ESI **The Erwin Schrödinger International Institute for Mathematical Physics** Pasteurgasse 6/7 A-1090 Wien, Austria

Rayleigh–type Isoperimetric Inequality with a Homogeneous Magnetic Field

László Erdős

Vienna, Preprint ESI 237 (1995) July 4, 1995

Supported by Federal Ministry of Science and Research, Austria
Available via http://www.esi.ac.at

Rayleigh-type isoperimetric inequality with a homogeneous magnetic field

László Erdős
Forschungsinstitut für Mathematik
ETH-Zentrum
CH-8092 Zürich
E-mail: `erdoes@math.ethz.ch`

June 20, 1995

Abstract

We prove that the two dimensional free magnetic Schrödinger operator, with a fixed constant magnetic field and Dirichlet boundary conditions on a planar domain with a given area, attains its smallest possible eigenvalue if the domain is a disk. We also give some rough bounds on the lowest magnetic eigenvalue of the disk.

AMS Subject Classification: 35J10, 35J20

Running title: Magnetic Rayleigh-type inequality.

1 Introduction

In his classical work [R] Lord Rayleigh conjectured three inequalities for basic physical quantities related to planar domains. The statements are that, among all planar domains with a fixed area, the disk minimizes: (i) the principal eigenvalue of the Dirichlet Laplacian (membrane); (ii) the principal eigenvalue of the Dirichlet biharmonic Laplacian (clamped plate); and (iii) the electrostatic capacity.

In the twenties, the membrane problem was solved independently by Faber ([F]) and Krahn ([K]); Pólya and Szegő gave a proof of the capacity problem twenty years later ([P-S]). The clamped plate problem remained open up to Nadirashvili's recent solution (see [N] and references therein for further backgrounds).

In this paper we propose a new isoperimetric statement, analogous to Rayleigh's problems above. The Laplacian in the membrane problem can be viewed as a free Schrödinger operator (with Dirichlet boundary conditions), and, therefore, the same question naturally arises if we consider the magnetic Laplacian (with a homogeneous magnetic field). Let $B > 0$ and let $\Omega \subset \mathbf{R}^2$ be a bounded domain (connected open set with smooth boundary), and let us denote by $\lambda(B, \Omega)$ the principal eigenvalue of the magnetic Schrödinger operator $(-i\nabla - A)^2$ on Ω with Dirichlet boundary conditions. The vectorpotential (gauge) $A : \Omega \to \mathbf{R}^2$ generates the constant magnetic field with strength B (i.e. rot $A = B$). It is well known that the spectrum is discrete and independent of the gauge choice. The area of Ω is denoted by $|\Omega|$. Our theorem is the following:

Theorem 1.1 *For any planar domain Ω and $B > 0$ we have*

$$\lambda(B, \Omega) \geq \lambda(B, D), \tag{1}$$

where D is the disk with $|D| = |\Omega|$. Equality occurs only if $\Omega = D$.

We are not aware of any appearance of this, to our opinion quite natural, question in the literature. Nevertheless, we think that both the statement and some ideas of the proof (especially Lemma 3.1 about the monotonicity of the principal eigenvalue of a special one dimensional quadratic form) are interesting.

2 Reduction to a radially symmetric problem

Consider the usual coordinate system in the plane and let $A(x,y)=(-By/2, Bx/2)$ be the linear gauge. Since $C_0^\infty(\Omega)$ is a form core for our operator, by variational principle

$$\lambda(B,\Omega)=\inf_{f\in C_0^\infty(\Omega),\|f\|=1}\int_\Omega |(-i\nabla - A)f|^2, \tag{2}$$

where $\|f\|:=\|f\|_{L^2(\Omega)}$.

We shall also need to consider the magnetic Schrödinger operator on a disk with a radially symmetric magnetic field (and Dirichlet boundary conditions). Let D be a disk with center at the origin and consider a bounded radial function $b(r)\geq 0$ (magnetic field). The corresponding (unique) radial gauge is given (in polar coordinates) by

$$A_{rad}=a(r)(-\sin\varphi,\cos\varphi), \tag{3}$$

where $a(r)=(1/r)\int_0^r b(\varrho)\varrho \mathrm{d}\varrho$, i.e. $a'(r)+a(r)/r=b(r)$.

