

## **Einstein Metrics in Projective Geometry**

**Andreas Čap  
A. Rod Gover  
Heather R. Macbeth**

Vienna, Preprint ESI 2373 (2012)

January 29, 2013

Supported by the Austrian Federal Ministry of Education, Science and Culture  
Available online at <http://www.esi.ac.at>

# EINSTEIN METRICS IN PROJECTIVE GEOMETRY

A. ČAP, A. R. GOVER & H. R. MACBETH

ABSTRACT. It is well known that pseudo-Riemannian metrics in the projective class of a given torsion free affine connection can be obtained from (and are equivalent to) the solutions of a certain overdetermined projectively invariant differential equation. This equation is a special case of a so-called first BGG equation. The general theory of such equations singles out a subclass of so-called normal solutions. We prove that non-degenerate normal solutions are equivalent to pseudo-Riemannian Einstein metrics in the projective class and observe that this connects to natural projective extensions of the Einstein condition.

AČ & ARG gratefully acknowledge support from the Royal Society of New Zealand via Marsden Grant 10-UOA-113; AČ gratefully acknowledges support by project P23244-N13 of the “Fonds zur Förderung der wissenschaftlichen Forschung” (FWF); AČ and HRM are grateful for the hospitality of the University of Auckland.

## 1. INTRODUCTION

Suppose that  $\nabla$  is a torsion-free connection on a manifold  $M^n$ ,  $n \geq 2$ , and consider its geodesics as unparametrised curves. The problem of whether these agree with the (unparametrised) geodesics of a pseudo-Riemannian metric is the classical problem of metrizability of projective structures which has attracted recent interest [4, 12, 16, 18, 19, 21].

Recall that torsion-free connections  $\nabla$  and  $\widehat{\nabla}$  are said to be projectively equivalent if they have the same geodesics as unparameterised curves. A projective structure on a manifold  $M$  (of dimension  $n \geq 2$ ) is a projective equivalence class  $p$  of connections.

As is usual in projective geometry, we write  $\mathcal{E}(1)$  for a choice of line bundle with  $(-2n - 2)$ nd power the square of the canonical bundle  $\Lambda^n T^*M$ . Observe that any connection  $\nabla \in p$  determines a connection on  $\mathcal{E}(1)$  as well as its real powers  $\mathcal{E}(w)$ ,  $w \in \mathbb{R}$ ; we call  $\mathcal{E}(w)$  the bundle of projective densities of weight  $w$ . Given any bundle  $\mathcal{B}$  we shall write  $\mathcal{B}(w)$  as a shorthand notation for  $\mathcal{B} \otimes \mathcal{E}(w)$ .

For simplicity here we suppose that  $M$  is connected and orientable. We say that a connection  $\nabla$  is *special* if it preserves a volume form  $\epsilon$  on  $M$ . On the other hand, suppose that  $\epsilon$  is a volume form on  $M$  and  $\widetilde{\nabla}$  is any connection on  $TM$ .

---

2000 *Mathematics Subject Classification*. Primary 53B10, 53A20, 53C29; Secondary 35Q76, 53A30.

*Key words and phrases*. projective differential geometry, Einstein metrics, conformal differential geometry.

Considering the induced connection on  $\Lambda^n T^*M$ , we can write  $\nabla\epsilon$  as  $\alpha\epsilon$  for some one-form  $\alpha \in \Omega^1(M)$ . Then one easily verifies that

$$\nabla_\xi\eta := \tilde{\nabla}_\xi\eta + \frac{1}{n+1}\alpha(\xi)\eta + \frac{1}{n+1}\alpha(\eta)\xi$$

is a connection in the projective class of  $\tilde{\nabla}$  for which  $\epsilon$  is parallel. Henceforth we use  $p$  to denote the equivalence class of projectively related special connections. For convenience we shall often use the Penrose abstract index notation and write  $\mathcal{E}^{(bc)}$  for the symmetric tensor power of the tangent bundle (otherwise written  $S^2(TM)$ ) and  $(\mathcal{E}_a^{(bc)})_0$  for the trace-free part of  $T^*M \otimes S^2(TM)$ .

Consider the differential operator

$$D_a : \mathcal{E}^{(bc)}(-2) \rightarrow (\mathcal{E}_a^{(bc)})_0(-2), \quad \text{given by } \sigma^{bc} \mapsto \text{trace-free}(\nabla_a\sigma^{bc}).$$

It is an easy exercise to verify that  $D$  is a projectively invariant differential operator in that it is independent of the choice  $\nabla \in p$ . Part of the importance of  $D$  derives from the following result due to Mikes and Sinjukov [18, 21].

