

**Self-Dual Noncommutative ϕ^4 -Theory
in Four Dimensions is a Non-Perturbatively Solvable
and Non-Trivial Quantum Field Theory**

**Harald Grosse
Raimar Wulkenhaar**

Vienna, Preprint ESI 2428 (2013)

June 17, 2013

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available online at <http://www.esi.ac.at>

Self-dual noncommutative ϕ^4 -theory in four dimensions is a non-perturbatively solvable and non-trivial quantum field theory

Harald GROSSE¹ and Raimar WULKENHAAR²

¹ *Fakultät für Physik, Universität Wien
Boltzmannngasse 5, A-1090 Wien, Austria*

² *Mathematisches Institut der Westfälischen Wilhelms-Universität
Einsteinstraße 62, D-48149 Münster, Germany*

Abstract

We study quartic matrix models with partition function $\mathcal{Z}[E, J] = \int dM \exp(\text{trace}(JM - EM^2 - \frac{\lambda}{4}M^4))$. The integral is over the space of Hermitian $\mathcal{N} \times \mathcal{N}$ -matrices, the external matrix E encodes the dynamics, $\lambda > 0$ is a scalar coupling constant and the matrix J is used to generate correlation functions. For E not a multiple of the identity matrix, we prove a universal algebraic recursion formula which gives all higher correlation functions in terms of the 2-point function and the distinct eigenvalues of E . The 2-point function itself satisfies a closed non-linear equation which must be solved case by case for given E . These results imply that if the 2-point function of a quartic matrix model is renormalisable by mass and wavefunction renormalisation, then the entire model is renormalisable and has vanishing β -function.

As main application we prove that Euclidean ϕ^4 -quantum field theory on four-dimensional Moyal space with harmonic propagation, taken at its self-duality point and in the infinite volume limit, is exactly solvable and non-trivial. This model is a quartic matrix model, where E has for $\mathcal{N} \rightarrow \infty$ the same spectrum as the Laplace operator in 4 dimensions. Using the theory of singular integral equations of Carleman type we compute (for $\mathcal{N} \rightarrow \infty$ and after renormalisation of E, λ) the free energy density (1/volume) $\log(\mathcal{Z}[E, J]/\mathcal{Z}[E, 0])$ exactly in terms of the solution of a non-linear integral equation. Existence of a solution is proved via the Schauder fixed point theorem.

The derivation of the non-linear integral equation relies on an assumption which we verified numerically for coupling constants $0 < \lambda \leq \frac{1}{\pi}$.

MSC2010: 81T16, 39B42, 45E05, 47H10

Keywords: quantum field theory in 4 dimensions; noncommutative geometry; exactly solvable models; matrix models; Schwinger-Dyson equations; singular integral equations

¹harald.grosse@univie.ac.at

²raimar@math.uni-muenster.de

Contents

1	Introduction	1
1.1	Perturbative, axiomatic and algebraic quantum field theory	2
1.2	Euclidean quantum field theory	2
1.3	Solvable models	3
1.4	Constructive methods	4
1.5	Noncommutative geometry	5
1.6	Matrix models	6
1.7	Outline of the paper	8
2	Ward identities in matrix models	10
2.1	Field-theoretical matrix models	11
2.2	Ward identity	12
2.3	Expansion into boundary components	13
3	Schwinger-Dyson equations	15
3.1	Strategy	15
3.2	Schwinger-Dyson equations for a ϕ^4 -model: $B = 1$ cycle	16
3.3	An algebraic recursion formula for a real field theory	17
3.4	The genus expansion	19
3.5	Graphical solution of the recursion formula in the planar sector	23
4	Self-dual noncommutative ϕ_4^4-theory	28
4.1	Definition of the matrix model	28
4.2	Renormalisation of the two-point function	29
4.3	Integral representation	31
4.4	The Carleman equation	32
4.5	The master equation	36
4.6	Higher correlation functions and effective coupling constant	42
4.7	Miscellaneous remarks	45
5	Conclusion and outlook	46
A	Correlation functions with two boundary components	48
A.1	Two cycles of odd length: Schwinger-Dyson equations	48
A.2	Two cycles of even length: Schwinger-Dyson equations	50
A.3	Remarks on functions with more than two cycles	52
A.4	The $B = 2$ sector of noncommutative ϕ_4^4 -theory	52
B	Perturbative expansion	55
	Acknowledgements	58
	References	58

1 Introduction

A rigorous construction of quantum field theories in *four dimensions* was not very successful so far. In this paper we show that for ϕ^4 -theory on four-dimensional Moyal space

with harmonic propagation, taken at critical frequency and in the infinite volume limit, much more is true: *The model is exactly solvable.*

We know that this is a toy model in so far as classical locality and Poincaré symmetry are not realised and the Minkowskian continuation of that Euclidean model needs to be investigated. On the other hand, the model

- carries an action of an infinite-dimensional symmetry group,
- is invariant under a duality transformation between position and momentum space [LS02],
- is almost scale invariant [DGMR07],
- is known to have a realisation as a matrix model (with non-constant kinetic term),
- has perturbatively an infinite number of divergent but renormalisable Feynman graphs [GW05b].

Each renormalised Feynman graph has subleading logarithmic terms which make the perturbatively renormalised correlation functions divergent at large energy. The model and its solution touch many aspects of quantum field theory which we recall in the next subsections. The reader in a hurry may jump to Sec. 1.7.

1.1 *Perturbative, axiomatic and algebraic quantum field theory*

Starting with the Lamb shift in the 1940s and culminating in the experimental tests of the Standard Model, perturbatively renormalised quantum field theory is an enormous phenomenological success. However, this success lacks a mathematical understanding. The perturbation series is at best an asymptotic expansion which cannot converge at physical coupling constants such as the electron charge $e^2 \approx \frac{1}{137}$. In addition there are physical effects such as confinement which seem out of reach for perturbation theory. Therefore, the development of a mathematical foundation of quantum field theory that permits non-trivial examples is one of the most urgent challenges in mathematical physics.

In the early 1950's, Gårding and Wightman gave a rigorous mathematical foundation to relativistic quantum field theory by casting the unquestionable physical principles (locality, covariance, stability, unitarity) into a set of axioms. These ideas were published years later [Wig56, WG64, SW64]. The difficulty to provide non-trivial examples to these axioms motivated the development of equivalent or more general frameworks such as Algebraic quantum field theory and Constructive/Euclidean quantum field theory. Algebraic quantum field theory shifts the focus from the field operators to the Haag-Kastler net of algebras assigned to open regions in space-time [HK64]. Fields merely provide coordinates on the algebra. Over the years this point of view turned out to be very fruitful [Haa92]. One important advantage over the axiomatic setup is the natural possibility to describe quantum field theory on curved space-time [BF09].

1.2 *Euclidean quantum field theory*

Of central importance for us is the Euclidean approach. Wightman functions admit an analytic continuation in time. At purely imaginary time they become the Schwinger functions [Sch59] of a Euclidean quantum field theory. Symanzik emphasised the powerful Euclidean-covariant functional integral representation [Sym64], which yields a Feynman-Kac formula of the heat kernel [Kac49]. In this way the Schwinger functions become the

moments of a statistical physics probability distribution. This tight connection between Euclidean quantum field theory and statistical physics led to a fruitful exchange of concepts and methods, most importantly that of the renormalisation group [WK74].

It is sometimes possible to rigorously prove the existence of a Euclidean quantum field theory or of a statistical physics model without knowing or using that this model derives from a true relativistic quantum field theory. This is, for instance, the case for the model constructed in this paper. Sufficient conditions on the Euclidean model which guarantee the Wightman axioms were first given by Nelson [Nel71]. These conditions based on Markov fields turned out to be too strong or inconvenient. Shortly later, Osterwalder and Schrader established a set of axioms [OS73, OS75] by which the Euclidean quantum field theory is equivalent to a Wightman theory. The most decisive axiom is reflection positivity which yields existence of the Hilbert space and a positive energy Hamiltonian. The Euclidean approach together with the Osterwalder-Schrader axioms turned out to be the key to construct relativistic quantum field theories in dimension less than four. Two successful methods were developed: The correlation inequality method [Gin70, GRS75] relies on positivity and monotonicity of Schwinger functions and is suitable for bosonic theories. The phase space expansion method [GJS74, GJ87, Riv91] works both for bosonic and fermionic theories. It uses lattice partitions and iterated cluster and Mayer expansions to control the ultraviolet limit of the theory. It can also typically establish that the Schwinger functions built constructively are the Borel sums of their ordinary perturbative series. The cluster expansion [GJS74] is also used to prove a mass gap in the spectrum of the Hamiltonian.

1.3 Solvable models

Under solvable models we understand models in quantum field theory or statistical physics where all correlation functions can be exactly evaluated in terms of “known” functions, at least in principle. With a few three-dimensional exceptions, solvable models were established only in one or two dimensions.

The first field-theoretical example was the Thirring model [Thi58] which describes a quartic self-interaction of a Dirac field in 1+1 dimensions. In the massless case, Johnson [Joh61] found the exact expression for the 2- and 4-point functions. This work was extended by Hagen [Hag67] and Klaiber [Kla67] to the explicit solution of any correlation function. The massive case is more complicated; we mention the construction of the S-matrix by Korepin [Kor79] using the Bethe ansatz [Bet31]. Another famous model that is exactly solvable is the Schwinger model [Sch62], or QED in 2 dimensions.

Many 2-dimensional models of statistical physics are known to be solvable. The first one was the 2-dimensional square-lattice critical Ising model [Isi25], which after important work by Kramers and Wannier [KW41] was solved by Onsager [Ons44] using the transfer matrix method. The list of more involved exactly solvable models contains the 6-vertex model (or ice model) solved by Lieb [Lie67], the 8-vertex model solved by Baxter [Bax71] and the hard hexagon model also solved by Baxter [Bax80]. The quantum inverse scattering method [Fad95] and Yang-Baxter equations [Jim90] are important tools for these achievements.

The Ising model turned out to be the prototype of a discrete series of solvable two-dimensional field theories: the Minimal Models in conformal field theory [BPZ84, FQS84].

Minimal Models correspond to highest-weight representations of the Virasoro algebra at central charge $c = 1 - \frac{6}{m(m+1)}$. Some of these models give non-trivial realisations of the Osterwalder-Schrader or Wightman axioms. A solvable model of different type in conformal quantum field theory is the Wess-Zumino-(Novikov-)Witten model [WZ71, Wit84] whose solutions are realised by affine Kac-Moody algebras.

1.4 Constructive methods

Constructing a model means to prove for a specific candidate Hamiltonian or Lagrangian the axioms of Wightman, Haag-Kastler or Osterwalder-Schrader. Thereby the Wightman or Schwinger functions, although not computed/solved explicitly, are shown to have the required properties. Constructing a model is less than solving it. However, most of the solved models are disguised free fields whereas the constructed models are true interacting field theories, unfortunately only in dimension < 4 . For a historical review (that we used already in the section on Euclidean quantum field theory) we refer to [Jaf00].

The first successfully constructed model is the ϕ^4 -model in 2 dimensions, ϕ_2^4 for short, constructed in a series of articles by Glimm-Jaffe [GJ68-72]. Whereas this was directly achieved in the Wightman or Haag-Kastler setup, the Euclidean approach was decisive for the generalisation to the $P(\phi)_2$ model [GJS74, GRS75], i.e. a two-dimensional scalar field with polynomial interaction. The construction of ϕ_3^4 [GJ73, FO76] turned out to be much harder. There is a stability problem which was overcome by renormalisation group ideas. Another problem not present in two dimensions is the inequivalence of representations of the canonical commutation relations for the interacting and the free field.

Perturbatively, these models ϕ_2^4 , ϕ_3^4 and $P(\phi)_2$ are all super-renormalisable. It is therefore not surprising that a construction of the ϕ_4^4 -model (which is perturbatively just renormalisable) faces far more problems. In $4+\epsilon$ dimensions, the ϕ^4 -model does not exist. As shown by Aizenman and Fröhlich [Aiz81, Frö82], the model is trivial, i.e. it only exists if the coupling constant vanishes. Related to triviality is the appearance of Landau poles [LAK54], first discovered for quantum electrodynamics (QED₄): The β -function which describes the running of the coupling constant when changing the scale could develop a singularity at finite momentum cut-off. This means that the theory is only consistent below that cut-off, or has to be trivial if the cut-off is removed. Perturbatively, the β -function of ϕ_4^4 -theory develops a Landau pole, which together with the triviality of $\phi_{4+\epsilon}^4$ strongly suggests (although a proof is still missing) that ϕ_4^4 cannot be constructed.

There is a variant of the Euclidean $\lambda\phi_4^4$ -model, the wrong-sign model $\lambda < 0$, which can be constructed if restricted to planar graphs [tHo82, Riv83]. Because of the wrong sign and the resulting instability, this Euclidean model does not give rise to a model of relativistic quantum field theory. We mention this model because it shares some aspects with the model we study and solve in this paper. Our model also restricts to the planar sector, but as result of the thermodynamic limit and not by hand. Our model is stable but violates locality in the traditional sense. It is unknown so far whether or not our model satisfies a non-local variant of the Osterwalder-Schrader axioms.

Recall that QED₄ is ruled out for the same Landau pole problem [LAK54]. It is therefore important that non-abelian Yang-Mills theory is asymptotically free [GW73, Pol73] and as such does not have a Landau pole (in perturbation theory). This means that Yang-Mills theory is a candidate for a constructive quantum field theory in four dimen-

sions. Unfortunately, Yang-Mills theory is too difficult so that its construction is an open problem [JW00]. A toy model of asymptotic freedom is the Gross-Neveu model [GN74], a generalisation of the Thirring model to N -component fermions. In two dimensions this model (GN_2 for short) is perturbatively renormalisable but not super-renormalisable. The construction of (GN_2) was achieved in [GK85, FMRS85] and constitutes the first example of a constructed just renormalisable quantum field theory.

1.5 *Noncommutative geometry*

Our present fundamental physics rests on two pillars: Quantum field theory and General relativity. One of the main questions in this area concerns the matching of these two concepts. In perturbative quantum field theory, the standard model which correctly describes all experimentally observed particles is renormalisable, whereas general relativity which correctly describes all observed gravity effects is not. The renormalisation group tells us that non-renormalisable interactions are scaled away. To put it differently, the existence of gravity means that space-time cannot be viewed as a differentiable manifold over all length scales. From the numerical value of the gravitational coupling constant the renormalisation group estimates the fundamental scale at which the manifold structure breaks down: it is the Planck scale 10^{-35} m. The most active approaches of quantum gravity all agree on the assumption that space-time is fundamentally different at the Planck scale; there is only disagreement about the structure which replaces it.

The approach to quantum gravity which is relevant for this paper is noncommutative geometry [Con94]. Noncommutative geometry has its roots in the mathematical description of quantum mechanics. It has been vastly developed over the years and achieved spectacular success in mathematics. Its relevance to physics comes from the fact that both Yang-Mills theory (as a classical field theory) and general relativity are unquestionably of geometrical origin. Noncommutative geometry achieved, on the level of classical field theories, a true unification of Yang-Mills theory, and even of the whole standard model, with general relativity [Con96, CC96]: General relativity put on a noncommutative space which is the product of a manifold with a discrete space is nothing but Yang-Mills theory. The astonishing picture which results is that the breakdown of the manifold structure of space-time already occurs at the much larger length scale given by the Compton length of the W and Z bosons of the standard model. This breakdown is mild; what at larger distances is a point of space-time becomes a couple of points at scales shorter than the standard model scale. But once accustomed to this idea it is natural to assume that passing to shorter and shorter distances there could be a cascade of phase transitions in which the manifold structure of space-time fades out more and more.

Quantum field theory as defined by the Wightman, Haag-Kastler or Osterwalder-Schrader axioms relies on the manifold structure of space-time. In giving up the manifold one has to adapt the axioms. A reasonable replacement is not yet available. One of the reasons is that whereas noncommutative geometry has a clear concept of compact manifolds with Euclidean signature [Con08], it is unknown how to describe non-compact manifolds with Lorentzian signature. Therefore, the focus has been on Euclidean quantum field theories on noncommutative manifolds. Because of their functional integral realisation, such models are easy to define: It suffices to specify a parametrisation of the fields and an action functional for them. Scalar fields, for instance, can classically be viewed

as sections of a vector bundle over the manifold. In noncommutative geometry, such sections form a projective module over an algebra, with the algebra itself being the simplest example. The traditional pointlike interaction of scalar fields is then translated into the product in a noncommutative algebra.

The simplest examples are given by deformations. This means that one takes the same continuous or smooth functions on the manifold as parametrisation but equipped with a deformed product \star . Rieffel showed [Rie93] that if the manifold carries an action of \mathbb{R}^d by translations, as it is the case for Euclidean space, it is possible to turn this group action into a deformed associative product. Doplicher-Fredenhagen-Roberts studied quantum space-time arising from space-time uncertainty relations at the Planck scale [DFR95], Filk derived the Feynman rules for deformed Euclidean space [Fil96], and in 1999 many authors showed perturbative one-loop renormalisability of a number of models. Shortly later Minwalla, van Raamsdonk and Seiberg discovered a severe problem at higher loop order, the so-called ultraviolet/infrared mixing [MVS00].

A possible way to cure this problem for the ϕ_4^4 -model has been found by us in previous work [GW05b]. It leads to an action functional which has 4 relevant/marginal operators $Z, \mu^2, \lambda, \Omega$ (instead of 3):

$$S = \int_{\mathbb{R}^4} dx \left(\frac{Z}{2} \phi(-\Delta + \Omega^2 \|2\Theta^{-1}x\|^2 + \mu^2)\phi + \frac{Z^2\lambda}{4} \phi \star \phi \star \phi \star \phi \right)(x). \quad (1.1)$$

Here Θ is the deformation matrix which defines the \star -product. We have been able to show that the resulting model is renormalisable up to all orders in perturbation theory. In addition, a new fixed point appears at $\Omega = 1$. At one-loop order the theory flows into this fixed point, leaving the ratio $\frac{\lambda}{\Omega^2}$ constant [GW04, GW05c]: In contrast to the usual ϕ_4^4 -theory, the running coupling constant is one-loop bounded. The group around Rivasseau then showed that for the critical case $\Omega = 1$ the β -function vanishes up to three loop order [DR07], and finally succeeded in proving $\beta = 0$ up to all orders in perturbation theory [DGMR07]. In this way, the Landau pole problem is tamed.

Vanishing of the β -function is often a consequence of integrability. It was therefore conjectured that the model (1.1) at critical frequency $\Omega = 1$ has a chance of construction. The construction was searched in two directions. There is the loop vertex expansion developed by Rivasseau [Riv07b] which combines the Hubbard-Stratonovich transform with the Brydges-Kennedy-Abdesselam-Rivasseau forest formula [BK87, AR94]. This method already proved useful in traditional constructive quantum field theory [MR08, RW12] and has been applied in [Wan12] to the two-dimensional version of (1.1). The second approach was proposed by us in [GW09]. By extending the Ward identities and Schwinger-Dyson equations used by Disertori-Gurau-Magnen-Rivasseau in [DGMR07], we derived a self-consistent non-linear integral equation for the renormalised 2-point function of the model (1.1). This is a non-perturbative equation, which we solved perturbatively up to third order in the coupling constant.

1.6 Matrix models

The large- \mathcal{N} limit of $SU(\mathcal{N})$ Yang-Mills theory led to matrix models with dominant planar diagrams [tHo74]. Since that observation, matrix models play an important rôle in mathematical physics: Triangulations of two-dimensional manifolds as simple models of

quantum gravity coupled to matter led to many studies of matrix models [DGZ95]. The IKKT [IKKT97] and BFSS [BFSS97] models are important as reductions of string theory models. Of great interest is the observation [BK90, DS90, GM90] that in the double scaling limit of matrix models a new phase transition occurs.

The partition function of a one-matrix model as toy model for 2D gravity is given by $\mathcal{Z} = \int dM \exp(-\sum_n \alpha_n \text{tr}(M^n))$, where the integral is over $(\mathcal{N} \times \mathcal{N})$ -Hermitian matrices and the α_n are scalar coefficients which may depend on \mathcal{N} . In the double scaling limit $\alpha_n = \mathcal{N}t_n$ and $\mathcal{N} \rightarrow \infty$, the partition function becomes a series in (t_n) which can be expressed in terms of the τ -function for the Korteweg-de Vries (KdV) hierarchy (see [DGZ95] for a review). There is another approach to topological gravity in which the partition function is a series in (t_n) with coefficients given by intersection numbers of complex curves. Witten conjectured [Wit91] that the partition functions of the two approaches coincide. This conjecture was proved by Kontsevich [Kon82] who achieved the computation of the intersection numbers in terms of weighted sums over ribbon graphs (fat Feynman graphs), which he proved to be generated from the Airy function matrix model (Kontsevich model)

$$\mathcal{Z}[E] = \frac{\int dM \exp\left(-\frac{1}{2}\text{tr}(EM^2) + \frac{i}{6}\text{tr}(M^3)\right)}{\int dM \exp\left(-\frac{1}{2}\text{tr}(EM^2)\right)}, \quad (1.2)$$

where $i^2 = -1$ and E is a positive Hermitian matrix related to the series (t_n) by $t_n = (2n-1)!! \text{tr}(E^{-(2n-1)})$. The large- \mathcal{N} limit of (1.2) gives the KdV evolution equation, thus proving Witten's conjecture.

The Rieffel deformation of Euclidean space can be described by a matrix product [GV88]. Langmann, Szabo and Zarembo showed [LSZ04] that the model for a complex scalar field Φ with interaction $(\bar{\Phi} \star \Phi)^2$ in an external magnetic field $B = \Theta^{-1}$ which is dual [LS02] to the deformation matrix gives rise to a matrix model with partition function $\int dM dM^\dagger \exp(-\text{tr}(EM^\dagger M - \lambda M^\dagger M M^\dagger M))$. They showed that in the large- \mathcal{N} limit the model is exactly solvable but trivial. In joint work with Steinacker [GS06, GS08], one of us used the relation of the Langmann-Szabo self-dual noncommutative $\phi^{\star 3}$ -model to the Kontsevich model to perform a non-perturbative renormalisation and to extract the running of the asymptotically free $\phi^{\star 3}$ -model in 6 dimensions.

In this paper we relate the Langmann-Szabo self-dual $\phi^{\star 4}$ -model (which in contrast to $\phi^{\star 3}$ is stable) in 4 dimensions to a matrix model with partition function

$$\mathcal{Z}[E, J, \lambda] = \frac{\int dM \exp(\text{tr}(JM) - \text{tr}(EM^2) - \frac{\lambda}{4}\text{tr}(M^4))}{\int dM \exp(-\text{tr}(EM^2) - \frac{\lambda}{4}\text{tr}(M^4))}. \quad (1.3)$$

The matrix E is fixed by identification with the noncommutative field theory, but our key results hold for general E . The external source J is used to generate correlation functions. We prove a universal algebraic recursion formula for all higher correlation functions in terms of the two-point function and the eigenvalues of E . The formula implies

that if the two-point function is renormalisable, then all higher correlation functions are renormalisable, too, with vanishing β -function.

For the specific case of the ϕ^{*4} -model in 4 dimensions we identify a scaling limit $\mathcal{N} \rightarrow \infty$ in which we can also prove existence of the two-point function. Thanks to the recursion formulae, this provides the exact solution of the model, which in contrast to the Langmann-Szabo-Zarembo model is non-trivial.

It is tempting to speculate that there might be relations to mathematical structures which are the analogues of KdV hierarchy and intersection theory of complex curves to which the Kontsevich model is related.

1.7 Outline of the paper

This paper is a far-reaching generalisation of our previous work [GW09]. We first define in Sec. 2 a general framework of field-theoretical matrix models for a compact operator M on a Hilbert space¹. Whereas the interaction is, as usual, the trace of a polynomial in M , the kinetic term is $\text{tr}(EM^2)$ for a selfadjoint unbounded operator E with compact resolvent. The action of the group of unitaries on Hilbert space induces an infinite number of Ward identities. In Theorem 2.3 we turn this Ward identity into a formula for the second derivative of the partition function. This is the most crucial step which goes far beyond previous glimpses [DGMR07, GW09] of that formula. It is then straightforward to algebraically derive in Sec. 3 the Schwinger-Dyson equation of *any* correlation function (i.e. including non-planar functions and with several boundary components) of the matrix model, and not only the planar 2- and 4-point functions as in [GW09].

In the infinite volume limit, the non-planar sector with genus $g > 0$ is suppressed, but there remain non-trivial sectors which have the topology of a sphere with B punctures/marked points/boundary components. Accordingly, the theory is given by the set of $(N_1 + \dots + N_B)$ -point functions, where the i^{th} boundary component carries N_i external sources. For $B = 1$ we prove that *the 2-point function $G_{|ab|}$ satisfies a non-linear equation for that function alone*. The matrix indices a, b label the eigenvalues $\{E_a\}$ of the external matrix E in (1.3), which may occur with multiplicities. For a real theory with $M = M^*$ we prove in Sec. 3.3 an algebraic recursion formula which computes any $(N \geq 4)$ -point function $G_{|b_0 \dots b_{N-1}|}$ in terms of $G_{|ab|}$ and the eigenvalues of the external matrix E . The solution for the 4-point function is remarkably simple, $G_{b_0 b_1 b_2 b_3} = (-\lambda) \frac{G_{b_0 b_1} G_{b_2 b_3} - G_{b_0 b_3} G_{b_2 b_1}}{(E_{b_0} - E_{b_2})(E_{b_1} - E_{b_3})}$.

The N -point function $G_{b_0 \dots b_{N-1}}$ has a similar structure, namely a sum of fractions with products of two-point functions in the numerator and differences of eigenvalues of E in the denominator. This structure can be visualised in a graphical manner, see Sec. 3.5. We put the indices b_0, \dots, b_{N-1} in cyclic order on a circle and symbolise a factor $G_{b_i b_j}$ by a chord between b_i, b_j and a factor $\frac{1}{E_{b_i} - E_{b_j}}$ by an arrow from b_i to b_j . This produces the non-crossing chord diagrams counted by the Catalan numbers, decorated by two oriented trees, one connecting all even vertices and one connecting all odd vertices, subject to the condition that any of the trees intersects the chords only in the vertices. The solution for $N \in \{4, 6, 8\}$ is given explicitly.