By standard elliptic theory it is clear that the infimum in (2) is attained, and the minimizer is a $C^\infty(\overline{\Omega})$ eigenfunction of the corresponding Schrödinger operator

$$(-i\nabla - A)^2 f=\lambda(B,\Omega)f, \quad f|_{\partial\Omega}=0. \tag{4}$$

Moreover, since the coefficients are analytic functions, $f\in A(\Omega)$, where $A(\Omega)$ denotes the set of analytic functions (in two variables) on Ω. Let us call a set analytic if it is the union of finitely many points and analytic line segments (level sets of a nonconstant analytic function). Obviously, $\{f=0\}\cap\Omega$ is analytic (we exclude the $f\equiv 0$ solution). Let $\Omega_0:=\Omega\setminus\{f=0\}$. Write $f=\psi\exp(i\varphi)$ with $\psi:\Omega_0\to(0,\infty)$, $\varphi:\Omega_0\to S^1$ smooth functions. In addition, $\psi^2\in A(\Omega)$, $\psi\not\equiv const$, therefore $\{\nabla\psi=0\}\cap\Omega$ is analytic. Clearly

$$|(-i\nabla - A)f|^2=|\nabla\psi|^2+|(\nabla\varphi - A)\psi|^2, \tag{5}$$

and let $w = (w_1, w_2) := A - \nabla\varphi$, with $\mathrm{rot}\, w = B$ on Ω_0. Considering that $|\nabla f|^2 = |\nabla\varphi|^2\psi^2 + |\nabla\psi|^2$, one obtains that $w^2\psi^2$ is bounded on Ω. In particular, $w\psi^2$ vanishes on the set $\{f = 0\} \equiv \{\psi = 0\}$.

Let $v = (v_1, v_2) := (w_2, -w_1)$, then by a simple integration by parts on each connected components of Ω_0 we obtain

$$\int_\Omega |\nabla\psi + v\psi|^2 = \int_\Omega |\nabla\psi|^2 + v^2\psi^2 - \psi^2 \mathrm{div}\, v. \tag{6}$$

Here we used that $\psi^2 v$ vanishes on the boundaries of the connected components of Ω_0. Since $\mathrm{div}\, v = \mathrm{rot}\, w = B$, we have, by (5) and (6),

$$\lambda(B, \Omega) \geq B + E(B, \Omega) \tag{7}$$

with

$$E(B,\Omega) := \inf\Big\{ \int_\Omega |\nabla\psi + v\psi|^2 \,:\, \psi \in C^\infty(\overline{\Omega}) \cap A(\Omega),\ \Omega_0 := \Omega \setminus \{\psi = 0\} \tag{8}$$

$$\psi \geq 0,\ \psi|_{\partial\Omega} = 0,\ \int_\Omega \psi^2 = 1,\ v \in C^\infty(\Omega_0, \mathbf{R}^2),\ \mathrm{div}\, v = B \text{ on } \Omega_0 \Big\}.$$

Note that $E(B, \Omega)$ is the lowest eigenvalue of the half of the Pauli operator, $(-i\nabla - A)^2 - B$.

The goal of this section is to prove the following

Proposition 2.1 *Let $D(0, R)$ be the disk, centered at the origin, with $|D| = \pi R^2 = |\Omega|$. Then*

$$E(B, \Omega) \geq \inf\left\{ \frac{2\pi \int_0^R (q'(r) + a(r)q(r))^2 r\mathrm{d}r}{2\pi \int_0^R q^2(r) r\mathrm{d}r} : \right. \tag{9}$$

$$\left. q(r) \in H^1((0,R), r\mathrm{d}r),\ q(R) = 0,\ q(r) \geq 0,\ 0 \leq a(r) \leq \frac{Br}{2} \textit{ for a.a. } r \right\}.$$

