}(1)$.

Theorem 3.2. *Consider a projective structure on a smooth manifold M of dimension n, consider a normal chart U on M, let $x_1, \dots, x_n$ be the corresponding normal coordinates, and let $\{\xi_1, \dots, \xi_n\}$ be the normal frame of $TM(-1)$ determined by U. Then we have:*

(1) For $r < n$, any normal solution of the first BGG operator defined on $\Lambda^r TM(-r)$ restricts on U to a linear combination with constant coefficients of the following sections

$$\begin{cases} \xi_{i_1} \wedge \cdots \wedge \xi_{i_r} & \text{for } 1 \leq i_1 < i_2 < \cdots < i_r \leq n \\ \sum_j x_j \xi_j \wedge \xi_{i_1} \wedge \cdots \wedge \xi_{i_{r-1}} & \text{for } 1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n \end{cases}$$

In the case of the homogeneous model S^n, ξ^i is the coordinate frame associated to a choice of basis of $\mathfrak{g}_-$, each of the sections listed above is the restriction of a solution, and they form a basis for the space of solutions of the first BGG operator on U.

(2) For $k \geq 2$, any normal solution of the (kth order) first BGG operator defined on $\mathcal{E}(k-1)$ is a homogeneous polynomial of degree $k-1$ in the generalised homogeneous coordinates $\{X^0, \dots, X^n\}$ determined by U. In the case of S^n, all such polynomials arise as solutions.

(3) For $k \geq 2$, any normal solution of the (kth order) first BGG operator acting on $\Lambda^r TM(k-r-1)$ can be written as a linear combination of the sections listed in (1) with coefficients which are homogeneous polynomials of degree $k-1$ in the generalised homogeneous coordinates $\{X^0, \dots, X^n\}$.

Proof. (1) For a matrix X contained in the subspace $\mathfrak{g}_{-1}$ of matrices from 3.1, one simply has $\exp(-X) = \mathbb{I} - X$, where $\mathbb{I}$ denotes the unit matrix. The action on the standard representation $\mathbb{R}^{n+1}$ is thus given by $\exp(-X)(e_0) = e_0 - \sum_{j=1}^n x_j e_j$ and $\exp(-X)(e_i) = e_i$ for $i > 0$, where the x_j are the components of X. Denoting by $\{s_0, \dots, s_n\}$ the normal frame determined by U and the basis $\{e_0, \dots, e_n\}$ of $\mathbb{R}^{n+1}$, Lemma 2.9 shows that any parallel section of the standard tractor bundle $\mathcal{T}M$ must be given by a linear combination with constant coefficients of the sections $s_0 - \sum_{j=1}^n x_j s_j$ and $s_1, \dots, s_n$. The projection to the quotient bundle $TM(-1)$ maps s_0 to zero and s_i to ξ_i for $i = 1, \dots, n$ (compare with Lemma 2.7). Hence any normal solution of the first BGG operator on this quotient bundle must be a linear combination with constant coefficients of $\xi_0 := \sum_{j=1}^n x_j \xi_j$ and $\xi_1, \dots, \xi_n$, which is our claim for $r = 1$.

Now the wedge products of the elements of the standard basis of $\mathbb{R}^{n+1}$ form a basis for $\Lambda^r \mathbb{R}^{n+1}$, and we conclude that any parallel section of $\Lambda^r \mathcal{T}M$ must then be a linear combination with constant coefficients of the wedge products of the sections $s_0, \dots, s_n$. But the projection of a wedge product of sections is just the wedge product of the projections of the individual sections. Expanding this for the wedge products involving ξ_0, we obtain the claimed list of sections. In the case of the homogeneous model, each of the sections s_i and thus any wedge product of these sections actually is parallel, which together with Proposition 2.8 implies the last part of (1).

(2) Let us look at the result of (1) in the case $r = n$. Then we get the sections $\xi_1 \wedge \dots \wedge \xi_n$ and (up to a sign that we may ignore) $x_i \xi_1 \wedge \dots \wedge \xi_n$ for $i = 1, \dots, n$. The quotient bundle $\Lambda^n TM(-n)$ is a line bundle which is trivialised over U by the section $\xi_1 \wedge \dots \wedge \xi_n$. Compatibility of normal sections with constructions on the inducing representations shows that this section has to be a normal frame for this bundle. Since $\Lambda^n \mathbb{R}^{n+1} \cong \mathbb{R}^{(n+1)*}$, we must have $\Lambda^n TM(-n) \cong \mathcal{E}(1)$ with the isomorphism mapping $\xi_1 \wedge \dots \wedge \xi_n$ to X^0. Hence we conclude that any normal solution of the first BGG operator on $\mathcal{E}(1)$ is a linear combination with constant coefficients of the generalised homogeneous coordinates, which is our claim for $k = 2$.