In Appendix A we extend these results to $(N_1 + N_2)$ -point functions. The Schwinger-Dyson equations for the (1+1)- and (2+2)-point functions are linear in the top degree

¹We write ϕ instead of M in Sec. 2. For this outline we write M to be consistent with (1.3).

and depend otherwise only on the N -point functions already known. Given these basic functions, any other (N_1+N_2) -point function with one $N_i \geq 3$ is again purely algebraic. This pattern continues to arbitrary $(N_1+\dots+N_B)$ -point functions: The basic functions with all $N_i \leq 2$ satisfy linear equations to be solved case by case, whereas the higher functions with one $N_i \geq 3$ are universal and purely algebraic.

The main consequence of these universal algebraic recursion formulae is that *if the 2-point function is renormalisable by wavefunction and mass renormalisation $E \mapsto Z(E + \frac{\mu^2 - \mu_{bare}^2}{2})$, $\lambda \mapsto Z^2\lambda$, and if this renders the basic functions with $N_i \leq 2$ finite, then the whole quartic matrix model with action $\text{tr}(EM^2 + \frac{\lambda}{4}M^4)$ is non-perturbatively renormalisable and has vanishing β -function* (Theorem 3.2). To the best of our knowledge such a statement for the quartic matrix model was not known before; it came in this generality completely unexpected for us.

So far this applies to any field-theoretical matrix model with cut-off. One example is the model (1.1) of a four-dimensional noncommutative quantum field theory which we recall in Sec. 4.1. By results of Gracia-Bondía and Várilly [GV88], the model has a matrix realisation. As shown by us and together with Gayral in [GW12, GW11], the model has, as a noncommutative geometry, a finite volume of diameter $\sqrt{\frac{\theta}{\Omega}}$, where θ is the deformation parameter which determines the \star -product. Since $\Omega = 1$ is fixed, going to the thermodynamic (infinite volume) limit amounts to the limit $\theta \rightarrow \infty$. The limit ($\Omega = 0$, $\theta \rightarrow \infty$) was already emphasised by Minwalla, Van Raamsdonk and Seiberg in [MVS00] and then further studied by Becchi, Giusto and Imbimbo in [BGI02, BGI03] in the renormalisation group framework. Becchi et al found for graphs in momentum space, and we confirm this for whole functions at $\Omega = 1$, that the non-planar sector is scaled away but there remains a non-trivial topology in form of spheres with several holes (they called them ‘swiss cheese’ in [BGI02]).

Moreover, and this is the key to all further progress, the infinite volume limit $\theta \rightarrow \infty$ turns the Schwinger-Dyson equation for the basic 2-point function into an *integral equation* (Sec. 4.3). Performing the wavefunction renormalisation in order to prepare for the continuum limit $\Lambda \rightarrow \infty$, this integral equation for the 2-point function G_{ab} at continuous “matrix indices” $a, b \in [0, \Lambda^2]$ is cubic in G_{ab} and has a singular integral kernel. In our previous work [GW09] we have already arrived at this point. The focus there was put on a perturbative solution up to third order in the coupling constant which revealed certain number-theoretic properties. In this paper we arrange the equation in different manner. In this way we recognise that the equation for G_{ab} is linear in the function with $a, b \neq 0$ and non-linear only in its boundary value G_{a0} . Even more, the resulting linear equation at given boundary G_{a0} is the well-studied *Carleman type singular integral equation* [Car22, Tri57, Mus65]. Using the known solution of that particular Riemann-Hilbert problem, we solve exactly the equation for G_{ab} in terms of G_{a0} . But because the 2-point function is symmetric, and $G_{0b} = G_{b0}$ is solved in terms of G_{a0} , this gives rise to a consistency equation $G = TG$

$$G_{b0} = \frac{1}{1+b} \exp \left(-\lambda \int_0^b dt \int_0^\infty dp \frac{(G_{p0})^2}{(\lambda\pi p G_{p0})^2 + (1 + tG_{p0} + \lambda\pi p \mathcal{H}_p^\infty[G_{\bullet 0}])^2} \right). \quad (1.4)$$

Here, \mathcal{H}_a^∞ is the *Hilbert transform* $\mathcal{H}_a^\infty[f(\bullet)] = \frac{1}{\pi} \lim_{\Lambda \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left(\int_0^{a-\epsilon} + \int_{a+\epsilon}^\Lambda \right) dp \frac{f(p)}{p-a}$.

The big progress over [GW09] is that the non-linear operator T is, although still complicated, an improvement operator: We prove in Sec. 4.5 that for $\lambda > 0$ the operator T satisfies the assumptions of the Schauder fixed point theorem which therefore guarantees that (1.4) has a solution $(b \mapsto G_{b0}) \in \mathcal{C}_0^1(\mathbb{R}_+)$ which is monotonously decreasing, positive, and vanishing together with its derivative at infinity.

To be precise, the consistency equation (1.4) relies on an assumption which we do not check. In an accompanying paper [GW13b] we study (1.4) numerically and validate this assumption for coupling constants $0 < \lambda \leq \lambda^* = \frac{1}{\pi}$. For $\lambda > \lambda^*$ we find that the required symmetry $G_{ab} = G_{ba}$ does not hold for the numerical solution of (1.4). We trace this back to the non-trivial solution [Tri57] of the homogeneous Carleman equation which we have put to zero in the derivation of (1.4). The additional term is related to a winding number [Mus65] for the boundary value of an analytic function. It is very well possible that at λ^* one enters another sheet of the logarithm which would provide a continuation of the solution to $\lambda > \frac{1}{\pi}$. Moreover, the non-trivial solution of the homogeneous Carleman equation also affects the uniqueness proof. All this is left for future investigation.

In sec. 4.6 we determine the effective coupling constant $\lambda_{eff} = -G_{0000}$ explicitly in terms of λ and the fixed point solution G_{b0} . In agreement with the general Theorem 3.2 that all renormalisable quartic matrix models have vanishing β -function, λ_{eff} differs only by a finite factor from the bare coupling λ . This confirms at non-perturbative level the perturbative result of Disertori, Gurau, Magnen and Rivasseau [DGMR07] that the self-dual noncommutative ϕ_4^4 -model has vanishing β -function.

Our solution for the 2-point function and the universal algebraic solutions for higher correlation functions can be seen as summation of infinitely many renormalised Feynman graphs. We check in appendix B that the perturbative expansion (at next-to-lowest order) of our exact results coincides with the Feynman graph computation.

We stress again that this is an exact solution of a Euclidean quantum field theory. But thanks to the explicit knowledge of all correlation functions it should be possible in not too distant future to decide whether or not the model satisfies reflection positivity. An affirmative answer is not impossible in view of the positivity and monotonicity properties of the 2-point function. We are convinced that in perturbation theory there is no chance to settle this question. We feel that getting thus far with this model was only possible because there is a deep mathematical structure behind which is still to be discovered. We hope that the novel methods we developed and the fascinating results we obtained might be relevant also for realistic quantum field theories in four dimensions.

2 Ward identities in matrix models

A classical scalar field on \mathbb{R}^d is (e.g.) a continuous function $\phi \in \mathcal{C}_0(\mathbb{R}^d)$ vanishing at ∞ . The space of such functions forms a commutative C^* -algebra and hence is realised as a bounded linear operator on Hilbert space. One natural translation of existence of the mass term $\int_{\mathbb{R}^d} dx \frac{m^2}{2} \phi^2(x)$ to *noncommutative operator algebras* is to require that ϕ^2 is a trace-class operator, which is the case if ϕ is a Hilbert-Schmidt compact operator on Hilbert space.

2.1 Field-theoretical matrix models

A matrix for us is a compact operator on Hilbert space H . For the Hilbert space $H = \mathbb{C}^n$ this is an ordinary $n \times n$ -matrix, for $H = \ell^2(\mathbb{N})$ an infinite matrix and for $H = L^2(\mathbb{R}^n)$ an integral kernel operator. Accordingly, we let I be a set of indices, which can be finite, countable or continuous.

We let \mathcal{A}_I be the algebra of matrices $\phi = (\phi_{ab})_{a,b \in I}$ associated with I , i.e. the vector space of maps $I \times I \ni (a, b) \mapsto \phi_{ab} \in \mathbb{C}$ equipped with the product $(\phi\psi)_{ab} = \sum_{c \in I} \phi_{ac}\psi_{cb}$. For continuous I the space \mathcal{A}_I is the space of integral kernels $\phi_{ab} \equiv \phi(a, b)$, and the above sum is a short-hand notation for the integral $\sum_{c \in I} \phi_{ac}\psi_{cb} \equiv \int_I d\mu(c) \phi(a, c)\psi(c, b)$ with respect to a measure μ on I . It might be convenient to also allow for weight factors in discrete sums, $(\phi\psi)_{ab} = \sum_{c \in I} \mu_c \phi_{ac}\psi_{cb}$.

Recall that for continuous I a matrix (ϕ_{ab}) defines a compact operator on $H = L^2(I, \mu)$ (in fact a Hilbert-Schmidt operator) if $(\phi_{ab}) \in L^2(I \times I, \mu \times \mu)$. The case of infinite discrete I is then a special case for the Dirac measure. For $\psi = (\psi_{ab})$ in the trace-class ideal of compact operators we let $\text{tr}(\psi) := \sum_a \psi_{aa}$ be the trace ($\text{tr}(\psi) := \sum_a \mu_a \psi_{aa}$ in presence of a measure), and $\phi_{ab}^* = \overline{\phi_{ba}}$ defines the adjoint matrix of ϕ .

We consider Euclidean quantum field theory for a single real scalar field $\phi = \phi^* \in \mathcal{A}_I$ living on a space of volume V . At a later point we will pass to densities in the infinite volume limit $V \rightarrow \infty$ so that we must carefully keep track of volume factors. The field theory is defined by an action functional

$$S[\phi] = V \text{tr}(E\phi^2 + P[\phi]) . \quad (2.1)$$

Here, $P[\phi]$ is a polynomial in ϕ with coefficients in \mathbb{R} , and $S_{int}[\phi] = V \text{tr}(P[\phi])$ would be a standard matrix model action. Being interested in a situation closer to field theory, we need an analogue of the kinetic term $\int_{\mathbb{R}^d} dx \frac{1}{2}\phi(x)(-\Delta\phi)(x)$ involving the Laplacian. We therefore let E to be an *unbounded selfadjoint operator on Hilbert space with compact resolvent*. This means that the resolvent $(E - i)^{-1}$ is a compact operator on H , i.e. a matrix, so that by little abuse of notation we can view E as a matrix, too. To make sense of $V \text{tr}(E\phi^2)$ we require ϕ to belong to an appropriate subspace of the compact operators such that $E\phi^2$ is trace-class. Note that the Kontsevich model [Kon82] with action (1.2) fits into this setting, with the resolvent of E being trace-class.

The action (2.1) gives rise to the partition function

$$\mathcal{Z}[J] = \int \mathcal{D}[\phi] \exp(-S[\phi] + V \text{tr}(\phi J)) , \quad (2.2)$$

where $\mathcal{D}[\phi] = \prod_{a,b \in I} d\phi_{ab}$ is the extension of the Lebesgue measure defined on finite-rank operators to the space of selfadjoint matrices $\phi \in \mathcal{A}_I$ for which $E\phi^2$ is trace-class. By $J \in \mathcal{A}_I$ we denote the external source. The partition function gives rise to connected correlation functions by

$$\langle \varphi_{a_1 b_1} \cdots \varphi_{a_n b_n} \rangle_c := \frac{\partial^n \mathcal{W}[J]}{\partial J_{b_1 a_1} \cdots \partial J_{b_n a_n}} , \quad \mathcal{W}[J] := \frac{1}{V} \log \mathcal{Z}[J] . \quad (2.3)$$

The functional $\mathcal{W}[J]$ is the free energy density which should have a limit for $V \rightarrow \infty$. Unless I is finite, the resulting index sums may diverge and require a renormalisation.

2.2 Ward identity

The action of the group $\mathcal{U}(H)$ of unitaries on Hilbert space induces an adjoint action of $U \in \mathcal{U}(H)$ on bounded operators, $\mathcal{A}_I \ni \phi \mapsto U\phi U^* \in \mathcal{A}_I$. We let $\phi^U := U\phi U^*$. Since the adjoint action preserves the space of selfadjoint matrices and ϕ is a dummy integration variable in the partition function, one has

$$\mathcal{Z}[J] = \int \mathcal{D}[\phi] \exp(-S[\phi] + V \operatorname{tr}(\phi J)) = \int \mathcal{D}[\phi^U] \exp(-S[\phi^U] + V \operatorname{tr}(\phi^U J)). \quad (2.4)$$

For finite matrices $A = A^*, B \in M_n(\mathbb{C})$, the Jacobian of the transformation $A \mapsto BAB^*$ is $\det \frac{\partial BAB^*}{\partial A} = (\det B \det B^*)^n$. In particular, $\det \frac{\partial UAU^*}{\partial A} = 1$ for $U \in U(n)$. Therefore, the transformation formula for $\phi^U[\phi]$ yields

$$0 = \int \mathcal{D}[\phi] \left\{ \exp(-S[\phi] + V \operatorname{tr}(\phi J)) - \exp(-S[U\phi U^*] + V \operatorname{tr}(U\phi U^* J)) \right\} \quad (2.5)$$

for any unitary U . Note that the integrand itself does not vanish because $V \operatorname{tr}(E\phi^2)$ and $V \operatorname{tr}(\phi J)$ are not invariant under $\phi \mapsto \phi^U$. Linearisation of (2.5) about $U = \operatorname{id}$ gives for the action $S[\phi] = V \operatorname{tr}(E\phi^2 + P[\phi])$ the *system of Ward identities*

$$0 = \int \mathcal{D}[\phi] \left(E\phi^2 - \phi^2 E - J\phi + \phi J \right) \exp(-S[\phi] + V \operatorname{tr}(\phi J)). \quad (2.6)$$

Note that (2.6) is a matrix equation, which means that for an infinite index set I there is an infinite number of (scalar) Ward identities. By adjoint action with U we can always achieve that the external matrix E is diagonal, $E_{mn} = E_m \delta_{mn}$. In this case the system takes the form

$$0 = \int \mathcal{D}[\phi] \sum_{n \in I} \left((E_p - E_a) \phi_{pn} \phi_{na} - J_{pn} \phi_{na} + J_{na} \phi_{pn} \right) \exp(-S[\phi] + V \operatorname{tr}(\phi J)). \quad (2.7)$$

Writing under the integral $\phi_{ab} = \frac{1}{V} \frac{\partial}{\partial J_{ba}}$, we finally obtain (this was first derived in [DGMR07]):

Proposition 2.1 *The partition function $\mathcal{Z}[J]$ of a real scalar matrix model with action $S[\phi] = V \operatorname{tr}(E\phi^2 + P[\phi])$, where $E_{mn} = E_m \delta_{mn}$ is diagonal and $P[\phi]$ a polynomial in ϕ , satisfies the system of Ward identities*

$$0 = \sum_{n \in I} \left(\frac{(E_p - E_a)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} - J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} \right), \quad a, p \in I. \quad (2.8) \quad \square$$

We shall make a technical

Assumption 2.2 *The map $\iota : I \rightarrow \mathbb{R}$ defined by $\iota(m) = E_m$ is injective.*

This assumption is in fact less restrictive than it seems. For fixed potential $\operatorname{tr}(P[\phi])$, the partition function is completely determined by the spectrum $\sigma(E)$ of E and the spectral density. Since E is selfadjoint and has compact resolvent, $\sigma(E)$ is discrete and has finite-dimensional eigenspaces. Moreover, by achieving E to be diagonal we have

already placed ourselves in an eigenbasis of E . The requirements on E thus imply that I is necessarily countable, and the chosen diagonalisation yields a bijection between I and the set $\{(\varepsilon, n_\varepsilon) : \varepsilon \in \sigma(E), n_\varepsilon \in \{1, 2, \dots, \dim(\ker(E - \varepsilon))\}\}$. This means that index summations over $m \in I$ of a function f which depends only on the spectrum of E but not on the multiplicity can be partitioned into

$$\sum_{m \in I} f(m) = \sum_{\varepsilon \in \sigma(E)} \sum_{n_\varepsilon=1}^{\dim \ker(E-\varepsilon)} f(\varepsilon, n_\varepsilon) = \sum_{\varepsilon \in \sigma(E)} \dim(\ker(E - \varepsilon)) \cdot f(\varepsilon).$$

Now the new summation over $\sigma(E) \ni \varepsilon$ with measure $\dim(\ker(E - \varepsilon))$ is automatically injective. In other words, the Assumption 2.2 already covers the general case if we include an appropriate measure in the index summation. In order to make use of Proposition 2.1 we need at least two indices $a \neq p$ with $E_a \neq E_p$. This means that the only case which we really exclude is the standard framework of matrix models $V \operatorname{tr}(E\phi^2) = V\mu^2 \operatorname{tr}(\phi^2)$ for which there are highly developed technologies [DGZ95]. Note that this $E = \mu^2 \operatorname{id}_H$ does not have compact resolvent on infinite-dimensional Hilbert spaces.

2.3 Expansion into boundary components

Feynman graphs in matrix models are ribbon graphs drawn on a genus- g Riemann surface with B boundary components (or punctures/holes/marked points/faces) to which the external legs are attached [GW05a]. Diagonality of the external matrix $E_{mn} = E_m \delta_{mn}$ guarantees that the matrix index is constant on every face (i.e. single line of the ribbon). This means that the right index b of a source J_{ab} coincides with the left index of another source J_{bc} , or of the same source J_{bb} . Accordingly, there is a decomposition of the free energy density $\mathcal{W}[J] = \frac{1}{V} \log \mathcal{Z}[J]$ into products of J -cycles $J_{p_1 p_2} J_{p_2 p_3} \cdots J_{p_{n-1} p_n} J_{p_n p_1} =: J_{P^n}$, where $P^n = p_1 \dots p_n$ stands for a collection of n indices. The decomposition according to the longest cycle reads

$$\mathcal{W}[J] = \mathcal{W}[0] + \sum_{L=1}^{\infty} \sum_{\substack{n_1, \dots, n_L=0 \\ n_L \geq 1}}^{\infty} \left(\prod_{j=1}^L \frac{1}{n_j! j^{n_j}} \right) \sum_{P_{i_j}^j \in I^j} G_{|P_1^1| \dots |P_{n_1}^1| \dots |P_1^L| \dots |P_{n_L}^L|} \prod_{j=1}^L \prod_{i_j=1}^{n_j} J_{P_{i_j}^j}. \quad (2.9)$$

The total number of J -cycles in a function $G_{|P_1^1| \dots |P_{n_1}^1| \dots |P_1^L| \dots |P_{n_L}^L|}$ with $N = \sum_{j=1}^L j n_j$ external sources is its number $B = n_1 + \dots + n_L$ of boundary components. For convenience we also list those terms of this decomposition which contain at most four sources:

$$\begin{aligned} \mathcal{W}[J] = & \mathcal{W}[0] + \sum_{p \in I} G_{|p|} J_{pp} + \sum_{p, q \in I} \left\{ \frac{1}{2} G_{|pq|} J_{pq} J_{qp} + \frac{1}{2} G_{|p|q|} J_{pp} J_{qq} \right\} \\ & + \sum_{p, q, r \in I} \left\{ \frac{1}{6} G_{|p|q|r|} J_{pp} J_{qq} J_{rr} + \frac{1}{2} G_{|p|qr|} J_{pp} J_{qr} J_{rq} + \frac{1}{3} G_{|pqr|} J_{pq} J_{qr} J_{rp} \right\} \\ & + \sum_{p, q, r, s \in I} \left\{ \frac{1}{24} G_{|p|q|r|s|} J_{pp} J_{qq} J_{rr} J_{ss} + \frac{1}{4} G_{|p|q|rs|} J_{pp} J_{qq} J_{rs} J_{sr} + \frac{1}{8} G_{|pq|rs|} J_{pq} J_{qp} J_{rs} J_{sr} \right. \\ & \left. + \frac{1}{3} G_{|p|qrs|} J_{pp} J_{qr} J_{rs} J_{sq} + \frac{1}{4} G_{|pqr|s|} J_{pq} J_{qr} J_{rs} J_{sp} \right\} + \mathcal{O}(J^6). \quad (2.10) \end{aligned}$$

In the sequel it is important to understand that the decomposition into cycles also persists for coinciding indices. For example, $G_{|pp|}J_{pp}J_{pp}$ is topologically different from $G_{|p|p|}J_{pp}J_{pp}$. We keep track of this distinction by a continuity argument. For that we use an appropriate embedding $I \subset \mathbb{R}^k$ and a corresponding extension of E_m to any \mathcal{C}^1 -function (continuously differentiable) on \mathbb{R}^k with local extrema outside I . Perturbatively, the coefficients $G_{|\dots|}$ are rational functions of E_m so that they become differentiable functions of the indices, too. Identifying the connected functions by functional derivative of \mathcal{W} with respect to J_{ab} it is then important to perform the derivative at generic indices, e.g.

$$G_{|pq|} = \left. \frac{\partial^2 \mathcal{W}}{\partial J_{pq} \partial J_{qp}} \right|_{J=0, q \neq p}, \quad G_{|p|q|} = \left. \frac{\partial^2 \mathcal{W}}{\partial J_{pp} \partial J_{qq}} \right|_{J=0, q \neq p}.$$

The case of coinciding indices is then obtained by continuity, $G_{|pp|} = \lim_{q \rightarrow p} G_{|pq|}$ and $G_{|p|p|} = \lim_{q \rightarrow p} G_{|p|q|}$.

We are now going to prove the following Ward identity, which is the key to solve our model.

Theorem 2.3 *For injective $m \mapsto E_m$, the partition function $\mathcal{Z}[J]$ of a real scalar matrix model with action $S[\phi] = V \operatorname{tr}(E\phi^2 + P[\phi])$, where $E_{mn} = E_m \delta_{mn}$ is diagonal and $P[\phi]$ a polynomial in ϕ , satisfies the system of Ward identities*

$$\sum_{n \in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an} \partial J_{np}} = \delta_{ap} (V^2 W_a^1[J] + V W_a^2[J]) \mathcal{Z} + \frac{V}{E_p - E_a} \sum_{n \in I} \left(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \right), \quad (2.11)$$

$$\begin{aligned} W_a^2[J] &:= \sum_{L=0}^{\infty} \sum_{n_1, \dots, n_L=0}^{\infty} \left(\prod_{j=1}^L \frac{1}{n_j! j^{n_j}} \right) \sum_{P_{i_j}^j \in I^j} \left(\prod_{j=1}^L \prod_{i_j=1}^{n_j} J_{P_{i_j}^j} \right) \times \\ &\quad \left(G_{|a|a|P_1^1| \dots |P_{n_L}^L|} + \sum_{n \in I} G_{|P_1^1| \dots |P_{n_1}^1| a n | P_1^2| \dots |P_{n_L}^L|} \right. \\ &\quad \left. + \sum_{k=3}^L \sum_{n, q_1, \dots, q_{k-3} \in I} G_{|P_1^1| \dots |P_{n_{k-1}}^{k-1}| n a n q_1 \dots q_{k-3} | P_1^k| \dots |P_{n_L}^L|} J_{n q_1} J_{q_1 q_2} \dots J_{q_{k-3} n} \right), \\ W_a^1[J] &:= \sum_{L, L'=0}^{\infty} \sum_{n_1, \dots, n_L, m_1, \dots, m_{L'}=0}^{\infty} \left(\prod_{j=1}^L \frac{1}{n_j! j^{n_j}} \right) \left(\prod_{k=1}^{L'} \frac{1}{m_k! k^{m_k}} \right) \times \\ &\quad \sum_{P_{i_j}^j \in I^j} \sum_{Q_{i_k}^k \in I^k} \left(\prod_{j=1}^L \prod_{i_j=1}^{n_j} J_{P_{i_j}^j} \right) \left(\prod_{k=1}^{L'} \prod_{l_k=1}^{m_k} J_{Q_{l_k}^k} \right) G_{|a|P_1^1| \dots |P_{n_L}^L|} G_{|a|Q_1^1| \dots |Q_{m_{L'}}^L|} \}. \end{aligned}$$

Proof. In $\sum_{n \in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an} \partial J_{np}} = \sum_{n \in I} \left(V \frac{\partial^2 \mathcal{W}[J]}{\partial J_{an} \partial J_{np}} + V^2 \frac{\partial \mathcal{W}[J]}{\partial J_{an}} \frac{\partial \mathcal{W}[J]}{\partial J_{np}} \right) \mathcal{Z}[J]$ we write

$$\frac{\partial^2 \mathcal{W}[J]}{\partial J_{an} \partial J_{np}} = \delta_{ap} W_a^2[J] + W_{ap}^{2,reg}[J], \quad \frac{\partial \mathcal{W}[J]}{\partial J_{an}} \frac{\partial \mathcal{W}[J]}{\partial J_{np}} = \delta_{ap} W_a^1[J] + W_{ap}^{1,reg}[J],$$

where $\delta_{ap} W_a^i[J]$ contain all terms in which the derivatives generate a factor δ_{ap} , and $W_{ap}^{i,reg}[J]$ are the remainders. Applying the functional derivatives to (2.9) we derive the

structure of W_a^1, W_a^2 as given in the Theorem. Namely, a δ_{ap} arises in $\frac{\partial^2 \mathcal{W}[J]}{\partial J_{an} \partial J_{np}}$ if (in that order)

- both $\frac{\partial}{\partial J_{an}}$ and $\frac{\partial}{\partial J_{np}}$ act on different J -cycles each of length 1, or
- both $\frac{\partial}{\partial J_{an}}$ and $\frac{\partial}{\partial J_{np}}$ act on the same J -cycle of length 2, or
- $\frac{\partial}{\partial J_{np}}$ acting on a J -cycle of length $k \geq 3$ gives the chain $J_{pq_0} J_{q_0 q_1} \dots J_{q_{k-3} n}$. The second derivative $\frac{\partial}{\partial J_{an}}$ then acts on J_{pq_0} in that chain.