**Theorem 1.1.** *Suppose that  $n \geq 2$  and  $\nabla$  is a special torsion-free connection on  $M$ . Then  $\nabla$  is projectively equivalent to a Levi-Civita connection if and only if there is a non-degenerate solution  $\sigma$  to the equation*

$$(1) \quad D\sigma = 0.$$

Here  $\sigma$  *non-degenerate* means that it is non-degenerate as a bilinear form on  $T^*M(1)$ . Our presentation of the Theorem here follows the treatment [12] of Eastwood-Matveev.

Let us write  $\epsilon_{a_1 a_2 \dots a_n}$  for the canonical section of  $\Lambda^n T^*M(n+1)$  which gives the tautological bundle map  $\Lambda^n TM \rightarrow \mathcal{E}(n+1)$ . Observe that each section  $\sigma^{ab}$  in  $\mathcal{E}^{(ab)}(-2)$  canonically determines a section  $\tau^\sigma \in \mathcal{E}(2)$ , by taking its determinant using  $\epsilon$ :

$$(2) \quad \sigma^{ab} \mapsto \tau^\sigma := \sigma^{a_1 b_1} \dots \sigma^{a_n b_n} \epsilon_{a_1 \dots a_n} \epsilon_{b_1 \dots b_n}.$$

We may form

$$(3) \quad \tau^\sigma \sigma^{ab}$$

and in the case that  $\sigma^{ab}$  is non-degenerate taking the inverse of this yields a metric that we shall denote  $g_{ab}^\sigma$ . This construction is clearly invertible and a metric  $g_{ab}$  determines a non-degenerate section  $\sigma^{ab} \in \mathcal{E}^{(ab)}(-2)$ . We are interested in the metric  $g^\sigma$  when  $\sigma$  is a solution to (1). Indeed, the Levi-Civita connection mentioned in the Theorem is the Levi-Civita connection for  $g^\sigma$ .

Now the projectively invariant differential operator  $D$  arises from a very general construction, namely as the first operator in a Bernstein–Gelfand–Gelfand (BGG) sequence. For the definition and general construction of these sequences see [10, 5]. For any first BGG equation there is a special class of solutions known as normal solutions, see [15]. These have striking properties, see [7, 8, 9, 13], but in general it is unclear how restrictive the normality condition is, and in particular how commonly normal solutions are available. The aim of this article is to analyse the

normality condition for solutions of (1). This needs only very basic facts on BGG sequences and simple elementary considerations, and furthermore the answer is significant and appealing. To explain these terms and prepare for that discussion we need some elements of tractor calculus, an invariant calculus for projective structures.

1.1. **Acknowledgements.** We are grateful for the referee's insightful comments.

## 2. PROJECTIVE TRACTOR CALCULUS

Consider the first jet prolongation  $J^1\mathcal{E}(1) \rightarrow M$  of the density bundle. By definition, its fiber over  $x \in M$  consists of all one-jets  $j_x^1\sigma$  of local smooth sections  $\sigma \in \Gamma(\mathcal{E}(1))$  defined in a neighborhood of  $x$ . Here for two sections  $\sigma$  and  $\tilde{\sigma}$  we have  $j_x^1\sigma = j_x^1\tilde{\sigma}$  if and only if in one — or equivalently any — local chart the sections  $\sigma$  and  $\tilde{\sigma}$  have the same Taylor–development in  $x$  up to first order. In particular, mapping  $j_x^1\sigma$  to  $\sigma(x)$  defines a surjective bundle map  $J^1\mathcal{E}(1) \rightarrow \mathcal{E}(1)$ , called the *jet projection*. If  $j_x^1\sigma$  lies in the kernel of this projection, so  $\sigma(x) = 0$  then the value  $\nabla\sigma(x) \in T_x^*M \otimes \mathcal{E}_x(1)$  is the same for all linear connections  $\nabla$  on the vector bundle  $\mathcal{E}(1)$ . This identifies the kernel of the jet projection with the bundle  $T^*M \otimes \mathcal{E}(1)$ . (See for example [20] for a general development of jet bundles.)

In an abstract index notation let us write  $\mathcal{E}_A$  for  $J^1\mathcal{E}(1)$  and  $\mathcal{E}^A$  for the dual vector bundle. Then we can view the jet projection as a canonical section  $X^A$  of the bundle  $\mathcal{E}^A \otimes \mathcal{E}(1) = \mathcal{E}^A(1)$ . Likewise, the inclusion of the kernel of this projection can be viewed as a canonical bundle map  $\mathcal{E}_a(1) \rightarrow \mathcal{E}_A$ , which we denote by  $Z_A^a$ . Thus the jet exact sequence (at 1-jets) is written in this context as

$$(4) \quad 0 \rightarrow \mathcal{E}_a(1) \xrightarrow{Z_A^a} \mathcal{E}_A \xrightarrow{X^A} \mathcal{E}(1) \rightarrow 0.$$

