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Multiplication of Conjugacy Classes, Colligations, and Characteristic Functions of Matrix Argument

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Multiplication of conjugacy classes, colligations, and characteristic functions of matrix argument

Yury A. Neretin[1]

We extend the classical construction of operator colligations and characteristic functions. Consider the group G of finite block unitary matrices of size $\alpha+\infty+\cdots+\infty$ (k times). Consider the subgroup $K=\mathrm{U}(\infty)$, which consists of block diagonal unitary matrices (with blocks 1 of size α and a matrix $u\in K$ repeated k times). It appears that there is a natural multiplication on the conjugacy classes $G//K$. We construct 'spectral data' of conjugacy classes, which visualize the multiplication and are sufficient for a reconstruction of a conjugacy class.

1 Characteristic functions

1.1. Notation. By $1=1_n$ we denote the unit matrix of order n. Let $\mathrm{Mat}(n)$ be the space of complex matrices of size $n\times n$, $\mathrm{GL}(n,\mathbb{C})$ the group of invertible matrices of order n, $\mathrm{U}(n)$ be the unitary subgroup in $\mathrm{GL}(n,\mathbb{C})$. By $\mathrm{GL}(\infty,\mathbb{C})$ we denote the group of finite infinite invertible matrices, i.e. matrices g such that $g-1$ has only finite number of non-zero elements. By $\mathrm{U}(\infty)$ we denote its subgroup consisting of unitary matrices. Denote by $\mathrm{Mat}(\infty)$ the space of all matrices, that are *finite in the same sense*[2].

1.2. Structure of the paper. This section contains preliminaries (results of [11]), construction of 'spectral data', and main Theorems 1.10, 1.11. Proofs are contained in Section 2.

1.3. Product of conjugacy classes. Fix integers $\alpha\geqslant 0$ and $m\geqslant 1$. Let $N>0$. Consider the space $\mathrm{Mat}(\alpha+mN)$. We write its elements as block matrices

$$g=\begin{pmatrix} a & b_1 & \dots & b_m\\ c_1 & d_{11} & \dots & d_{1m}\\ \vdots & \vdots & \ddots & \vdots\\ c_m & d_{m1} & \dots & d_{mm}\end{pmatrix} \tag{1.1}$$

of size $\alpha+N+\cdots+N$. For an element $u\in\mathrm{GL}(n,\mathbb{C})$ we denote by $\iota(u)$ the following matrix

$$\iota(u)=\begin{pmatrix} 1_\alpha & 0 & 0 & \dots & 0\\ 0 & u & 0 & \dots & 0\\ 0 & 0 & u & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & u\end{pmatrix}. \tag{1.2}$$

Denote by

$$\mathcal{M}_N=\mathcal{M}_N^{\alpha,m}:=\mathrm{Mat}(\alpha+mN)//GL(N,\mathbb{C})$$

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[2]For instance 1_∞ is finite and 0 not.

the set of conjugacy classes of $\mathrm{Mat}(\alpha + mN)$ by $\mathrm{GL}(n, \mathbb{C})$, i.e.,

$$g \sim \iota(u) g \iota(u)^{-1}. \tag{1.3}$$

We call elements of this set by *colligations*.

Usually, we will omit superscripts α, m from $\mathcal{M}_N^{\alpha,m}$.

Next, we formulate the definition of a colligation in another form. Consider the space $V = \mathbb{C}^m$ and $Z_N = \mathbb{C}^N$. Then an operator g can be regarded as an operator

$$\mathbb{C}^\alpha \oplus \bigl(V \otimes Z_N\bigr) \to \mathbb{C}^\alpha \oplus \bigl(V \otimes Z_N\bigr)$$

defined up to a conjugation by an element of $\mathrm{GL}(N, \mathbb{C})$ acting in Z_N.

We wish to define a canonical multiplication

$$\mathcal{M}_{N_1} \times \mathcal{M}_{N_2} \to \mathcal{M}_{N_1+N_2}$$

(α, m are fixed). First, we consider the case $m = 2$. Then the multiplication is given by

$$g \circ h = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{pmatrix} \circ \begin{pmatrix} p & q_1 & q_2 \\ c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} a & b_1 & 0 & b_2 & 0 \\ c_1 & d_{11} & 0 & d_{12} & 0 \\ 0 & 0 & 1_{N_2} & 0 & 0 \\ c_2 & d_{21} & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 1_{N_2} \end{pmatrix} \begin{pmatrix} p & 0 & q_1 & 0 & q_2 \\ 0 & 1_{N_2} & 0 & 0 & 0 \\ c_1 & 0 & d_{11} & 0 & d_{12} \\ 0 & 0 & 0 & 1_{N_2} & 0 \\ c_2 & 0 & d_{21} & 0 & d_{22} \end{pmatrix} =$$

$$= \left(\begin{array}{c|cccccc} aa' & b_1 & ab'_1 & & b_1 & ab'_1 \\ \hline c_1 a' & d_{11} & c_1 b'_1 & & d_{12} & c_1 b'_2 \\ c'_1 & 0 & d'_{11} & & 0 & d'_{12} \\ & & & & & \\ c_2 a' & d_{21} & c_2 b'_1 & & d_{22} & c_2 b'_2 \\ c'_2 & 0 & d'_{21} & & 0 & d'_{22} \end{array}\right)$$