In $\frac{\partial \mathcal{W}[J]}{\partial J_{an}} \frac{\partial \mathcal{W}[J]}{\partial J_{np}}$, the only possibility to generate a δ_{ap} is to let

- both $\frac{\partial}{\partial J_{an}}$ and $\frac{\partial}{\partial J_{np}}$ act on J -cycles each of length 1.

The combinatorial factors in (2.9) are reproduced after a shift. It is straightforward to also write down the remainders $W_{ap}^{i,reg}[J]$ in explicit form; but as we do not need this we refrain from listing the result.

We consider the case $a \neq p$. On one hand this implies $\delta_{ap} W_a^i = 0$, on the other hand we may divide the expression in Proposition 2.1 by $\frac{E_p - E_a}{V} \neq 0$ to obtain

$$a \neq p \Rightarrow \sum_{n \in I} (V^2 W_{ap}^{1,reg} + V W_{ap}^{2,reg}) \mathcal{Z} = \sum_{n \in I} \frac{V}{E_p - E_a} \left(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \right). \quad (2.12)$$

It is now important to realise that the identity (2.12) also extends to $p = a$. Namely, the partition function and its derivatives such as $W_{ap}^{1,reg}, W_{ap}^{2,reg}$ are completely determined by the entries E_m in the external matrix. Extending $m \mapsto E_m$ to a differentiable function, the rhs of (2.12) is at least continuous at $p \rightarrow a$ by l'Hôpital's rule and agrees with the lhs for which there is no obstruction to go to $p = a$. This finishes the proof. \square

3 Schwinger-Dyson equations

3.1 Strategy

A standard procedure in quantum field theory consists in expressing the fields ϕ in the interaction $S_{int}[\phi] = V \text{tr}(P[\phi])$ of the partition function (2.2) as derivatives with respect to the sources,

$$\mathcal{Z}[J] = \int \mathcal{D}[\phi] \exp \left(- S_{int} \left[\frac{1}{V} \frac{\partial}{\partial J} \right] \right) \exp \left(- V \text{tr}(E\phi^2) + V \text{tr}(\phi J) \right).$$

In this formulation the Gaussian ϕ -integration can formally be carried out and gives

$$\mathcal{Z}[J] = C \exp \left(- S_{int} \left[\frac{1}{V} \frac{\partial}{\partial J} \right] \right) \exp \left(\frac{V}{2} \langle J, J \rangle_E \right), \quad \langle J, J \rangle_E := \sum_{m,n \in I} \frac{J_{mn} J_{nm}}{E_m + E_n}. \quad (3.1)$$

The constant (possibly ill-defined) prefactor C will be omitted. The power-series expansion of $e^{-S_{int}[\frac{1}{V} \frac{\partial}{\partial J}]}$ gives rise to Feynman graphs, but since the expansion loses the distinction between $e^{-S_{int}[\frac{1}{V} \frac{\partial}{\partial J}]}$ and $e^{+S_{int}[\frac{1}{V} \frac{\partial}{\partial J}]}$, the expansion does not converge. A better strategy is to keep $e^{-S_{int}[\frac{1}{V} \frac{\partial}{\partial J}]}$ intact and instead let functional derivatives $\phi_{pq} = \frac{1}{V} \frac{\partial}{\partial J_{qp}}$

act on \mathcal{W} in order to produce the connected functions G in the expansion (2.9). These “external” derivatives combine with the “internal” derivatives in $e^{-S_{int}[\frac{1}{V}\frac{\partial}{\partial J}]}$ to certain identities called *Schwinger-Dyson equations*. It turns out that for each function G the Ward identity of Theorem 2.3 can be used at an intermediate step to generate new relations.

We shall demonstrate this for the regular (i.e. $B = 1$) two-point function $G_{|ab|}$. As usual we impose $a \neq b$ so that $0 = (\frac{\partial \mathcal{Z}}{\partial J_{ab}})|_{J=0}$. We have

$$\begin{aligned} G_{|ab|} &= \frac{1}{V} \frac{\partial^2(\log \mathcal{Z})[J]}{\partial J_{ba} \partial J_{ab}} \Big|_{J=0} = \frac{1}{V} \frac{1}{\mathcal{Z}[0]} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{ba} \partial J_{ab}} \Big|_{J=0} - \frac{1}{V} \frac{1}{\mathcal{Z}[0]} \frac{\partial \mathcal{Z}[J]}{\partial J_{ba}} \Big|_{J=0} \frac{1}{\mathcal{Z}[0]} \frac{\partial \mathcal{Z}[J]}{\partial J_{ab}} \Big|_{J=0} \\ &= \frac{1}{V} \frac{1}{\mathcal{Z}[0]} \left\{ \frac{\partial}{\partial J_{ba}} e^{-S_{int}[\frac{1}{V}\frac{\partial}{\partial J}]} \frac{\partial}{\partial J_{ab}} e^{\frac{V}{2}\langle J, J \rangle_E} \right\}_{J=0} \\ &= \frac{1}{E_a + E_b} + \frac{1}{(E_a + E_b)\mathcal{Z}[0]} \left\{ \left(\phi_{ab} \frac{\partial(-S_{int})}{\partial \phi_{ab}} \right) \left[\frac{1}{V} \frac{\partial}{\partial J} \right] \right\} \mathcal{Z}[J] \Big|_{J=0}. \end{aligned} \quad (3.2)$$

For the most general interaction $S_{int}[\phi] = V \sum_{k=3}^K \sum_{p_1, \dots, p_k \in I} \frac{\lambda_k}{k} \phi_{p_1 p_2} \cdots \phi_{p_{k-1} p_k} \phi_{p_k p_1}$ we have

$$\left(\phi_{ab} \frac{\partial(-S_{int})}{\partial \phi_{ab}} \right) \left[\frac{1}{V} \frac{\partial}{\partial J} \right] = \sum_{k=3}^K \sum_{n, p, p_1, \dots, p_{k-3} \in I} \frac{(-\lambda_k)}{V^{k-1}} \frac{\partial^{k-2}}{\partial J_{pp_1} \dots \partial J_{p_{k-3} b} \partial J_{ba}} \frac{\partial^2}{\partial J_{an} \partial J_{np}}, \quad (3.3)$$

i.e. in any part we encounter the two-fold derivative $\frac{\partial^2}{\partial J_{an} \partial J_{np}}$ known from Theorem 2.3. This appearance of $\frac{\partial^2}{\partial J_{an} \partial J_{np}}$ is a general feature of any connected function G .

3.2 Schwinger-Dyson equations for a ϕ^4 -model: $B = 1$ cycle

We now specify to the case $S_{int}[\phi] = V \frac{\lambda_4}{4} \text{tr}(\phi^4)$. We first evaluate (3.2) using (3.3) and Theorem 2.3, taking $a \neq b$ into account and the fact that $G_{|q|} = 0$:

$$\begin{aligned} G_{|ab|} &= \frac{1}{E_a + E_b} + \frac{(-\lambda_4)}{V^3(E_a + E_b)\mathcal{Z}[0]} \left\{ \frac{\partial^2((V^2 W_a^1[J] + V W_a^2[J])\mathcal{Z}[J])}{\partial J_{ab} \partial J_{ba}} \right. \\ &\quad \left. + \sum_{p, n \in I} \frac{V}{E_p - E_a} \frac{\partial^2}{\partial J_{pb} \partial J_{ba}} \left(J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \right) \right\}_{J=0} \\ &= \frac{1}{E_a + E_b} + \frac{(-\lambda_4)}{(E_a + E_b)} \left\{ \frac{1}{V} G_{|ab|} \left(G_{|a|a|} + \sum_{n \in I} G_{|an|} \right) \right. \\ &\quad \left. + \frac{1}{V^2} \left(G_{|a|a|ab|} + \sum_{n \in I} G_{|an|ab|} + G_{|aaab|} + G_{|baba|} \right) \right. \\ &\quad \left. + \frac{1}{V} \sum_{p \in I} \frac{G_{|ab|} - G_{|pb|}}{E_p - E_a} + \frac{1}{V} \frac{G_{|a|b|} - G_{|b|b|}}{E_b - E_a} \right\}. \end{aligned} \quad (3.4)$$

To obtain the last line one has to use $\frac{\partial \mathcal{Z}}{\partial J_{an}} = V \mathcal{Z}[0] (G_{|an|} J_{na} + \delta_{an} \sum_{q \in I} G_{|a|q|} J_{qq} + \mathcal{O}(J^2))$ and $\frac{\partial \mathcal{Z}}{\partial J_{np}} = V \mathcal{Z}[0] (G_{|pn|} J_{pn} + \delta_{np} \sum_{q \in I} G_{|p|q|} J_{qq} + \mathcal{O}(J^2))$.

In a similar manner we obtain for pairwise different indices a, b_i the following equation for the regular (i.e. single-cycle) ($N \geq 4$)-point function:

$$G_{|ab_1\dots b_{N-1}|} = \frac{(-\lambda_4)}{V^3(E_a + E_b)} \mathcal{Z}[0] \left\{ \frac{\partial^N ((V^2 W_a^1[J] + V W_a^2[J]) \mathcal{Z}[J])}{\partial J_{ab_1} \partial J_{b_1 b_2} \dots \partial J_{b_{N-2} b_{N-1}} \partial J_{b_{N-1} a}} \right. \\ \left. + \sum_{p, n \in I} \frac{V}{E_p - E_a} \frac{\partial^N}{\partial J_{pb_1} \partial J_{b_1 b_2} \dots \partial J_{b_{N-2} b_{N-1}} \partial J_{b_{N-1} a}} \left(J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \right) \right\}_{J=0}.$$

Since all indices a, b_i are pairwise different, the first line gives $V^2 W_a^1[0] + V W_a^2[0]$ times the application of all derivatives to $\mathcal{Z}[J]$ plus $\mathcal{Z}[0]$ times the application of all derivatives to $(V^2 W_a^1[J] + V W_a^2[J])$. In the first case there is actually only a contribution from the application of all derivatives to $\log \mathcal{Z}[J] = V W[J]$, because the indices are pairwise different. The second line is a sum of terms where one of the derivatives acts on J_{pn} minus the action of $J_{b_{N-1} a}$ on J_{na} , with all other derivatives acting on $\frac{\partial \mathcal{Z}[J]}{\partial J_{an}}$ and $\frac{\partial \mathcal{Z}[J]}{\partial J_{np}}$, respectively. Depending on the Kronecker- δ arising in this way, the derivatives acting on $\mathcal{Z}[J]$ either form a single cycle or two cycles. For the derivative $\frac{\partial^N \mathcal{Z}[J]}{\partial J_{pb_1} \partial J_{b_1 b_2} \dots \partial J_{b_{N-2} b_{N-1}} \partial J_{b_{N-1} p}}$ special care is needed in the cases $p = b_k$. Either these are the necessarily arising cases in derivatives which form a single cycle, or the derivatives form two cycles $\frac{\partial^k \mathcal{Z}[J]}{\partial J_{b_k b_1} \partial J_{b_1 b_2} \dots \partial J_{b_{k-1} b_k}}$ and $\frac{\partial^{N-k} \mathcal{Z}[J]}{\partial J_{b_k b_{k+1}} \dots \partial J_{b_{N-2} b_{N-1}} \partial J_{b_{N-1} b_k}}$ with one common index b_k . The single-cycle cases reduce to its action on $\log \mathcal{Z}[J]$. For the case of two cycles there are two possibilities: Either both act on a two-cycle contribution of $\log \mathcal{Z}[J]$, or on one-cycle contributions of $\frac{1}{2!} (\log \mathcal{Z}[J])^2$. With these remarks we find, noting that each $\log \mathcal{Z}$ contributes a factor V ,

$$G_{|ab_1\dots b_{N-1}|} \\ = \frac{(-\lambda_4)}{E_a + E_{b_1}} \left\{ \frac{1}{V} \left(G_{|a|a|} + \sum_{n \in I} G_{|an|} \right) G_{|ab_1\dots b_{N-1}|} \right. \\ \left. + \frac{1}{V^2} \left(G_{|a|a|ab_1\dots b_{N-1}|} + \sum_{n \in I} G_{|an|ab_1\dots b_{N-1}|} + G_{|aaab_1\dots b_{N-1}|} + \sum_{k=1}^{N-1} G_{|b_1\dots b_k ab_k \dots b_{N-1} a|} \right) \right. \\ \left. - \frac{1}{V} \sum_{p \in I} \frac{G_{|pb_1\dots b_{N-1}|} - G_{|ab_1\dots b_{N-1}|}}{E_p - E_a} - \frac{1}{V} \sum_{k=1}^{N-1} \frac{G_{|b_1\dots b_k|b_{k+1}\dots b_{N-1} b_k|} - G_{|b_1\dots b_k|b_{k+1}\dots b_{N-1} a|}}{E_{b_k} - E_a} \right. \\ \left. - \sum_{l=1}^{\frac{N-2}{2}} G_{|b_1\dots b_{2l}|} \frac{G_{|b_{2l+1}\dots b_{N-1} b_{2l}|} - G_{|b_{2l+1}\dots b_{N-1} a|}}{E_{b_{2l}} - E_a} \right\}. \quad (3.5)$$

In Appendix A we derive the Schwinger-Dyson equations for functions with $B = 2$ boundary components.

3.3 An algebraic recursion formula for a real field theory

The generating functional $\mathcal{W}[J]$ for connected N -point functions is real. For real fields $\phi = \phi^*$ also the source J is real, i.e. $J = J^*$ or $J_{pq} = \overline{J_{qp}}$. Selfadjointness of E implies $E_a = \overline{E_a}$. Therefore, the expansion coefficients of $\mathcal{W}[J]$ in (2.9) are all real, $G_{|P_1^1| \dots |P_{n_L}^L|} = \overline{G_{|P_1^1| \dots |P_{n_L}^L|}}$. This is clear in perturbation theory where the $G_{|P_1^1| \dots |P_{n_L}^L|}$ are

represented by ribbon graphs made of propagators $\frac{1}{E_a + E_b}$ and vertices $(-\lambda_4)$. It follows non-perturbatively from the reality of the equations (3.4), (3.5) and their $(B \geq 2)$ -analogues, possibly after a decoupling of these equations by genus expansion introduced in section 3.4. These considerations imply for the $B = 1$ sector

$$\sum_{p_0, \dots, p_{N-1} \in I} G_{|p_0 \dots p_{N-1}|} J_{p_0 p_1} \cdots J_{p_{N-2} p_{N-1}} J_{p_{N-1} p_0} = \sum_{p_0, \dots, p_{N-1} \in I} G_{|p_0 \dots p_{N-1}|} J_{p_0 p_{N-1}} J_{p_{N-1} p_{N-2}} \cdots J_{p_1 p_0}.$$

Renaming the indices $p_k \leftrightarrow p_{N-k}$ for $k \in \{1, 2, \dots, \frac{N}{2}\}$ we notice that N -point functions are not only invariant under cyclic permutations but also *invariant under reversal of the order of its indices* (or orientation reversal), and this extends to arbitrary $B \geq 1$:

$$G_{|p_0^1 p_1^1 \dots p_{N_1-1}^1 \dots | p_0^B p_1^B \dots p_{N_B-1}^B|} = G_{|p_0^1 p_{N_1-1}^1 \dots p_1^1 \dots | p_0^B p_{N_B-1}^B \dots p_1^B|}. \quad (3.6)$$

The equation (3.6) is an empty condition if all $N_i \leq 2$. But as soon as $N_i > 2$ for at least one $1 \leq i \leq B$ it allows us to completely solve $G_{|p_0^1 p_1^1 \dots p_{N_1-1}^1 \dots | p_0^B p_1^B \dots p_{N_B-1}^B|}$ in terms of the functions having all $N_i \leq 2$. Here we demonstrate this for $B = 1$. The case $B = 2$ and an outlook to $B > 2$ is given in Appendix A.

Starting point is (3.5) which we multiply by $(E_a + E_{b_1})$. From the resulting equation we subtract the equation obtained by the same procedure but with renamed indices $b_k \leftrightarrow b_{N-k}$ for $1 \leq k \leq \frac{N}{2}$. From (3.6) we conclude $(E_a + E_{b_1})G_{|ab_1 \dots b_{N-1}|} - (E_a + E_{b_{N-1}})G_{|ab_{N-1} \dots b_1|} = (E_{b_1} - E_{b_{N-1}})G_{|ab_1 \dots b_{N-1}|}$. On the rhs of (3.5) all terms not having E_{b_k} in the denominator cancel by (3.6), leaving

$$\begin{aligned} (E_{b_1} - E_{b_{N-1}})G_{|ab_1 \dots b_{N-1}|} &= \frac{\lambda_4}{V} \sum_{k=1}^{N-1} \left(\frac{G_{|b_1 \dots b_k | b_{k+1} \dots b_{N-1} b_k|} - G_{|b_1 \dots b_k | b_{k+1} \dots b_{N-1} a|}}{E_{b_k} - E_a} \right. \\ &\quad \left. - \frac{G_{|b_{N-1} \dots b_{N-k} | b_{N-k-1} \dots b_1 b_{N-k}|} - G_{|b_{N-1} \dots b_{N-k} | b_{N-k-1} \dots b_1 a|}}{E_{b_{N-k}} - E_a} \right) \\ &\quad + \lambda_4 \sum_{l=1}^{\frac{N-2}{2}} \left(G_{|b_1 \dots b_{2l}|} \frac{G_{|b_{2l+1} \dots b_{N-1} b_{2l}|} - G_{|b_{2l+1} \dots b_{N-1} a|}}{E_{b_{2l}} - E_a} \right. \\ &\quad \left. - G_{|b_{N-1} \dots b_{N-2l}|} \frac{G_{|b_{N-2l-1} \dots b_1 b_{N-2l}|} - G_{|b_{N-2l-1} \dots b_1 a|}}{E_{b_{N-2l}} - E_a} \right). \end{aligned}$$

Rearranging the sums to common denominators $E_{b_k} - E_a$ and taking (3.6) into account, we conclude:

Theorem 3.1 *The (unrenormalised) connected $(N \geq 4)$ -point function at $B = 1$ of a real scalar matrix model with action $S = V \text{tr} (E\phi^2 + \frac{\lambda_4}{4}\phi^4)$ is for injective $m \mapsto E_m$ given by the algebraic recursion formula*

$$\begin{aligned} G_{|b_0 b_1 \dots b_{N-1}|} &= (-\lambda_4) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})} \\ &\quad + \frac{(-\lambda_4)}{V} \sum_{k=1}^{N-1} \frac{G_{|b_0 b_1 \dots b_{k-1} | b_k b_{k+1} \dots b_{N-1}|} - G_{|b_k b_1 \dots b_{k-1} | b_0 b_{k+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_k})(E_{b_1} - E_{b_{N-1}})}. \quad (3.7) \quad \square \end{aligned}$$

The equation (3.4) for $G_{|ab|}$ is not algebraic. It involves sums over the index set I which in general will diverge for infinite I . One would try to absorb these divergences in a field redefinition $\phi \mapsto \sqrt{Z}\phi$ in the initial action $S = V \text{tr}(E\phi^2 + \frac{\lambda_4}{4}\phi^4)$ and a mass renormalisation of the smallest eigenvalue of E . This amounts to replace $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu_{\text{bare}}^2}{2})$ and $\lambda_4 \mapsto Z^2\lambda_4$. Remarkably, these replacements leave (3.7) invariant. Anticipating a similar recursion formula for $B = 2$ we can formulate this observation as follows:

Theorem 3.2 *Given a real scalar matrix model with action $S = V \text{tr}(E\phi^2 + \frac{\lambda_4}{4}\phi^4)$ and $m \mapsto E_m$ injective. If the basic functions $G_{|p_1|\dots|p_{2k}|p_{2k+1}p_{2k+2}|\dots|p_{N-1}p_N|}$ with at most two sources per cycle are turned finite by renormalisation $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu_{\text{bare}}^2}{2})$ and $\lambda_4 \mapsto Z^2\lambda_4$, then all $(N \geq 4)$ -point functions $G_{|b_0b_1\dots b_{N-1}|}$ are finite (at least for pairwise different indices) without further need of a renormalisation of λ_4 . This means that all such ϕ^4 -matrix models have vanishing β -function.*

All higher correlation functions are given by algebraic recursion formulae in terms of renormalised basic functions $G_{|p_1|\dots|p_{2k}|p_{2k+1}p_{2k+2}|\dots|p_{N-1}p_N|}$. \square

There remains some worry about $G_{|b_0b_1\dots b_{N-1}|}$ at coinciding indices. In perturbation theory these are rational functions of E_{b_i} so that the $E_{b_i} - E_{b_j}$ in the denominator of (3.7) cancel. In a thermodynamic limit $V \rightarrow \infty$ and $\frac{1}{V} \sum_{i \in I}$ finite one can hope that the basic functions $G_{|p_1|\dots|p_{2k}|p_{2k+1}p_{2k+2}|\dots|p_{N-1}p_N|}$ become smooth functions of continuous indices so that the limit to coinciding indices can exist.

Observe that the limit $V \rightarrow \infty$ eliminates the second line of (3.7). We show in section 3.4 that this agrees with the restriction to the planar sector. In section 3.5 we give a graphical solution of the planar part of (3.7). This involves non-crossing chord diagrams counted by the Catalan numbers.

We recall that Disertori, Gurau, Magnen and Rivasseau proved in [DGMR07] that the β -function of self-dual noncommutative ϕ_4^4 -theory vanishes to all orders in perturbation theory. We have generalised the strategy of [DGMR07] to a very general class of ϕ^4 -matrix models and proved by Theorem 3.2 that *all these models have a non-perturbatively vanishing β -function, provided that the two-point function is renormalisable.*

3.4 The genus expansion

Our strategy will be to perform the infinite volume limit $V \rightarrow \infty$ in such a manner that the the densitised summation $\frac{1}{V} \sum_{p \in I}$ has a limit. This is achieved by relating a cut-off in the index sum over $p \in I$ to the volume. This limit scales away all terms in (3.4), (3.5) and (3.7) with more inverse powers of V than the number of index summations. We argue in this subsection that the contributions which are scaled away in this limit are precisely the non-planar parts of genus $g \geq 1$. These arguments are not necessary to understand the rest of the paper so that the reader may jump to (3.10) and (3.11) which summarise the scaling limit of (3.4) and (3.7). In fact the direct scaling limit ($V \rightarrow \infty$, $\frac{1}{V} \sum_{p \in I}$ finite) avoids the infinite sum over all genera ≥ 1 which can at best be expected to be Borel summable².

Perturbatively, the connected functions $G_{|P_1^1|\dots|P_n^L|}$ are expanded into ribbon graphs, the matrix analogue of Feynman graphs. A ribbon graph represents a two-dimensional

²We thank the referee for this remark.

simplicial complex and as such encodes the genus g of a Riemann surface on which the ribbon graph can be drawn. We thus decompose the connected functions according to the genus, $G_{|P_1^1| \dots |P_{n_L}^L|} = \sum_{g=0}^{\infty} G_{|P_1^1| \dots |P_{n_L}^L|}^{(g)}$. We are not going to discuss the convergence of this series. We shall see in Proposition 3.5 that this expansion can be viewed as formal power series in V^{-2g} . Understanding the resummation of this series for large enough V would be an interesting problem of constructive physics².

The Schwinger-Dyson equations of subsection 3 describe how a given connected function $G_{|a|P_1^1| \dots |P_{n_L}^L|}$ or $G_{|abp_1 \dots p_{k-2}|P_1^{k+1}| \dots |P_{n_L}^L|}$ is represented as a contraction of the special (planar) vertex $(\frac{\partial(-S_{int})}{\partial\phi_{aa}})$ or $(\phi_{ap_{k-2}} \dots \phi_{p_1b} \frac{\partial(-S_{int})}{\partial\phi_{ab}})$, which carries the external source J_{aa} or J_{ab} , to a smaller subgraph. This contraction may increase the genus. We have investigated a similar problem in [GW05a] and use these results to identify the topology of the various contributions to the Schwinger-Dyson equations. We can summarise the outcome of [GW05a] as follows:

Proposition 3.3 *i) Attaching one leg of a (planar) vertex to a connected subgraph does not change the genus. If this attachment is to a J -cycle of length 1, then the other legs of the distinguished vertex form a cycle of its own. If the attachment is to a J -cycle of length ≥ 2 , then the other legs insert in cyclic order into the connected chain.*

ii) Joining two legs of different cycles of a connected graph increases the genus by 1. At the same time, these two cycles are connected to a single one, unless both are of length 1 in which case both cycles disappear.

iii) Joining two legs of the same cycle of a connected graph does not change the genus. The cycle disappears if before the contraction it was of length 2. If the original cycle was of length $k \geq 3$ and neighbored legs are joint, then the remaining legs form a single cycle of length $k - 2$. If the original cycle was of length $k \geq 4$ and legs are joint which are not neighbours, then the joining splits the original cycle into two, one consisting of the ‘inner’ legs relative to the contraction, the other one of the ‘outer’ legs. \square

We now identify the topology of individual parts of Ward identity and Schwinger-Dyson equations. We start with the two parts $W_a^1[J]$, $W_a^2[J]$ in Theorem 2.3: $W_a^2[J]$ arises by insertion of $(\phi^2)_{pa}$ into a connected function of \mathcal{W} , and $W_a^1[J]$ arises by joining a pair of disjoint connected functions from $\mathcal{W} \cdot \mathcal{W}$ via the vertex $(\phi^2)_{pa}$. In the application to Schwinger-Dyson equation later on it will be important that $(\phi^2)_{pa}$ is part of a larger vertex $(\phi^2)_{pa}(\phi^{\otimes r})_{(b_i)}$ similar to (3.3). The r legs of $(\phi^{\otimes r})_{(b_i)}$ have to be taken into account in the cycle structure. We find:

1. $G_{|a|P_1^1| \dots |P_{n_L}^L|}$ arises by contracting two cycles of the same graph, each of length 1, via $(\phi^2)_{pa}$. We are in the situation of Proposition 3.3.ii) so that the genus *first increases by 1*. But if $(\phi^2)_{pa}$ is actually part of a larger vertex $(\phi^2)_{pa}(\phi^r)_{ab}$, then the remainder $(\phi^r)_{ab}$ forms a cycle of its own although notationally not visible in $G_{|a|P_1^1| \dots |P_{n_L}^L|}$. If these derivatives in the cycle $(\phi^r)_{ab}$ contract to h of the cycles $|P_1^1|, \dots, |P_{n_L}^L|$, then the genus *further increases by h* . The total increase of the genus is thus by $h + 1$.