We write  $\mathcal{E}_A = \mathcal{E}(1) \oplus \mathcal{E}_a(1)$  to summarise the composition structure in (4). As mentioned, any connection  $\nabla \in p$  determines a connection on  $\mathcal{E}(1)$ , and this is precisely a splitting of (4). Thus given such a choice we have the direct sum decomposition  $\mathcal{E}_A \stackrel{\nabla}{=} \mathcal{E}(1) \oplus \mathcal{E}_a(1)$  with respect to which we define a connection by

$$(5) \quad \nabla_a^{\mathcal{T}^*} \begin{pmatrix} \sigma \\ \mu_b \end{pmatrix} := \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + P_{ab} \sigma \end{pmatrix}.$$

Here  $P_{ab}$  is the projective Schouten tensor and, with  $R_{ab}{}^c{}_d$  denoting the curvature of  $\nabla$ , is related to the Ricci tensor  $R_{ab} := R_{ca}{}^c{}_b$  by  $(n-1)P_{ab} = R_{ab}$ . It turns out that (5) is independent of the choice  $\nabla \in p$ , and so  $\nabla^{\mathcal{T}^*}$  is determined canonically by the projective structure  $p$ . We have followed the construction of [2], but this *cotractor connection* is due to [22]. It is equivalent to the normal Cartan connection (of [11]) for the Cartan structure of type  $(G, P)$ , see [6]. Thus we shall also term  $\mathcal{E}_A$  the *cotractor bundle*, and we note the dual *tractor bundle*  $\mathcal{E}^A$  (or in index free notation  $\mathcal{T}$ ) has canonically the dual *tractor connection*: in terms of a splitting dual to that above this is given by

$$(6) \quad \nabla_a^{\mathcal{T}} \begin{pmatrix} \nu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \nu^b + \rho \delta_a^b \\ \nabla_a \rho - P_{ab} \nu^b \end{pmatrix}.$$

Now consider  $\mathcal{E}^{(BC)} = S^2\mathcal{T}$ . It follows immediately that this has the composition series

$$\mathcal{E}^{(bc)}(-2) \oplus \mathcal{E}^b(-2) \oplus \mathcal{E}(-2),$$

and the normal tractor connection is given on  $S^2\mathcal{T}$  by

$$(7) \quad \nabla_a^{\mathcal{T}} \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma^{bc} + \delta_a^b \mu^c + \delta_a^c \mu^b \\ \nabla_a \mu^b + \delta_a^b \rho - P_{ac} \sigma^{bc} \\ \nabla_a \rho - 2P_{ab} \mu^b \end{pmatrix}.$$

**2.1. The Kostant codifferential.** The tractor connection on  $S^2\mathcal{T}$ , and more generally on a tractor bundle  $\mathcal{V}$ , which is formed by tensorial constructions from  $\mathcal{E}^A$  and  $\mathcal{E}_A$ , extends to the covariant exterior derivative on  $\mathcal{V}$ -valued forms. Thus one so obtains a twisting of the de Rham sequence by  $\mathcal{V}$ , and this is central in the usual construction of BGG sequences. At the next stage of the construction, a second ingredient is needed, as follows.

Note that from (4) it follows that there is a canonical (projectively invariant) map

$$(8) \quad \mathbb{X} : T^*M \rightarrow \text{End}(\mathcal{T}) \quad \text{given by} \quad u_b \mapsto X^A Z_B^b u_b.$$

Since sections of  $\text{End}(\mathcal{T})$  act on any tractor bundle in the obvious (tensorial) way, we thus obtain via  $\mathbb{X}$  a canonical action of  $T^*M$  on any tractor bundle  $\mathcal{V}$ . This induces a sequence of natural bundle maps

$$\partial^* : \Lambda^k T^*M \otimes \mathcal{V} \rightarrow \Lambda^{k-1} T^*M \otimes \mathcal{V}, \quad k = 1, \dots, n,$$

on  $\mathcal{V}$ -valued differential form bundles, but going in the opposite direction to the twisted de Rham sequence. This is a special case of a Kostant codifferential and satisfies  $\partial^* \circ \partial^* = 0$ , so it leads to natural subquotient bundles  $H_k(M, \mathcal{V}) := \ker(\partial^*) / \text{im}(\partial^*)$ . (The notation for these bundles is due to the fact that they are induced by certain Lie algebra homology groups, but this is not relevant for our purposes.)