Pass from $m = 2$ to arbitrary m is evident. However we formulate a formal definition, which is valid for all m. We extend the operator g to an operator $\widetilde{g}$ in

$$\mathbb{C}^\alpha \oplus \bigl(V \otimes Z_{N_1+N_2}\bigr) = \Bigl[\mathbb{C}^\alpha \oplus \bigl(V \otimes Z_{N_1}\bigr)\Bigr] \oplus \Bigl[V \otimes Z_{N_2}\Bigr]$$

This operator acts as g on the first summand and as 1 on the second summand. Next, we extend h to an operator $\widehat{h}$ acting in the same way in the space

$$\mathbb{C}^\alpha \oplus \bigl(V \otimes Z_{N_1+N_2}\bigr) = \Bigl[\mathbb{C}^\alpha \oplus \bigl(V \otimes Z_{N_2}\bigr)\Bigr] \oplus \Bigl[V \otimes Z_{N_1}\Bigr],$$

and evaluate $g \circ \widehat{h}$.

Proposition 1.1 *The $\circ$-multiplication is associative, i.e., for any N_1, N_2, N_3 and $g_1 \in \mathcal{M}_{N_1}$, $g_2 \in \mathcal{M}_{N_2}$, $g_3 \in \mathcal{M}_{N_3}$ the following equality holds:*

$$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3).$$

This is more or less obvious (see also [11]).

1.4. Infinite-dimensional version. Denote by $\mathcal{M}_\infty)$ the set of conjugacy classes on $\mathrm{Mat}(\infty)$ with respect to $\mathrm{GL}(\infty, \mathbb{C})$. Then $\mathcal{M}(\infty)$ is a semigroup with respect to $\circ$.

1.5. Variants. Denote $\mathcal{GL}_N$ conjugacy classes of $\mathrm{GL}(\alpha + mN, \mathbb{C})$ with respect to $\mathrm{GL}(N, \mathbb{C})$, by $\mathcal{U}_N$ the conjugacy classes of $\mathrm{U}(\alpha+mN)$ with respect to $\mathrm{U}(N)$, in all cases we consider the equivalence (1.3). Thus we get the operations

$$\begin{aligned} \mathcal{GL}_{N_1} \times \mathcal{GL}_{N_2} &\to \mathcal{GL}_{N_1+N_2}; \\ \mathcal{GL}_\infty \times \mathcal{GL}_\infty &\to \mathcal{GL}_\infty; \\ \mathcal{U}_{N_1} \times \mathcal{U}_{N_2} &\to \mathcal{U}_{N_1+N_2}; \\ \mathcal{U}_\infty \times \mathcal{U}_\infty &\to \mathcal{U}_\infty. \end{aligned}$$

1.6. The origin of $\circ$-multiplication. The spaces $\mathcal{M}_N$ and $\mathcal{U}_n$ are classical topics of the spectral theory of non-self-adjoint operators (see [2], [3], [18], [6]) and of the system theory (see e.g., [4], Chapter 'Realizations'). In 70s a family of operations of this type appeared in representation theory of infinite-dimensional groups, see [14], [10], Section IX.4. For instance, there is a semigroup structure on double cosets

$$\mathrm{O}(\infty) \setminus \mathrm{U}(\alpha + \infty)/\mathrm{O}(\infty).$$

Moreover this semigroup acts in $\mathrm{O}(\infty)$-fixed vectors of unitary representations of $\mathrm{U}(\alpha+\infty)$. Big zoo of operation on $K\backslash G/K$ (with 'small' subgroup K arose in [11], [12] (our $\mathrm{U}(\alpha + m\infty)//\mathrm{U}(\infty)$ is inside this zoo). For another explanation, see [13].

The purpose of this work is to obtain spectral date for $\mathrm{U}(\alpha + mN)//\mathrm{U}(n)$ and $\mathrm{U}(\alpha + m\infty)//\mathrm{U}(\infty)$ visualizing $\circ$-multiplications.

Notice that the spectral data for collections of matrices were widely discussed, see different approaches in surveys [1], [8]. It seems that our approach produces another kind of spectral data. On the other hand, it extends classical approach for $m = 1$.

1.7. Categorical quotient. Notice that the spaces $\mathcal{M}_N$ are non-Hausdorff. There are many ways to construct Hausdorff spaces from set-theoretical X/G quotients. We prefer the following.

Let a reductive group G act on an affine algebraic variety X. Consider the algebra $\mathbb{C}(X)^G$ of G-invariant regular functions on X. The *categorical quotient* is the set of maximal ideals of $\mathbb{C}(X)^G$.