2. $G_{P_1^1|\dots|P_{n_1}^1|an|P_1^2|\dots|P_{n_L}^L|}$ arises by joining both legs of a cycle of length 2 via $(\phi^2)_{pa}$. According to Proposition 3.3.iii), the genus *first is unchanged*. The original J -cycle disappears in this way, but the remaining part $(\phi^{\otimes r})_{(b_i)}$ of the vertex forms again a cycle of its own. If these derivatives in $(\phi^{\otimes r})_{(b_i)}$ contract to h of the cycles $|P_1^1|, \dots, |P_{n_L}^L|$, then the genus *further increases by h* .
3. $G_{P_1^1|\dots|P_{n_k-2}^{k-2}|nanq_1\dots q_{k-3}|P_1^k|\dots|P_{n_L}^L|}$ also has unchanged genus according to Proposition 3.3.iii) after connection of the cycle of length $k \geq 3$ via $(\phi^2)_{pa}$. However, the final cycle structure is more involved: Assume the vertex is $\phi_{pn}\phi_{na}(\phi^{\otimes r})_{(b_i)}$ and to be inserted into the cycle $J_{q_{k-1}q_k}J_{q_kq_1}J_{q_1q_2}\dots J_{q_{k-2}q_{k-1}}$. We first contract ϕ_{pn} with $J_{q_{k-2}q_{k-1}}$, giving a $\delta_{nq_{k-2}}\delta_{pq_{k-1}}$. According to Proposition 3.3.i), the legs of the vertex insert into the cycle, giving

$$J_{pq_k}J_{q_kq_1}J_{q_1q_2}\dots J_{q_{k-3}n}\phi_{na}(\phi^{\otimes r})_{(b_i)}. \quad (*)$$

In the second step ϕ_{na} is contracted with J_{pq_k} , giving a $\delta_{ap}\delta_{nq_k}$. Since $k \geq 3$ and $r \geq 1$, the two legs ϕ_{na} and J_{pq_k} are not neighbours, so that according to Proposition 3.3.iii) the cycle (*) is split into two cycles $J_{nq_1}J_{q_1q_2}\dots J_{q_{k-3}n}$ and $(\phi^{\otimes r})_{(b_i)}$. When $G_{P_1^1|\dots|P_{n_k-2}^{k-2}|nanq_1\dots q_{k-3}|P_1^k|\dots|P_{n_L}^L|}$ is used in a Schwinger-Dyson equation, the J -derivatives encoded by $(\phi^{\otimes r})_{(b_i)}$ necessarily connect to $h \geq 1$ different J -cycles, and this increases the genus by h .

4. $G_{|a|P_1^1|\dots|P_{n_L}^L|G_{|a|Q_1^1|\dots|Q_{m_L'}^{L'}|}$ arises by connecting two length-1 cycles of different subgraphs via $(\phi^2)_{pa}$. According to Proposition 3.3.i), the genus is unchanged. The remaining part $(\phi^{\otimes r})_{(b_i)}$ of the total vertex $(\phi^2)_{pa}(\phi^{\otimes r})_{(b_i)}$ forms a cycle of its own. In Schwinger-Dyson equations, this cycle connects to $h \geq 1$ other cycles in $G_{|a|P_1^1|\dots|P_{n_L}^L|G_{|a|Q_1^1|\dots|Q_{m_L'}^{L'}|}$, thus increasing the genus by h .
5. $J_{pn}\frac{\partial Z[J]}{\partial J_{an}} - J_{na}\frac{\partial Z[J]}{\partial J_{np}}$. Here we have to expand Z as polynomial in connected functions G and apply the above discussion case by case.

For the single-cycle two-point Schwinger-Dyson equation (3.4) we thus have:

$$\begin{aligned} G_{|ab|}^{(g)} &= \frac{\delta_{g0}}{E_a + E_b} - \frac{\lambda_4}{(E_a + E_b)} \left(\frac{1}{V} \sum_{g'+g''+1=g} G_{|ab|}^{(g')} G_{|a|a|}^{(g'')} + \frac{1}{V} \sum_{n \in I} \sum_{g'+g''=g} G_{|ab|}^{(g')} G_{|an|}^{(g'')} \right. \\ &\quad \left. + \frac{1}{V^2} \left(G_{|a|a|ab|}^{(g-2)} + \sum_{n \in I} G_{|an|ab|}^{(g-1)} + G_{|aaab|}^{(g-1)} + G_{|baba|}^{(g-1)} \right) \right) \\ &\quad + \frac{\lambda_4}{(E_a + E_b)} \left(\frac{1}{V} \sum_{p \in I} \frac{G_{|ab|}^{(g)} - G_{|pb|}^{(g)}}{E_a - E_p} + \frac{1}{V} \frac{G_{|a|b|}^{(g-1)} - G_{|b|b|}^{(g-1)}}{E_a - E_b} \right). \end{aligned} \quad (3.8)$$

The very last term $\frac{G_{|a|b|}^{(g-1)} - G_{|b|b|}^{(g-1)}}{E_a - E_b}$ has genus increased by 1 because two different cycles are connected by the external vertex. Similarly, the equation (3.5) for the single-cycle ($N \geq 4$)-point function has the expansion

$$\begin{aligned}
& G_{|ab_1 \dots b_{N-1}|}^{(g)} \\
&= \frac{(-\lambda_4)}{E_a + E_{b_1}} \left\{ \frac{1}{V} \sum_{g'+g''=g} \sum_{n \in I} G_{|an|}^{(g')} G_{|ab_1 \dots b_{N-1}|}^{(g'')} + \frac{1}{V} \sum_{g'+g''+1=g} G_{|a|a|}^{(g')} G_{|ab_1 \dots b_{N-1}|}^{(g'')} \right. \\
&+ \frac{1}{V^2} \left(G_{|a|a|ab_1 \dots b_{N-1}|}^{(g-2)} + \sum_{n \in I} G_{|an|ab_1 \dots b_{N-1}|}^{(g-1)} + G_{|aaab_1 \dots b_{N-1}|}^{(g-1)} + \sum_{k=1}^{N-1} G_{|b_k ab_k b_{k+1} \dots b_{N-1} ab_1 \dots b_{k-1}|}^{(g-1)} \right) \\
&- \frac{1}{V} \sum_{p \in I} \frac{G_{|pb_1 \dots b_{N-1}|}^{(g)} - G_{|ab_1 \dots b_{N-1}|}^{(g)}}{E_p - E_a} - \frac{1}{V} \sum_{k=1}^{N-1} \frac{G_{|b_1 \dots b_k | b_{k+1} \dots b_{N-1} b_k|}^{(g-1)} - G_{|b_1 \dots b_k | b_{k+1} \dots b_{N-1} a|}^{(g-1)}}{E_{b_k} - E_a} \\
&- \left. \sum_{l=1}^{\frac{N-2}{2}} \sum_{g'+g''=g} G_{|b_1 \dots b_{2l}|}^{(g')} \frac{G_{|b_{2l+1} \dots b_{N-1} b_{2l}|}^{(g'')} - G_{|b_{2l+1} \dots b_{N-1} a|}^{(g'')}}{E_{b_{2l}} - E_a} \right\}. \tag{3.9}
\end{aligned}$$

The same expansion extends to (3.7). In particular, we have proved

Proposition 3.4 *The (unrenormalised) planar connected N -point functions $G_{|ab_1 \dots b_{N-1}|}^{(0)}$ at $B = 1$ in a real ϕ^4 -matrix model with injective external matrix E satisfy the recursive system of equations*

$$G_{|ab|}^{(0)} = \frac{1}{E_a + E_b} - \frac{\lambda_4}{(E_a + E_b)V} G_{|ab|}^{(0)} \sum_{p \in I} G_{|ap|}^{(0)} + \frac{\lambda_4}{(E_a + E_b)V} \sum_{p \in I} \frac{G_{|pb|}^{(0)} - G_{|ab|}^{(0)}}{E_p - E_a}, \tag{3.10}$$

$$G_{|b_0 b_1 \dots b_{N-1}|}^{(0)} = (-\lambda_4) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|}^{(0)} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|}^{(0)} - G_{|b_{2l} b_1 \dots b_{2l-1}|}^{(0)} G_{|b_0 b_{2l+1} \dots b_{N-1}|}^{(0)}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}. \tag{3.11}$$

In the scaling limit ($V \rightarrow \infty$, $\frac{1}{V} \sum_{p \in I}$ finite), the full function and the genus-0 function satisfy the same equation so that $\lim_{\frac{1}{V} \sum_{p \in I} \text{finite}}^{V \rightarrow \infty} G_{|ab_1 \dots b_{N-1}|}^{(0)} = \lim_{\frac{1}{V} \sum_{p \in I} \text{finite}}^{V \rightarrow \infty} G_{|ab_1 \dots b_{N-1}|}$. \square

The equation (3.10) is a remarkable *closed equation for the planar connected regular two-point function alone*. This was already established in [GW09] by a different method, and versions of (3.10) for self-dual noncommutative ϕ^4 -theory at $a = b = 0$ have already been derived in [DGMR07]. We have generalised the ideas of [DGMR07] to a large class of matrix models and obtained a self-consistent equation for $G_{|ab|}^{(0)}$ for *any* matrix indices a, b . Usual Schwinger-Dyson techniques give an *open* system of equations relating the 2-point function to the 4-point function, the 4-point function to the 6-point function, and so on. In our setting, the Ward identity allowed us to break this system apart. There is now one closed non-linear equation for the 2-point function alone and then a recursive formula for the exact solution of all ($N > 2$)-point functions.

We stress that the two-point function is by definition symmetric, $G_{|ab|}^{(0)} = G_{|ba|}^{(0)}$, although this is not manifest in (3.10)!

We observe in Appendix A that planar functions with $B = 2$ cycles scale with $\frac{1}{V}$ for $V \rightarrow \infty$. Inserted into (3.8) and (3.9) we see that functions with ($g = 1, B = 1$) then scale with $\frac{1}{V^2}$. This holds in general: the whole non-planar sector with genus $g \geq 1$ is scaled away for $V \rightarrow \infty$.

Proposition 3.5 *In the scaling limit ($V \rightarrow \infty, \frac{1}{V} \sum_{p \in I} \text{finite}$), the $(N_1 + \dots + N_B)$ -point function of genus g has a scaling*

$$G_{|b_1^1 \dots b_{N_1}^1 | \dots | b_1^B \dots b_{N_B}^B|}^{(g)} = V^{1-B-2g} G_{|b_1^1 \dots b_{N_1}^1 | \dots | b_1^B \dots b_{N_B}^B|}^g, \quad (3.12)$$

where $G_{|b_1^1 \dots b_{N_1}^1 | \dots | b_1^B \dots b_{N_B}^B|}^g$ has a finite limit for $V \rightarrow \infty$.

Proof. We proceed by induction in g and B . The step from $(B = 1, g = 0)$ to $(B = 2, g = 0)$ is verified in (A.3) and (A.8). Then (3.8) and (3.9) verify the step from $(B = 1, g = 0)$ and $(B = 2, g = 0)$ to $(B = 2, g = 0)$. Knowing that these matrix models have a universal scaling degree in the combination $B + 2g$ [GW05b]) we confirm (3.12) in the general case. \square

Recall that the genus expansion reads $G_{|b_1^1 \dots b_{N_1}^1 | \dots | b_1^B \dots b_{N_B}^B|} = \sum_{g=0}^{\infty} G_{|b_1^1 \dots b_{N_1}^1 | \dots | b_1^B \dots b_{N_B}^B|}^{(g)}$. Expressing $G^{(g)}$ in terms of the functions G^g on the lhs of (3.12) which have a limit for $V \rightarrow \infty$ we see that a genus- g contribution is suppressed by a factor V^{-2g} . There is even a better argument why the whole non-planar sector with $g > 0$ is scaled away: As noticed in Propositions 3.4, A.1 and A.2, the whole function and its genus-0 part satisfy the same equation.

Note that (3.12) also suggests that N -point functions with $B > 1$ are suppressed by a factor $V^{-(B-1)}$ over the one-cycle functions. This is definitely true for matrix model correlation functions. We give in section 4.7 an outlook to ongoing work in which we study for the ϕ^4 -model on Moyal space the passage from matrix representation to position space. It turns out that all planar functions with $(B \geq 1, g = 0)$ do contribute to position space correlation functions, whereas the non-planar sector remains suppressed.

3.5 Graphical solution of the recursion formula in the planar sector

In the limit $V \rightarrow \infty$ the second line of (3.7) is absent, so that we obtain an algebraic recursion formula in the $B = 1$ sector alone. For $N \in \{4, 6\}$ we thus obtain

$$G_{|abcd|}^{(0)} = (-\lambda_4) \frac{G_{|ab|}^{(0)} G_{|cd|}^{(0)} - G_{|ad|}^{(0)} G_{|bc|}^{(0)}}{(E_a - E_c)(E_b - E_d)}, \quad (3.13)$$

$$\begin{aligned} G_{|b_0 \dots b_5|}^{(0)} &= (-\lambda_4) \left\{ \frac{G_{|b_0 b_1|}^{(0)} G_{|b_2 b_3 b_4 b_5|}^{(0)} - G_{|b_1 b_2|}^{(0)} G_{|b_3 b_4 b_5 b_0|}^{(0)}}{(E_{b_0} - E_{b_2})(E_{b_1} - E_{b_5})} + \frac{G_{|b_0 b_1 b_2 b_3|}^{(0)} G_{|b_4 b_5|}^{(0)} - G_{|b_1 b_2 b_3 b_4|}^{(0)} G_{|b_5 b_0|}^{(0)}}{(E_{b_0} - E_{b_4})(E_{b_1} - E_{b_5})} \right\} \\ &= (-\lambda_4)^2 \left\{ \frac{G_{|b_0 b_1|}^{(0)} G_{|b_2 b_5|}^{(0)} G_{|b_4 b_3|}^{(0)}}{e_{b_0 b_2} e_{b_2 b_4} \cdot e_{b_1 b_5} e_{b_5 b_3}} + \frac{G_{|b_0 b_3|}^{(0)} G_{|b_2 b_1|}^{(0)} G_{|b_4 b_5|}^{(0)}}{e_{b_4 b_0} e_{b_0 b_2} \cdot e_{b_5 b_3} e_{b_3 b_1}} + \frac{G_{|b_0 b_5|}^{(0)} G_{|b_2 b_3|}^{(0)} G_{|b_4 b_1|}^{(0)}}{e_{b_2 b_4} e_{b_4 b_0} \cdot e_{b_3 b_1} e_{b_1 b_5}} \right. \\ &\quad + G_{|b_0 b_1|}^{(0)} G_{|b_2 b_3|}^{(0)} G_{|b_4 b_5|}^{(0)} \left(\frac{1}{e_{b_0 b_2} e_{b_2 b_4} \cdot e_{b_1 b_3} e_{b_3 b_5}} + \frac{1}{e_{b_0 b_4} e_{b_4 b_2} \cdot e_{b_1 b_3} e_{b_3 b_5}} + \frac{1}{e_{b_0 b_4} e_{b_4 b_2} \cdot e_{b_1 b_5} e_{b_5 b_3}} \right) \\ &\quad \left. + G_{|b_0 b_5|}^{(0)} G_{|b_2 b_1|}^{(0)} G_{|b_4 b_3|}^{(0)} \left(\frac{1}{e_{b_0 b_4} e_{b_4 b_2} \cdot e_{b_5 b_3} e_{b_3 b_1}} + \frac{1}{e_{b_0 b_2} e_{b_2 b_4} \cdot e_{b_5 b_3} e_{b_3 b_1}} + \frac{1}{e_{b_0 b_2} e_{b_2 b_4} \cdot e_{b_5 b_1} e_{b_1 b_3}} \right) \right\}, \quad (3.14) \end{aligned}$$

where $e_{b_i b_j} := E_{b_i} - E_{b_j}$. The denominators arising in (3.14) are not unique; we rearrange them using the identity

$$\frac{1}{e_{b_i b_j} e_{b_j b_k}} + \frac{1}{e_{b_j b_k} e_{b_k b_i}} + \frac{1}{e_{b_k b_i} e_{b_i b_j}} = 0. \quad (3.15)$$

We introduce a graphical representation for a contribution to $G_{|b_0\dots b_{N-1}|}^{(0)}$. We put the indices b_0, \dots, b_{N-1} in cyclic order on a circle. A factor $G_{|b_i b_j|}^{(0)}$ is drawn as a *thick* chord between b_i and b_j , and a factor $\frac{1}{E_{b_m} - E_{b_n}}$ as a *thin* arrow from b_m to b_n . Reversing the arrow thus changes the sign. Later on we connect the arrows to trees where it will be necessary to specify the root as a thick dot. The corresponding graphical description of (3.13) reads

$$G_{|b_0 b_1 b_2 b_3|}^{(0)} = (-\lambda_4) \left\{ \begin{array}{c} \text{Diagram 1} \\ + \\ \text{Diagram 2} \end{array} \right\}. \quad (3.16)$$

The two graphs are turned into each other by a rotation of $\frac{2\pi}{4}$ and possibly reversion of both arrows.

Our rules give the following graphical representation of the six-point function (3.14):

$$G_{|b_0\dots b_5|}^{(0)} = (-\lambda_4)^2 \left\{ \begin{array}{c} \text{Diagram 1} \\ + \\ \text{two rotations} \\ \text{by } \frac{2\pi k}{6} \\ k \in \{1, 2\} \end{array} \right\} + (-\lambda_4)^2 \left\{ \begin{array}{c} \left(\text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \right) \\ + \text{rotation} \\ \text{by } \frac{2\pi}{6} \end{array} \right\}. \quad (3.17)$$

We first notice that only the 5 *non-crossing pairings* of the cyclic indices $b_0 b_1 b_2 b_3 b_4 b_5$ appear. The remaining pairing $G_{b_0 b_3} G_{b_1 b_4} G_{b_2 b_5}$ has a self-crossing. The non-crossing pairings of $2n$ cyclic indices are counted by the Catalan number $C_n = \frac{(2n)!}{n!(n+1)!}$.

Note that the 4 possible products of even/odd paths starting with root b_0/b_1 are distributed over the pairings in such a manner that the paths intersect the pairings only in the vertices and every product of paths appears exactly once. Further rotation of the first line by $\frac{2\pi \cdot 3}{6}$ and of the second line by $\frac{2\pi \cdot 2}{6}$ give the same graphs after common reversal of the arrows and use of the identity (3.15).

These observations generalise to any planar N -point function at $B = 1$:

Proposition 3.6 *The N -point function $G_{|b_0\dots b_{N-1}|}^{(0)}$ is a sum with prefactors $\pm(-\lambda_4)^{\frac{N}{2}-1}$ of graphs with vertices b_0, \dots, b_{N-1} put in cyclic order on a circle and (fat) edges between b_k, b_l representing a factor $G_{|b_k b_l|}^{(0)}$ and (thin) arrows from b_k to b_l representing a factor $\frac{1}{E_{b_k} - E_{b_l}}$. We call b_{2i} an even vertex and b_{2i+1} an odd vertex. In every such graph we have (or can achieve via (3.15)):*

1. *Every even vertex is paired (by a fat chord) with exactly one odd vertex, and these chords do not cross.*
2. *Taking any even vertex as a root, there is a an oriented rooted tree between all even vertices. This even tree has no self-intersection and intersects the fat chord only at the vertices.*

3. Taking any odd vertex as a root, there is an oriented rooted tree between all odd vertices. This odd tree has no self-intersection and intersects the fat chords only at the vertices.

Proof. By (3.11), the N -point function $G_{|b_0 \dots b_{N-1}|}^{(0)}$ arises by connections of $2l$ -point functions with $(N - 2l)$ -point functions via an even arrow and an odd arrow. We proceed by induction and assume validity of the Proposition for the $2l$ - and $(N - 2l)$ -point functions, starting with $G_{|b_0 b_1|}^{(0)}$ which is represented by one chord without any arrows. We can symbolise (3.11) (without prefactor $(-\lambda_4)$) as follows:

$$G_{|b_0 \dots b_{N-1}|}^{(0)} = \sum_{l=1}^{\frac{N-2}{2}} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right)$$

Here, the \star locates the vertices and the dashed line separates the $2l$ -point function from the $(N - 2l)$ -point function of which we do not show their inner chords and trees.

- 1) By induction in one of these smaller functions every even vertex is paired with exactly one odd vertex. These pairings do not cross in the smaller function and since they do not cross the dashed cut, there cannot be a crossing between chords of the two smaller functions.
- 2) All even vertices in the $2l$ -point function and in the $(N - 2l)$ -point functions are connected by a tree which by induction has no self-intersection and intersects the chords only in the vertices. The additional arrow from b_0 to b_l connects both trees of l and $\frac{N}{2} - l$ vertices to a larger tree of $\frac{N}{2}$ vertices. It is geometrically clear that the additional arrow intersects both the two smaller even trees and the chords precisely in the vertices b_0 and b_l .
- 3) All odd vertices in the $2l$ -point function and in the $(N - 2l)$ -point functions are connected by a tree which by induction has no self-intersection and intersects the chords only in the vertices. The additional arrow from b_1 to b_{N-1} connects both trees of l and $\frac{N}{2} - l$ vertices to a larger tree of $\frac{N}{2}$ vertices. This larger tree does not have self-intersections, but in general intersects the chord emanating from b_0 . Since a cyclic permutation of the indices interchanges even and odd vertices and hence even and odd trees, by 2) it must be possible to use the identity (3.15) to rearrange the odd tree in such a way that the possible intersection with the chord emanating from b_0 can be avoided. \square