What we have actually done here was construct a frame for the cotractor bundle $\mathcal{T}^*M$, over U, such that any parallel section of $\mathcal{T}^*M$ must be a linear combination with constant coefficients of the frame elements and such that the projections of the frame elements to the quotient bundle $\mathcal{E}(1)$ are exactly the generalised homogeneous coordinates X^i. Now we immediately conclude that the symmetric power $S^{k-1}\mathcal{T}^*M$ has $\mathcal{E}(k-1) \cong S^{k-1}\mathcal{E}(1)$ as its natural quotient. The symmetric products of $k-1$ elements of our frame for $\mathcal{T}^*M$ form a frame for $S^{k-1}\mathcal{T}^*M$ such that any parallel section must be a linear combination of the frame elements with constant coefficients. Projecting a symmetric product of sections to $\mathcal{E}(k-1)$, one of course obtains the product of the projections of the individual sections. This completes the proof of (2).

(3) This now immediately follows since from (1) and (2) we can form a frame of $S^{k-1}\mathcal{T}^*M \otimes \Lambda^r\mathcal{T}M$ such that any parallel section of this bundle must be a linear combination with constant coefficients of the frame elements. As we have noticed above, the first BGG operator defined on $\Lambda^r TM(k-r-1)$ comes from the subbundle corresponding to $S^{k-1}\mathbb{R}^{(n+1)*} \odot \Lambda^r\mathbb{R}^{n+1}$. Of course, we can construct a frame of this subbundle which consists of linear combinations of tensor products of elements of the frames of the two factors as constructed in (1) and (2). The projection of such a tensor product is again the tensor product of the projections of the factors, which implies the result. □

3.3. **A more involved example for projective structures.** The result in part (3) of Theorem 3.2 is not optimal, since only certain linear combinations of the elements of the frame of $S^{k-1}\mathcal{T}^*M\otimes\Lambda^r\mathcal{T}M$ constructed in the proof will actually lie in $S^{k-1}\mathcal{T}^*M \odot \Lambda^r\mathcal{T}M$. Going into more details on the decomposition of tensor products, one can improve the result. Using similar considerations, one can extend the results of part (1) to operators defined on more complicated bundles, and we discuss an example of this. We want to describe solutions of the first BGG operators on the bundle $S^2T^*M(4)$, which is of order one. To formulate the result, it will be better to first recast the result of Theorem 3.2 for $r = n-1$ in terms of the bundle $T^*M(2)$. We have chosen this example since both these results are of interest for Riemannian geometry, see 3.4 below.

We can identify $\Lambda^{n-1}\mathbb{R}^{n+1}$ with $\Lambda^2\mathbb{R}^{(n+1)*}$. It then follows immediately that for the bundle $\Lambda^2\mathcal{T}^*M$, the irreducible quotient is $T^*M(2)$ while the subbundle contained in there is $\Lambda^2T^*M(2)$. Now consider the normal frame $\{\varphi_1,\dots,\varphi_n\}$ for $T^*M(2)$. Under the isomorphism $\Lambda^{n-1}TM(n-1)\cong T^*M(2)$ the element φ_i corresponds of course (up to a sign which is not relevant for us) to the wedge product of the ξ_j for $j \neq i$. Using this, we can read off the following from part (1) of Theorem 3.2:

Corollary 3.3. *In terms of a normal frame $\{\varphi_1,\dots,\varphi_n\}$ for $T^*M(2)$ and the corresponding normal coordinates $x_1,\dots,x_n$, any normal solution of the first BGG operator on $T^*M(2)$ can be written as a linear combination with constant coefficients of the forms φ_i for $i = 1,\dots,n$ and $x_i\varphi_j - x_j\varphi_i$ for $i < j$. On the homogeneous model the φ_i are coordinate forms, each of the listed forms is a solution, and they form a basis for the space of all solutions.*

We can use an analogue of homogenising (as in projective algebraic geometry) to present the result in a more uniform way. Let us start with a normal frame $\{\tilde{\varphi}_1,\dots,\tilde{\varphi}_n\}$ for the bundle $T^*M(1)$. Of course, we simply get $\varphi_i = X^0\tilde{\varphi}_i$ for $i = 1,\dots,n$, where $X^0 \in \Gamma(\mathcal{E}(1))$ is the non–vanishing section defining the normal frame, see 3.1. Since the product x_iX^0 is just the homogeneous coordinate X^i, we can write the remaining sections from the corollary as $X^i\tilde{\varphi}_j - X^j\tilde{\varphi}_i$ for $1 \leq i < j \leq n$. Even more uniformly, we can put $\tilde{\varphi}_0 = 0$, and then obtain all sections as $X^i\tilde{\varphi}_j - X^j\tilde{\varphi}_i$ for $0 \leq i < j \leq n$. We can now use this result to give a complete list of potential normal solutions of the first BGG operator defined on $S^2T^*M(4)$.