Remark. The expression in the nominator on the right hand side of (9) is just

$$\int_\Omega |(-i\nabla - A_{rad})\tilde{q}|^2 - \tilde{b}\tilde{q}^2 \tag{10}$$

where $\tilde{q}(x,y) := q(\sqrt{x^2+y^2})$ and similarly for $\tilde{b}$. This is the quadratic form of the corresponding two dimensional Pauli operator, $((-i\nabla - A_{rad})^2 - \tilde{b})$, on the spin-up subspace and in the zero angular momentum sector. The proposition tells us that the lowest eigenvalue of the Pauli operator on Ω with a constant field B can be minorized by that of the Pauli operator on the disk D with a (possibly nonhomogeneous) magnetic field whose flux on each disk $D(0,r)$ ($0 \le r \le R$) is not bigger than that of the constant B field (i.e. $2\pi a(r) r \le B\pi r^2$). Moreover, it also shows that the lowest eigenvalue of the Pauli operator on the disk is attained in the zero angular momentum sector. In the next section we show (Lemma 3.1) that this lowest eigenvalue, in fact, decreases if we pointwise increase the radial gauge (i.e we increase the flux on each $D(0,r)$). This will finish the proof of Theorem 1.1.

Proof of Proposition 2.1. The key idea is that we shall minimize the functional (8) on a larger set of admissible pairs of functions (ψ, v). Namely, we do not require $\operatorname{div} v$ to be equal to B at every point, we only require that the integral of $\operatorname{div} v$ on the level sets of ψ be equal B-times the area of this set. Therefore, obviously,

$$E(B,\Omega) \ge \inf\Big\{ \int_\Omega |\nabla\psi + v\psi|^2 \;:\; \psi \in C^\infty(\overline{\Omega}) \cap A(\Omega),\ \Omega_0 := \Omega \setminus \{\psi = 0\}, \tag{11}$$

$$\psi \ge 0,\ \psi|_{\partial\Omega} = 0,\ \int_\Omega \psi^2 = 1,\ v \in C^\infty(\Omega_0, \mathbf{R}^2),\ \int_{\{\psi > c\}} \operatorname{div} v = B|\{\psi > c\}| \text{ for a.a. } c \Big\}.$$

The set $\{\psi > c\} \subset \Omega$ consists of finitely many open sets with piecewise smooth (even analytic) boundary. By Stokes theorem, $\int_{\{\psi>c\}} \operatorname{div} v = -\int_{\{\psi=c\}} v \cdot n$, where $n := \nabla\psi/|\nabla\psi|$ is the inner unit normal vector to the curve $\{\psi = c\}$ (in fact, this set can be a union of finitely many curves, but for simplicity, we refer to them as level curves). Since ψ^2 is analytic, any of these curves shrink to a point only for finitely many c values (this can happen only when $\nabla\psi = 0$), and since $\{\nabla\psi = 0\}$ is analytic, the unit normal vector is well defined and given by $\nabla\psi/|\nabla\psi|$ along all but finitely many curves with a possible finitely many exceptional points. Let $\Omega_1 := \Omega_0 \setminus \{\nabla\psi = 0\}$.

Fix ψ for the moment and notice that replacing v by $(v\cdot n)n$ does not effect the divergence condition in (11), the integral does not increase (in fact, it decreases unless v were parallel with $\nabla\psi$ at almost all points), and the vectorfield $(v\cdot n)n$ is smooth on Ω_1. Thus, it is enough to consider $v=-\varphi\nabla\psi$ with some real smooth function φ, which is, perhaps, undefined on an analytic set. Therefore

$$E(B,\Omega)\geq \inf\left\{\int_\Omega |\nabla\psi-\varphi\psi\nabla\psi|^2\,:\,\psi\in C^\infty(\overline{\Omega})\cap A(\Omega),\ \Omega_1:=\Omega\setminus(\{\psi=0\}\cup\{\nabla\psi=0\}),\right.$$

$$\left.\psi\geq 0,\ \psi|_{\partial\Omega}=0,\ \int_\Omega\psi^2=1,\ \varphi\in C^\infty(\Omega_1,\mathbf{R}),\ \int_{\{\psi=c\}}\varphi|\nabla\psi|=B|\{\psi>c\}|\text{ for a.a. }c\right\},\quad (12)$$

where the divergence condition is rewritten using Stokes theorem.