Proposition 3.6 determines the planar connected N -point function up to the problem to identify the product of even and odd trees associated with a given non-crossing chord diagram. Due to (3.15) this product cannot be unique. We leave this question of canonical

trees for future investigation. We finish this section with the eight-point function which shows that not all possible trees do actually arise:

$$\begin{aligned}
& G_{|b_0 \dots b_7|}^{(0)} \\
&= (-\lambda) \left\{ \frac{G_{|b_0 b_1|}^{(0)} G_{|b_2 b_3 b_4 b_5 b_6 b_7|}^{(0)} - G_{|b_1 b_2|}^{(0)} G_{|b_3 b_4 b_5 b_6 b_7 b_0|}^{(0)}}{(E_{b_0} - E_{b_2})(E_{b_1} - E_{b_5})} + \frac{G_{|b_0 b_1 b_2 b_3|}^{(0)} G_{|b_4 b_5 b_6 b_7|}^{(0)} - G_{|b_1 b_2 b_3 b_4|}^{(0)} G_{|b_5 b_6 b_7 b_0|}^{(0)}}{(E_{b_0} - E_{b_4})(E_{b_1} - E_{b_5})} \right. \\
&\quad \left. + \frac{G_{|b_0 b_1 b_2 b_3 b_4 b_5|}^{(0)} G_{|b_6 b_7|}^{(0)} - G_{|b_1 b_2 b_3 b_4 b_5 b_6|}^{(0)} G_{|b_7 b_0|}^{(0)}}{(E_{b_0} - E_{b_6})(E_{b_1} - E_{b_5})} \right\} \\
&= (-\lambda)^3 \left\{ G_{|b_0 b_1|}^{(0)} G_{|b_2 b_7|}^{(0)} G_{|b_4 b_5|}^{(0)} G_{|b_6 b_3|}^{(0)} \left(\frac{1}{e_{b_0 b_2} e_{b_2 b_6} e_{b_6 b_4} \cdot e_{b_1 b_7} e_{b_7 b_3} e_{b_3 b_5}} \right) \right. \\
&+ G_{|b_0 b_3|}^{(0)} G_{|b_2 b_1|}^{(0)} G_{|b_4 b_7|}^{(0)} G_{|b_6 b_5|}^{(0)} \left(\frac{1}{e_{b_2 b_0} e_{b_0 b_4} e_{b_4 b_6} \cdot e_{b_1 b_3} e_{b_3 b_7} e_{b_7 b_5}} \right) \\
&+ G_{|b_0 b_5|}^{(0)} G_{|b_2 b_3|}^{(0)} G_{|b_4 b_1|}^{(0)} G_{|b_6 b_7|}^{(0)} \left(\frac{1}{e_{b_2 b_4} e_{b_4 b_0} e_{b_0 b_6} \cdot e_{b_3 b_1} e_{b_1 b_5} e_{b_5 b_7}} \right) \\
&+ G_{|b_0 b_7|}^{(0)} G_{|b_2 b_5|}^{(0)} G_{|b_4 b_3|}^{(0)} G_{|b_6 b_1|}^{(0)} \left(\frac{1}{e_{b_4 b_2} e_{b_2 b_6} e_{b_6 b_0} \cdot e_{b_3 b_5} e_{b_5 b_1} e_{b_1 b_7}} \right) \\
&+ G_{|b_0 b_1|}^{(0)} G_{|b_2 b_7|}^{(0)} G_{|b_4 b_3|}^{(0)} G_{|b_6 b_5|}^{(0)} \left(\frac{1}{e_{b_0 b_2} e_{b_2 b_4} e_{b_4 b_6} \cdot e_{b_1 b_7} e_{b_7 b_3} e_{b_3 b_5}} + \frac{1}{e_{b_0 b_2} e_{b_2 b_4} e_{b_4 b_6} \cdot e_{b_1 b_7} e_{b_7 b_5} e_{b_5 b_3}} \right. \\
&\quad \left. + \frac{1}{e_{b_0 b_2} e_{b_2 b_6} e_{b_6 b_4} \cdot e_{b_1 b_7} e_{b_7 b_5} e_{b_5 b_3}} \right) \\
&+ G_{|b_0 b_3|}^{(0)} G_{|b_2 b_1|}^{(0)} G_{|b_4 b_5|}^{(0)} G_{|b_6 b_7|}^{(0)} \left(\frac{1}{e_{b_2 b_0} e_{b_0 b_4} e_{b_4 b_6} \cdot e_{b_1 b_3} e_{b_3 b_5} e_{b_5 b_7}} + \frac{1}{e_{b_2 b_0} e_{b_0 b_6} e_{b_6 b_4} \cdot e_{b_1 b_3} e_{b_3 b_5} e_{b_5 b_7}} \right. \\
&\quad \left. + \frac{1}{e_{b_2 b_0} e_{b_0 b_6} e_{b_6 b_4} \cdot e_{b_1 b_3} e_{b_3 b_7} e_{b_7 b_5}} \right) \\
&+ G_{|b_0 b_7|}^{(0)} G_{|b_2 b_3|}^{(0)} G_{|b_4 b_1|}^{(0)} G_{|b_6 b_5|}^{(0)} \left(\frac{1}{e_{b_2 b_4} e_{b_4 b_6} e_{b_6 b_0} \cdot e_{b_3 b_1} e_{b_1 b_5} e_{b_5 b_7}} + \frac{1}{e_{b_2 b_4} e_{b_4 b_6} e_{b_6 b_0} \cdot e_{b_3 b_1} e_{b_1 b_7} e_{b_7 b_5}} \right. \\
&\quad \left. + \frac{1}{e_{b_2 b_4} e_{b_4 b_0} e_{b_0 b_6} \cdot e_{b_3 b_1} e_{b_1 b_7} e_{b_7 b_5}} \right) \\
&+ G_{|b_0 b_1|}^{(0)} G_{|b_2 b_5|}^{(0)} G_{|b_4 b_3|}^{(0)} G_{|b_6 b_7|}^{(0)} \left(\frac{1}{e_{b_4 b_2} e_{b_2 b_6} e_{b_6 b_0} \cdot e_{b_3 b_5} e_{b_5 b_7} e_{b_7 b_1}} + \frac{1}{e_{b_4 b_2} e_{b_2 b_0} e_{b_0 b_6} \cdot e_{b_3 b_5} e_{b_5 b_7} e_{b_7 b_1}} \right. \\
&\quad \left. + \frac{1}{e_{b_4 b_2} e_{b_2 b_0} e_{b_0 b_6} \cdot e_{b_3 b_5} e_{b_5 b_1} e_{b_1 b_7}} \right) \\
&+ G_{|b_0 b_7|}^{(0)} G_{|b_2 b_1|}^{(0)} G_{|b_4 b_5|}^{(0)} G_{|b_6 b_3|}^{(0)} \left(\frac{1}{e_{b_4 b_6} e_{b_6 b_0} e_{b_0 b_2} \cdot e_{b_5 b_3} e_{b_3 b_7} e_{b_7 b_1}} + \frac{1}{e_{b_4 b_6} e_{b_6 b_0} e_{b_0 b_2} \cdot e_{b_5 b_3} e_{b_3 b_1} e_{b_1 b_7}} \right. \\
&\quad \left. + \frac{1}{e_{b_4 b_6} e_{b_6 b_2} e_{b_2 b_0} \cdot e_{b_5 b_3} e_{b_3 b_1} e_{b_1 b_7}} \right) \\
&+ G_{|b_0 b_1|}^{(0)} G_{|b_2 b_3|}^{(0)} G_{|b_4 b_7|}^{(0)} G_{|b_6 b_5|}^{(0)} \left(\frac{1}{e_{b_6 b_4} e_{b_4 b_0} e_{b_0 b_2} \cdot e_{b_5 b_7} e_{b_7 b_1} e_{b_1 b_3}} + \frac{1}{e_{b_6 b_4} e_{b_4 b_2} e_{b_2 b_0} \cdot e_{b_5 b_7} e_{b_7 b_1} e_{b_1 b_3}} \right. \\
&\quad \left. + \frac{1}{e_{b_6 b_4} e_{b_4 b_2} e_{b_2 b_0} \cdot e_{b_5 b_7} e_{b_7 b_3} e_{b_3 b_1}} \right) \\
&+ G_{|b_0 b_5|}^{(0)} G_{|b_2 b_1|}^{(0)} G_{|b_4 b_3|}^{(0)} G_{|b_6 b_7|}^{(0)} \left(\frac{1}{e_{b_6 b_0} e_{b_0 b_2} e_{b_2 b_4} \cdot e_{b_7 b_5} e_{b_5 b_1} e_{b_1 b_3}} + \frac{1}{e_{b_6 b_0} e_{b_0 b_2} e_{b_2 b_4} \cdot e_{b_7 b_5} e_{b_5 b_3} e_{b_3 b_1}} \right. \\
&\quad \left. + \frac{1}{e_{b_6 b_0} e_{b_0 b_4} e_{b_4 b_2} \cdot e_{b_7 b_5} e_{b_5 b_3} e_{b_3 b_1}} \right) \\
&+ G_{|b_0 b_7|}^{(0)} G_{|b_2 b_3|}^{(0)} G_{|b_4 b_5|}^{(0)} G_{|b_6 b_1|}^{(0)} \left(\frac{1}{e_{b_0 b_6} e_{b_6 b_2} e_{b_2 b_4} \cdot e_{b_7 b_1} e_{b_1 b_3} e_{b_3 b_5}} + \frac{1}{e_{b_0 b_6} e_{b_6 b_4} e_{b_4 b_2} \cdot e_{b_7 b_1} e_{b_1 b_3} e_{b_3 b_5}} \right. \\
&\quad \left. + \frac{1}{e_{b_0 b_6} e_{b_6 b_4} e_{b_4 b_2} \cdot e_{b_7 b_1} e_{b_1 b_5} e_{b_5 b_3}} \right) \\
&+ G_{|b_0 b_1|}^{(0)} G_{|b_2 b_3|}^{(0)} G_{|b_4 b_5|}^{(0)} G_{|b_6 b_7|}^{(0)} \left(\frac{1}{e_{b_0 b_6} e_{b_6 b_4} e_{b_4 b_2} \cdot e_{b_1 b_7} e_{b_7 b_5} e_{b_5 b_3}} + \frac{1}{e_{b_0 b_6} e_{b_6 b_4} e_{b_4 b_2} \cdot e_{b_1 b_7} e_{b_7 b_3} e_{b_3 b_5}} \right. \\
&\quad + \frac{1}{e_{b_0 b_6} e_{b_6 b_4} e_{b_4 b_2} \cdot e_{b_1 b_3} e_{b_3 b_7} e_{b_7 b_5}} + \frac{1}{e_{b_0 b_2} e_{b_2 b_4} e_{b_4 b_6} \cdot e_{b_1 b_3} e_{b_3 b_5} e_{b_5 b_7}} + \frac{1}{e_{b_0 b_2} e_{b_2 b_6} e_{b_6 b_4} \cdot e_{b_1 b_3} e_{b_3 b_5} e_{b_5 b_7}} \\
&\quad + \frac{1}{e_{b_0 b_6} e_{b_6 b_2} e_{b_2 b_4} \cdot e_{b_1 b_3} e_{b_3 b_5} e_{b_5 b_7}} + \frac{1}{e_{b_0 b_2} e_{b_2 b_6} e_{b_6 b_4} \cdot e_{b_1 b_3} e_{b_3 b_7} e_{b_7 b_5}} + \frac{1}{e_{b_0 b_6} e_{b_6 b_2} e_{b_2 b_4} \cdot e_{b_1 b_7} e_{b_7 b_3} e_{b_3 b_5}} \\
&\quad \left. + \frac{1}{e_{b_0 b_6} e_{b_6 b_2} e_{b_2 b_4} \cdot e_{b_1 b_3} e_{b_3 b_7} e_{b_7 b_5}} \right)
\end{aligned}$$

$$\begin{aligned}
& + G_{|b_0 b_7|}^{(0)} G_{|b_2 b_1|}^{(0)} G_{|b_4 b_3|}^{(0)} G_{|b_6 b_5|}^{(0)} \left(\frac{1}{e_{b_0 b_6} e_{b_6 b_4} e_{b_4 b_2} \cdot e_{b_7 b_5} e_{b_5 b_3} e_{b_3 b_1}} + \frac{1}{e_{b_0 b_6} e_{b_6 b_2} e_{b_2 b_4} \cdot e_{b_7 b_5} e_{b_5 b_3} e_{b_3 b_1}} \right. \\
& + \frac{1}{e_{b_0 b_2} e_{b_2 b_6} e_{b_6 b_4} \cdot e_{b_7 b_5} e_{b_5 b_3} e_{b_3 b_1}} + \frac{1}{e_{b_0 b_2} e_{b_2 b_4} e_{b_4 b_6} \cdot e_{b_7 b_1} e_{b_1 b_3} e_{b_3 b_5}} + \frac{1}{e_{b_0 b_2} e_{b_2 b_4} e_{b_4 b_6} \cdot e_{b_7 b_1} e_{b_1 b_5} e_{b_5 b_3}} \\
& + \frac{1}{e_{b_0 b_2} e_{b_2 b_4} e_{b_4 b_6} \cdot e_{b_7 b_5} e_{b_5 b_1} e_{b_1 b_3}} + \frac{1}{e_{b_0 b_2} e_{b_2 b_6} e_{b_6 b_4} \cdot e_{b_7 b_1} e_{b_1 b_5} e_{b_5 b_3}} + \frac{1}{e_{b_0 b_6} e_{b_6 b_2} e_{b_2 b_4} \cdot e_{b_7 b_5} e_{b_5 b_1} e_{b_1 b_3}} \\
& \left. + \frac{1}{e_{b_0 b_2} e_{b_2 b_6} e_{b_6 b_4} \cdot e_{b_7 b_5} e_{b_5 b_1} e_{b_1 b_3}} + \frac{1}{e_{b_0 b_2} e_{b_0 b_4} e_{b_4 b_6} \cdot e_{b_7 b_5} e_{b_7 b_3} e_{b_3 b_1}} \right) \}. \quad (3.18)
\end{aligned}$$

We have used (3.15) to bring the denominators for the last two products of $G_{b_i b_j}$ into a form which (in the sum) is symmetric in the even and odd indices and has a common sign. A novel feature compared with the six-point function is the appearance of trees with branches $\frac{1}{e_{b_0 b_6} e_{b_0 b_4} e_{b_4 b_2}} \cdot \frac{1}{e_{b_1 b_3} e_{b_1 b_5} e_{b_5 b_7}}$ and $\frac{1}{e_{b_0 b_2} e_{b_0 b_4} e_{b_4 b_6}} \cdot \frac{1}{e_{b_7 b_5} e_{b_7 b_3} e_{b_3 b_1}}$. Trees become unavoidable in the 12-point function where some pairings do not permit a non-crossing rooted path. It seems natural to require that paired roots carry the same number of branches. If we furthermore require that in 8-point functions only the root has more than one branch, then the representation given in (3.18) seems distinguished. Its graphical representation reads

$$\begin{aligned}
G_{|b_0 \dots b_7|}^{(0)} = & (-\lambda)^3 \left\{ \begin{array}{c} \text{three rotations} \\ \text{by } \frac{2\pi k}{8}, \\ k \in \{1, 2, 3\} \end{array} \right\} \\
& + (-\lambda)^3 \left\{ \begin{array}{c} \text{seven rotations} \\ \text{by } \frac{2\pi k}{8}, \\ k \in \{1, 2, 3, 4, 5, 6, 7\} \end{array} \right\} \\
& + (-\lambda)^3 \left\{ \begin{array}{c} \text{one} \\ \text{rotation} \\ \text{by } \frac{2\pi}{8} \end{array} \right\}. \quad (3.19)
\end{aligned}$$

Again we obtain all $C_4 = 14$ non-crossing pairings of the cyclic external indices b_0, \dots, b_7 which give the products of $G_{|b_i b_j|}^{(0)}$.

4 Self-dual noncommutative ϕ_4^4 -theory

4.1 Definition of the matrix model

In order to improve the problems of four-dimensional quantum field theory it was suggested to include “gravity effects” through deforming space-time. A simple example for such a deformation is the Moyal space, which arises by deformation of the Fréchet algebra of Schwarz class functions by the action of the translation group [Rie93],

$$(f \star g)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{dy dk}{(2\pi)^d} f\left(x + \frac{1}{2}\Theta k\right) g(x + y) e^{i\langle k, y \rangle}, \quad f, g \in \mathcal{S}(\mathbb{R}^d). \quad (4.1)$$

Here, $\Theta = -\Theta^t \in M_d(\mathbb{R})$ is a skew-symmetric matrix. Although not required by the general theory, for our purpose we need Θ to be of full rank, which implies that d is even. The algebra $\mathcal{A}_\Theta = (\mathcal{S}(\mathbb{R}^d), \star)$ is a pre- C^* -algebra [GGISV03, Prop. 2.14]. The standard \mathbb{R}^d -Lebesgue integral defines a linear functional which is positive $\int_{\mathbb{R}^d} (f \star \bar{f})(x) \geq 0$ and tracial $\int_{\mathbb{R}^d} (f \star g)(x) = \int_{\mathbb{R}^d} (g \star f)(x)$. Therefore, GNS-construction gives rise to a Hilbert space $H = H_1 \otimes H_2$ on which there are commuting representations $\pi \otimes \text{id}$ of \mathcal{A}_Θ and $\text{id} \otimes \pi^{op}$ of the opposite algebra \mathcal{A}_Θ^{op} . Restricted to H_1 , any element of \mathcal{A}_Θ is a compact operator on H_1 , in fact even a trace-class operator.

This means that we can define field-theoretical matrix models on $\mathcal{A}_\Theta \ni \phi$ by specifying the polynomial $P[\phi]$ and the external matrix E . We take $d = 4$ and $P[\phi] = \frac{\lambda_4}{4}\phi^4$. The unbounded operator E should play the rôle of the Laplacian. However, the Laplacian Δ on \mathbb{R}^d does not have compact resolvent. This non-compactness of Δ is the reason why field theories based on action functionals such as

$$\int_{\mathbb{R}^d} dx \left(\frac{1}{2}\phi(-\Delta + \mu^2)\phi + \frac{\lambda_4}{4}\phi \star \phi \star \phi \star \phi \right)(x)$$

have bad properties (UV/IR-mixing [MVS00]).

In our previous work [GW05b] we found a way to handle this problem. We realised that extending the Laplacian to the harmonic oscillator Hamiltonian (which has compact resolvent!), the field theory defined by the action

$$S = 64\pi^2 \int d^4x \left(\frac{1}{2}\phi(-\Delta + \Omega^2\|2\Theta^{-1}x\|^2 + \mu^2)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi \right)(x) \quad (4.2)$$

is renormalisable to all orders of perturbation theory. The perturbative renormalisability was proved by renormalisation group techniques [GW05b] and multi-scale analysis, both in matrix representation [RVW06] and position space [GMRV06]. The model is covariant under the Langmann-Szabo duality transformation [LS02]. It becomes self-dual at $\Omega = 1$, a point at which the β -function vanishes to all orders in perturbation theory [DGMR07]. Certain variants of this model have also been treated, see [Riv07a] for a review.

Writing the action functional (4.2) in the matrix base of the Moyal space, see [GW05b, GW05a], one obtains a *non-local* matrix model in the sense that the ϕ -bilinear term is a tensor product $\text{tr}(\hat{E} \cdot (\phi \otimes \phi))$. We have studied the power-counting behaviour of such matrix models in [GW05a]. By a sophisticated analysis of the tensor-product external matrix \hat{E} we proved in [GW05b] the perturbative renormalisability of the model (4.2). The action simplifies enormously at the self-duality point $\Omega = 1$ where it becomes an

ordinary matrix model in the sense of section 2 with diagonal external matrix E . In terms of the *bare* quantities, which involves the bare mass μ_{bare} and the wave function renormalisation $\phi \mapsto Z^{\frac{1}{2}}\phi$, the action functional becomes [GW05b, eqs. (2.5)+(2.6)]

$$S[\phi] = V \left(\sum_{\underline{m}, \underline{n} \in \mathbb{N}_{\mathcal{N}}^2} E_{\underline{m}} \phi_{\underline{m}\underline{n}} \phi_{\underline{n}\underline{m}} + \frac{S_{int}[\phi]}{V} \right), \quad V := \left(\frac{\theta}{4} \right)^2 \quad (4.3a)$$

$$E_{\underline{m}} = Z \left(\frac{4}{\theta} |\underline{m}| + \frac{\mu_{bare}^2}{2} \right), \quad \frac{S_{int}[\phi]}{V} = \frac{Z^2 \lambda}{4} \sum_{\underline{m}, \underline{n}, \underline{k}, \underline{l} \in \mathbb{N}_{\mathcal{N}}^2} \phi_{\underline{m}\underline{n}} \phi_{\underline{n}\underline{k}} \phi_{\underline{k}\underline{l}} \phi_{\underline{l}\underline{m}}. \quad (4.3b)$$

We have made use of the fact that, after an appropriate coordinate transformation in \mathbb{R}^4 , the only non-vanishing components of Θ are $\Theta_{12} = -\Theta_{21} = \Theta_{34} = -\Theta_{43} =: \theta$. We have also absorbed the ground energy of the harmonic oscillator into a redefinition of μ_{bare} . The prefactor $64\pi^2$ in (4.2) is introduced in order to identify the volume $V = (\frac{\theta}{4})^2$ instead of the usual factor $(2\pi\theta)^2$ arising from the integral of the matrix basis. All summation indices $\underline{m}, \underline{n}, \dots$ belong to \mathbb{N}^2 , with $|\underline{m}| := m_1 + m_2$. The symbol $\mathbb{N}_{\mathcal{N}}^2$ refers to a cut-off in the matrix size which later on will be made more precise. The scalar field is real, $\phi_{\underline{m}\underline{n}} = \overline{\phi_{\underline{n}\underline{m}}}$. We then know from Theorem 3.2 that there is no need of a renormalisation $\lambda = Z_4 \lambda_{ren}$ of the coupling constant.

4.2 Renormalisation of the two-point function

We now focus on the Schwinger-Dyson equation (3.10) for the planar regular connected two-point function $G_{|\underline{a}\underline{b}|}^{(0)}$. From (4.3b) we read off $\lambda_4 = Z^2 \lambda$. The summation over $\underline{p} \in I := \mathbb{N}_{\mathcal{N}}^2$ in (3.10) diverges if we remove the cut-off $\mathbb{N}_{\mathcal{N}}^2 \rightarrow \mathbb{N}^2$ and, therefore, requires renormalisation. Renormalisation is defined by *normalisation conditions* for the first Taylor coefficients of the *one-particle irreducible (1PI)* function $\Gamma_{\underline{a}\underline{b}}$ related to $G_{|\underline{a}\underline{b}|}^{(0)}$ according to

$$G_{|\underline{a}\underline{b}|}^{(0)} = (H_{\underline{a}\underline{b}} - \Gamma_{\underline{a}\underline{b}})^{-1}, \quad H_{\underline{a}\underline{b}} := E_{\underline{a}} + E_{\underline{b}} = Z \left(\frac{4}{\theta} (|\underline{a}| + |\underline{b}|) + \mu_{bare}^2 \right). \quad (4.4)$$

It is then an easy exercise to rewrite (3.10) into the following equation for $\Gamma_{\underline{a}\underline{b}}$:

$$\Gamma_{\underline{a}\underline{b}} = -\frac{\lambda Z^2}{V} \sum_{\underline{p} \in \mathbb{N}_{\mathcal{N}}^2} \left(\frac{1}{H_{\underline{a}\underline{p}} - \Gamma_{\underline{a}\underline{p}}} + \frac{1}{H_{\underline{p}\underline{b}} - \Gamma_{\underline{p}\underline{b}}} \right) + \frac{\lambda Z}{V} \sum_{\underline{p} \in \mathbb{N}_{\mathcal{N}}^2} \frac{1}{(H_{\underline{p}\underline{b}} - \Gamma_{\underline{p}\underline{b}})} \frac{\Gamma_{\underline{p}\underline{b}} - \Gamma_{\underline{a}\underline{b}}}{\theta (|\underline{p}| - |\underline{a}|)}. \quad (4.5)$$

We require

$$\begin{aligned} \Gamma_{\underline{a}\underline{b}} &= Z \mu_{bare}^2 - \mu^2 + (Z - 1) \frac{4}{\theta} (|\underline{a}| + |\underline{b}|) + \Gamma_{\underline{a}\underline{b}}^{ren}, \\ \Gamma_{\underline{0}\underline{0}}^{ren} &= 0 \quad (\partial \Gamma^{ren})_{\underline{0}\underline{0}} = 0, \end{aligned} \quad (4.6)$$

where $\partial \Gamma^{ren}$ is any of the (discrete) derivatives with respect to a_1, a_2, b_1, b_2 . This implies

$$\frac{1}{G_{|\underline{a}\underline{b}|}^{(0)}} = H_{\underline{a}\underline{b}} - \Gamma_{\underline{a}\underline{b}} = \frac{4}{\theta} (|\underline{a}| + |\underline{b}|) + \mu^2 - \Gamma_{\underline{a}\underline{b}}^{ren}. \quad (4.7)$$

Hence, μ is the renormalised mass, and both $G_{|ab|}^{(0)}$ and Γ_{ab}^{ren} should be regular if the cut-off in the matrix indices is removed. Inserted into (4.5) we find

$$\begin{aligned}
& Z\mu_{bare}^2 - \mu^2 + (Z-1)\frac{4}{\theta}(|\underline{a}| + |\underline{b}|) + \Gamma_{\underline{ab}}^{ren} \\
&= -\frac{\lambda}{V} \sum_{\underline{p} \in \mathbb{N}_{\mathcal{N}}^2} \frac{Z^2}{\frac{4}{\theta}(|\underline{a}| + |\underline{p}|) + \mu^2 - \Gamma_{\underline{ap}}^{ren}} \\
&- \frac{\lambda}{V} \sum_{\underline{p} \in \mathbb{N}_{\mathcal{N}}^2} \left(\frac{Z}{\frac{4}{\theta}(|\underline{b}| + |\underline{p}|) + \mu^2 - \Gamma_{\underline{pb}}^{ren}} - \frac{Z}{\frac{4}{\theta}(|\underline{b}| + |\underline{p}|) + \mu^2 - \Gamma_{\underline{pb}}^{ren}} \frac{\Gamma_{\underline{pb}}^{ren} - \Gamma_{\underline{ab}}^{ren}}{\frac{4}{\theta}(|\underline{p}| - |\underline{a}|)} \right). \quad (4.8)
\end{aligned}$$

Notice the different exponents of Z in the two tadpoles $\sum_{\underline{p}} \frac{1}{\frac{4}{\theta}(|\bullet| + |\underline{p}|) + \mu^2 - \Gamma_{\underline{p}\bullet}^{ren}}$! Separating the first Taylor term (by putting $|\underline{a}| = |\underline{b}| = 0$) we obtain

$$\begin{aligned}
& Z\mu_{bare}^2 - \mu^2 \\
&= -\frac{\lambda}{V} \sum_{\underline{p} \in \mathbb{N}_{\mathcal{N}}^2} \frac{Z^2}{\frac{4}{\theta}|\underline{p}| + \mu^2 - \Gamma_{0\underline{p}}^{ren}} - \frac{\lambda}{V} \sum_{\underline{p} \in \mathbb{N}_{\mathcal{N}}^2} \left(\frac{Z}{\frac{4}{\theta}|\underline{p}| + \mu^2 - \Gamma_{\underline{p}0}^{ren}} - \frac{Z}{\frac{4}{\theta}|\underline{p}| + \mu^2 - \Gamma_{\underline{p}0}^{ren}} \frac{\Gamma_{\underline{p}0}^{ren}}{\frac{4}{\theta}|\underline{p}|} \right),
\end{aligned}$$

which inserted into (4.8) gives

$$\begin{aligned}
& (Z-1)\frac{4}{\theta}(|\underline{a}| + |\underline{b}|) + \Gamma_{\underline{ab}}^{ren} \\
&= -\frac{\lambda}{V} \sum_{\underline{p} \in \mathbb{N}_{\mathcal{N}}^2} \left(\frac{Z^2}{\frac{4}{\theta}(|\underline{a}| + |\underline{p}|) + \mu^2 - \Gamma_{\underline{ap}}^{ren}} - \frac{Z^2}{\frac{4}{\theta}|\underline{p}| + \mu^2 - \Gamma_{0\underline{p}}^{ren}} \right) \\
&- \frac{\lambda}{V} \sum_{\underline{p} \in \mathbb{N}_{\mathcal{N}}^2} \left(\frac{Z}{\frac{4}{\theta}(|\underline{b}| + |\underline{p}|) + \mu^2 - \Gamma_{\underline{pb}}^{ren}} - \frac{Z}{\frac{4}{\theta}|\underline{p}| + \mu^2 - \Gamma_{\underline{p}0}^{ren}} \right. \\
&\quad \left. - \frac{Z}{\frac{4}{\theta}(|\underline{b}| + |\underline{p}|) + \mu^2 - \Gamma_{\underline{pb}}^{ren}} \frac{\Gamma_{\underline{pb}}^{ren} - \Gamma_{\underline{ab}}^{ren}}{\frac{4}{\theta}(|\underline{p}| - |\underline{a}|)} + \frac{Z}{\frac{4}{\theta}|\underline{p}| + \mu^2 - \Gamma_{\underline{p}0}^{ren}} \frac{\Gamma_{\underline{p}0}^{ren}}{\frac{4}{\theta}|\underline{p}|} \right). \quad (4.9)
\end{aligned}$$