Proposition 3.4. *Consider a normal frame $\{\varphi_{ij} : 1 \leq i \leq j \leq n\}$ for the bundle $S^2T^*M(4)$ on a manifold endowed with a projective structure. Then any normal solution of the first BGG operator defined on this bundle can be written as a linear combination* *with constant coefficients of the following sections:*

$$
\begin{array}{ll}
\varphi_{ij} & i \leq j \\
x_k\varphi_{ij} - x_j\varphi_{ik} & i \leq j < k \\
x_k\varphi_{ij} - x_i\varphi_{jk} & i < j \leq k \\
x_jx_k\varphi_{ii} - x_ix_k\varphi_{ij} - x_ix_j\varphi_{ik} + {x_i}^2\varphi_{jk} & i < j \leq k \\
x_kx_\ell\varphi_{ij} - x_jx_k\varphi_{i\ell} - x_ix_\ell\varphi_{jk} + x_ix_j\varphi_{k\ell} & i < j < k \leq \ell \\
x_jx_\ell\varphi_{ik} - x_jx_k\varphi_{i\ell} - x_ix_\ell\varphi_{jk} + x_ix_k\varphi_{j\ell} & i < j \leq k < \ell
\end{array}
$$

On the homogeneous model, each of these sections is a solution, and they form a basis for the space of all solutions.

Proof. We have seen above that the P–irreducible quotient of the G–irreducible representation $\Lambda^2\mathbb{R}^{(n+1)*}$ induces the bundle $T^*M(2)$. Consequently, the P–irreducible quotient of $S^2(\Lambda^2\mathbb{R}^{(n+1)*})$ induces $S^2T^*M(4)$. Elementary representation theory shows that $S^2(\Lambda^2\mathbb{R}^{(n+1)*})$ splits into two irreducible components. The smaller of those is mapped isomorphically to $\Lambda^4\mathbb{R}^{(n+1)*}$ by the wedge product of forms, which easily implies that it is contained in the kernel of the projection to the P–irreducible quotient. Hence the right tractor bundle to start with is induced by the Cartan product, which is the kernel of the wedge product.

To obtain Corollary 3.3 we have used a local frame for $\Lambda^2\mathcal{T}^*M$, whose elements we number as ψ_{ab} with $0 \leq a < b \leq n$. In the notation of that corollary, the projections of the elements of the frame are given by $\Pi(\psi_{0i}) = \varphi_i$ for $i = 1, \dots, n$ and $\Pi(\psi_{ij}) = (x_i\varphi_j - x_j\varphi_i)$ for $1 \leq i < j \leq n$. On the homogeneous model, the ψ_{ab} form a basis for the space of parallel sections, while in general any parallel section is a linear combination with constant coefficient of these sections.

If we order pairs of indices in some way, the symmetric products $\psi_{ab} \vee \psi_{cd}$ with $ab \leq cd$ form a frame for $S^2(\Lambda^2\mathcal{T}^*M)$. Now we can modify this frame in such a way that some of its elements lie in the kernel of the wedge product while the remaining ones project isomorphically to a frame for $\Lambda^4\mathcal{T}^*M$. Ignoring the latter elements, we arrive at a frame for our tractor bundle, which is a basis for the space of parallel sections on the homogeneous model, while in general any parallel section is a linear combinations with constant coefficients of elements of the frame.

Working this out, we see that the frame in question arises from the following elements

$$
\begin{array}{ll}
\psi_{ab} \vee \psi_{ac} & 0 \leq a < b \leq c \leq n \\
\psi_{ab} \vee \psi_{cd} + \psi_{ac} \vee \psi_{bd} & 0 \leq a < b \leq c < d \leq n \\
\psi_{ad} \vee \psi_{bc} + \psi_{ac} \vee \psi_{bd} & 0 \leq a < b < c \leq d \leq n
\end{array}
$$

Now the projections of a symmetric product are just the symmetric products of the projections of the individual factors, and after recombining some elements we arrive at the claimed list. □

3.4. **Applications to (pseudo–)Riemannian geometry.** We will always use the term "Riemannian" to also include metrics of indefinite signature. Indeed, the results we discuss here are independent of the signature in question. It is a standard technique in Riemannian geometry to study conformal changes of metrics or, more technically speaking, to study the conformal structure induced by a Riemannian metric. Properties of Riemannian manifolds depending only on the induced conformal class are then considered as particularly robust. However, via its Levi–Civita connection, a Riemannian metric on a smooth manifold also determines a projective structure. The distinguished paths of this structure are the geodesic paths of the metric, so this point of view in particular captures aspects related to geodesics. Note that a Riemannian metric gives rise to a canonical volume density and thus to a trivialisation of all projective density bundles.

We believe that the projective structure induced by a Riemannian metric has by far not been used up to its potential. This approach seems particularly promising since several of the fundamental differential equations of Riemannian geometry admit a projectively invariant interpretation. This was pointed out and studied in [16]. In particular, this is true for the infinitesimal automorphism equation, which is among the examples we study below. Here we want to continue exploring this point of view. In particular, we want to point out that normality as a solution of a projective first BGG operator gives rise to highly interesting conditions on solutions of some important natural differential equations in Riemannian geometry.

Returning to projective structures, we can explicitly interpret the first BGG operators of order one on symmetric powers of the cotangent bundle. From the description in 3.2, is is clear that these must map sections of $S^kT^*M(w)$ to sections of $S^{k+1}T^*M(w')$ for appropriate weights w and w'. It then follows easily that in terms of a chosen connection in the projective class, the operator is just given by symmetrising the covariant derivative (and the weight is chosen in such a way that this does not depend on the choice of the connection in the class).