By the co-area formula and Hölder inequality

$$\int_\Omega |\nabla\psi|^2(1-\varphi\psi)^2=\int_0^\infty \mathrm{d}c\int_{\{\psi=c\}}(1-\varphi c)^2|\nabla\psi|\geq \quad (13)$$

$$\geq\int_0^\infty \mathrm{d}c\ \frac{\left(\int_{\{\psi=c\}}(1-\varphi c)|\nabla\psi|\right)^2}{\int_{\{\psi=c\}}|\nabla\psi|}=\int_0^\infty \mathrm{d}c\ \frac{\left(\int_{\{\psi=c\}}|\nabla\psi|-cB|\{\psi>c\}|\right)^2}{\int_{\{\psi=c\}}|\nabla\psi|},$$

and equality occurs only if φ is constant on each $\{\psi=c\}$ (for almost all c). Therefore the energy functional does not increase if we replace φ by

$$\varphi_0(x):=\frac{B|\{\psi>c\}|}{\int_{\{\psi=c\}}|\nabla\psi|}\quad (14)$$

on the set $\{x\,:\,\psi(x)=c\}$, which is an admissible function in (12) (notice that the level curves in Ω_1 are genuine curves, not points). Let $\Phi(c):=\varphi_0(\psi^{-1}(c))$. Let C_*^∞ be the set of real valued functions on $\mathbf{R}$ which are well defined and smooth with the exception of finitely many points; clearly $\Phi(c)\in C_*^\infty$. Therefore

$$E(B,\Omega)\geq \inf\left\{\int_\Omega |\nabla\psi-\Phi(\psi)\psi\nabla\psi|^2\,:\,\psi\in C^\infty(\overline{\Omega})\cap A(\Omega),\right.$$

$$\psi \geq 0,\ \psi|_{\partial\Omega} = 0,\ \int_\Omega \psi^2 = 1,\ \Phi(c) := B|\{\psi > c\}| \left(\int_{\{\psi=c\}} |\nabla\psi| \right)^{-1} \text{ for a.a. } c \Bigg\}. \tag{15}$$

Using the isoperimetric inequality (between the area and the perimeter of a domain, which is valid even if the domain has several components) and co-area formula again, we have

$$|\Omega| \geq |\{\psi > c\}| = \int_c^\infty \mathrm{d}\xi \int_{\{\psi=\xi\}} \frac{1}{|\nabla\psi|} \geq \int_c^\infty \frac{L^2(\{\psi=\xi\})}{\int_{\{\psi=\xi\}} |\nabla\psi|}\,\mathrm{d}\xi \geq \int_c^\infty \frac{4\pi|\{\psi>\xi\}|}{\int_{\{\psi=\xi\}} |\nabla\psi|}\,\mathrm{d}\xi, \tag{16}$$

which shows, in particular, that Φ is integrable ($L(\cdot)$ denotes the length of the corresponding one dimensional level curve). Therefore, we can set $\Lambda(c) := \int_c^\infty \Phi(\xi)\,\mathrm{d}\xi$, $\Lambda' = -\Phi$, and we arrive at

$$E(B,\Omega) \geq \inf \Bigg\{ \int_\Omega |\nabla(\psi e^{\Lambda(\psi)})|^2 e^{-2\Lambda(\psi)} :\ \psi \in C^\infty(\overline{\Omega}) \cap A(\Omega), \tag{17}$$

$$\psi \geq 0,\ \psi|_{\partial\Omega} = 0,\ \int_\Omega \psi^2 = 1,\ \Lambda(c) := \int_c^\infty B|\{\psi > \xi\}| \left(\int_{\{\psi=\xi\}} |\nabla\psi| \right)^{-1} \mathrm{d}\xi \Bigg\}.$$

Note that, by definition, $\Lambda \in C_*^\infty$ is a nonnegative, strictly monotone decreasing, continuous function on the range of ψ.

Let $h := \Lambda(\psi)$, then $\nabla h = \Lambda'(\psi)\nabla\psi$. Clearly, for any $b > 0$ (with finitely many exceptions)

$$\int_{\{h<b\}} \operatorname{div} h \leq \int_{\{h=b\}} |\nabla h| = -\int_{\{\psi=\Lambda^{-1}(b)\}} \Lambda'(\psi)|\nabla\psi| = B|\{\psi > \Lambda^{-1}(b)\}| = B|\{h < b\}|. \tag{18}$$