In the case of  $\mathcal{V} = S^2\mathcal{T}$ , which is relevant for our purposes, the end of this sequence has the form

$$\begin{pmatrix} \mathcal{E}^{(ab)}(-2) \\ \mathcal{E}^a(-2) \\ \mathcal{E}(-2) \end{pmatrix} \xleftarrow{\partial^*} \begin{pmatrix} \mathcal{E}_a^{(bc)}(-2) \\ \mathcal{E}_a^b(-2) \\ \mathcal{E}_a(-2) \end{pmatrix} \xleftarrow{\partial^*} \begin{pmatrix} \mathcal{E}_{[ab]}^{(cd)}(-2) \\ \mathcal{E}_{[ab]}^c(-2) \\ \mathcal{E}_{[ab]}(-2) \end{pmatrix},$$

where we have used a vector notation analogous to (7). From the general theory (or indeed the formula (8)) it follows that  $\partial^*$  maps each row in some column to the row below in the next column to the left, and that all the bundle maps are natural. All we need to now here is the following result.

**Lemma 2.1.** *In terms of the composition series  $\mathcal{E}_a^{(bc)}(-2) \oplus \mathcal{E}_a^b(-2) \oplus \mathcal{E}_a(-2)$  for  $T^*M \otimes S^2\mathcal{T}$  we have*

$$\text{im}(\partial^*) = (\mathcal{E}_a^b)_0(-2) \oplus \mathcal{E}_a(-2) \subset (\mathcal{E}_a^{(bc)})_0(-2) \oplus (\mathcal{E}_a^b)_0(-2) \oplus \mathcal{E}_a(-2) = \ker(\partial^*)$$

*Proof.* From what we know about  $\partial^*$ , we see that  $\partial^* : T^*M \otimes S^2\mathcal{T} \rightarrow S^2\mathcal{T}$  can only induce some multiples of the trace maps  $\mathcal{E}_a^b(-2) \rightarrow \mathcal{E}(-2)$  and  $\mathcal{E}_a^{(bc)}(-2) \rightarrow \mathcal{E}^c(-2)$  on the two upper slots. Likewise, in the next step, there can only be multiples of the canonical trace maps applied to the two upper slots.

Now there are some simple general facts about the homology of the  $\partial^*$ -sequence, see e.g. [3]. These imply that the homology in degree zero coincides with the irreducible quotient bundle  $\mathcal{E}^{(ab)}(-2)$  and the homology in degree one is  $(\mathcal{E}_a^{(bc)})_0(-2)$ . This implies that all the bundle maps from above are actually non-zero multiples of the trace maps, and hence the claim.  $\square$

### 3. BGG SEQUENCES AND NORMAL SOLUTIONS

Let us write  $\Pi : \mathcal{E}^{(BC)} \rightarrow \mathcal{E}^{(bc)}(-2)$  for the canonical projectively invariant map onto the quotient; explicitly this is given by  $H^{BC} \mapsto Z_B^b Z_C^c H^{BC}$ . The key step to the construction of BGG sequences is the construction of a differential splitting to the tensorial operator on sections induced by this projection. Phrased for the case of  $S^2\mathcal{T}$ , this reads as

**Proposition 3.1.** *For a smooth section  $\sigma$  of  $\mathcal{E}^{(bc)}(-2)$  there is a unique smooth section  $L(\sigma)$  of  $\mathcal{E}^{(AB)}$  such that  $\Pi(L(\sigma)) = \sigma$  and  $\partial^*(\nabla L(\sigma)) = 0$ . This defines a projectively invariant differential operator  $L$ , and  $D(\sigma)$  is given by projecting  $\nabla(L(\sigma))$  to the quotient bundle  $\ker(\partial^*)/\text{im}(\partial^*) \cong (\mathcal{E}_a^{(bc)})_0(-2)$ .*

In the special case needed here, this can also be proved by a direct computation. Indeed, given  $\sigma = \sigma^{ab}$  we can add components  $\mu^b$  and  $\rho$  and then use that the two top slots of (7) have to be tracefree to deduce that  $\mu^b = \frac{1}{n+1} \nabla_i \sigma^{ib}$  and

$$\rho = \frac{1}{n} (\nabla_i \mu^i + P_{ij} \sigma^{ij}) = \frac{1}{n(n+1)} (\nabla_i \nabla_j + (n+1)P_{ij}) \sigma^{ij}.$$

This proves the first claim and then projective invariance of  $L$  can be verified by a direct computation. It is also evident then, that projecting to  $\ker(\partial^*)/\text{im}(\partial^*)$ , one exactly obtains the tracefree part of the top slot, which equals  $D(\sigma)$ .

In particular, we see that  $\sigma$  is a solution of  $D$  if and only if  $\nabla L(\sigma)$  is actually a section of the subbundle  $\text{im}(\partial^*) \subset \ker(\partial^*)$ . This suggests how the subclass of normal solutions is defined.

**Definition 3.2.** A solution  $\sigma$  of the metricity equation  $D(\sigma) = 0$  is said to be *normal* if  $\nabla L(\sigma) = 0$ .