If we look at the projective structure induced by a Riemannian metric, then as noted above the weights do not play a role, so one obtains the standard Killing operators on symmetric tensors. In particular, for $k = 1$ its solutions (which are one–forms) can be interpreted as vector fields using the metric, and then they are exactly the infinitesimal isometries. For $k = 2$, one obtains (via the metric) symmetric Killing tensors of valence two which are important in several parts of Riemannian geometry and integrable systems. The solutions which are normal in the projective sense form an interesting subclass of Killing vectors respectively Killing tensors, which, to our knowledge, has not been studied in any detail so far.

We start with the case $k = 1$, phrase things in terms of vector fields rather than one–forms, and use the Penrose abstract index notation.

Proposition 3.5. *Let (M, g) be a Pseudo–Riemannian manifold of dimension $n \geq 2$ with Levi–Civita connection ∇. Then a vector field $\xi = \xi^k$ on M is a Killing field if and only if the associated one–form $\psi_i := g_{ij}\xi^j$ is a solution of the projective first BGG operator defined on $T^*M(2)$.*

If $n = 2$ then any such solution is normal in the projective sense. If $n \geq 3$, then let $R_{ij}{}^k{}_\ell$ be the Riemann curvature tensor of g, $R_{ij} := R_{ki}{}^k{}_j$ its Ricci curvature. A solution ψ_i is normal in the projective sense if and only if

$$R_{ij}{}^k{}_\ell\psi_k = \tfrac{2}{n-1}\psi_{[i}R_{j]\ell}. \tag{3.1}$$

In particular, any Killing vector field ξ which is normal in the projective sense can be written as a linear combination with constant coefficients of the vector fields obtained via the metric from the forms listed in Corollary 3.3.

Proof. In this proof we use significantly more information on BGG sequences and the normalisation condition for parabolic geometries than outlined in section 2. The necessary facts can be found in [12] and [29].

We have noted above that the first BGG operator on $T^*M(2)$ is given by taking one covariant derivative and then symmetrising the two indices. It is well known that applying this to the one–form obtained from a vector field via the metric one obtains the Killing equation.

We have noted in 3.3 that the tractor bundle inducing this operator is $\Lambda^2\mathcal{T}^*M$, the second exterior power of the dual of the standard tractor bundle. We have also seen there that the composition series of this tractor bundle consists of just two parts, a subbundle isomorphic to $\Lambda^2T^*M(2)$ and the irreducible quotient $T^*M(2)$. We write Π for the projection map to this irreducible quotient. Likewise, each of the bundles $\Lambda^kT^*M\otimes\Lambda^2\mathcal{T}^*M$ of k–forms with values in our tractor bundle has a composition series with two factors, a subbundle isomorphic to $\Lambda^kT^*M\otimes\Lambda^2T^*M(2)$, the quotient by which is isomorphic to $\Lambda^kT^*M\otimes T^*M(2)$.

In the construction of the BGG operators, one uses the so–called splitting operator $S:\Gamma(T^*M(2))\to\Gamma(\Lambda^2\mathcal{T}^*M)$ and the bundle maps

$$\partial^*:\Lambda^kT^*M\otimes\Lambda^2\mathcal{T}^*M\to\Lambda^{k-1}T^*M\otimes\Lambda^2\mathcal{T}^*M.$$

The curvature κ of the canonical Cartan connection can be interpreted as a two–form on M with values in the bundle $\mathfrak{sl}(\mathcal{T}M)$. In particular, the values of κ act on any tractor bundle. Given a section ψ of the bundle $T^*M(2)$, we can first apply the splitting operator and then act with the curvature on the result to obtain a two–form with values in $\Lambda^2\mathcal{T}^*M$, which is usually denoted by $\kappa\bullet S(\psi)$. Now it follows from the general theory (see [4, 30, 32]) that normality of the solution ψ is equivalent to the fact that $\partial^*(\kappa\bullet S(\psi))=0$.

Since ∂^* preserves homogeneity, it vanishes on the subbundle

$$\Lambda^2T^*M\otimes\Lambda^2T^*M(2)\subset\Lambda^kT^*M\otimes\Lambda^2\mathcal{T}^*M$$

and thus factors through the quotient $\Lambda^2T^*M\otimes T^*M(2)$ and its values lie in the subbundle $T^*M\otimes\Lambda^2T^*M(2)\subset T^*M\otimes\Lambda^2\mathcal{T}^*M$. Hence ∂^* acts between two isomorphic completely reducible bundles, which both split into two non–isomorphic irreducible summands, so it has to act by a multiple of the identity on each of the two summands. It also follows from general results that ∂^* must map onto the subbundle in question so both these multiples must be non–zero.

From the well known form of κ (see e.g. section 5.1.1 of [29]) and the fact that $\Pi(S(\psi))=\psi$, we conclude that the projection of $\kappa\bullet S(\psi)$ to the quotient $\Lambda^2T^*M\otimes T^*M(2)$ must be a non–zero multiple of $C_{ij}{}^k{}_\ell\psi_k$, where $C_{ij}{}^k{}_\ell$ denotes the projective Weyl–curvature. If $n=2$ then it is well known that the projective Weyl–curvature vanishes identically, so normality follows. For $n\geq 3$, the projective Weyl curvature satisfies the first Bianchi–identity, so $C_{ij}{}^k{}_\ell\psi_k$ lies in the kernel of the alternation over the three lower indices. This condition characterises exactly one of the two irreducible summands mentioned above, so we conclude that normality of ψ is equivalent to $C_{ij}{}^k{}_\ell\psi_k=0$.