Let $\Theta(x) := \Lambda^{-1}(x)e^x$, defined on the range of $\Lambda(\psi)$. Since ψ is bounded, Θ is so. Then

$$\int_\Omega |\nabla(\psi e^{\Lambda(\psi)})|^2 e^{-2\Lambda(\psi)} = \int_\Omega |\nabla(\Theta(h))|^2 e^{-2h} = \int_\Omega (\Theta'(h))^2 e^{-2h} |\nabla h|^2 \tag{19}$$

$$= \int_0^\infty \mathrm{d}b\ (\Theta'(b))^2 e^{-2b} \int_{\{h=b\}} |\nabla h| = B \int_0^\infty (\Theta'(b))^2 e^{-2b} F(b)\,\mathrm{d}b,$$

with $F(b) := |\{h < b\}|$ (in the last step we used (18)). We can also express $\int_\Omega \psi^2$ with the help of the functions Θ and F as follows:

$$\int_\Omega \psi^2 = \int_\Omega (\Lambda^{-1}(h))^2 = \int_\Omega (\Theta(h)e^{-h})^2 \tag{20}$$

$$= \int_0^\infty \mathrm{d}b\, e^{-2b}\Theta^2(b) \int_{\{h=b\}} \frac{1}{|\nabla h|} = \int_0^\infty e^{-2b}\Theta^2(b)F'(b)\,\mathrm{d}b,$$

where, in the last step, the co-area formula was used again

$$F(b) = |\{h < b\}| = \int_0^b \mathrm{d}\xi \int_{\{h=\xi\}} \frac{1}{|\nabla h|} \tag{21}$$

to calculate $F'(b)$.

Now we determine the constraints for Θ and F. Starting from ψ, by construction, Θ and F are smooth functions (with possible finitely many exceptional points), F is strictly monotonically increasing (notice that $h \in C_*^\infty$ is continuous) and $F(0) = 0$. Moreover, we claim, that if $F(b) < |\Omega|$, then $F'(b) \geq 4\pi/B$. For, similarly to (16), by (21), a Hölder inequality and (18)

$$F'(b) = \int_{\{h=b\}} \frac{1}{|\nabla h|} \geq \frac{L^2(\{h=b\})}{\int_{\{h=b\}} |\nabla h|} \geq \frac{4\pi|\{h<b\}|}{B|\{h<b\}|}. \tag{22}$$

This means that there exists a smallest value of $b = b_0$, such that $F(b_0) = |\Omega|$, $b_0 \leq B|\Omega|/(4\pi)$. Since $\psi = 0$ at the boundary of Ω, $\Lambda(0) = b_0$, therefore $\Theta(b_0) = 0$.

Putting all these information together, we obtain that

$$E(B,\Omega) \geq \inf \left\{ \frac{B\int_0^{b_0} (\Theta'(b))^2 e^{-2b} F(b)\,\mathrm{d}b}{\int_0^{b_0} \Theta^2(b) e^{-2b} F'(b)\,\mathrm{d}b} \;:\; F,\,\Theta\, \in C_*^\infty,\ \Theta \geq 0,\ \Theta \text{ bounded} \right. \tag{23}$$

$$\left. F \text{ strictly monotone},\ F'(b) \geq \frac{4\pi}{B} \text{ for a.a. } 0 < b < b_0,\ F(0) = 0,\ F(b_0) = |\Omega|,\ \Theta(b_0) = 0 \right\}.$$

Now we consider the disk $D = D(0,R)$ with $|D| = |\Omega|$. For any admissible (F,Θ) pair (for which the energy integral in (23) is finite), we shall construct admissible functions $q(r)$ and $a(r)$ for the right hand side of (9), which give a lower (or equal) energy.

Let $h^*(r) := F^{-1}(\pi r^2)$, $a(r) := (h^*)'(r)$, then (for almost all r)

$$0 \leq a(r) = (h^*)'(r) = \frac{2\pi r}{F'(F^{-1}(\pi r^2))} \leq \frac{Br}{2}. \tag{24}$$

Next, we define $q(r) := \Theta(h^*(r))\exp(-h^*(r))$. Clearly $h^*(0) = 0$, $h^*(R) = b_0$, $q(R) = 0$ and

$$2\pi\int_0^R (q'(r) + a(r)q(r))^2 r\mathrm{d}r = 2\pi\int_0^R |(q(r)e^{h^*(r)})'|^2 e^{-2h^*(r)} r\mathrm{d}r = 2\pi\int_0^R |(\Theta(h^*(r)))'|^2 e^{-2h^*(r)} r\mathrm{d}r \tag{25}$$

$$= \int_D (\Theta'(h^*))^2 e^{-2h^*}|\nabla h^*|^2 = \int_0^{b_0} \mathrm{d}b\,(\Theta'(b))^2 e^{-2b}\int_{\{h^*=b\}} |\nabla h^*| \le \int_0^{b_0} B(\Theta'(b))^2 e^{-2b} F(b)\,\mathrm{d}b,$$

where, in the last step, we used that $\int_{\{h^*=b\}} |\nabla h^*| \le B\pi r^2 = BF(b)$ by (24) and the definition of h^*. In particular, we obtain from (25) that $q \in H^1((0,R), r\mathrm{d}r)$ (notice that q is bounded).