Observe that for a parallel section  $s$  of  $S^2\mathcal{T}$ , Proposition 3.1 implies that  $s = L(\Pi(s))$ , so normal solutions of the equation (1) are in bijective correspondence with parallel sections of the tractor bundle  $S^2\mathcal{T}$ , so this gives a connection to the holonomy of the tractor connection.

The question then is, in the case that  $\sigma$  is non-degenerate, what normality implies for the metric structure  $g^\sigma$ . The answer in this case is elegant and important. Here if  $n = 2$  we take *Einstein* to mean constant Gaussian curvature.

**Theorem 3.3.** *A non-degenerate solution  $\sigma$  of the metricity equation (1) is normal if and only if the corresponding metric  $g^\sigma$  is an Einstein metric.*

*Proof.* Assume that  $\sigma$  is a non-degenerate solution of the metricity equation and  $g^\sigma$  is the corresponding metric. From above we have the formula for computing  $L(\sigma)$ . By projective invariance there is no loss if we calculate in the scale  $\tau^\sigma$ , meaning we use the Levi-Civita connection  $\nabla$  of  $g^\sigma$ . It follows from the discussion in Section 1 that this has the congenial consequence

$$\nabla\sigma = 0.$$

Moreover, the corresponding tensor  $P_{ab}$  is a non-zero multiple of the Ricci-tensor of  $g^\sigma$ . Now we get  $\mu^b = 0$  and  $\rho = \frac{1}{n}P_{ij}\sigma^{ij}$ , so the latter is just a multiple of the scalar curvature. Hence from (7) we see that  $\nabla L(\sigma)$  has zero in the top slot, the trace-free part of  $P_{ai}\sigma^{bi}$  in the middle slot and  $\frac{1}{n}\nabla_a(P_{ij}\sigma^{ij})$  in the bottom slot.

By the non-degeneracy of  $\sigma$ ,  $\nabla L(\sigma) = 0$  implies that  $P_{ab}$  must be some multiple of  $g_{ab}^\sigma$ , which for  $n \geq 3$  is precisely the Einstein condition. Conversely, assuming this and  $n \geq 3$ , the scalar curvature is constant whence  $\nabla L(\sigma) = 0$ . On the other hand if  $n = 2$  the result follows immediately from the described form of the bottom slot.  $\square$

From the formula for  $L(\sigma)$  in the proof, we also see that for any non-degenerate solution  $\sigma$  of the metricity equation, the bilinear form on each fiber of  $\mathcal{E}_A$  induced by the section  $L(\sigma)$  of  $\mathcal{E}^{(AB)}$  is non-degenerate if and only if the scalar curvature is non-zero in that point. In particular, if  $\sigma$  is a non-degenerate normal solution, then  $L(\sigma)$  is non-degenerate in this sense if and only if  $g^\sigma$  is not Ricci flat.

Our results give the perspective that the so-called Beltrami theorem, i.e. the characterization of projectively flat metrics, is a special case of the link with Einstein structure described above.

**Corollary 3.4.** *Let  $M$  be a smooth manifold of dimension  $n \geq 2$  and let  $g$  be a pseudo-Riemannian metric on  $M$  such that the projective structure determined by  $g$  is locally projectively flat. Then  $g$  has constant sectional curvature.*

*Proof.* Tautologically,  $g$  determines a solution of the metricity equation for the projective structure determined by  $g$ . But it is well known that on locally flat structures, any solution of a first BGG-operator is normal (see e.g. [10, (3) of Lemma 2.7]). Hence by Theorem 3.3,  $g$  is Einstein and so has constant scalar curvature. Together with the vanishing of the projective Weyl curvature and Cotton tensor, implied by projective flatness, this shows that  $g$  has constant sectional curvature.  $\square$

Note that we include the above well known result primarily to illustrate that Theorem 1.1 may be viewed as generalisation of this. The proof of Corollary 3.4 here may be viewed as simply a repackaging of that given in [12, Corollary 5.6].

## 4. RELATIONS TO OTHER KNOWN RESULTS

**4.1. A prolongation connection for the metricity equation.** A crucial point about the proof of Theorem 3.3, that we have given above, is that apart from general facts on first BGG operators it only needs very simple computations. Thus the argument has the scope to generalise easily to significantly more complicated cases. In the special case of the metricity equation (which has been very well studied) one may also deduce Theorem 3.3 directly from results of [12], as we now discuss. (However, the detailed analysis of the metricity equation done in that reference would be much more complicated to generalize.)

Since the operator  $D$  is a linear partial differential operator of finite type, one knows in general that it can be equivalently written in first order closed form. Based on ideas on BGG sequences, this has been done for a large class of cases (including  $D$ ) in [3]. There it is shown that solutions of these equations are in one-to-one correspondence with parallel sections for some linear connection on an auxiliary bundle. In general such connections are not unique and it is difficult to prolong invariant equations by invariant connections. In the special case of the metricity equation, this is precisely what has been done in [12]: the authors construct a projectively invariant connection on  $S^2\mathcal{T}$  whose parallel sections are in bijective correspondence with solutions of the metricity equation.