We consider categorical quotients $[\mathcal{M}_N]$ of $\mathrm{Mat}(\alpha + mN)$ by $\mathrm{GL}(n, \mathbb{C})$ and $[\mathcal{GL}_N]$ of $\mathrm{GL}(\alpha + mN, \mathbb{C})$ by $\mathrm{GL}(n, \mathbb{C})$.

Notice that regular functions on $\mathrm{Mat}(\alpha+mN)$ are polynomials. The algebra of regular functions on $\mathrm{GL}(\alpha+mN, \mathbb{C})$ is generated by polynomials and $\det(g)^{-1}$.

Proposition 1.2 a) *The natural map* $[\mathcal{M}_N] \to [\mathcal{M}_{N+1}]$ *is injective.*

b) *The natural map* $[\mathcal{GL}_N] \to [\mathcal{GL}_{N+1}]$ *is injective for* $N > m\alpha$. *Moreover, for* $N > m\alpha$ *we have* $[\mathcal{GL}_N] = [\mathcal{M}_N]$.

c) *For* $N > m\alpha$ *the natural map* $\mathcal{U}_N \to \mathcal{U}_{N+1}$ *is injective.*

The statement is proved in Subsection 2.4.

This allows to define the infinite-dimensional 'categorical quotient' $[\mathcal{M}_\infty]$. On the other hand this shows that the topological quotient $\mathcal{U}_\infty$ is Hausdorff.

Proposition 1.3 *The* $\circ$*-multiplication determines a map of categorical quotients,*

$$[\mathcal{M}^{\alpha,m}_{N_1}] \times [\mathcal{M}^{\alpha,m}_{N_2}] \to [\mathcal{M}^{\alpha,m}_{N_1+N_2}].$$

The statement is proved in Subsection 2.5.

1.8. Characteristic functions. Let $g \in \mathrm{Mat}(N)$. Let S range in $\mathrm{Mat}(m)$. We write the following equation:

$$\begin{pmatrix} q \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a & b_1 & \dots & b_m \\ c_1 & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & d_{m1} & \dots & d_{mm} \end{pmatrix} \begin{pmatrix} p \\ s_{11}x_1 + \dots + s_{1m}x_m \\ \vdots \\ s_{m1}x_1 + \dots + s_{mm}x_m \end{pmatrix}. \tag{1.4}$$

Eliminating variables $x_1, \dots, x_n$, we get a dependence

$$q = \chi_g(S)\, p,$$

where $\chi_g(S)$ is a rational matrix-valued function, $\mathrm{Mat}(m) \to \mathrm{Mat}(\alpha)$. We call χ_g by *the characteristic function* of g.

The following statements were obtained in [11].

Proposition 1.4 *The characteristic function of* g *depends only on a* $\mathrm{GL}(N,\mathbb{C})$*-conjugacy class of* g (*and not on* g *itself*).

Proposition 1.5 *Consider the natural map* $I_N : \mathcal{M}_N \to \mathcal{M}_{N+1}$. *Then*

$$\chi_{I_N g} = \chi_g(S).$$

This is obvious.

Theorem 1.6

$$\chi_{g\circ h(S)} = \chi_g(S)\chi_h(S).$$

Theorem 1.7 *Let* g *be unitary. Then*

a) *If* $\|S\| \leqslant 1$, *then* $\|\chi_g(S)\| \leqslant 1$.

b) *If* S *is unitary, then* $\chi_g(S)$ *is unitary.*

Thus we get a multivariate analog of inner functions (see [5], [16]).

Denoting $\widetilde{S} := S \otimes 1_N$, we represent (1.4) as

$$q = ap + b\widetilde{S}x$$
$$x = cp + d\widetilde{S}x$$

Therefore

$$x = (1 - d\widetilde{S})^{-1}p$$

and

$$\chi_g(S) = a + b\widetilde{S}(1 - d\widetilde{S})^{-1}c.$$

Proposition 1.8

$$\det \chi_g(S) = \frac{\det \begin{pmatrix} a & -b\widetilde{S} \\ c & 1 - d\widetilde{S} \end{pmatrix}}{\det(1 - d\widetilde{S})} \tag{1.5}$$

To prove this, we apply the formula for a determinant of block matrix to the numerator of (1.5).

1.9. The language of Grassmannians. Now we reformulate the definition of characteristic function. For a linear space W denote by $\mathrm{Gr}(W)$ the set of all subspaces in W. For an even-dimensional space denote by $\mathrm{Gr}^{1/2}(W)$ the set of subspaces having dimension $\frac{1}{2}\dim W$.

Consider the Grassmannian $\mathrm{Gr}^{1/2}(V \oplus V)$ in the linear space $V \oplus V$. For any $L \subset \mathrm{Gr}^{1/2}(V \oplus V)$ consider the subspace $L \otimes Z_N \subset V \otimes Z_N$. We write the equation

$$\begin{pmatrix} q \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a & b_1 & \dots & b_m \\ c_1 & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & d_{m1} & \dots & d_{mm} \end{pmatrix} \begin{pmatrix} p \\ y_1 \\ \vdots \\ y_m \end{pmatrix}. \tag{1.6}$$

Consider the set $\mathcal{X}_g(L)$ of all (q,p) such that there are $(x,y) \in L \otimes Z_m$ satisfying (1.6). The set $\mathcal{X}_g(L)$ is a subspace in $\mathbb{C}^\alpha \oplus \mathbb{C}^\alpha$. The following statement is straightforward.