Now the projective Weyl tensor is given as the tracefree part (with respect to the Ricci type contraction) of the curvature of any connection in the projective class. In the case of the projective structure determined by a Riemannian metric, we can use the Riemann curvature tensor. The Ricci type contraction of this is just the classical Ricci tensor and hence is symmetric. Then the well known formula for the projective Weyl tensor shows that

$$C_{ij}{}^k{}_\ell=R_{ij}{}^k{}_\ell-\frac{2}{n-1}\delta^k_{[i}R_{j]\ell}.$$

Hooking $\psi_k=\xi^ag_{ak}$ into this expression we immediately get (3.1). □

Finally we note that we may also interpret the result of Proposition 3.4 in a Riemannian context: Given a section $\xi=\xi^{ab}$ of the bundle S^2TM on a Riemannian manifold, one can use the metric to lower the two indices and then consider the equation $\nabla_{(a}\xi_{bc)} = 0$. Solutions of this equation are called Killing–2–tensors and they play a crucial role in the study of symmetries of mechanical systems. So similarly as discussed for Killing fields above, solutions which satisfy the condition of projective normality can be obtained from linear combinations with constant coefficients of the sections listed in Proposition 3.4. Thus the condition of projective normality should be very interesting for Riemannian geometry. Using results from section 5.3 of [29], this condition can be made explicit along similar lines as the one for Killing forms. Since the result is rather involved, we do not write it out explicitly here.

3.5. **Example 2: Conformal structures.** We next apply our results to (oriented) pseudo–Riemannian conformal structures of arbitrary signature. As we shall see, this looks a bit more complicated than the projective case, but making our results explicit is still straightforward. In the conformal case, the first BGG operators coming from tractor bundles induced by fundamental representations, are the operator governing Einstein rescalings and the conformal Killing equations on differential forms. So these are of strong interest in conformal geometry and are studied intensively.

By classical results going back to E. Cartan, oriented conformal structures of signature (p, q) in dimension $n = p + q \geq 3$ are equivalent to parabolic geometries of type (G, P), where $G = SO(p + 1, q + 1)$ and $P \subset G$ is the stabiliser of an isotropic ray in the standard representation $\mathbb{R}^{p+1,q+1}$ of G. To pass to matrix representations, one usually takes the standard basis of $\mathbb{R}^{p+1,q+1}$ numbered as $e_0, \dots, e_{n+1}$ and defines the inner product by $\langle e_0, e_{n+1}\rangle = \langle e_{n+1}, e_0\rangle = 1$, $\langle e_i, e_i\rangle = \varepsilon_i$ for $i = 1, \dots, n$ and all other inner products vanishing. Here $\varepsilon_i = 1$ for $i = 1, \dots, p$ and $\varepsilon_i = -1$ for $i = p + 1, \dots, n$. So $e_1, \dots, e_n$ form an orthonormal basis for the standard inner product of signature (p, q) on an n–dimensional subspace and the other two elements are a light cone basis for the orthocomplement of that subspace. For this choice of basis, one obtains

$$\mathfrak{g} = \mathfrak{so}(p + 1, q + 1) = \left\{ \begin{pmatrix} a & Z & 0 \\ X & A & -\mathbb{I}_{p,q}Z^t \\ 0 & -X^t\mathbb{I}_{p,q} & -a \end{pmatrix} \right\},$$

where $a \in \mathbb{R}$, $X \in \mathbb{R}^n$, $Z \in \mathbb{R}^{n*}$, $A \in \mathfrak{so}(p, q)$, and $\mathbb{I}_{p,q}$ is the diagonal matrix with entries ε_i, see section 1.6.2 of [12]. The grading corresponding to the parabolic subgroup P is of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with $\mathfrak{g}_{-1}$ spanned by the elements X, $\mathfrak{g}_0$ by a and A, and $\mathfrak{g}_1$ by Z. In particular, we can simply use the components of X as coordinates on $\mathfrak{g}_- = \mathfrak{g}_{-1}$. Then we compute

$$\exp \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ -x_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -x_n & 0 & \dots & 0 & 0 \\ 0 & \varepsilon_1 x_1 & \dots & \varepsilon_n x_n & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ -x_1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -x_n & 0 & \dots & 1 & 0 \\ \frac{\sum \varepsilon_i x_i^2}{2} & \varepsilon_1 x_1 & \dots & \varepsilon_n x_n & 1 \end{pmatrix} \tag{3.2}$$

From this, we can immediately read off the basic result on parallel sections of the standard tractor bundle, which by definition is induced by the standard representation $\mathbb{R}^{p+1,q+1}$.