**Theorem 4.1.** [12] *The solutions to (1) are in one-to-one correspondence with solutions of the following system:*

$$(9) \quad \nabla_a \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} + \frac{1}{n} \begin{pmatrix} 0 \\ W_{ac}{}^b{}_d \sigma^{cd} \\ -2Y_{abc} \sigma^{bc} \end{pmatrix} = 0.$$

Here

$$(10) \quad Y_{abc} := \nabla_a P_{bc} - \nabla_b P_{ac}$$

is the *projective Cotton tensor*, and  $W_{ac}{}^b{}_d$  is the *projective Weyl tensor* (i.e. the completely trace-free part of the full curvature). There are some sign differences compared to [12], as in that source the authors have used a splitting of the tractor bundle different to that in [2].

Of course, one expects to be able to recover Theorem 3.3. from (9). Indeed this is so. This claim amounts to showing that the second tractor term in the display vanishes if and only if  $g_\sigma^{ab}$  is (the inverse of) an Einstein metric. But it is straightforward to show that, calculating in the scale  $\nabla^g$ , we have

$$W_{ac}{}^b{}_d g^{cd} = \frac{n}{n-1} \cdot \Phi_{ac} g^{bc},$$

where  $\Phi_{ab}$  is the trace-free part of the Ricci tensor of  $\nabla^g$ . Thus when  $n \geq 3$  the term  $W_{ac}{}^b{}_d g^{cd}$  certainly vanishes if and only if  $g$  is Einstein.

For the cases  $n \geq 3$  it remains only to verify that

$$Y_{abc} g^{bc} = 0$$



if  $g$  is Einstein. But then  $g$  Einstein implies that  $P_{ab} = \lambda g_{ab}$ , with  $\lambda$  constant. Thus from (10) it follows at once that  $Y_{abc} = 0$ . Finally for projective manifolds of dimension  $n = 2$  the tensor  $W_{ac}{}^b{}_d$  is identically zero, so we get no information at that stage. On the other hand in this dimension (and since we calculate in the scale  $\nabla^g$ )  $P_{ab}$  is a multiple of the Gauss curvature  $K$  times the metric, thus  $Y_{abc} = 0$  if and only if  $K$  is constant.

**Remark 4.2.** It should also be mentioned that there are also links with the work [14] of Kiosak-Matveev. In that source, the authors consider the implications of having two distinct Levi-Civita connections in a projective class, at least one of which is Einstein. In this setting a specialisation of the system given in (7) (see equations (8), (24), and (32) in [14]) is used to prove a number of interesting results, including that if one of two projectively related Levi-Civita connections is Einstein, then so is the other, which was originally shown in [17]. As pointed out by the referee, this implies that if one solution of (1) is normal then so are all others. In [14] it is also shown (see Theorem 2) that in dimension 4 if two such solutions are linearly independent, then the structure is projectively flat. These results should be visible using the tools developed here and we shall take that up elsewhere.

**4.2. Projective holonomy.** The parallel section of  $S^2\mathcal{T}$  determined by a normal solution of the metricity equation can be interpreted as a reduction of projective holonomy, i.e. the holonomy of the standard tractor connection. In the case that this parallel section is pointwise non-degenerate (see also the next subsection) this falls into the cases studied in the insightful work [1] of S. Armstrong. In this reference it is shown (without discussing the related BGG equations) that, on a set of generic points, this yields an Einstein metric whose Levi-Civita connection lies in the projective class.

**4.3. Another first BGG equation and Klein-Einstein structures.** There is a projectively invariant differential operator

$$K : \mathcal{E}(2) \rightarrow \mathcal{E}_{(abc)}(2)$$

with leading term  $\nabla_{(a}\nabla_b\nabla_{c)}$ , for any  $\nabla \in p$ . This is another first BGG operator, but note that it looks very different to the (first order) metricity operator  $D$ . Nevertheless normal solutions satisfying suitable non-degeneracy conditions are again equivalent to Einstein metrics, cf. Theorem 3.3. This is proved in [7, Section 3.3] where, among other things, the normal solutions are used to define projective compactifications of certain Einstein manifolds; one case, that we term a Klein-Einstein structure, is both a curved generalisation of the Klein model of hyperbolic space and a projectively compact analogue of a Poincaré-Einstein manifold. The latter is a conformally compact negative Einstein manifold.