The next step consists in differentiating (4.9) at 0 with respect to a_i and b_i in order to get a self-consistent system of equations for $Z, \Gamma_{\underline{ab}}^{ren}$. It is now crucial that (4.9) depends only on the sums $|\underline{a}| = a_1 + a_2$, $|\underline{b}| = b_1 + b_2$ and $|\underline{p}| = p_1 + p_2$ of indices. This degeneracy is first passed on to the bare two-point function $\Gamma_{\underline{ab}}$. But then the derivatives $\frac{\partial}{\partial a_i}|_{|\underline{a}|, |\underline{b}|=0} = \frac{\partial}{\partial |\underline{a}|}|_{|\underline{a}|, |\underline{b}|=0}$ and $\frac{\partial}{\partial b_i}|_{|\underline{a}|, |\underline{b}|=0} = \frac{\partial}{\partial |\underline{b}|}|_{|\underline{a}|, |\underline{b}|=0}$ respect the degeneracy, so that also the renormalised two-point function $\Gamma_{\underline{ab}}^{red}$ depends only on the 1-norms $|\underline{a}| = a_1 + a_2$ and $|\underline{b}| = b_1 + b_2$. Consequently, we can replace the index sums by $\sum_{\underline{p} \in \mathbb{N}_{\mathcal{N}}^2} f(|\underline{p}|) = \sum_{|\underline{p}|=0}^{\mathcal{N}} (|\underline{p}|+1)f(|\underline{p}|)$, because there are $|\underline{p}|+1$ possibilities³ to write $|\underline{p}|$ as a sum $|\underline{p}| = p_1 + p_2$ with $p_1, p_2 \in \mathbb{N}$. The cut-off $|\underline{p}| \leq \mathcal{N} \in \mathbb{N}$ then specifies the previously unclear symbol $\underline{p} \in \mathbb{N}_{\mathcal{N}}^2$. In summary, (4.9) takes with $V = (\frac{\theta}{4})^2$ from (4.3a) the form

$$\begin{aligned}
& (Z-1)\frac{4}{\theta}(|\underline{a}| + |\underline{b}|) + \Gamma_{\underline{ab}}^{ren} \\
&= -\lambda \left(\frac{4}{\theta}\right)^2 \sum_{|\underline{p}|=0}^{\mathcal{N}} (|\underline{p}|+1) \left(\frac{Z^2}{\frac{4}{\theta}(|\underline{a}| + |\underline{p}|) + \mu^2 - \Gamma_{\underline{ap}}^{ren}} - \frac{Z^2}{\frac{4}{\theta}|\underline{p}| + \mu^2 - \Gamma_{0\underline{p}}^{ren}} \right)
\end{aligned}$$

³In d dimensions the measure would be $\binom{|\underline{p}| + \frac{d}{2} - 1}{\frac{d}{2} - 1}$.

$$\begin{aligned}
& -\lambda\left(\frac{4}{\theta}\right)^2 \sum_{|\underline{p}|=0}^{\mathcal{N}} (|\underline{p}| + 1) \left(\frac{Z}{\frac{4}{\theta}(|\underline{b}| + |\underline{p}|) + \mu^2 - \Gamma_{\underline{p}\underline{b}}^{\text{ren}}} - \frac{Z}{\frac{4}{\theta}|\underline{p}| + \mu^2 - \Gamma_{\underline{p}0}^{\text{ren}}} \right. \\
& \left. - \frac{Z}{\frac{4}{\theta}(|\underline{b}| + |\underline{p}|) + \mu^2 - \Gamma_{\underline{p}\underline{b}}^{\text{ren}}} \frac{\Gamma_{\underline{p}\underline{b}}^{\text{ren}} - \Gamma_{\underline{a}\underline{b}}^{\text{ren}}}{\frac{4}{\theta}(|\underline{p}| - |\underline{a}|)} + \frac{Z}{\frac{4}{\theta}|\underline{p}| + \mu^2 - \Gamma_{\underline{p}0}^{\text{ren}}} \frac{\Gamma_{\underline{p}0}^{\text{ren}}}{\frac{4}{\theta}|\underline{p}|} \right). \quad (4.10)
\end{aligned}$$

4.3 Integral representation

We study a particular limit in which the self-dual noncommutative ϕ_4^4 -model converges to a large- \mathcal{N} limit of a certain matrix model. This limit is defined by sending $\theta, \mathcal{N} \rightarrow \infty$ such that

$$\frac{4}{\theta}\mathcal{N} = \mu^2(1 + \mathcal{Y})\Lambda^2 = \text{const}. \quad (4.11)$$

Here $\mathcal{Y} > -1$ is a real number which we identify later in order to simplify our equation. The number $(1 + \mathcal{Y})$ can be seen as a finite wavefunction renormalisation. This large- \mathcal{N} limit (4.11) plays the rôle of a *thermodynamic* (infinite volume⁴) *limit* because it turns the discrete norms $|\underline{a}|, |\underline{b}|, |\underline{p}|$ of matrix indices into real numbers $a, b, p \in [0, \Lambda^2]$ defined by

$$\frac{4}{\theta}|\underline{a}| =: \mu^2(1 + \mathcal{Y})a, \quad \frac{4}{\theta}|\underline{b}| =: \mu^2(1 + \mathcal{Y})b, \quad \frac{4}{\theta}|\underline{p}| =: \mu^2(1 + \mathcal{Y})p, \quad \Gamma_{\underline{a}\underline{b}}^{\text{ren}} =: \mu^2\Gamma_{ab}. \quad (4.12)$$

In the very end we also have to remove the cut-off Λ^2 in a *continuum limit* $\Lambda \rightarrow \infty$ (thus relaxing $\Lambda = \text{const}$ in (4.11)). It is important that $\Gamma_{\underline{a}\underline{b}}^{\text{ren}}$ only depends on $|\underline{a}|$ and $|\underline{b}|$ so that it converges to a function $[0, \Lambda^2]^2 \ni (a, b) \mapsto \Gamma_{ab} \in \mathbb{R}$. In the limit (4.11), a sum over $|\underline{p}|$ converges to

$$\frac{4}{\theta} \sum_{|\underline{p}|=0}^{\mathcal{N}} f\left(\frac{4}{\theta}|\underline{p}|\right) \xrightarrow{(4.11)} \mu^2(1 + \mathcal{Y}) \int_0^{\Lambda^2} dp f(\mu^2(1 + \mathcal{Y})p).$$

Therefore, the equation (4.10) converges to the following *integral equation* where the mass μ^2 drops out (we work with densities):

$$\begin{aligned}
& (Z - 1)(1 + \mathcal{Y})(a + b) + \Gamma_{ab} \\
& = -\lambda(1 + \mathcal{Y})^2 \int_0^{\Lambda^2} p dp \left(\frac{Z^2}{(a + p)(1 + \mathcal{Y}) + 1 - \Gamma_{ap}} - \frac{Z^2}{p(1 + \mathcal{Y}) + 1 - \Gamma_{0p}} \right) \\
& - \lambda(1 + \mathcal{Y})^2 \int_0^{\Lambda^2} p dp \left(\frac{Z}{(b + p)(1 + \mathcal{Y}) + 1 - \Gamma_{pb}} - \frac{Z}{p(1 + \mathcal{Y}) + 1 - \Gamma_{p0}} \right. \\
& \left. - \frac{Z}{(b + p)(1 + \mathcal{Y}) + 1 - \Gamma_{pb}} \frac{\Gamma_{pb} - \Gamma_{ab}}{(1 + \mathcal{Y})(p - a)} + \frac{Z}{p(1 + \mathcal{Y}) + 1 - \Gamma_{p0}} \frac{\Gamma_{p0}}{p(1 + \mathcal{Y})} \right). \quad (4.13)
\end{aligned}$$

The powerful analytical tools available for integral equations are a huge advantage over the discrete equations. On the other hand, in regions of the space of continuous parameters

⁴See [GW11, GW12] where it is shown that the spectral action behind the model under consideration yields for $\Omega = 1$ an 8-dimensional finite volume proportional to θ^4 .

where Γ varies slowly we can expect the integral equation to be a good approximation for the discrete equation (4.10) also for finite θ .

It turns out that the terms with Z^2 -coefficient in (4.13) need a subtle discussion. It is highly convenient to eliminate them via the equation resulting from (4.13) at $b = 0$:

$$\begin{aligned} & (Z - 1)(1 + \mathcal{Y})a + \Gamma_{a0} \\ &= -\lambda(1 + \mathcal{Y})^2 \int_0^{\Lambda^2} p dp \left(\frac{Z^2}{(a + p)(1 + \mathcal{Y}) + 1 - \Gamma_{ap}} - \frac{Z^2}{p(1 + \mathcal{Y}) + 1 - \Gamma_{0p}} \right) \\ &+ \lambda(1 + \mathcal{Y}) \int_0^{\Lambda^2} dp \frac{Z}{p(1 + \mathcal{Y}) + 1 - \Gamma_{p0}} \frac{a\Gamma_{p0} - p\Gamma_{a0}}{(p - a)}. \end{aligned} \quad (4.14)$$

The difference between (4.13) and (4.14), expressed in terms of the dimensionless function

$$G_{ab} := \mu^2 G_{|ab|}^{(0)} = \frac{1}{(a + b)(1 + \mathcal{Y}) + 1 - \Gamma_{ab}}, \quad (4.15)$$

reads

$$Z(1 + \mathcal{Y})b - \frac{1}{G_{ab}} + \frac{1}{G_{a0}} = \lambda(1 + \mathcal{Y}) \int_0^{\Lambda^2} p dp Z \frac{\frac{G_{pb}}{G_{ab}} - \frac{G_{p0}}{G_{a0}}}{p - a}. \quad (4.16)$$

From (4.16) we obtain the desired equation for Z^{-1} by putting $a = 0$, dividing by $Z(1 + \mathcal{Y})b$ and going to the limit $b \rightarrow 0$, where $\lim_{b \rightarrow 0} \frac{1}{b} \left(\frac{1}{G_{0b}} - 1 \right) = 1 + \mathcal{Y}$ and $\lim_{b \rightarrow 0} G_{0b} = 1$ are used:

$$Z^{-1} = 1 - \lambda(1 + \mathcal{Y}) \int_0^{\Lambda^2} dp G_{p0} - \lambda \lim_{b \rightarrow 0} \int_0^{\Lambda^2} dp \frac{G_{pb} - G_{p0}}{b}.$$

It will be convenient to choose \mathcal{Y} as

$$\mathcal{Y} := -\lambda \lim_{b \rightarrow 0} \int_0^{\Lambda^2} dp \frac{G_{pb} - G_{p0}}{b} \quad \Rightarrow \quad Z^{-1} = (1 + \mathcal{Y}) \left(1 - \lambda \int_0^{\Lambda^2} dp G_{p0} \right). \quad (4.17)$$

Of course it remains to verify that $-1 < \mathcal{Y} < \infty$. Multiplication of (4.16) by $\frac{G_{ab}}{b(1 + \mathcal{Y})Z}$ and elimination of Z^{-1} via (4.17) gives

$$(G_{ab} - G_{a0}) \left(1 + \frac{1}{bG_{a0}} \right) + G_{a0} = \frac{\lambda}{b} \int_0^{\Lambda^2} dp \left(\frac{p(G_{pb} - G_{p0}) - a(G_{ab} - G_{a0}) \frac{G_{p0}}{G_{a0}}}{p - a} \right). \quad (4.18)$$

4.4 The Carleman equation

We make now the crucial assumption that G_{ab} is *Hölder-continuous*, i.e.

$$|G_{pb} - G_{ab}| \leq C_b |p - a|^{\eta_b} \quad \forall 0 \leq a \neq p \leq \Lambda^2, \quad (4.19)$$

for some constants $C_b, \eta_b > 0$. Under this assumption we may replace the singular integrals by their Cauchy principal values:

$$\int_0^{\Lambda^2} dp \frac{G_{pb} - G_{ab}}{p - a} = \lim_{\epsilon \rightarrow 0} \left(\int_0^{a-\epsilon} + \int_{a+\epsilon}^{\Lambda^2} \right) dp \frac{G_{pb} - G_{ab}}{p - a}. \quad (4.20)$$

Introducing the *finite Hilbert transform*

$$\mathcal{H}_a^\Lambda[f(\bullet)] := \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \left(\int_0^{a-\epsilon} + \int_{a+\epsilon}^{\Lambda^2} \right) \frac{f(p)}{p-a}, \quad (4.21)$$

we can rewrite the equation (4.18) as

$$\left(\frac{b}{a} + \frac{1 + \lambda\pi a \mathcal{H}_a^\Lambda[G_{\bullet 0}]}{aG_{a0}} \right) D_{ab} - \lambda\pi \mathcal{H}_a^\Lambda[D_{\bullet b}] = -G_{a0}, \quad (4.22a)$$

$$D_{ab} := a \frac{G_{ab} - G_{a0}}{b}. \quad (4.22b)$$

The equation (4.22a) is now a standard *singular integral equation of Carleman type* [Car22, Tri57]. We cite from Tricomi's book [Tri57]:

Proposition 4.1 ([Tri57, §4.4]) *Let $h \in \mathcal{C}(]-1, 1[)$ and $f \in L^p(]-1, 1[)$ for some $p > 1$. Then the singular integral equation*

$$h(x)\varphi(x) - \lambda\pi \mathcal{H}_a[\varphi(\bullet)] = f(x), \quad x \in]-1, 1[, \quad (4.23)$$

has the solution

$$\varphi(x) = \frac{\sin(\vartheta(x))}{\lambda\pi} \left(f(x) \cos(\vartheta(x)) + e^{\mathcal{H}_x[\vartheta]} \mathcal{H}_x[e^{-\mathcal{H}\cdot[\vartheta]} f(\bullet) \sin(\vartheta(\bullet))] + \frac{Ce^{\mathcal{H}_x[\vartheta]}}{1-x} \right), \quad (4.24)$$

$$\vartheta(x) = \arctan_{[0, \pi]} \left(\frac{\lambda\pi}{h(x)} \right), \quad \sin(\vartheta(x)) = \frac{|\lambda\pi|}{\sqrt{(h(x))^2 + (\lambda\pi)^2}}, \quad \cos(\vartheta(x)) = \frac{\sin(\vartheta(x))}{\tan(\vartheta(x))},$$

where the Hilbert transform integrates over $]-1, 1[$, and C is an arbitrary constant.

The angle $\vartheta(x)$ obeys the identities [Tri57, §4.4(28)] and [Tri57, §4.4(18)],

$$e^{-\mathcal{H}_x[\vartheta]} \cos(\vartheta(x)) + \mathcal{H}_x[e^{-\mathcal{H}\cdot[\vartheta]} \sin(\vartheta(\bullet))] = 1, \quad (4.25a)$$

$$e^{\mathcal{H}_x[\vartheta]} \cos(\vartheta(x)) - \mathcal{H}_x[e^{\mathcal{H}\cdot[\vartheta]} \sin(\vartheta(\bullet))] = 1. \quad (4.25b)$$

The solution (4.24) and the identities (4.25a) and (4.25b) are unchanged if transformed via $p = \frac{\Lambda^2}{2}(1+x)$ to $p \in [0, \Lambda^2]$ instead of $x \in [-1, 1]$. We make the following decisive

Assumption 4.2 $C = 0$.

This assumption (or rather choice) will be discussed at the end of Sec. 4.5 and in Sec. 5. Under this assumption the solution of (4.22a) is

$$D_{ab} = -\frac{\sin(\vartheta_b(a))}{\lambda\pi} \left(G_{a0} \cos(\vartheta_b(a)) + e^{\mathcal{H}_a^\Lambda[\vartheta_b]} \mathcal{H}_a^\Lambda[e^{-\mathcal{H}\cdot[\vartheta_b]} G_{\bullet 0} \sin(\vartheta_b(\bullet))] \right), \quad (4.26a)$$

$$\vartheta_b(a) = \arctan_{[0, \pi]} \left(\frac{\lambda\pi a G_{a0}}{1 + bG_{a0} + \lambda\pi a \mathcal{H}_a^\Lambda[G_{\bullet 0}]} \right). \quad (4.26b)$$

The form (4.26a) of the solution of (4.22a) is not very useful. We can simplify it enormously noting that (4.26b) is, for $b = 0$, also a Carleman-type singular integral equation

$$\lambda\pi \cot \vartheta_0(a) G_{a0} - \lambda\pi \mathcal{H}_a^\Lambda[G_{a0}] = \frac{1}{a}. \quad (4.27)$$

This equation has the following solution:

Lemma 4.3 $G_{a0} = e^{\mathcal{H}_a^\Lambda[\vartheta_0] - \mathcal{H}_0^\Lambda[\vartheta_0]} \frac{\sin(\vartheta_0(a))}{|\lambda|\pi a}$.

Proof. By (4.24) the solution of (4.27) is for $C = 0$

$$G_{a0} = \frac{\sin(\vartheta_0(a))}{\lambda\pi} \left(\frac{\cos(\vartheta_0(a))}{a} + e^{\mathcal{H}_a^\Lambda[\vartheta_0]} \mathcal{H}_a^\Lambda \left[\frac{e^{-\mathcal{H}_\bullet^\Lambda[\vartheta_0]} \sin(\vartheta_0(\bullet))}{\bullet} \right] \right).$$

Rational fraction expansion $\mathcal{H}_a^\Lambda \left[\frac{f(\bullet)}{\bullet} \right] = \frac{1}{a} \left(\mathcal{H}_a^\Lambda[f(\bullet)] - \mathcal{H}_0^\Lambda[f(\bullet)] \right)$ and the identity (4.25a) yield the assertion, where $\vartheta_0(0) = 0$ for $\lambda > 0$ and $\vartheta_0(0) = \pi$ for $\lambda < 0$, hence $\cos \vartheta_0(0) = \text{sign}(\lambda)$, are used. \square

For the Hilbert transform occurring in (4.26a) and for the investigation of the four-point function later on we need the following addition theorem:

Lemma 4.4 For all $0 \leq a, b, d \leq \Lambda^2$ one has

$$\lambda\pi a \sin(\vartheta_d(a) - \vartheta_b(a)) = (b - d) \sin \vartheta_b(a) \sin \vartheta_d(a). \quad (4.28)$$

Proof. This follows from insertion of (4.26b) into $\cot \vartheta_b(a) - \cot \vartheta_d(a)$. \square

This Lemma has several important consequences:

Corollary 4.5 1. For $\lambda > 0$, the function $b \mapsto \vartheta_b(a)$ is monotonously decreasing for any fixed $a > 0$. In particular, $\vartheta_b(a) < \vartheta_0(a)$ for any $0 < a, b \leq \Lambda^2$.

2. For $\lambda < 0$, the function $b \mapsto \vartheta_b(a)$ is monotonously increasing for any fixed $a > 0$. In particular, $\vartheta_b(a) > \vartheta_0(a)$ for any $0 < a, b \leq \Lambda^2$.

Proof. By continuity we can choose $|b - d|$ small enough so that $\tau_{abd} := \vartheta_d(a) - \vartheta_b(a) \in] -\frac{\pi}{2}, \frac{\pi}{2}[$. Then τ_{abd} and $\sin \tau_{abd}$ have the same sign. Let $b > d$. Then from (4.28) we conclude $\vartheta_d > \vartheta_b$ for $\lambda > 0$ and $\vartheta_d < \vartheta_b$ for $\lambda < 0$. \square

The next step is the analogue of Lemma 4.3 for the Hilbert transform occurring in (4.26a):

Lemma 4.6

$$\mathcal{H}_a^\Lambda [e^{-\mathcal{H}_\bullet^\Lambda[\vartheta_b]} G_{\bullet 0} \sin(\vartheta_b(\bullet))] = \frac{\text{sign}(\lambda)}{b} e^{-\mathcal{H}_0^\Lambda[\vartheta_0]} \left(e^{\mathcal{H}_a^\Lambda[\vartheta_0 - \vartheta_b]} \cos(\vartheta_0(a) - \vartheta_b(a)) - 1 \right).$$

Proof. We insert $G_{\bullet 0}$ from Lemma 4.3 into $\mathcal{H}_a^\Lambda [e^{-\mathcal{H}_\bullet^\Lambda[\vartheta_b]} G_{\bullet 0} \sin(\vartheta_b(\bullet))]$ and obtain with (4.28):

$$\mathcal{H}_a^\Lambda [e^{-\mathcal{H}_\bullet^\Lambda[\vartheta_b]} G_{\bullet 0} \sin(\vartheta_b(\bullet))] = \frac{\text{sign}(\lambda)}{b} e^{-\mathcal{H}_0^\Lambda[\vartheta_0]} \mathcal{H}_a^\Lambda \left[e^{\mathcal{H}_\bullet^\Lambda[\vartheta_0 - \vartheta_b]} \sin(\vartheta_0(\bullet) - \vartheta_b(\bullet)) \right].$$

Now the identity (4.25b) yields the assertion. \square

This allows us to prove the following remarkable formula for G_{ab} :

Theorem 4.7 *The renormalised planar connected two-point function G_{ab} of self-dual noncommutative ϕ_4^4 -theory (with continuous indices arising as limit $\theta \rightarrow \infty$) satisfies (under the assumption 4.2) the equation*

$$G_{ab} = \frac{\sin(\vartheta_b(a))}{|\lambda|\pi a} e^{\mathcal{H}_a^\Lambda[\vartheta_b] - \mathcal{H}_0^\Lambda[\vartheta_0]} . \quad (4.29)$$

In particular, $G_{ab} \geq 0$ for all $a, b \in [0, \Lambda^2]$.

Proof. Insertion of (4.22b) into (4.26a) gives

$$G_{ab} = G_{a0} - b \frac{\sin(\vartheta_b(a))}{\lambda\pi a} \left(G_{a0} \cos(\vartheta_b(a)) + e^{\mathcal{H}_a^\Lambda[\vartheta_b]} \mathcal{H}_a^\Lambda \left[e^{-\mathcal{H}_a^\Lambda[\vartheta_b]} G_{\bullet 0} \sin(\vartheta_b(\bullet)) \right] \right) .$$

We insert Lemma 4.3 and Lemma 4.6:

$$\begin{aligned} G_{ab} = & e^{-\mathcal{H}_0^\Lambda[\vartheta_0]} \frac{\sin(\vartheta_0(a))}{|\lambda|\pi a} \left\{ e^{\mathcal{H}_a^\Lambda[\vartheta_0]} - b \frac{\sin(\vartheta_b(a))}{\lambda\pi a} \left\{ e^{\mathcal{H}_a^\Lambda[\vartheta_0]} \cos(\vartheta_b(a)) \right. \right. \\ & \left. \left. + e^{\mathcal{H}_a^\Lambda[\vartheta_b]} \frac{\lambda\pi a}{b \sin(\vartheta_0(a))} \left(e^{\mathcal{H}_a^\Lambda[\vartheta_0 - \vartheta_b]} \cos(\vartheta_0(a) - \vartheta_b(a)) - 1 \right) \right\} \right\} . \end{aligned}$$

We express $\cos(\vartheta_0(a) - \vartheta_b(a))$ by its addition theorem and combine the sin-sin part with $e^{\mathcal{H}_a^\Lambda[\vartheta_0]}$ after the first “{”:

$$\begin{aligned} G_{ab} = & e^{-\mathcal{H}_0^\Lambda[\vartheta_0]} \frac{\sin(\vartheta_0(a))}{|\lambda|\pi a} \left\{ e^{\mathcal{H}_a^\Lambda[\vartheta_0]} \cos^2(\vartheta_b(a)) + e^{\mathcal{H}_a^\Lambda[\vartheta_b]} \frac{\sin(\vartheta_b(a))}{\sin(\vartheta_0(a))} \right. \\ & \left. - e^{\mathcal{H}_a^\Lambda[\vartheta_0]} \sin(\vartheta_b(a)) \cos(\vartheta_b(a)) \left(\frac{b}{\lambda\pi a} + \cot \vartheta_0(a) \right) \right\} . \end{aligned}$$

Now (4.26b) implies $\frac{b}{\lambda\pi a} + \cot \vartheta_0(a) = \cot \vartheta_b(a)$, and then the above equation collapses to (4.29). \square

Lemma 4.8 *One has*

$$\mathcal{Y} = \lambda \int_0^{\Lambda^2} dp \frac{(G_{p0})^2}{(\lambda\pi p G_{p0})^2 + (1 + \lambda\pi p \mathcal{H}_p^\Lambda[G_{\bullet 0}])^2} . \quad (4.30)$$

Proof. To compute $\mathcal{Y} = -\lambda \int_0^{\Lambda^2} \frac{dp}{p} D_{p0} = -\lambda\pi \mathcal{H}_0^\Lambda[D_{\bullet 0}]$ in (4.17) we use the *convolution theorem* for the finite Hilbert transform [Tri57, §4.3(4)]

$$\mathcal{H}_x \left[\phi_1(\bullet) \mathcal{H}_\bullet[\phi_2] + \phi_2(\bullet) \mathcal{H}_\bullet[\phi_1] \right] = \mathcal{H}_x[\phi_1(\bullet)] \mathcal{H}_x[\phi_2(\bullet)] - \phi_2(x) \phi_1(x) . \quad (4.31)$$

With D_{p0} given by (4.26a), we have to identify $x = 0$, $\phi_1(a) = e^{\mathcal{H}_a^\Lambda[\vartheta_0]} \sin \vartheta_0(a)$ and $\phi_2(a) = e^{-\mathcal{H}_a^\Lambda[\vartheta_0]} G_{a0} \sin(\vartheta_0(a))$. With $\sin \vartheta_0(0) = 0$ we obtain

$$\begin{aligned} \mathcal{Y} = & \mathcal{H}_0^\Lambda \left[\sin(\vartheta_0(\bullet)) G_{\bullet 0} \cos(\vartheta_0(\bullet)) \right] - \mathcal{H}_0^\Lambda \left[\mathcal{H}_\bullet^\Lambda \left[\sin(\vartheta_0) e^{\mathcal{H}_\bullet^\Lambda[\vartheta_0]} \right] e^{-\mathcal{H}_\bullet^\Lambda[\vartheta_0]} G_{\bullet 0} \sin(\vartheta_0(\bullet)) \right] \\ & + \mathcal{H}_0^\Lambda \left[\sin(\vartheta_0(\bullet)) e^{\mathcal{H}_\bullet^\Lambda[\vartheta_0]} \right] \mathcal{H}_0^\Lambda \left[e^{-\mathcal{H}_\bullet^\Lambda[\vartheta_0]} G_{\bullet 0} \sin(\vartheta_0(\bullet)) \right] . \end{aligned}$$