Proposition 3.6. *Let $(M, [g])$ a smooth manifold endowed with a conformal pseudo–Riemannian structure of signature (p, q) and let $\mathcal{T}M$ be the standard tractor bundle. For a normal chart $U \subset M$ with corresponding normal coordinates $x_1, \dots, x_n$ let $\{s_0, \dots, s_{n+1}\}$ be the normal frame of $\mathcal{T}M$ corresponding to the standard basis of $\mathbb{R}^{p+1,q+1}$.*

Then any parallel section of $\mathcal{T}|_U$ is a linear combination with constant coefficients of the sections $\tilde{s}_0 := s_0 - \sum x_i s_i + (\frac{1}{2}\sum \varepsilon_i x_i^2)s_{n+1}$, $\tilde{s}_i := s_i + \varepsilon_i x_i s_{n+1}$ for $i = 1, \dots, n$ and $\tilde{s}_{n+1} := s_{n+1}$.

Viewing $\mathbb{V} := \mathbb{R}^{p+1,q+1}$ as a representation of the parabolic subgroup P, there is an obvious P–invariant filtration of the form $\mathbb{V} = \mathbb{V}^{-1} \supset \mathbb{V}^0 \supset \mathbb{V}^1$. Here $\mathbb{V}^1$ is the isotropic line stabilised by P and $\mathbb{V}^0$ is its orthogonal space. Our choice of the quadratic form was made in such a way that $\mathbb{V}^1 = \mathbb{R}\cdot e_0$ while $\mathbb{V}^0$ is spanned by $e_0, e_1, \dots, e_n$. Now $\mathbb{V}/\mathbb{V}^0$ is a (non–trivial) one–dimensional representation of P (spanned by the image of e_{n+1} in the quotient). The natural line bundle induced by this representation is a density bundle, usually denoted by $\mathcal{E}[1]$. The quotient $\mathbb{V}^0/\mathbb{V}^1$ is n–dimensional (spanned by the images of $e_1, \dots, e_n$). The bundle induced by this quotient turns out to be $T^*M[1] = T^*M \otimes \mathcal{E}[1]$.

Passing to associated bundles, the filtration of $\mathbb{V}$ induces a filtration $\mathcal{T}M = \mathcal{T}^{-1}M \supset \mathcal{T}^0 M \supset \mathcal{T}^1 M$ by smooth subbundles such that $\mathcal{T}M/\mathcal{T}^0 M \cong \mathcal{E}[1]$ and $\mathcal{T}^0 M/\mathcal{T}^1 M \cong T^*M[1]$. From the construction it is clear that the normal frame $\{s_0, \dots, s_{n+1}\}$ from Proposition 3.6 has the property that s_0 is a normal frame for $\mathcal{T}^1 M$, while $\{s_0, \dots, s_n\}$ form a normal frame for $\mathcal{T}^0 M$. Further, Lemma 2.7 shows that the natural projection $\mathcal{T}M \to \mathcal{T}M/\mathcal{T}^0 M \cong \mathcal{E}[1]$ maps $s_0, \dots, s_n$ to zero and s_{n+1} to a normal frame of $\mathcal{E}[1]$. Likewise, the projection $\mathcal{T}^0 M \to \mathcal{T}^0 M/\mathcal{T}^1 M \cong T^*M[1]$ annihilates s_0 and maps $s_1, \dots, s_n$ to a normal frame of $T^*M[1]$. Using this, we can define the conformal version of generalised homogeneous coordinates.

Definition 3.7. Let $(M, [g])$ be a smooth manifold endowed with a pseudo–Riemannian conformal structure of signature (p, q). Let $U \subset M$ be a normal chart and let X^0 be the nowhere vanishing section of $\mathcal{E}[1]|_U$ defining a normal frame for this bundle. Then we define the *generalised homogeneous coordinates* for M on U as the sections $X^0, \dots, X^{n+1}$ of $\mathcal{E}[1]$ defined by $X^i := \varepsilon_i x_i X^0$ for $i = 1, \dots, n$ and $X^{n+1} = \frac{\sum_j \varepsilon_j x_j^2}{2} X^0$.

Observe that the generalised homogeneous coordinates satisfy the relation $2X^0X^{n+1} + \sum_{i=1}^n \varepsilon_i (X^i)^2 = 0$ (the products can be interpreted as sections of $\mathcal{E}[2]$). This exactly corresponds to the defining equation of the light cone in $\mathbb{R}^{p+1,q+1}$ for our choice of inner product. This is very natural in view of the fact that the homogeneous model of our geometry is the space of isotropic rays in $\mathbb{R}^{p+1,q+1}$, which is isomorphic to $S^p \times S^q$.

From the filtration of the standard representation one can immediately read off the filtration for its exterior powers. Looking at $\Lambda^r\mathbb{V}$, one gets a P–invariant subspace spanned by the wedge products which involve e_0 but not e_{n+1}. This is contained in a P–invariant subspace spanned by the wedge products which either involve e_0 or don't involve e_{n+1}. The P–irreducible quotient of $\Lambda^r\mathbb{V}$ is spanned by the images of all elements of the form $e_{i_1} \wedge \dots \wedge e_{i_{r-1}} \wedge e_{n+1}$ with $1 \le i_1 < \dots < i_{r-1} \le n$. This shows that the P–irreducible quotient of the tractor bundle $\Lambda^r\mathcal{T}$ is isomorphic to $\Lambda^{r-1}T^*M[r]$. In particular, for $r = 2$, one gets $T^*M[2] \cong TM$.