This interesting link is easily explained from our current perspective. The parallel tractor  $H$  arising in connection with normal solutions to the equation  $K\tau = 0$  is a section of  $S^2\mathcal{T}^*$ , and thus, if non-degenerate, it is equivalent to a unique parallel section of  $S^2\mathcal{T}$ , namely  $H^{-1}$ . Let us say that any solution  $\sigma$  of the metricity

equation is *algebraically generic* if the corresponding section  $L(\sigma) \in \Gamma(S^2\mathcal{T})$  is everywhere non-degenerate (for normal solutions on connected manifolds this is equivalent to non-degenerate at one point). From the explicit description of the splitting operator  $L$ , as given in Section 3, we have that on the locus where  $\sigma$  itself is non-degenerate this non-degeneracy condition is equivalent to the non-vanishing of the scalar curvature of  $g^\sigma$ . Similarly we shall say solutions of the  $K\tau = 0$  equation are algebraically generic if the corresponding section of  $S^2\mathcal{T}^*$  is everywhere non-degenerate.

Now using this terminology, combined with machinery developed in [7], the observations above are easily rephrased as statement in terms of the affine connections and related structures in the projective class. Doing this we arrive at the following result, which in part generalises Theorem 3.3.

**Theorem 4.3.** *Normal algebraically generic solutions  $\sigma$  of the metricity equation are naturally in one-to-one correspondence with normal algebraically generic solutions to the equation  $K\tau = 0$ . This is via the non-linear map  $\sigma \mapsto \tau = H_{AB}X^AX^B$ , where  $H_{AB}$  is the section of  $S^2\mathcal{T}^*$  inverse to  $(L(\sigma))$ . For the inverse map: given a normal algebraically generic solution  $\tau$  to  $K\tau = 0$  then, on the open set where  $\tau$  is not zero, the corresponding solution of the metricity equation is  $(\tau P_{ab}^\tau)^{-1}$ . Here  $P_{ab}^\tau$  is the Schouten tensor for the connection  $\nabla^\tau \in \mathfrak{p}$  preserving  $\tau$ .*

Note that  $\tau$  as given above in Theorem 4.3 agrees with  $\tau^\sigma$ , as defined in (2). This is easily seen to be true, up to a constant factor, as they have a common zero locus, and where non-vanishing are both parallel for the Einstein Levi-Civita connection.

This Theorem means that we can at once import the results from [7] for normal solutions to the  $K$  equation and apply these to generic normal solutions of the metricity equation. In particular we have the following statement available.

**Corollary 4.4.** *If  $\sigma$  is an algebraically generic normal solution of the metricity equation then  $\sigma$  is non-degenerate (and determines a metric  $g^\sigma$ ) on an open dense subset of  $M$ . The set where  $\sigma$  is degenerate is the zero locus of  $\tau^\sigma$ , and if non-empty is a smoothly embedded hypersurface (not necessarily connected) with a canonically induced non-degenerate conformal structure. This hypersurface is separating and the signature of  $\sigma$  changes as the hypersurface is crossed.*

We have not attempted to be complete here; further results are available by translating in an obvious way the results from [7, Theorem 3.2].

**Remark 4.5.** The bundle  $\Lambda^{n+1}\mathcal{T}^*$  is parallelisable by the projective tractor connection. Let us select a “tractor volume form”  $\eta$ . That is  $\eta \in \Gamma(\Lambda^{n+1}\mathcal{T}^*)$ ,  $\eta$  is non-trivial and  $\nabla\eta = 0$ . Given any section  $Q$  of  $S^2\mathcal{T}$  we may take its determinant using  $\eta$ ; let us denote this  $\det(Q)$ . For a section of  $\sigma^{ab} \in \Gamma(\mathcal{E}^{(ab)}(-2))$  the condition that  $\sigma$  is algebraically generic is exactly to say that  $\det(L(\sigma))$  is nowhere zero. On the other hand if  $\sigma$  is a solution of the metricity equation then, on the locus where  $\sigma$  is non-degenerate,  $\det(L(\sigma))$  agrees with the scalar curvature of  $g^\sigma$  up to a non-zero constant; let us assume  $\eta$  is chosen so that this constant 1. Thus, for solutions of (1),  $\det(L(\sigma))$  is a natural extension of the scalar curvature of  $g^\sigma$  to

a quantity that is defined everywhere on  $M$ , even though  $g^\sigma$  may not be available globally. In particular for normal solutions  $\det(L(\sigma))$  is a constant. On the other hand the system consisting of (1) plus  $\det(L(\sigma)) = \text{constant}$  is a weakening of the normality condition, and it provides a projective analogue of the conformal almost scalar constant equation [13] from conformal geometry.

**Remark 4.6.** Interestingly there is a further link with article [14] (cf. Remark 4.2). The equation  $K\tau = 0$  in Theorem 4.3 may be specialised to the case that the background connection  $\nabla$  is an Einstein Levi-Civita connection. The result is the symmetric part of the equation (34) of [14]. (The other part of (34) is then a differential consequence of this part and the Einstein condition.)