Proposition 1.9 *Let L be a graph of an operator $S : V \to V$. Assume that S is a non-singular point of χ_g. Then $\mathcal{X}_g(L)$ is the graph of the operator $\chi_g(S) : \mathbb{C}^\alpha \to \mathbb{C}^\alpha$.*

Thus $\mathcal{X}_g$ is a rational map $\mathrm{Gr}^{1/2}(V \oplus V) \to \mathrm{Gr}^{1/2}(\mathbb{C}^\alpha \oplus \mathbb{C}^\alpha)$.

Notice that the map $\mathrm{Gr}^{1/2}(V \oplus V) \to \mathrm{Gr}(\mathbb{C}^\alpha \oplus \mathbb{C}^\alpha)$ is well-defined on the whole $\mathrm{Gr}^{1/2}(V \oplus V)$.

1.10. Distinguished divisor. A characteristic function does not determines an element of $\mathcal{M}_N$. Indeed, consider the following matrix of size $\alpha + (k+l) + (k+l)$

$$\begin{pmatrix} a & b_1 & 0 & b_2 & 0 \\ c_1 & d_{11} & 0 & d_{12} & 0 \\ 0 & 0 & e_{11} & 0 & e_{12} \\ c_2 & d_{21} & 0 & d_{22} & 0 \\ 0 & 0 & e_{21} & 0 & e_{22} \end{pmatrix} \in \mathcal{M}^{\alpha,2}_{k+l}.$$

Then its characteristic functions does not depend on the matrix $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$. Therefore additional invariants are necessary.

Let $g \in \mathrm{Mat}(n)$. We write the equation

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_{11} & \dots & d_{1m} \\ \vdots & \ddots & \vdots \\ d_{m1} & \dots & d_{mm} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \tag{1.7}$$

and consider the set ξ_g of all S such that this equation has a nonzero solution. Clearly, *all poles of χ_g are contained in ξ_g*.

It is more convenient to reformulate the definition in a more precise form. we consider the polynomial p_g on $\mathrm{Mat}(m)$ determined by the equation

$$p_g(S) = \det(1 - d\widetilde{S}) =$$
$$= \left(1_{mN} - \begin{pmatrix} d_{11} & \dots & d_{1m} \\ \vdots & \ddots & \vdots \\ d_{m1} & \dots & d_{mm} \end{pmatrix} \begin{pmatrix} s_{11} \cdot 1_N & \dots & s_{1m} \cdot 1_N \\ \vdots & \ddots & \vdots \\ s_{m1} \cdot 1_N & \dots & s_{mm} \cdot 1_N \end{pmatrix} \right)$$

Denote by $P_g(S)$ the denominator of $\det \chi_g(S)$ (we assume that the fraction is reduced). By 1.5, $P_g(S)$ is a divisor of $p_g(S)$. Denote

$$\pi_g(S) := \frac{p_g(S)}{P_g(S)}.$$

The set ξ_g is the set of zeros of the polynomial p_g. Moreover, it is more natural to consider the set ξ_g as a divisor (see, e.g. [7]), i.e. we decompose $p_g(S)$ as a product of irreducible factors

$$p_g(S) = \prod_i h_i(S)^{v_i},$$

where h_i are pairwise distinct. Then we consider collection of hypersurfaces[3] $h_i(S) = 0$ with assign multiplicities v_i.

[3] of complex codimension 1.

Evidently,

$$p_{g\circ h}(S) = p_g(S)\, p_h(S).$$

Equivalently,

$$\xi_{g\circ h} = \xi_g + \xi_h.$$

The sign '+' means that we consider union of hypersurfaces taking in accounts their multiplicities.

Let us pass to the language of Grassmannians. Consider another matrix coordinate on Grassmannian $\Lambda = S^{-1}$. Then the equation for the divisor passes to the form

$$\det(d - S) = \left(\begin{pmatrix} d_{11} & \dots & d_{1m} \\ \vdots & \ddots & \vdots \\ d_{m1} & \dots & d_{mm} \end{pmatrix} - \begin{pmatrix} \lambda_{11}\cdot 1_N & \dots & \lambda_{1m}\cdot 1_N \\ \vdots & \ddots & \vdots \\ \lambda_{m1}\cdot 1_N & \dots & \lambda_{mm}\cdot 1_N \end{pmatrix} \right) = 0$$

Two equations

$$\det(1 - dS) = 0, \qquad \det(d - \Lambda) = 0$$

determine a divisor Ξ_g in $\mathrm{Gr}^{1/2}(V\oplus V)$. Indeed, we have two charts in $\mathrm{Gr}^{1/2}(V\oplus V)$, one consists of graphs of operators $V\oplus 0 \to 0\oplus V$, another from graphs of operators $0\oplus V \to V\oplus 0$. These charts does not cover the whole Grassmannian, but the complement has codimension 2. Therefore any hypersurface in $\mathrm{Gr}^{1/2}(V\oplus V)$ has an intersection with one of charts. In fact all hypersurfaces in $\mathrm{Gr}^{1/2}(V\oplus V)$ are observable in the chart S, except the hypersurface $\det\Lambda = 0$ (it is a complement to the chart S).