Using (4.25b) this equation collapses with $\cos \vartheta_b(0) = \text{sign}(\lambda)$ to

$$\begin{aligned} \mathcal{Y} &= \text{sign}(\lambda) \mathcal{H}_0^\Lambda \left[e^{-\mathcal{H}_0^\Lambda[\vartheta_0] + \mathcal{H}_0[\vartheta_0]} G_{\bullet 0} \sin(\vartheta_0(\bullet)) \right] \\ &= \frac{1}{\lambda \pi^2} \int_0^{\Lambda^2} dp \frac{\sin^2(\vartheta_0(p))}{p^2} = \lambda \int_0^{\Lambda^2} \frac{dp}{(\lambda \pi p)^2 (1 + \cot^2(\vartheta_0(p)))}, \end{aligned}$$

where the second line follows from Lemma 4.3. Now the assertion follows with (4.26b). \square

It follows that $\mathcal{Y} > 0$ for $\lambda > 0$ and $-1 < \mathcal{Y} < 0$ for $\lambda < 0$ with $|\lambda|$ small enough. Now observe that (4.17), Lemma 4.3 and (4.25b) give

$$\begin{aligned} Z^{-1} &= (1 + \mathcal{Y}) \left(1 - \text{sign}(\lambda) \frac{e^{-\mathcal{H}_0^\Lambda[\vartheta_0]}}{\pi} \int_0^{\Lambda^2} \frac{dp}{p} \sin \vartheta_0(p) e^{\mathcal{H}_p^\Lambda[\vartheta_0]} \right) \\ &= (1 + \mathcal{Y}) \left(1 - \text{sign}(\lambda) e^{-\mathcal{H}_0^\Lambda[\vartheta_0]} \mathcal{H}_0^\Lambda[\sin \vartheta_0(\bullet) e^{\mathcal{H}_\bullet^\Lambda[\vartheta_0]}] \right) = (1 + \mathcal{Y}) \text{sign}(\lambda) e^{-\mathcal{H}_0^\Lambda[\vartheta_0]}. \end{aligned} \quad (4.32)$$

It therefore follows that $\text{sign}(Z^{-1}) = \text{sign}(\lambda)$ so that the case $\lambda < 0$ leads to an unstable action functional. *We therefore discard the case $\lambda < 0$ in the sequel.*

4.5 The master equation

The identity (4.29) allows us to compute G_{ab} once G_{a0} is known, but (4.29) *alone is not sufficient* to determine G_{a0} . This is because we ignored so far the equation (4.14) which in terms of (4.15) and after insertion of (4.17) reads

$$a - \frac{1}{G_{a0}} + 1 = -\lambda(1 + \mathcal{Y})Z \int_0^{\Lambda^2} p dp (G_{ap} - G_{0p}) - \lambda \int_0^{\Lambda^2} dp \frac{a - a \frac{G_{p0}}{G_{a0}}}{(p - a)}. \quad (4.33)$$

We introduce the Hilbert transform and insert (4.29) and (4.32):

$$1 + a - \lambda \pi a \cot \vartheta_0(a) = - \int_0^{\Lambda^2} p dp \left(\frac{\sin \vartheta_p(a)}{\pi a} e^{\mathcal{H}_a^\Lambda[\vartheta_p]} - \lim_{a \rightarrow 0} \frac{\sin \vartheta_p(a)}{\pi a} e^{\mathcal{H}_0^\Lambda[\vartheta_p]} \right) - \lambda \pi a \mathcal{H}_a^\Lambda[1]. \quad (4.34)$$

We write $\sin \vartheta_p(a) = (1 + \cot^2 \vartheta_p(a))^{-\frac{1}{2}} = (1 + (\frac{p}{\lambda \pi a} + \cot \vartheta_0(a))^2)^{-\frac{1}{2}}$ and use the following limit for $\lambda > 0$:

$$\lim_{a \rightarrow 0} \left(\frac{\sin(\vartheta_p(a))}{\pi a} \right) = \cos \vartheta_p(0) \lim_{a \rightarrow 0} \frac{\vartheta_p(a)}{a} = \frac{\lambda}{(1 + p)}. \quad (4.35)$$

This gives:

Proposition 4.9 *The function $\mathcal{T}_a := \lambda \pi a \cot \vartheta_0(a)$, with $\mathcal{T}_0 = 1$, is determined by*

$$\begin{aligned} \mathcal{T}_a &= 1 + a + \lambda \lim_{\epsilon \rightarrow 0} \left(\int_0^{a-\epsilon} + \int_{a+\epsilon}^{\Lambda^2} \right) dp \left(\frac{a}{p - a} + \frac{p \exp \left(\mathcal{H}_a^\Lambda \left[\arctan \frac{\lambda \pi \bullet}{p + \mathcal{T}_\bullet} \right] \right)}{\sqrt{(\lambda \pi a)^2 + (p + \mathcal{T}_a)^2}} \right. \\ &\quad \left. - \frac{p \exp \left(\mathcal{H}_0^\Lambda \left[\arctan \frac{\lambda \pi \bullet}{p + \mathcal{T}_\bullet} \right] \right)}{1 + p} \right). \end{aligned} \quad (4.36) \quad \square$$

This is a non-linear self-consistency equation whose solution yields the input data $\vartheta_0(a)$ and $\vartheta_b(a)$ to evaluate the two-point function G_{ab} by Theorem 4.7.

Unfortunately, (4.36) is too complicated for further analysis. We therefore derive an alternative equation from the symmetry $G_{b0} = \lim_{a \rightarrow 0} G_{ab}$. This requirement together with Theorem 4.7 gives:

Proposition 4.10 *The renormalised planar regular two-point function G_{b0} is determined by the self-consistency equation*

$$G_{b0} = \frac{1}{1+b} \exp \left(-\lambda \int_0^b dt \int_0^{\Lambda^2} dp \frac{(G_{p0})^2}{(\lambda \pi p G_{p0})^2 + (1 + t G_{p0} + \lambda \pi p \mathcal{H}_p^\Lambda[G_{\bullet 0}])^2} \right), \quad (4.37)$$

which is well-defined in the class of continuously differentiable functions on $[0, \Lambda^2]$.

Proof. The limit $a \rightarrow 0$ in (4.29) yields with (4.35)

$$G_{b0} \equiv \lim_{a \rightarrow 0} G_{ab} = \frac{1}{1+b} e^{-\mathcal{H}_0^\Lambda[\vartheta_0 - \vartheta_b]} = \frac{1}{1+b} \exp \left(-\frac{1}{\pi} \int_0^{\Lambda^2} \frac{dp}{p} (\vartheta_0(p) - \vartheta_b(p)) \right). \quad (4.38)$$

Together with Corollary 4.5 this formula already establishes that for $\lambda > 0$ the function $b \mapsto G_{b0}$ is monotonously decreasing and maps $[0, \Lambda^2]$ into $[0, 1]$. The explicit formula (4.37) follows from

$$-\frac{1}{\pi} \int_0^{\Lambda^2} \frac{dp}{p} (\vartheta_0(p) - \vartheta_b(p)) = \frac{1}{\pi} \int_0^{\Lambda^2} \frac{dp}{p} \int_0^b dt \frac{d\vartheta_t(p)}{dt}$$

and (4.26b) after exchange of the p, t -integrals.

The equation (4.37) is meaningful on the class of continuously differentiable functions. Namely, if $G_{\bullet 0} \in \mathcal{C}^1[0, \Lambda^2]$, then

$$\begin{aligned} \lambda \pi p \mathcal{H}_p^\Lambda[G_{\bullet 0}] &= \lambda p \lim_{\epsilon \rightarrow 0} \left(\int_0^{p-\epsilon} + \int_{p+\epsilon}^{\Lambda^2} \right) dq G_{q0} \frac{d}{dq} \log |q - p| \\ &= \lambda p \left(G_{\Lambda^2 0} \log(\Lambda^2 - p) - \log p - \int_0^{\Lambda^2} dq G'_{q0} \log |q - p| \right). \end{aligned}$$

This shows that $p \mapsto \lambda \pi p \mathcal{H}_p^\Lambda[G_{\bullet 0}]$ is a continuous function on $[0, \Lambda^2]$. Since $\lim_{p \rightarrow \Lambda^2} \lambda \pi p \mathcal{H}_p^\Lambda[G_{\bullet 0}] = -\infty$ and $\lambda \pi p \mathcal{H}_p^\Lambda[G_{\bullet 0}]$ appears in the denominator together with bounded functions, the integrand in (4.37) extends to a continuous function on $[0, b] \times [0, \Lambda^2] \ni (t, p)$. By the fundamental theorem of calculus we have

$$\frac{dG_{b0}}{db} = -G_{b0} \left(\frac{1}{1+b} + \lambda \int_0^{\Lambda^2} dp \frac{(G_{p0})^2}{(\lambda \pi p G_{p0})^2 + (1 + b G_{p0} + \lambda \pi p \mathcal{H}_p^\Lambda[G_{\bullet 0}])^2} \right). \quad (4.39)$$

Since the rhs is a smooth function of $b \in [0, \Lambda^2]$, continuous derivatives of G_{b0} exists inductively to any order. \square

Observe from (4.39) and (4.30) that $\frac{dG_{b0}}{db} \Big|_{b=0} = -(1 + \mathcal{Y})$.

The next task is to prove that the master equation (4.37) has a solution for any $\lambda > 0$ and Λ^2 sufficiently large, including the continuum limit $\Lambda \rightarrow \infty$. This will be achieved by the Schauder fixed point theorem:

Theorem 4.11 (Schauder) *Let X be a Banach space, $K \subset X$ a convex subset and T a continuous mapping of K into itself. If $T(K)$ is contained in a compact subset of K , then T has a fixed point.*

The main road to establish compactness for X being the space of continuous functions on a compact Hausdorff space is the Arzelà-Ascoli theorem. There exist generalisations to differentiable functions and to locally compact Hausdorff spaces. The setup relevant for our situation is contained in [CCMX11] that we specify to our case. Let

$$\mathcal{C}_0(\mathbb{R}_+) := \{f : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ continuous and bounded, } \lim_{t \rightarrow \infty} f(t) = 0\}$$

$$\mathcal{C}_0^1(\mathbb{R}_+) := \{f : \mathbb{R}_+ \rightarrow \mathbb{R} : f, f' \text{ continuous and bounded, } \lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} f'(t) = 0\}$$

be the spaces of continuous and continuously differentiable functions on \mathbb{R}_+ , respectively, that vanish (together with their derivative) at ∞ . These are Banach spaces with norms $\|f\|_\infty = \sup_{t \in \mathbb{R}_+} |f(t)|$ and $\|f\|'_\infty = \sup_{t \in \mathbb{R}_+} |f(t)| + \sup_{t \in \mathbb{R}_+} |f'(t)|$, respectively. One has

Proposition 4.12 ([Avr69, Lemme 1]) *A subset $K \subset \mathcal{C}_0(\mathbb{R}_+)$ is relatively compact if and only if it is uniformly bounded, equicontinuous on \mathbb{R}_+ and uniformly convergent at ∞ .*

We recall that a subset $K \subset X$ is relatively compact in the topology induced from X if its closure \bar{K} is compact. A subset K of continuous functions on a metric space (M, d) is equicontinuous if for every $\epsilon > 0$ there is a uniform $\delta > 0$ such that $|f(t) - f(s)| < \epsilon$ for all $f \in K$ and all $s, t \in M$ with $d(s, t) < \delta$. For subsets of $\mathcal{C}_0^1(\mathbb{R}_+)$ one has:

Proposition 4.13 ([CCMX11, Thm. 3.1]) *A subset $K \subset \mathcal{C}_0^1(\mathbb{R}_+)$ is relatively compact if and only if*

1. $K' := \{f' : f \in K\}$ is relatively compact in $\mathcal{C}_0(\mathbb{R}_+)$,
2. For every $\epsilon > 0$ there is a $L > 0$ with $\|f\|_{[L, \infty[} < \epsilon$ for all $f \in K$.

The following operator T_λ is well-defined on the subset of positive functions in $\mathcal{C}_0^1(\mathbb{R}_+) \cap L^q(\mathbb{R}_+)$, for any $1 \leq q < \infty$:

$$(T_\lambda f)(b) := \frac{1}{1+b} \exp \left(-\lambda \int_0^b dt \int_0^\infty \frac{dp}{(\lambda \pi p)^2 + \left(t + \frac{1 + \lambda \pi p \mathcal{H}_p^\infty[f(\bullet)]}{f(p)}\right)^2} \right). \quad (4.40)$$

The Hilbert transform maps differentiable functions to continuous functions on $]0, \infty[$, with logarithmic divergence at $p = 0$. To see this we write for $p > 0$

$$\mathcal{H}_p^\infty[f(\bullet)] = \frac{1}{\pi} \int_0^{2p} dq \frac{f(q) - f(p)}{q - p} + \frac{1}{\pi} \int_{2p}^\infty dq \frac{f(q)}{q - p}, \quad (4.41)$$

using the fact that $\mathcal{H}_p^{\sqrt{2p}}[1] = 0$. The second integral is bounded by Hölder's inequality, and its derivative with respect to p exists. In the first integral we have

$$\begin{aligned} & \int_0^{2(p+\delta)} dq \frac{f(q) - f(p+\delta)}{q - (p+\delta)} - \int_0^{2p} dq \frac{f(q) - f(p)}{q - p} \\ &= \int_{2p}^{2(p+\delta)} dq \frac{f(q) - f(p+\delta)}{q - (p+\delta)} + \int_0^{2p} dq \left(\frac{\delta}{q - (p+\delta)} \frac{f(q) - f(p)}{q - p} + \frac{f(p+\delta) - f(p)}{q - (p+\delta)} \right). \end{aligned}$$

Using the mean value theorem $f(p) - f(q) = (p - q)f'(\xi)$ for some ξ between p and q , this expression tends to zero for $\delta \rightarrow 0$. Since $\lim_{p \rightarrow 0} p\mathcal{H}_p^\infty[f(\bullet)] = 0$, the integrand in (4.40) is a rational function of positive continuous functions on \mathbb{R}_+ with denominator separated from zero. Therefore the p -integral exists and is convergent for large p . The fundamental theorem of calculus guarantees that $(T_\lambda f)$ is differentiable.

The following even stronger result holds:

Lemma 4.14 *Let $\lambda > 0$ and assume that $f \in C_0^1(\mathbb{R}_+)$ satisfies*

$$(A1) \quad f(0) = 1,$$

$$(A2) \quad 0 < f(b) \leq \frac{1}{1+b},$$

$$(A3) \quad 0 \leq -\left(\frac{1}{b+1} + \frac{f'(b)}{f(b)}\right) \leq C_\lambda,$$

for some $C_\lambda \geq 0$. Let $P_\lambda > 0$ be the unique solution of $2\lambda P_\lambda^2(1 + C_\lambda)e^{C_\lambda P_\lambda} = 1$. Then

$$(C1) \quad (T_\lambda f)(0) = 1,$$

$$(C2) \quad 0 < (T_\lambda f)(b) \leq \frac{1}{1+b},$$

$$(C3) \quad 0 \leq -\left(\frac{1}{b+1} + \frac{(T_\lambda f)'(b)}{(T_\lambda f)(b)}\right) \leq \frac{1}{2} + \frac{1}{\lambda\pi^2 P_\lambda},$$

$$(C4) \quad \left|\frac{(T_\lambda f)''(b)}{(T_\lambda f)(b)}\right| \leq \frac{23}{4} + \frac{2}{\pi} + \frac{7+8\pi}{2} \frac{1}{(\lambda\pi^2 P_\lambda)^2}.$$

Proof. (C1) and (C2) are obvious from (4.40). Integrating (A3) with initial condition (A1) one has $\frac{e^{-C_\lambda b}}{1+b} \leq f(b) \leq \frac{1}{1+b}$ which is even stronger than (A2). In analogy to (4.39) we have

$$-\left(\frac{(T_\lambda f)'(b)}{(T_\lambda f)(b)} + \frac{1}{1+b}\right) = \lambda \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(b + \frac{1+\lambda\pi p\mathcal{H}_p^\infty[f(\bullet)]}{f(p)}\right)^2}. \quad (4.42)$$

The rhs of (4.42) is positive, and it remains to show the bound (C3). Since $f > 0$ and $f' \leq 0$ we have the following inequalities for the integrals in (4.41):

$$\begin{aligned} 0 &\leq \int_{2p}^\infty dq \frac{f(q)}{q-p} \leq \frac{1}{p+1} \log \frac{1+2p}{p}, \\ 0 &\geq \int_0^{2p} dq \frac{f(q) - f(p)}{q-p} \geq 2p \inf_{q \in [0, 2p]} f'(q) \geq -2p(C_\lambda + 1). \end{aligned}$$

Collecting these results and using $\frac{e^{-C_\lambda b}}{1+b} \leq f(b) \leq \frac{1}{1+b}$ we conclude

$$\frac{1 + \lambda\pi p\mathcal{H}_p^\infty[f(\bullet)]}{f(p)} \geq (1+p) \left(1 - 2\lambda p^2(1 + C_\lambda)e^{C_\lambda p}\right). \quad (4.43a)$$

For fixed $C_\lambda \geq 0$ the continuous function $\mathbb{R}_+ \ni p \mapsto 2\lambda p^2(1 + C_\lambda)e^{C_\lambda p}$ is monotonously increasing and maps \mathbb{R}_+ to itself. Let P_λ be the unique solution of $2\lambda P_\lambda^2(1 + C_\lambda)e^{C_\lambda P_\lambda} = 1$. By the implicit function theorem, this gives rise to a globally defined differentiable function

$\mathbb{R}_+ \ni C_\lambda \mapsto P_\lambda$ which is monotonously decreasing. These monotonicity properties lead to the inequality

$$0 < p < P_\lambda \quad \Rightarrow \quad \frac{1 + \lambda\pi p \mathcal{H}_p^\infty[f(\bullet)]}{f(p)} \geq 1 - \frac{p^2}{P_\lambda^2} \geq 0. \quad (4.43b)$$

We now split the integral (4.42) at P_λ where for $0 \leq p \leq P_\lambda$ we use (4.43b) whereas for $p \geq P_\lambda$ we simply use $(b + \frac{1+\lambda\pi p \mathcal{H}_p^\infty[f(\bullet)]}{f(p)})^2 \geq 0$:

$$\begin{aligned} 0 \leq -\left(\frac{(T_\lambda f)'(b)}{(Tf)(b)} + \frac{1}{1+b}\right) &\leq \lambda \int_0^{P_\lambda} \frac{dp}{(\lambda\pi p)^2 + (1 - \frac{p^2}{P_\lambda^2})^2} + \lambda \int_0^{P_\lambda} \frac{dp}{(\lambda\pi p)^2} \\ &= \lambda P_\lambda \int_0^1 \frac{dx}{(\lambda\pi P_\lambda)^2 x^2 + (1-x^2)^2} + \frac{1}{\lambda\pi^2 P_\lambda} \\ &\leq \lambda P_\lambda \int_0^1 \frac{dx (1+x^2)}{(\lambda\pi P_\lambda)^2 x^2 + (1-x^2)^2} + \frac{1}{\lambda\pi^2 P_\lambda} = \frac{1}{2} + \frac{1}{\lambda\pi^2 P_\lambda}. \end{aligned} \quad (4.44)$$

We have used the integral $\int_0^1 dx \frac{1+x^2}{a^2 x^2 + (1-x^2)^2} = \frac{1}{2} \int_0^\infty dt \frac{\cosh \frac{t}{2}}{(\frac{a^2}{2} - 1) + \cosh t} = \frac{\pi}{2a}$ which can be looked up in [GR94, §3.517.1, §8.702]. This finishes the proof of (C3).

Another derivative of (4.42) gives

$$\begin{aligned} (T_\lambda f)''(b) = (Tf)(b) &\left\{ \left(\frac{1}{1+b} + \lambda \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + (b + \frac{1+\lambda\pi p \mathcal{H}_p^\infty[f(\bullet)]}{f(p)})^2} \right)^2 \right. \\ &\left. + \frac{1}{1+b} + \lambda \int_0^\infty dp \frac{2(b + \frac{1+\lambda\pi p \mathcal{H}_p^\infty[f(\bullet)]}{f(p)})}{\left((\lambda\pi p)^2 + (b + \frac{1+\lambda\pi p \mathcal{H}_p^\infty[f(\bullet)]}{f(p)})^2 \right)^2} \right\}. \end{aligned} \quad (4.45)$$

We thus get with (C3) the estimate

$$\begin{aligned} \frac{|(T_\lambda f)''(b)|}{(T_\lambda f)(b)} &\leq \left(\frac{3}{2} + \frac{1}{\lambda\pi^2 P_\lambda}\right)^2 + 1 + 2\lambda \int_0^\infty \frac{dp}{\left((\lambda\pi p)^2 + (b + \frac{1+\lambda\pi p \mathcal{H}_p^\infty[f(\bullet)]}{f(p)})^2 \right)^{\frac{3}{2}}} \\ &\leq \left(\frac{3}{2} + \frac{1}{\lambda\pi^2 P_\lambda}\right)^2 + 1 + 4\pi \left(\frac{1}{\lambda\pi^2 P_\lambda}\right)^2 + 2\lambda P_\lambda \int_0^1 \frac{dx}{\left((\lambda\pi P_\lambda)^2 x^2 + (1-x^2)^2 \right)^{\frac{3}{2}}}. \end{aligned}$$

We have with [GR94, §3.517.1, §8.702, §9.131.1, §9.122.1]

$$\begin{aligned} \int_0^1 \frac{(1+x^4) dx}{(a^2 x^2 + (1-x^2)^2)^{\frac{3}{2}}} &= \frac{1}{\sqrt{8}} \int_0^\infty dt \frac{\cosh t}{\left((\frac{a^2}{2} - 1) + \cosh t \right)^{\frac{3}{2}}} \\ &= \frac{3\pi}{4a^2} {}_2F_1\left(-\frac{1}{2}, \frac{3}{2} \middle| 1 - \frac{a^2}{4}\right) = \frac{3\pi}{8a} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2} \middle| 1 - \frac{4}{a^2}\right) \leq \frac{1+a}{a^2}. \end{aligned}$$

Together with $2\alpha \leq 1 + \alpha^2$ we confirm (C4). \square

It is of fundamental importance that we can always choose C_λ such that $\frac{1}{2} + \frac{1}{\lambda\pi^2 P_\lambda} < C_\lambda$. Namely,

$$\begin{aligned} \frac{1}{2} + \frac{1}{\lambda\pi^2 P_\lambda} < C_\lambda &\Leftrightarrow 2\lambda P_\lambda^2 \left(\frac{3}{2} + \frac{1}{\lambda\pi^2 P_\lambda} \right) e^{(\frac{1}{2} + \frac{1}{\lambda\pi^2 P_\lambda})P_\lambda} < 2\lambda P_\lambda^2 (1 + C_\lambda) e^{C_\lambda P_\lambda} := 1 \\ &\Leftrightarrow P_\lambda e^{\frac{1}{\lambda\pi^2}} \cdot \left(3\lambda P_\lambda + \frac{2}{\pi^2} \right) e^{\frac{P_\lambda}{2}} < 1. \end{aligned} \quad (4.46a)$$

The condition is satisfied for

$$P_\lambda = \frac{e^{-\frac{1}{\lambda\pi^2}}}{\sqrt{1+4\lambda}} \Rightarrow P_\lambda e^{\frac{1}{\lambda\pi^2}} \cdot \left(3\lambda P_\lambda + \frac{2}{\pi^2} \right) e^{\frac{P_\lambda}{2}} < 0.8303 < 1. \quad (4.46b)$$

Note that the inverse solution for C_λ is huge for $0 < \lambda \ll 1$.

We thus consider the following subset of $\mathcal{C}_0^1(\mathbb{R}_+)$:

$$\mathcal{K}_\lambda := \left\{ f \in \mathcal{C}_0^1(\mathbb{R}_+) : f(0) = 1, \quad 0 < f(b) \leq \frac{1}{1+b}, \quad 0 \leq -f'(b) \leq \left(\frac{1}{1+b} + C_\lambda \right) f(b) \right\} \quad (4.47)$$

with C_λ the solution of $2\lambda P_\lambda^2 (1 + C_\lambda) e^{C_\lambda P_\lambda} = 1$ at $P_\lambda = \frac{\exp(-\frac{1}{\lambda\pi^2})}{\sqrt{1+4\lambda}}$. We have established the following facts:

(K1) T_λ maps \mathcal{K}_λ into \mathcal{K}_λ by Lemma 4.14.(C1)+(C2)+(C3) and (4.46).

(K2) $(T_\lambda(\mathcal{K}_\lambda))' \subset \mathcal{C}_0(\mathbb{R}_+)$ is relatively compact. Namely, $(T_\lambda(\mathcal{K}_\lambda))' \subset \mathcal{C}_0(\mathbb{R}_+)$ is equicontinuous by Lemma 4.14.(C4), uniformly bounded as subset of \mathcal{K}_λ and uniformly convergent by Lemma 4.14.(C3)+(C2), thus relatively compact by Proposition 4.12.

(K3) $\|g\|_{[\frac{1}{\epsilon}, \infty[} < \epsilon$ for all $g \in T_\lambda(\mathcal{K}_\lambda)$ by Lemma 4.14.(C2).

(K4) For the closure we have $\overline{T_\lambda(\mathcal{K}_\lambda)} \subset \mathcal{K}_\lambda$: Functions $g = \lim_{k \rightarrow \infty} g_k$ with $g_k \in T_\lambda(\mathcal{K}_\lambda)$ on the closure can at most exceed \mathcal{K}_λ in $g(b) = 0$ for some b . But $0 \leq -g'(b) \leq (\frac{1}{1+b} + C_\lambda)g(b)$ and $g(0) = 1$ imply $\frac{e^{-C_\lambda b}}{1+b} \leq g(b) \leq \frac{1}{1+b}$ so that also g is strictly positive.