The first BGG operators corresponding to the fundamental representations are all well known and intensively studied in the literature. The first BGG operator on $\mathcal{E}[1]$ is the second order operator governing *almost Einstein* scales, see e.g. [22]. An important feature of this operator is that on any conformal manifold, all its solutions are automatically normal. For $r \ge 2$, the first BGG operator on $\Lambda^{r-1}T^*M[k]$ coming from the rth exterior power of the standard tractor bundle is the *conformal Killing operator* on $(r-1)$–forms, see e.g. [24, 37]. This is the first order operator given by taking one covariant derivative and then projecting to the highest weight component. Solutions of this operator are called *conformal Killing forms* and normal solutions are called *normal conformal Killing forms*, see [34].

Finally, in parallel to the projective case, there are some first BGG operators of higher order which are easy to describe. Starting from the symmetric power $S^k\mathcal{T}$ one obtains the P–irreducible quotient $\mathcal{E}[k]$ and the first BGG operator on this bundle is of order $k+1$. Likewise, forming the Cartan product $S^{k-1}\mathbb{V}\odot\Lambda^r\mathbb{V}$ one obtains a G–representation with P–irreducible quotient inducing the bundle $\Lambda^{r-1}T^*M[r+k-1]$ and the first BGG operator defined on this bundle is of order k.

Theorem 3.8. *Let $(M,[g])$ be a smooth manifold endowed with a pseudo–Riemannian conformal structure of signature (p,q). Let $U\subset M$ be a normal chart and $x_1,\dots,x_n$ the corresponding normal coordinates.*

(1) Any normal solution of the first BGG operator on $\mathcal{E}[k]$ over U is given as a homogeneous polynomial of degree k in the generalised homogeneous coordinates $X^0,\dots,X^{n+1}$. In particular, any almost Einstein scale on M restricts on U to a linear combination with constant coefficients of the generalised homogeneous coordinates. On the homogeneous model $S^p\times S^q$ any homogeneous polynomial in the generalised homogeneous coordinates defines a solution.

(2) Let $\{\xi_1,\dots,\xi_n\}$ be a normal frame of the tangent bundle over U. Then any normal conformal Killing vector field on U is a linear combination with constant coefficients of the fields $\xi_1,\dots,\xi_n$, $\sum_{i=1}^n x_i\xi_i$, $\varepsilon_jx_j\xi_i-\varepsilon_ix_i\xi_j$ for $1\le i<j\le n$, and $\frac{1}{2}(\sum_{\ell=1}^n\varepsilon_\ell x_\ell^2)\xi_i+\varepsilon_ix_i\sum_{j=1}^n x_j\xi_j$. On the homogeneous model, the ξ_i are a coordinate frame and the vector fields listed above form a basis for the space of conformal Killing fields.

(3) Let $\{\varphi_1,\dots,\varphi_n\}$ be a normal frame of the bundle T^*M over U and for $r\ge 2$ let us denote by $\varphi_{i_1\dots i_r}$ the section $(X^0)^{r+1}\varphi_{i_1}\wedge\dots\wedge\varphi_{i_r}$ of the bundle $\Lambda^rT^*M[r+1]$. Then any normal conformal Killing r–form on M is a linear combination with constant coefficients of the following forms (with a hat denoting omission)

$$\varphi_{i_1\dots i_r},\quad \textstyle\sum_{j=1}^n x_j\varphi_{ji_1\dots i_{r-1}},\quad \sum_{j=1}^{r+1}(-1)^{j-1}\varepsilon_{i_j}x_{i_j}\varphi_{i_1\dots\widehat{i_j}\dots i_{r+1}}\quad \text{and}$$
$$(-1)^r\tfrac{1}{2}(\textstyle\sum_\ell\varepsilon_\ell x_\ell^2)\varphi_{i_1\dots i_r}+\sum_{j=1}^r(-1)^{r-j}\varepsilon_{i_j}x_{i_j}\left(\sum_\ell x_\ell\varphi_{\ell i_1\dots\widehat{i_j}\dots i_r}\right),$$

where in each case the $(i_1,\dots)$ runs through all strictly increasing sequences of numbers between 1 and n.

On the homogeneous model, the forms in the list constitute a basis for the space of all conformal Killing forms of degree r.

(4) For $r\ge 1$ and $k\ge 2$, any normal solution of the first BGG operator on the bundle $\Lambda^rT^*M[r+k]$ (which is of order k) can be written as a linear combination of the sections listed in (2) (for $r=1$) respectively (3) with coefficients which are homogeneous polynomials of degree $k-1$ in the generalised homogeneous coordinates $X^0,\dots,X^{n+1}$ from Definition 3.7.