Similarly, we calculate the norm of $q(r)$:

$$2\pi\int_0^R q^2(r)\, r\,\mathrm{d}r = \int_D \Theta^2(h^*)e^{-2h^*} \tag{26}$$

$$= \int_0^{b_0} \mathrm{d}b\,\Theta^2(b)e^{-2b}\int_{\{h^*=b\}} \frac{1}{|\nabla h^*|} = \int_0^{b_0} \Theta^2(b)e^{-2b}F'(b)\,\mathrm{d}b$$

using (24) in the last step. This completes the proof of Proposition 2.1. □

3 Comparison lemma for the radial case

In this section we prove the following comparison result which, in addition to Proposition 2.1 will complete the proof of Theorem 1.1.

Lemma 3.1 *Let $0 \le a_1(r) \le a_2(r) \le Cr$ be two functions on $[0,R]$ with some constant $C > 0$. Then, for*

$$E(a(r), D) := \inf\left\{2\pi\int_0^R (q'(r) + a(r)q(r))^2\, r\mathrm{d}r \;:\; q \in H^1((0,R), r\mathrm{d}r),\right. \tag{27}$$

$$\left. 2\pi\int_0^R q^2(r)\, r\mathrm{d}r = 1,\;\; q(R) = 0,\;\; q \ge 0\right\},$$

we have $E(a_2(r), D) \le E(a_1(r), D)$. Equality occurs if and only if $a_1(r) = a_2(r)$ almost everywhere.

Remark. In terms of the Pauli operator (see (10)) the statement of this lemma is quite natural. The Pauli operator on $\mathbf{R}^2$ in the radial gauge has a positive, zero-energy eigenfunction which is well localized around the origin. The localization is stronger if the field is larger. Therefore the spectral shift due to imposing Dirichlet boundary conditions on the boundary of the disk is smaller in case of a stronger field (see Appendix).

Proof of Lemma 3.1. We shall prove that infinitesimally increasing the function $a(r)$, the eigenvalue $E(a(r), D)$ strictly decreases. Let q be the minimizer of the variational problem for $E(a(r), D)$, let $h(r) := \int_0^r a(s)\,\mathrm{d}s$ and $p(r) := q(r)e^{h(r)}$. Note that p minimizes the Dirichlet integral $\int_0^R |\nabla p|^2 \mathrm{d}\mu(r)$ with respect to the measure $\mathrm{d}\mu(r) = e^{-2h(r)} r\mathrm{d}r$ (with Neumann boundary condition at 0 and Dirichlet b.c. at R), in particular $p > 0$ and $q > 0$ on $[0, R)$.

Clearly q satisfies the corresponding Schrödinger equation in a weak sense

$$-q'' - \frac{q'}{r} + \left(a^2 - a' - \frac{a}{r}\right) q = Eq, \tag{28}$$

(with $E = E(a(r), D) > 0$, where the positivity of E follows from the fact that it could be zero only if p were constant, but then $q(R) = 0$ were impossible). This implies that the function $p = qe^h$ satisfies the following equation (in a weak sense)

$$p'' + \left(\frac{1}{r} - 2h'(r)\right) p' = -Ep \tag{29}$$

Since q' is square-integrable with respect to the measure $r\mathrm{d}r$, there is a sequence $r_n \to 0$ such that $|q'(r_n)|r_n \to 0$. By the bound on $a(r) = h'(r)$ and $|q(r)| \leq \int_r^R |q'(s)|\mathrm{d}s \leq \|q\|_{H^1(r\,\mathrm{d}r)}(\int_r^R s^{-1}\mathrm{d}s)^{1/2} \leq (const)(|\log r|)^{1/2}$, we have $|p'(r_n)|r_n \to 0$ along the same sequence. Consider the function $t(r) := p'(r)re^{-2h(r)}$. By (29), this function has a negative distributional derivative, and it goes to zero along a sequence r_n converging to zero. Therefore, $t(r) < 0$ for $r > 0$, i.e. $0 > p' = (q' + aq)e^h$ on $(0, R)$.