The degree of the polynomial p_g for generic g is mN. If the divisor Ξ_g contains the component $\det\Lambda = 0$ with multiplicity l, then the degree of p_g is $m(N - l)$.

1.11. Characteristic functions for $\mathcal{M}_\infty$. We have a natural embedding

$$I_N : \mathcal{M}_N \to \mathcal{M}_{N+1}$$

induced by the embedding $Z_N \to Z_{N+1}$. Evidently,

$$\chi_{I_N g}(S) = \chi_g(S).$$

Therefore, we can define a characteristic function for elements of $\mathcal{M}_\infty = \cup_N \mathcal{M}_N$. On the other hand, we can repeat the definition for $N = \infty$, this produces the same result.

Next,

$$p_{I_N g}(S) = p_g(S)\cdot \det(1 - S).$$

Denoting by δ the divisor $\det(1 - S) = 0$ we get

$$\Xi_{I_N g} = \Xi_g + \delta.$$

Thus we can define a 'divisor' $\Xi_g \subset \mathrm{Gr}^{1/2}(V \oplus V)$ for $g \in \mathcal{M}_\infty$. Notice that its component δ has multiplicity ∞. Except δ, we have finite number of components of finite multiplicity.

REMARK. In the same way, $\xi_{I_N g} = \xi_g + \delta$. We can define ξ_g for $g \in \mathcal{M}_\infty$. However, acting in this way we loss information concerning the multiplicity of the divisor $\det \Lambda = 0$. $\square$

1.12. Central extension. Denote by Γ the multiplicative group of all rational functions $\chi : \mathrm{Mat}(m) \to \mathrm{Mat}(\alpha)$ such that $\chi(0) = 1_\alpha$. Denote by Δ the Abelian additive group of all divisors in $\mathrm{Gr}^{1/2}(V \oplus V)$. We send

$$\mathcal{M}_\infty^{\alpha,m} \to \Gamma \times \Delta.$$

However, the image of this map does not split to a product. The reason is Proposition 1.8. Indeed, let pass to the data $(\chi_g(S), \pi_g(S))$. Then

$$\pi_{g\circ h}(S) = \pi_g(S)\pi_h(S) \cdot c_{g,h}(S),$$

where

$$c_{g,h}(S) = \frac{P_g(S)P_h(S)}{P_{g\circ h}(S)}$$

is a nontrivial cocycle (it is responsible for cancellations of poles of $\chi_g(S)$ and zeros of $\chi_h(S)$ in the product $\chi_{g\circ h}(S)$.

1.13. More invariants. Let $j = 1, 2, \ldots$. For $g \in \mathrm{GL}(\infty, \mathbb{C})$ we consider the direct sum

$$g^{[j]} := g \oplus \cdots \oplus g$$

of j copies of g. It acts in the space

$$\Big[\mathbb{C}^\alpha \oplus (V \otimes Z_N)\Big] \oplus \cdots \oplus \Big[\mathbb{C}^\alpha \oplus (V \otimes Z_N)\Big] = \mathbb{C}^{j\alpha} \oplus \big((V \oplus \cdots \oplus V) \otimes Z_N\big) = \\ = \mathbb{C}^{j\alpha} \oplus \big((V \otimes \mathbb{C}^j) \otimes Z_N\big)$$

Thus we get an embedding

$$\mathcal{M}_N^{\alpha,m} \to \mathcal{M}_N^{j\alpha,jm}$$

It is compatible with $\circ$-multiplication.

Now for a given $g \in \mathcal{M}_N^{\alpha,m}$ we get a collection of characteristic functions

$$\chi_g,\, \chi_{g^{[2]}}, \chi_{g^{[3]}}, \ldots; \qquad \chi_{g^{[j]}} : \mathrm{Mat}(jm) \to \mathrm{Mat}(j\alpha) \tag{1.8}$$

and the collection of divisors

$$\Xi_g,\, \Xi_{g^{[2]}}, \xi_{g^{[3]}}, \ldots; \qquad \Xi_{g^{[j]}} \subset \mathrm{Gr}\big((V \otimes \mathbb{C}^j) \,\oplus\, (V \otimes \mathbb{C}^j)\big). \tag{1.9}$$

Theorem 1.10 a) *For any α, m, N the characteristic functions* (1.8) *and divisors* (1.9) *uniquely determine an element of the categorical quotient* $[\mathcal{M}_N^{\alpha,m}]$

b) *For any $N > \alpha m$ the characteristic functions* (1.8) *and divisors* (1.9) *uniquely determine an element of the categorical quotient* $[\mathcal{GL}_N^{\alpha,m}]$

Proof is contained in Subsections 2.1–2.2.