Proof. Since different normal frames for the normal chart U are obtained from each other via linear combinations with constant coefficients, it suffices to prove each of the claims for one normal frame of the bundle in question. We start with the normal frame $\{s_0,\dots,s_{n+1}\}$ for the standard tractor bundle $\mathcal{T}M$ from Proposition 3.6. As we have observed already, the projection $\mathcal{T}M\to\mathcal{E}[1]$ maps $s_0,\dots,s_n$ to zero and s_{n+1} to a nowhere vanishing section X^0 of $\mathcal{E}[1]$, which constitutes a normal frame. Now the sections $\tilde{s}_i$ from Proposition 3.6 project to the generalised homogeneous coordinates $X^0,\dots,X^{n+1}$, which implies (1) for $k=1$.

To obtain (1) in the case $k \geq 2$, we just have to observe that Proposition 3.6 immediately implies that any parallel section of the bundle $S^k\mathcal{T}M$ must be a linear combination with constant coefficients of the symmetric products $\tilde{s}_{i_1} \vee \cdots \vee \tilde{s}_{i_k}$. Projecting such a product to the quotient bundle $\mathcal{E}[k]$, one obtains the product of the projections of the individual factors.

(2) The wedge products $s_i \wedge s_j$ with $0 \leq i < j \leq n+1$ form a normal frame for the bundle $\Lambda^2\mathcal{T}M$. The projection to the irreducible quotient bundle $T^*M[2] \cong TM$ annihilates a wedge product if either $i = 0$ or $j < n+1$ and maps the sections $s_i \wedge s_{n+1}$ for $i = 1, \dots n$ to a normal frame of TM and we use this frame as $\{\xi_1, \dots, \xi_n\}$. On the other hand, Proposition 3.6 implies that any parallel section of $\Lambda^2\mathcal{T}M$ must be a linear combination with constant coefficients of the sections $\tilde{s}_i \wedge \tilde{s}_j$ for $0 \leq i < j \leq n+1$. Expanding the formulae for the $\tilde{s}_\ell$ from Proposition 3.6 and using the observations on projections above, one obtains the list in (2).

(3) Here we consider the wedge products $s_{i_1} \wedge \cdots \wedge s_{i_{r+1}}$ which form a normal frame for $\Lambda^{r+1}\mathcal{T}M$. Projecting to the quotient bundle $\Lambda^r T^*M[r+1]$ kills any wedge product with $i_1 = 0$ or $i_{r+1} < n+1$. The wedge products $s_{i_1} \wedge \cdots \wedge s_{i_r} \wedge s_{n+1}$ with $1 \leq i_1 < \cdots < i_r \leq n$ descend to a normal frame of $\Lambda^r T^*M[r+1]$, which we use as $\varphi_{i_1 \dots i_r}$. By naturality of normal frames, this is of the form claimed in (3) for $\varphi_i = (X^0)^{-2}\xi_i$ with the vector fields ξ_i from (2). Again by Proposition 3.6, any parallel section of $\Lambda^{r+1}\mathcal{T}M$ is a linear combination with constant coefficients of the wedge products $\tilde{s}_{i_1} \wedge \cdots \wedge \tilde{s}_{i_{r+1}}$ and expanding this using the formula for the $\tilde{s}$ from that proposition and then projecting leads to the list in (3).

(4) As we have observed above, the (kth order) first BGG operator on $\Lambda^r T^*M[r+k]$ comes from the tractor bundle induced by $S^{k-1}\mathbb{V} \odot \Lambda^{r+1}\mathbb{V} \subset S^{k-1}\mathbb{V} \otimes \Lambda^{r+1}\mathbb{V}$. Thus this part can be proved exactly as the projective counterpart in part (3) of Theorem 3.2. □

Remark 3.9. As in the projective case, one can use homogenisation to present the results in (2) and (3) in a more uniform way. For the case of conformal Killing vectors, we start with a normal frame $\{\bar{\xi}_1, \dots, \bar{\xi}_n\}$ for the bundle $T^*M[1] \cong TM[-1]$, and define $\bar{\xi}_{n+1} := -\sum_{i=1}^n x_i \bar{\xi}_i$. Then we can write the sections in (2) as $X^0 \bar{\xi}_i$ for $i = 1, \dots, n+1$ and $X^j \bar{\xi}_i - X^i \bar{\xi}_j$ for $1 \leq i < j \leq n+1$.

For the conformal Killing forms in (3), one similarly looks at elements $\bar{\varphi}_{i_1 \dots i_r} := (X^0)^r \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_r}$ for $1 \leq i_1 < \cdots < i_r \leq n$ of the bundle $\Lambda^r T^*M[r]$. For $1 \leq i_1 < \cdots < i_{r-1} \leq n$, one then defines $\bar{\varphi}_{i_1 \dots i_r n+1} := \sum_\ell x_\ell (X^0)^r \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_{r-1}} \wedge \varphi_\ell$. Then the forms in (3) can be written as $X^0 \bar{\varphi}_{i_1 \dots i_r}$ and $\sum_{j=0}^r (-1)^{j-1} X^{i_j} \bar{\varphi}_{i_0 \dots \widehat{i_j} \dots i_r}$ where the i_ℓ in both cases run through all strictly increasing sequences of integers between 1 and $n+1$.

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