Now consider $a^{(\varepsilon)} := a + \varepsilon\chi$ with some $0 \leq \chi(r) \leq (const)r$, $\chi \not\equiv 0$ and small ε. Using q as a trial function in the variational problem for $E(a^{(\varepsilon)}(r), D)$, a short calculation gives, that

$$E(a^{(\varepsilon)}(r), D) \leq E(a(r), D) + 2\varepsilon \int_0^R \chi(r)q(r)(q'(r) + a(r)q(r))\, r\mathrm{d}r + O(\varepsilon^2). \tag{30}$$

Since the coefficient of ε is strictly negative, we obtain that pointwise increasing $a(r)$ strictly decreases $E(a(r), D)$. This completes the proof of Lemma 3.1. $\square$

Proof of Theorem 1.1. Proposition 2.1 and Lemma 3.1 proves (1) (choose $a_1(r) = a(r)$ and $a_2(r) = Br/2$ in Lemma 3.1). For equality in (1), according to Lemma 3.1, one needs $a(r) = Br/2$, which, in turn (see (24)), implies that $F'(b) = 4\pi/B$ (for almost all b). But then we have equality in (22), i.e. the level curves of $\{h = b\}$ must be circles, so are those of ψ (almost all b). Since $\psi \in A(\Omega)$, if almost all of its level curves are circles, then all of them are so, therefore $\partial\Omega = \{\psi = 0\}$ is a circle. $\square$

Appendix

A Estimates on the spectral shift

We supplement our theorem by giving some rough bounds on $\lambda(B, D)$. Let $\lambda(B, R) := \lambda(B, D(0, R))$, then, by scaling, $\lambda(B, R) = L^2\lambda(BL^{-2}, RL)$ (for any $L \geq 0$), i.e. there is only one effective parameter in the problem: BR^2. So from now on we consider only $\lambda(R) := \lambda(1, R)$.

Proposition A.1 *The following bounds hold for the rescaled eigenvalue:*

$$1 + C_1 R^{-2} e^{-R^2} \leq \lambda(1, R) \leq 1 + C_2(R^{-2} + R^2)e^{-R^2/8} \tag{31}$$

with some universal, explicit constants.

Remark. As it is easy to see from the following proof, the lower bound can be improved to $1 + C_1(\varepsilon)R^{-2}\exp\left(-(\frac{1}{2}+\varepsilon)R^2\right)$.

Proof. Consider the radial gauge with $a(r) = Br/2$ with $B = 1$ (see (3)). Proposition 2.1 and Lemma 3.1 show that $\lambda(R)$ is the smallest eigenvalue of the Schrödinger operator $H_0 = -\partial_r^2 - \frac{1}{r}\partial_r + r^2/4$ on $L^2((0,R), r\mathrm{d}r)$ with Dirichlet boundary condition at $r = R$ (and free b.c. at the origin). This is the same as the lowest eigenvalue of the two dimensional harmonic oscillator, $-\Delta + r^2/4$ on $D(0,R)$, which is $1 + 2\mathcal{E}(R)$, where $\mathcal{E}(R)$ is the lowest eigenvalue of the one dimensional shifted harmonic oscillator $H = -\partial_x^2 + x^2/4 - 1/2$ on $[-R, R]$ with Dirichlet b.c.

The upper bound on $\mathcal{E}(R)$ is easy; we simply take a trial function f constructed from the Gaussian eigenfunction $(2\pi)^{-1/4}\exp(-x^2/4)$ of the unrestricted oscillator by cutting it off using straight line segments on $R/2 \le |x| \le R$. Simple calculation shows that $\mathcal{E}(R) \le (const)(R^{-2} + R^2)\exp(-R^2/8)$.

For the lower bound, we use the Birman-Schwinger principle. Let U be a potential which is zero on $[-R, R]$ and infinite otherwise, then, instead of the Dirichlet b.c., we can consider $H + U$ on the whole space.