Theorem 1.11 a) *For any $N > \alpha m$ the characteristic functions* (1.8) *and divisors* (1.9) *uniquely determine an element of the space $\mathcal{U}_N^{\alpha,m}$ of conjugacy classes.*

b) *The characteristic functions* (1.8) *and divisors* (1.9) *uniquely determine an element of the space $\mathcal{U}_\infty^{\alpha,m}$ of conjugacy classes.*

Proof is contained in Subsection 2.3.

1.14. Relations between characteristic functions of different levels.

Proposition 1.12 a) *Let S_1, S_2 be matrices of sizes mj_1, mj_2 respectively. Then*

$$\chi_{g^{[j_1+j_2]}}\left(\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}\right) = \chi_{g^{[j_1]}}(S_1) \oplus \chi_{g^{[j_2]}}(S_2) \tag{1.10}$$

b) *For an invertible matrix H of size j denote by $\widetilde{H}$ the operator*

$$\widetilde{H} = H \otimes 1_j$$

in $V \otimes \mathbb{C}^j$. Then

$$\chi_{g^{[j]}}(\widetilde{H} S \widetilde{H}^{-1}) = H \chi_{g^{[j]}}(S) H^{-1}$$

c) *The divisor $\Xi_{g^{[j]}}$ is invariant with respect to the transformations of*

$$\mathrm{Gr}^{1/2}\big((V \otimes \mathbb{C}^j) \oplus (V \otimes \mathbb{C}^j)\big)$$

induced by transformations $\widetilde{H} \oplus \widetilde{H}$.

d) *Consider the natural embedding*

$$\mathrm{Gr}^{1/2}\big((V \otimes \mathbb{C}^{j_1}) \oplus (V \otimes \mathbb{C}^{j_1})\big) \times \mathrm{Gr}^{1/2}\big((V \otimes \mathbb{C}^{j_2}) \oplus (V \otimes \mathbb{C}^{j_2})\big) \to \\ \to \mathrm{Gr}^{1/2}\big((V \otimes \mathbb{C}^{j_1+j_2}) \oplus (V \otimes \mathbb{C}^{j_1+j_2})\big)$$

(*for a pair of subspaces L_1, L_2 we assign their direct sum $L_1 \oplus L_2$ in the target space*). *We have the following coincidence of divisors*

$$\Xi_{g^{[j_1+j_2]}} \bigcap \Big[\mathrm{Gr}^{1/2}\big((V \otimes \mathbb{C}^{j_1}) \oplus (V \otimes \mathbb{C}^{j_1})\big) \times \mathrm{Gr}^{1/2}\big((V \otimes \mathbb{C}^{j_2}) \oplus (V \otimes \mathbb{C}^{j_2})\big)\Big] = \\ = \Xi_{g^{[j_1]}} \times \mathrm{Gr}^{1/2}\big((V \otimes \mathbb{C}^{j_2}) \oplus (V \otimes \mathbb{C}^{j_2})\big) + \mathrm{Gr}^{1/2}\big((V \otimes \mathbb{C}^{j_1}) \oplus (V \otimes \mathbb{C}^{j_1})\big) \times \Xi_{g^{j_2]}}$$

2 Invariants

2.1. Invariants for $\mathcal{M}_N^{\alpha,m}$. Denote by $b_j[k]$ the k-th row of the matrix b_j, by $c_i[l]$ the l-th column of the matrix c_i. Let us regard $\mathrm{Mat}(\alpha + mN)$ as a linear space with action of $\mathrm{GL}(N, \mathbb{C})$. A point of the space is a collection of m^2 matrices d_{ij}, αm vectors $c_i[l]$, and αm of covectors $b_j[k]$. Generators of the algebra of invariants are known (see, e.g., [17], Section 11.8.1). The algebra is generated by the following polynomials

$$\operatorname{tr} d_{i_1j_1} d_{i_2j_2} \dots d_{i_nj_n}, \tag{2.1}$$
$$c_i[l] d_{i_1j_1} d_{i_2j_2} \dots d_{i_nj_n} b_j[k], \tag{2.2}$$
$$a_{ij}. \tag{2.3}$$

We wish to show that all the generators can be expressed in terms of Taylor coefficients of $\chi_{g^{[j]}}(S)$ and $p_{g^{[j]}}(S)$ at zero.

First, consider the expression

$$\ln p_{g^{[j]}}(S) = \ln\det(1 - g^{[j]}\widetilde{S}), \tag{2.4}$$

where S is an operator in $V \otimes \mathbb{C}^j$.