#### REFERENCES

- [1] S. Armstrong, *Projective holonomy. I. Principles and properties*, Ann. Global Anal. Geom., **33** (2008), 47–69.
- [2] T.N. Bailey, M.G. Eastwood, and A.R. Gover, *Thomas’s structure bundle for conformal, projective and related structures*, Rocky Mountain J. Math. **24** (1994), 1191–1217.
- [3] T. Branson, A. Čap, M.G. Eastwood, and A.R. Gover, *Prolongations of geometric overdetermined systems*, Internat. J. Math. **17** (2006), 641–664.
- [4] R. Bryant, M.G. Eastwood, and M. Dunajski, *Metrisability of two-dimensional projective structures*, J. Differential Geom. **83** (2009), 465–499.
- [5] D.M.J. Calderbank, and T. Diemer, *Differential invariants and curved Bernstein-Gelfand-Gelfand sequences*, J. Reine Angew. Math. **537** (2001), 67–103.
- [6] A. Čap, and A.R. Gover, *Tractor calculi for parabolic geometries*, Trans. Amer. Math. Soc. **354** (2002), 1511–1548.
- [7] A. Čap, A.R. Gover, and M. Hammerl, *Projective BGG equations, algebraic sets, and compactifications of Einstein geometries*, J. London Math. Soc. (2), **86** (2012), 433–454. Doi: 10.1112/jlms/jds002.
- [8] A. Čap, A.R. Gover, and M. Hammerl, *Holonomy reductions of Cartan geometries and curved orbit decompositions*, arXiv:1103.4497.
- [9] A. Čap, A.R. Gover, and M. Hammerl, *Normal BGG solutions and polynomials*, Internat. J. Math., **23** (2012), DOI 10.1142/S0129167X12501170. arXiv:1201.0799.
- [10] A. Čap, J. Slovák, and V. Souček, *Bernstein-Gelfand-Gelfand sequences*, Ann. of Math., (2) **154** (2001), 97–113.
- [11] Cartan, *Sur les variétés à connexion projective*, Bull. Soc. Math. France **52** (1924), 205–241.
- [12] M.G. Eastwood, and V. Matveev, *Metric connections in projective differential geometry in “Symmetries and overdetermined systems of partial differential equations”*, 339–350, IMA Vol. Math. Appl., 144, Springer, New York, 2008.
- [13] A.R. Gover, *Almost Einstein and Poincare-Einstein manifolds in Riemannian signature*, J. Geometry and Physics, **60**, (2010), 182–204.
- [14] V. Kiosak, and V. Matveev, *Complete Einstein metrics are geodesically rigid*, Comm. Math. Phys., **289**, (2009), 383–400.
- [15] F. Leitner, *Conformal Killing forms with normalisation condition*, Rend. Circ. Mat. Palermo (2) Suppl. No. **75** (2005) 279–292.
- [16] R. Liouville, *Sur les invariants de certaines équations différentielles et sur leurs applications*, Jour. de l’Ecole Polytechnique, **Cah.59** (1889) 7–76.
- [17] J. Mikes, *Geodesic mappings of Einstein spaces* (Russian), Mat. Zametki **28** (1980), no. 6, 935–938, English translation in Math. Notes **28** (1980), no. 6, 922–924 (1981).

- [18] J. Mikes, *Geodesic mappings of affine-connected and Riemannian spaces*, Jour. Math. Sci. **78** (1996) 311–333.
- [19] P. Nurowski, *Projective vs metric structures*, Journal of Geometry and Physics, **62** (2012), 657–674.
- [20] R.S. Palais, Seminar on the Atiyah-Singer index theorem. With contributions by M. F. Atiyah, A. Borel, E. E. Floyd, R. T. Seeley, W. Shih and R. Solovay. Annals of Mathematics Studies, No. 57 Princeton University Press, Princeton, N.J. 1965 x+366 pp.
- [21] N.S. Sinjukov, *geodesic mappings of Riemannian spaces*, (Russian), “Nauka,” Moscow 1979.
- [22] T.Y. Thomas, *Announcement of a projective theory of affinely connected manifolds*, Proc. Nat. Acad. Sci., **11** (1925), 588–589.

A.Č.: FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, NORDBERGSTR. 15, 1090 WIEN, AUSTRIA, A.R.G.: DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF AUCKLAND, PRIVATE BAG 92019, AUCKLAND 1142, NEW ZEALAND;; MATHEMATICAL SCIENCES INSTITUTE, AUSTRALIAN NATIONAL UNIVERSITY, ACT 0200, AUSTRALIA, H.R.M.: DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544, USA

*E-mail address:* `Andreas.Cap@univie.ac.at`

*E-mail address:* `r.gover@auckland.ac.nz`

*E-mail address:* `macbeth@math.princeton.edu`