Suppose we can show that for some $\eta > 0$ the number of eigenvalues of $H + U$ below η is smaller than 1 (i.e. zero). Then the lowest eigenvalue of $H + U$ is at least η. For $E > 0$, let

$$K_{\eta,E} := |U - \eta - E|_-^{1/2} \frac{1}{H+E} |U - \eta - E|_-^{1/2} \tag{32}$$

be the Birman-Schwinger kernel ($|\cdot|_-$ denotes the negative part), then

$$\begin{aligned} \#\{\text{ ev's of } H+U \text{ below } \eta\} &= \#\{\text{ ev's of } H+U-\eta-2E \text{ below } -2E\} \\ &\le \#\{\text{ ev's of } K_{\eta,E} \text{ above } 1\} \le \mathrm{Tr}\,(K_{\eta,E})^N \end{aligned} \tag{33}$$

for any $N \ge 1$. But, using a well known trace inequality $(\mathrm{Tr}(A^{1/2}BA^{1/2})^N \le \mathrm{Tr}(A^N B^N))$,

$$\mathrm{Tr}\,(K_{\eta,E})^N \le \mathrm{Tr}\,|U - \eta - E|_-^N \left(\frac{1}{H+E}\right)^N \tag{34}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+E)^N} (\eta+E)^N \int_{-R}^{R} |\varphi_n(x)|^2 \mathrm{d}x,$$

where φ_n's are the normalized eigenfunctions of H (with eigenvalue n). We know that $\varphi_0(x) = (2\pi)^{-1/4} \exp(-x^2/4)$, thus

$$\int_{-R}^{R} \varphi_0^2(x)\mathrm{d}x \le 1 - 2cR_*^{-1} e^{-R^2/2} \tag{35}$$

with some absolute constant c and $R_* := \max(1, R)$. For $n > 0$, we simply estimate $\int_{-R}^{R} |\varphi_n|^2$ by one.

Therefore

$$\mathrm{Tr}\,(K_{\eta,E})^N \le \left(\frac{\eta+E}{E}\right)^N (1 - 2cR_*^{-1} e^{-R^2/2}) + \sum_{n=1}^{\infty} \left(\frac{\eta+E}{n+E}\right)^N . \tag{36}$$

Choose $E := c_0 \eta N \exp(R^2)$ with a large enough c_0 (depending on c), such that $e^{\xi/c_0} \le 1 + c\xi/100$ for any $0 \le \xi \le 1$. Then $(1+\eta/E)^N$ is smaller than than $\exp(c_0^{-1}\exp(-R^2)) \le 1 + c\exp(-R^2)/100 \le 1 + cR_*^{-1}\exp(-R^2/2)$, so the first term on the right hand side of (36) is smaller than $1 - cR_*^{-1}\exp(-R^2/2)$. The second term is smaller than $cR_*^{-1}\exp(-R^2/2)$ if $\eta \le (1 + c_0 N \exp(R^2))^{-1}$ and $N \ge c_1 \cdot \max(R^2, 1)$ with c_1 depending on c_0. Therefore, if $\eta \le c_2 R^{-2} \exp(-R^2)$, with some universal constant, then $\mathrm{Tr}\,(K_{\eta,E})^N < 1$ for some suitable N, which completes the proof. $\square$

Acknowledgement: This problem was originated in the Erwin Schrödinger Institute, Vienna, where the author got interested in isoperimetrical problems under the influence of Professor Nadirashvili. The work was done in the stimulating environment and with the financial support of the Forschungsinstitut für Mathematik, ETH, Zürich. The author is especially grateful to Professors A.-S. Sznitman and Y. Pinchover for fruitful discussions.

References

[F] Faber, G.: *Beweis, daß unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt.* Sitzungsberichte der Bayerischen Akademie der Wissenschaften (1923), 169-172.

[K] Krahn, E.: *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises.* Math. Ann. **94** (1924), 97-100.

[N] Nadirashvili, N.: *Rayleigh's conjecture on the principal frequency of the clamped plate.* ESI Preprint No. 29 (1993).

[P-S] Pólya, G., Szegő, G.: *Inequalities for the capacity of a condenser.* Amer. J. of Math. **67** (1945), 1-32.

[R] Lord Rayleigh, *The Theory of Sound* 2nd. edition, London, 1894/96.