It is convenient to think that matrix elements s_{ij}^{kl} of S depend on 4 indexes: indices $i, j \leqslant m$ are responsible for an operator in the space $\mathbb{C}^m$, and $k, l \leqslant j$ for operators in the space $\mathbb{C}^j$. In a neighborhood of $S = 0$ we have the following expansion

$$\ln\det(1 - d^{[j]}\widetilde{S}) = \sum_{n>0} \frac{(-1)^n}{n} \operatorname{tr}(d^{[j]}\widetilde{S})^n =$$
$$= \sum_{\substack{n \geqslant 0,\\ \varphi_1,\dots,\varphi_n,\psi_1,\dots,\psi_n \leqslant m,\\ \mu_1,\dots,\mu_n \leqslant j}} \frac{(-1)^n}{n} s^{\mu_1,\mu_2}_{\varphi_1,\psi_1} s^{\mu_2,\mu_3}_{\varphi_2,\psi_2} \dots s^{\mu_n,\mu_1}_{\varphi_n,\psi_n} \times$$
$$\times \operatorname{tr} d_{\psi_1,\varphi_2} d_{\psi_2,\varphi_3} \dots d_{\psi_n,\varphi_1}. \tag{2.5}$$

Generally, Taylor coefficients (in $s^{\mu\nu}_{\varphi,psi}$) of this series are sums of traces. However, assume that all elements of the sequence

$$\mu_1, \dots, \mu_n$$

are pairwise distinct. Then the coefficient at

$$s^{\mu_2,\mu_3}_{\varphi_2,\psi_2} \dots s^{\mu_n,\mu_1}_{\varphi_n,\psi_n}$$

is

$$\operatorname{tr} d_{\psi_1,\varphi_2} d_{\psi_2,\varphi_3} \dots d_{\psi_n,\varphi_1}.$$

We observe that all the invariants (2.1) are contained in the set of Taylor coefficients of (2.4).

Next we expand the characteristic function

$$\chi_{g^{[j]}}(S) = a^{[j]} + b^{[j]}\widetilde{S}(1 - d^{[j]}S)^{-1},$$

Here

$$a^{[j]} = a \otimes 1_j, \qquad b^{[j]} = b \otimes 1_j, \qquad c^{[j]} = c \otimes 1_j, \quad d^{[j]} = d \otimes 1_j.$$

Substituting $S=0$, we get invariants (2.3) Next, expand

$$X := b^{[j]}\widetilde{S}(1-d^{[j]}\widetilde{S})^{-1}c^{[j]} = \sum_{n=0}^{\infty} b^{[j]}\widetilde{S}(d^{[j]}\widetilde{S})^{n} \tag{2.6}$$

in a Taylor series. The operator X acts in $\mathbb{C}^m \otimes \mathbb{C}^j$. We enumerate its matrix elements as $x_{kl}^{\gamma\delta}$, where k, $l \leqslant \alpha$ and γ, $\delta \leqslant j$. The matrix elements are

$$x_{kl}^{\gamma\delta} = \sum_{\substack{n\geqslant 0\\ \varphi_2,\dots,\varphi_{n+1}\leqslant m\\ \psi_1,\dots,\psi_n\leqslant m\\ \mu_1,\dots,\mu_{n+1}\leqslant j}} s^{\mu_1,\mu_2}_{\gamma,\psi_1} s^{\mu_2,\mu_3}_{\varphi_2,\psi_2}\dots s^{\mu_n,\mu_{n+1}}_{\varphi_{n+1},\delta}\times \\ \times b_\gamma[k] d_{\psi_1,\varphi_2} d_{\psi_2,\varphi_3}\dots d_{\psi_n,\varphi_{n+1}} c_\delta[l]$$

Again, if we take a term of the Taylor series at

$$s^{\mu_1,\mu_2}_{\gamma,\psi_1} s^{\mu_2,\mu_3}_{\varphi_2,\psi_2}\dots s^{\mu_n,\mu_{n+1}}_{\varphi_{n+1},\delta}$$

with pairwise distinct $\mu_1, \dots, \mu_{n+1}$, then factors $d_{\dots}$ in the coefficient and their order are are uniquely determined.

This proves Theorem 1.10.a.

2.2. Invariants for invertible matrices. Now consider the space $\mathcal{GL}_N^{\alpha,m}$. Invariant regular functions on this space have the form $p(g)/\det(g)^k$, where $p(g)$ is a polynomial satisfying

$$p\bigl(\iota(u)g\iota(u)^{-1}\bigr) = \det(g)^k, \qquad \text{where } u \in \mathrm{GL}(N,\mathbb{C}),$$

$iota(u)$ is given by (1.2). Therefore, $p(g)$ is invariant with respect to $\mathrm{SL}(N,\mathbb{C})$. In comparison with (2.1)–(2.3) we get new invariants (see [17], Section 11.8.1). First, consider an N-plet of pairwise distinct vectors $c_i[l]$, compose a matrix with such rows. Its determinant is an invariant of $\mathrm{SL}(N,\mathbb{C})$. In the same way we compose determinants from columns of c_j. These determinants are additional invariants.

However, we can do this only if $N \leqslant \alpha m$ (the total number of possible rows). Thus for $N > \alpha m$ there are no additional invariants.

This proves Theorem 1.10.b.

2.3. Invariants for unitary group. Consider invariants of $\mathrm{U}(N)$ on $\mathrm{U}(\alpha+mN)$. Consider the algebra $\mathcal{A}$ of functions on $\mathrm{U}(n)$ generated by matrix elements and $\det(g)^{-1}$. By the Peter-Weyl theorem it is dense in the algebra of continuous functions.

Consider two orbits $\mathcal{O}_1$, $\mathcal{O}_2 \subset \mathrm{U}(\alpha+mN)$ of $\mathrm{U}(N)$, consider an invariant continuous function φ separating these orbits. Consider a function $\psi \in \mathcal{A}$ approximating φ, say

$$\psi\Big|_{\mathcal{O}_1} \leqslant a < b \leqslant \psi\Big|_{\mathcal{O}_2}$$

We have

$$\psi(g) = \frac{p(g)}{(\det g)^k},$$

where p is polynomial. The average

$$f(g) = \int_{\mathrm{U}(N)} \psi\big(\iota(h)\, g\, \iota(h)^{-1}\big)\, dh$$

has the same form $\frac{q(g)}{(\det g)^k}$ with polynomial q. Therefore f has a regular continuation to the whole group $\mathrm{GL}(\alpha + mN, \mathbb{C})$. If $N > \alpha m$, then all such functions are polynomials in Taylor coefficients of $\chi_g(S)$ and $p_g(S)$.

2.4. Change of N. Next, we wish to show that the map $I_N : [\mathcal{M}_N^{\alpha,m}] \to [\mathcal{M}_{N+1}^{\alpha,m}]$ is an embedding(Proposition 1.2). The map I_N replaces

$$\begin{aligned}
a &\to a \\
b_i &\to \begin{pmatrix} b_i & 0 \end{pmatrix} \\
c_j &\to \begin{pmatrix} c_j \\ 0 \end{pmatrix} \\
d_{ij} &\to \begin{pmatrix} d_{ij} & 0 \\ 0 & 0 \end{pmatrix}, \qquad i \neq j \\
d_{ii} &\to \begin{pmatrix} d_{ii} & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

Take generators (2.1)–(2.3) of invariants of $g \in \mathrm{Mat}(\alpha + m(N+1))$ and restrict them to $\mathrm{Mat}(\alpha + mN)$. We get the same expressions in all the cases except traces of the form

$$\operatorname{tr} d_{i_1 i_1} d_{i_2 i_2} \dots d_{i_n i_n},$$

they are shifted by 1. Therefore, restricting a collection of generators of $\mathrm{GL}(N+1, \mathbb{C})$-invariants of $\mathrm{Mat}(\alpha + mN)$ we get a collection of generators for algebra of $\mathrm{GL}(N, \mathbb{C})$-invariants on $\mathrm{Mat}(\alpha + mN)$.

Thus we proved the desired statement.

2.5. Categorical quotient. Proposition 1.3 follows from Theorem 1.10 (invariants of a $\circ$-products are uniquely determined by invariants of factors).

2.6. Proof of Proposition 1.12. The statement a) is obvious, d) is straightforward. To derive the rest, we set $j = 2$. The statement c) follows from

$$\det\left(1_{2mN} - \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}\right) =$$

$$= \det\left(1_{2mN} - \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}^{-1}\right)$$

For simplicity we write h_{ij} instead of $h_{ij} \cdot 1_m \otimes 1_N$ and s_{ij} instead of $s_{ij} \otimes 1_N$. But d commutes with h_{ij} and we get c).

Next, we write the value of the characteristic function at $\widetilde{H}S\widetilde{H}^{-1}$,

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}^{-1} \times$$
$$\times \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}^{-1} \right)^{-1} \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

We keep in mind that h_{ij} are scalars, for this reason

$$\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} h_{11} \cdot 1_{mN} & h_{12} \cdot 1_{mN} \\ h_{21} \cdot 1_{mN} & h_{22} \cdot 1_{mN} \end{pmatrix} = \begin{pmatrix} h_{11} \cdot 1_{\alpha} & h_{12} \cdot 1_{\alpha} \\ h_{21} \cdot 1_{\alpha} & h_{22} \cdot 1_{\alpha} \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \quad \text{etc.}$$

Applying this several times we get

$$\begin{pmatrix} h_{11} \cdot 1_{\alpha} & h_{12} \cdot 1_{\alpha} \\ h_{21} \cdot 1_{\alpha} & h_{22} \cdot 1_{\alpha} \end{pmatrix} \times$$
$$\times \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \right)^{-1} \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \right) \times$$
$$\times \begin{pmatrix} h_{11} \cdot 1_{\alpha} & h_{12} \cdot 1_{\alpha} \\ h_{21} \cdot 1_{\alpha} & h_{22} \cdot 1_{\alpha} \end{pmatrix}^{-1}$$

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