(K5) \mathcal{K}_λ is convex: We have $\mu f_1 + (1 - \mu)f_2 \in \mathcal{K}_\lambda$ for any $f_1, f_2 \in \mathcal{K}_\lambda$ and $0 \leq \mu \leq 1$ by definition of \mathcal{K}_λ .

(K6) $T_\lambda : \mathcal{K}_\lambda \rightarrow \mathcal{K}_\lambda$ is continuous, i.e. $\|T_\lambda f - T_\lambda \tilde{f}\|'_\infty < \epsilon$ for $\|f - \tilde{f}\|'_\infty < \delta$: By the uniform convergence Lemma 4.14.(C2)+(C3) with $\frac{1}{2} + \frac{1}{\lambda\pi^2 P_\lambda} \leq C_\lambda$ it suffices to restrict the sup-norms to the compact interval $[0, \frac{6+3C_\lambda}{\epsilon}]$. Since continuous functions on compact intervals are uniformly continuous, it suffices that for fixed t the map $\mathcal{K}_\lambda \ni f \mapsto \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + (t + \frac{1+\lambda\pi p \mathcal{H}_p^\infty[f(\bullet)]}{f(p)})^2} \in \mathbb{R}_+$ is continuous. Since the integrand is bounded by $(\frac{1}{\lambda\pi p})^2$, we can for $\lambda > 0$ restrict the integral to a compact interval $[0, \Lambda^2(\epsilon)]$. Since the Hilbert transform is continuous according to the discussion in (4.41) and rational functions of continuous functions with denominator separated

from zero are continuous, the whole function $\mathcal{K}_\lambda \ni f \mapsto \frac{1}{(\lambda\pi p)^2 + \left(t + \frac{1 + \lambda\pi p \mathcal{H}_p^\infty[f(\bullet)]}{f(p)}\right)^2} \in \mathbb{R}_+$ is continuous for every p and hence uniformly continuous on $[0, \Lambda^2(\epsilon)]$. This implies $\|T_\lambda f - T_\lambda \tilde{f}\|'_\infty < \epsilon$ for f, \tilde{f} sufficiently close in \mathcal{K}_λ .

(K2) and (K3) are the conditions in Proposition 4.13 which imply that $T_\lambda(\mathcal{K}_\lambda) \subset \mathcal{C}_0^1(\mathbb{R}_+)$ is relatively compact. Hence $\overline{T_\lambda(\mathcal{K}_\lambda)} \subset \mathcal{C}_0^1(\mathbb{R}_+)$ is compact, with $\overline{T_\lambda(\mathcal{K}_\lambda)} \subset \mathcal{K}_\lambda$ by (K1) and (K4). This means that T_λ maps \mathcal{K}_λ into a compact subset of \mathcal{K}_λ . The properties (K5) and (K6) verify the other conditions in the Schauder fixed point theorem (Theorem 4.11), which thus guarantees that the map $T_\lambda : \mathcal{K}_\lambda \rightarrow \mathcal{K}_\lambda$ has a fixed point G_{b_0} :

Theorem 4.15 *For any $\lambda > 0$, the equation*

$$G_{b_0} = \frac{1}{1+b} \exp \left(-\lambda \int_0^b dt \int_0^\infty dp \frac{(G_{p0})^2}{(\lambda\pi p G_{p0})^2 + (1 + tG_{p0} + \lambda\pi p \mathcal{H}_p^\infty[G_{\bullet 0}])^2} \right) \quad (4.48)$$

has a solution $G_{b_0} \in \mathcal{C}_0^1(\mathbb{R}_+)$. This solution is automatically smooth, monotonously decreasing with uniformly bounded derivative $-\left(\frac{1}{1+b} + C_\lambda\right)G_{b_0} \leq \frac{dG_{b_0}}{db} \leq 0$ and pointwise bounded by $0 < G_{b_0} \leq \frac{1}{1+b}$. At $b = 0$ one has $G_{00} = 1$ and $\left.\frac{dG_{b_0}}{db}\right|_{b=0} = -(1 + \mathcal{Y})$. \square

In retrospect the Theorem justifies the assumption (4.19) on Hölder continuity of G_{ab} .

It is clear that by restricting to $[0, \Lambda^2]$ and ignoring the behaviour at ∞ , also (4.37) has a solution $G_{\bullet 0}^\Lambda \in \mathcal{C}^1[0, \Lambda^2]$. Note however that this proof works for fixed Λ^2 (including $\Lambda^2 = \infty$) but without control on the limit. This means that our existence proof does not imply the (highly plausible) existence of the limit $\lim_{\Lambda \rightarrow \infty} G_{b_0}^\Lambda$ and the equality with G_{b_0} .

The solution G_{b_0} of (4.48) provides $\vartheta_b(a)$ via (4.26b) and then gives the complete two-point function G_{ab} according to (4.29) and all higher correlation function via Theorem 3.1.

If we also knew uniqueness of the solution, then the resulting unique ϑ_0 would be the only candidate for a solution of (4.36). But before addressing the uniqueness question one has to understand the possible non-trivial solutions of the homogeneous Carleman equation which we ignored by Assumption 4.2. This is a project of its own.

4.6 Higher correlation functions and effective coupling constant

Higher correlation functions of the noncommutative ϕ_4^4 -model in matrix representation are obtained from the algebraic recursion formula (3.7) after specification to the parameters and index sets (4.3b). We are interested in the limit $V \rightarrow \infty$ subject to (4.11). According to (4.12) we have $\lim_{V \rightarrow \infty} E_a - E_b = Z\mu^2(1 + \mathcal{Y})(a - b)$. This suggests to absorb the mass dimension as follows:

$$G_{ab_1 \dots b_{N-1}} := \lim_{V \rightarrow \infty} \mu^{3N-4} G_{|ab_1 \dots b_{N-1}|}^{(0)}, \quad (4.49)$$

which for $N = 2$ is compatible with (4.7) and (4.15). With $\lambda_4 = Z^2\lambda$ we obtain from (3.7) in the limit $V \rightarrow \infty$ the recursion formula

$$G_{b_0 b_1 \dots b_{N-1}} = \frac{(-\lambda)}{(1 + \mathcal{Y})^2} \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{b_0 b_1 \dots b_{2l-1}} G_{b_{2l} b_{2l+1} \dots b_{N-1}} - G_{b_{2l} b_1 \dots b_{2l-1}} G_{b_0 b_{2l+1} \dots b_{N-1}}}{(b_0 - b_{2l})(b_1 - b_{N-1})}. \quad (4.50)$$

This gives

$$G_{abcd} = \frac{(-\lambda)}{(1 + \mathcal{Y})^2} \frac{G_{ab}G_{cd} - G_{ad}G_{bc}}{(a - c)(b - d)} \quad (4.51)$$

and so on.

Of particular interest is the *effective coupling constant* $\lambda_{eff} = -G_{0000}$. Indeed, we have $\lambda_{eff} = \Gamma_{0000}$ where the 1PI function Γ_{abcd} is obtained by amputation of the connected two-point functions from G_{abcd} and a change of sign, i.e. $\Gamma_{abcd} = -\frac{G_{abcd}}{G_{ab}G_{bc}G_{cd}G_{da}}$. The direct computation from (4.51) involves up to second derivatives of G_{ab} at $a = b = 0$ which are difficult to control. We therefore use a different method based on the $g = 0$ sector of the original equation (3.9). Specifying to the parameters and index sets (4.3b) of self-dual noncommutative ϕ^4 -theory we obtain in terms of (4.12) and (4.49) the formula

$$\begin{aligned} & \frac{Z^{-1}}{G_{ab_1}} G_{ab_1 \dots b_{N-1}} - \lambda \left(\frac{4}{\theta \mu^2} \right)^2 \sum_{|p|=0}^{\mathcal{N}} (|p| + 1) \frac{G_{pb_1 \dots b_{N-1}} - \frac{G_{pb_1}}{G_{ab_1}} G_{ab_1 \dots b_{N-1}}}{\frac{4}{\theta \mu^2} (|p| - |\underline{a}|)} \\ &= \lambda \sum_{l=1}^{\frac{N-2}{2}} G_{b_1 \dots b_{2l}} \frac{G_{b_{2l+1} \dots b_{N-1} b_{2l}} - G_{b_{2l+1} \dots b_{N-1} a}}{\frac{4}{\theta \mu^2} (|b_{2l}| - |\underline{a}|)}, \end{aligned} \quad (4.52)$$

where p, a, b_k are viewed as functions of $|p|, |\underline{a}|, |b_k|$ according to (4.12). In the limit (4.11) this equation converges to an integral equation which under the assumption that $G_{ab_1 \dots b_{N-1}}$ is Hölder-continuous can be rearranged as

$$\begin{aligned} & \frac{G_{ab_1 \dots b_{N-1}}}{G_{ab_1}} \left(\frac{Z^{-1}}{1 + \mathcal{Y}} + \lambda \pi \mathcal{H}_a^\Lambda [\bullet G_{\bullet b_1}] \right) - \lambda \pi \mathcal{H}_a^\Lambda [\bullet G_{\bullet b_1 \dots b_{N-1}}] \\ &= \frac{\lambda}{(1 + \mathcal{Y})^2} \sum_{l=1}^{\frac{N-2}{2}} G_{b_1 \dots b_{2l}} \frac{G_{b_{2l+1} \dots b_{N-1} b_{2l}} - G_{b_{2l+1} \dots b_{N-1} a}}{b_{2l} - a}. \end{aligned} \quad (4.53)$$

Using the previous definitions and identities (4.22b), (4.22a) and (4.17), the prefactor involving Z^{-1} is treated as follows:

$$\begin{aligned} & \frac{Z^{-1}}{1 + \mathcal{Y}} + \lambda \pi \mathcal{H}_a^\Lambda [\bullet G_{\bullet b_1}] \\ &= \frac{Z^{-1}}{1 + \mathcal{Y}} + \lambda \pi b_1 \mathcal{H}_a^\Lambda [D_{\bullet b_1}] + \lambda \pi \mathcal{H}_a^\Lambda [\bullet G_{\bullet 0}] \\ &= b_1 D_{ab_1} \left(\frac{b_1}{a} + \frac{1 + \lambda \pi a \mathcal{H}_a^\Lambda [G_{\bullet 0}]}{a G_{a0}} \right) + b_1 G_{a0} + \frac{Z^{-1}}{1 + \mathcal{Y}} + \lambda \pi \mathcal{H}_a^\Lambda [\bullet G_{\bullet 0}] \\ &= a G_{ab_1} \left(\frac{b_1}{a} + \frac{1 + \lambda \pi a \mathcal{H}_a^\Lambda [G_{\bullet 0}]}{a G_{a0}} \right) + \underbrace{\frac{Z^{-1}}{1 + \mathcal{Y}} - 1 + \lambda \pi \mathcal{H}_a^\Lambda [(\bullet - a) G_{\bullet 0}]}_{=0}. \end{aligned} \quad (4.54)$$

Inserted back into (4.53) we obtain for $(a G_{ab_1 \dots b_{N-1}})$ again a Carleman equation

$$\begin{aligned} & \left(\frac{b_1}{a} + \frac{1 + \lambda \pi a \mathcal{H}_a^\Lambda [G_{\bullet 0}]}{a G_{a0}} \right) (a G_{ab_1 \dots b_{N-1}}) - \lambda \pi \mathcal{H}_a^\Lambda [\bullet G_{\bullet b_1 \dots b_{N-1}}] \\ &= \frac{\lambda}{(1 + \mathcal{Y})^2} \sum_{l=1}^{\frac{N-2}{2}} G_{b_1 \dots b_{2l}} \frac{G_{b_{2l+1} \dots b_{N-1} b_{2l}} - G_{b_{2l+1} \dots b_{N-1} a}}{b_{2l} - a}. \end{aligned} \quad (4.55)$$

Remarkably, the Carleman equation (4.55) has the same homogeneous part as the equation (4.22a) for the two-point function D_{ab_1} , only the inhomogeneity is different. Its solution is given by Proposition 4.1, again under the assumption $C = 0$, as:

$$G_{ab_1\dots b_{N-1}} = \frac{\lambda}{(1+\mathcal{Y})^2} \sum_{l=1}^{\frac{N-2}{2}} G_{b_1\dots b_{2l}} \frac{\sin \vartheta_{b_1}(a)}{\lambda\pi a} \left(\frac{G_{b_{2l+1}\dots b_{N-1}b_{2l}} - G_{b_{2l+1}\dots b_{N-1}a}}{b_{2l} - a} \cos(\vartheta_{b_1}(a)) \right. \\ \left. + e^{\mathcal{H}_a^\Lambda[\vartheta_{b_1}]} \mathcal{H}_a^\Lambda \left[e^{-\mathcal{H}_a^\Lambda[\vartheta_{b_1}]} \sin(\vartheta_{b_1}(\bullet)) \frac{G_{b_{2l+1}\dots b_{N-1}b_{2l}} - G_{b_{2l+1}\dots b_{N-1}\bullet}}{b_{2l} - \bullet} \right] \right). \quad (4.56)$$

Consistency of our method implies that the solutions of (4.56) agree with the algebraic solutions (4.50), a fact that we have explicitly verified⁵ for $N = 4$.

We use (4.56) to compute $G_{0000} = \lim_{a \rightarrow 0} G_{a0000}$, taking $G_{00} = 1$, $\lim_{a \rightarrow 0} \frac{G_{a0} - 1}{a} = -(1 + \mathcal{Y})$ and $\cos \vartheta_0(0) = \text{sign}(\lambda)$ into account:

$$\lambda_{\text{eff}} = - \lim_{a \rightarrow 0} \frac{\sin \vartheta_0(a)}{(1+\mathcal{Y})^2 \pi a} \left(\frac{G_{a0} - 1}{a} \cos(\vartheta_0(a)) + e^{\mathcal{H}_a^\Lambda[\vartheta_0]} \mathcal{H}_a^\Lambda \left[e^{-\mathcal{H}_a^\Lambda[\vartheta_0]} \sin(\vartheta_0(\bullet)) \frac{G_{\bullet 0} - 1}{\bullet} \right] \right) \\ = \lim_{a \rightarrow 0} \frac{\sin \vartheta_0(a)}{(1+\mathcal{Y})\pi a} \left(\text{sign}(\lambda) + \mathcal{H}_0^\Lambda \left[e^{\mathcal{H}_0^\Lambda[\vartheta_0] - \mathcal{H}_0^\Lambda[\vartheta_0]} \sin(\vartheta_0(\bullet)) \frac{1 - G_{\bullet 0}}{(1+\mathcal{Y})\bullet} \right] \right).$$

Recalling $\lim_{a \rightarrow 0} \frac{\sin \vartheta_0(a)}{\pi a} = |\lambda|$ from (4.35) and Lemma 4.3, we obtain

$$\lambda_{\text{eff}} = \frac{\lambda}{1+\mathcal{Y}} + \frac{1}{(1+\mathcal{Y})^2} \mathcal{H}_0^\Lambda \left[\frac{1 - G_{\bullet 0}}{G_{\bullet 0}} \frac{\sin^2 \vartheta_0(\bullet)}{\pi(\bullet)^2} \right].$$

Spelling out the Hilbert transform, taking (4.30) into account and going to the limit $\Lambda \rightarrow \infty$, we have proved:

Proposition 4.16 *The effective coupling constant $\lambda_{\text{eff}} = -G_{0000}$ of self-dual noncommutative ϕ_4^A -theory is given in terms of the bare coupling constant λ by the following equivalent formulae*

$$\lambda_{\text{eff}} = \frac{\lambda}{1+\mathcal{Y}} + \frac{\lambda^2}{(1+\mathcal{Y})^2} \int_0^\infty dp \frac{1 - G_{p0}}{pG_{p0}} \frac{\sin^2 \vartheta_0(p)}{(\lambda\pi p)^2} \\ = \lambda \left\{ 1 + \frac{\lambda}{(1+\mathcal{Y})} \int_0^\infty dp \frac{\left(\frac{1 - G_{p0}}{(1+\mathcal{Y})p} - G_{p0} \right) G_{p0}}{(\lambda\pi p G_{p0})^2 + (1 + \lambda\pi p \mathcal{H}_p^\infty[G_{\bullet 0}])^2} \right\}. \quad (4.57)$$

By Theorem 4.15, the change $\lambda_{\text{eff}} \mapsto \lambda$ is only a finite renormalisation of the bare coupling constant in response to an infinite change of scales, which means that the QFT model has a non-perturbatively vanishing β -function. \square

We recall that vanishing of the β -function at any order of perturbation theory was proved by Disertori, Gurau, Magnen and Rivasseau in [DGMR07]. A decisive step in

⁵The proof is given in appendix A of the preliminary versions v2,v3 of this paper on arXiv. We have suppressed this appendix in the final version.

their proof was to neglect a term in [DGMR07, eq. (4.18)] which in our notation reads $\sum_{p \in \mathbb{N}_N^2} \frac{1}{|p|} G_{pm0m}^{(0)}$. This is perturbatively justified, but as far as we can see, this argument cannot be used non-perturbatively; at least it is not obvious. Our derivation of (4.57) shows that the omission of the mentioned term is not necessary in the non-perturbative treatment. By Theorem 3.2 much more is true: the β -function of any renormalisable ϕ^4 -matrix model with action $\text{tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)$ vanishes identically.

4.7 Miscellaneous remarks

1. We have shown in (3.12) that N -point functions with $B > 1$ seem to be suppressed by a factor $V^{-(B-1)}$ over the one-cycle functions. We show in [GW13a] that there is a reasonable (and interesting) limit in which the sector ($B > 1, g = 0$) survives. In the matrix basis of the Moyal plane, individual matrix elements of correlation functions have Gaussian decay $e^{-\|x\|^2/\theta}$ for $\|x\| \rightarrow \infty$. Such functions belong to Schwartz space where V is the correct volume. At the end we are interested in a particle interpretation with plane wave asymptotics. These plane waves are in the unital algebra. According to [GW11, GW12], *the spectral dimension of unital functions is doubled*, i.e. the volume is V^2 instead of V . The assembly of plane waves from Gauß packets involves sums over matrix indices which increase the volume from V to V^2 . There is one such factor V per cycle so that *all* correlation functions arising from $\frac{1}{\sqrt{V^2}} \log \mathcal{Z}[J]$ have a finite limit for $V \rightarrow \infty$ as soon as we assemble plane waves. In this situation the contributions from $B > 1$ will survive.

We recall that the $\theta \rightarrow \infty$ limit of the model at $\Omega = 0$ was studied by Becchi, Giusto and Imbimbo [BGI02, BGI03]. They already established the suppression of the non-planar sector (also noticed in [MVS00]) and presence of functions $B > 1$ (“swiss cheese”) in the limit $\theta \rightarrow \infty$.

2. Functional derivatives of the original Ward identity (2.8) express the index integral of an N -point function in terms of other functions. Putting $p = b_2$, $a = b_N$ and applying the derivatives with respect to $\frac{\partial^{N-2}}{\partial J_{b_2 b_3} \dots \partial J_{b_{N-1} b_N}}$, we obtain from (2.8)

$$\begin{aligned} & (E_{b_2} - E_{b_N}) \left(\sum_{n \in I} G_{|nb_2 \dots b_N|} + \sum_{k=2}^N G_{|b_2 \dots b_k | b_k \dots b_N|} + \sum_{l=1}^{\frac{N-2}{2}} G_{|b_2 \dots b_{2l+1}|} G_{|b_{2l+1} \dots b_N|} \right) \\ &= G_{|b_N b_3 \dots b_{N-1}|} - G_{|b_2 b_3 \dots b_{N-1}|}. \end{aligned} \quad (4.58)$$

Inserting (4.3b) and the rescaling (3.12) we see that the functions with $B = 2$ cycles increase the genus by 1. The restriction to the planar sector $g = 0$ thus leads with (4.12) to the equation

$$\begin{aligned} & Z(1 + \mathcal{Y})(b_2 - b_N) \left((1 + \mathcal{Y})^2 \int_0^{\Lambda^2} p dp G_{pb_2 \dots b_N} + \sum_{l=1}^{\frac{N-2}{2}} G_{b_2 \dots b_{2l+1}} G_{b_{2l+1} \dots b_N} \right) \\ &= G_{|b_N b_3 \dots b_{N-1}|} - G_{|b_2 b_3 \dots b_{N-1}|}. \end{aligned} \quad (4.59)$$

It is interesting to verify (4.59) for $N = 4$. From the explicit formula (4.50) we obtain

$$Z(1 + \mathcal{Y})\lambda\pi(G_{bc}\mathcal{H}_c^\Lambda[\bullet G_{\bullet d}] - G_{dc}\mathcal{H}_c^\Lambda[\bullet G_{\bullet b}]) + Z(1 + \mathcal{Y})(b-d)G_{bc}G_{cd} = G_{dc} - G_{bc}, \quad (4.60)$$

in agreement with (4.54).

3. One may speculate that for large indices $a, b \gg 1$ there is a scaling relation $G_{sa, sb} = s^\eta G_{ab}$. Such a relation would by (4.50) result in $G_{sb_0, \dots, sb_{N-1}} = s^{\frac{N-4}{2}\eta} G_{b_0 \dots b_{N-1}}$.
4. Taking the difference between (4.16) and the same equation at $b \mapsto c$ we have

$$(b - c) - \frac{1}{Z(1 + \mathcal{Y})} \left(\frac{1}{G_{ab}} - \frac{1}{G_{ac}} \right) = \lambda \int_0^{\Lambda^2} p dp \frac{\frac{G_{pb}}{G_{ab}} - \frac{G_{pc}}{G_{ac}}}{p - a}. \quad (4.61)$$

We divide by $b - c$ and go to the limit $b = c$. In terms of $G'_{ac} := \lim_{b \rightarrow c} \frac{G_{ab} - G_{ac}}{b - c}$ we have with (4.17)

$$\left(\frac{1 + \lambda\pi\mathcal{H}_a^\Lambda[\bullet G_{\bullet b}] - \lambda\pi\mathcal{H}_0^\Lambda[\bullet G_{\bullet 0}]}{G_{ab}} \right) G'_{ab} - \lambda\pi\mathcal{H}_a^\Lambda[\bullet G'_{\bullet b}] = -G_{ab}.$$

We insert (4.29) and obtain with (4.25b):

$$\lambda\pi \cot \vartheta_b(a) (aG'_{ab}) - \lambda\pi\mathcal{H}_a^\Lambda[\bullet G'_{\bullet b}] = -G_{ab}.$$

Recalling $\lambda\pi a \cot \vartheta_b(a) = b + \frac{1 + \lambda\pi a \mathcal{H}_a^\Lambda[G_{\bullet 0}]}{G_{a0}}$ we have $\frac{d}{db}(\lambda\pi a \cot \vartheta_b(a)) = 1$. This allows us to write down the Carleman equation for the n -th derivative $G_{ab}^{(n)} := \frac{d^n}{db^n} G_{ab}$

$$\lambda\pi \cot \vartheta_b(a) (aG_{ab}^{(n)}) - \lambda\pi\mathcal{H}_a^\Lambda[\bullet G_{\bullet b}^{(n)}] = -nG_{ab}^{(n-1)} + \frac{\delta_{n0}}{Z(1 + \mathcal{Y})}. \quad (4.62)$$

The case $n = 0$ follows directly from (4.29), (4.25b) and (4.32).

5 Conclusion and outlook

This paper was originally intended to achieve the solution of ϕ^4 -quantum field theory on four-dimensional Moyal space with harmonic propagation along the lines we proposed in [GW09]. In developing the necessary tools we realised that the mathematical structures extend to general quartic matrix models with action $S = V \text{tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)$ for a real matrix-valued field $\phi = \phi^* = (\phi_{pq})_{p, q \in I}$, where the external matrix E encodes the dynamics and the number $V > 0$ represents the volume. We proved that in a scaling limit $V \rightarrow \infty$ and $\frac{1}{V} \sum_{p \in I}$ finite, all such matrix models have a 2-point function satisfying a closed non-linear equation, and that all higher correlation functions are obtained from a universal algebraic recursion formula in terms of the 2-point function and the eigenvalues of E . The remarkable and completely unexpected conclusion is that *all* renormalisable quartic matrix models have a vanishing β -function, i.e. they are almost scale-invariant — a fact that was perturbatively established in [DGMR07] for the noncommutative ϕ_4^4 -model.

We feel that getting thus far was only possible because there is a deep mathematical structure behind. Our observations could be another facet of the close connection between integrability and scale invariance. This point deserves further investigation, in particular in view relations to other classes of integrable models. It would also be interesting to know whether there is more than a formal similarity with the cubic matrix model of Kontsevich [Kon82].

This paper achieves the non-perturbative solution of a toy model of *four-dimensional* quantum field theory — of the ϕ_4^4 -model on noncommutative Moyal space in the limit $\theta \rightarrow \infty$. The solution of the correlation functions can be viewed as summation of infinitely many renormalised Feynman graphs (see also Appendix B). The main tools for these achievements are Ward identities in field-theoretical matrix models, the resulting Schwinger-Dyson equations and the solution theory of the Carleman singular integral equation. The 2-point function plays a central rôle in our approach. It is expressed in terms of a single function G of one variable given as solution (that we proved to exist) of a non-linear integral equation.

We are left with a few problems:

- We have ignored so far the possible non-trivial solutions of the homogeneous Carleman equation. The numerical treatment in [GW13b] justifies this for coupling constants $0 < \lambda \leq \frac{1}{\pi}$. This needs a rigorous confirmation, which is only possible if the whole space of solutions of the Carleman equation is taken into account. The investigation of the consistency requirements should then select the right point in the solution space. In this way it should be possible to continue the solution of the model to $\lambda > \frac{1}{\pi}$. It seems plausible that the true solution changes its sign for $\lambda > \frac{1}{\pi}$. This could be interpreted as the impossibility of an infinite correlation length so that $\lambda = \frac{1}{\pi}$ might be the end point of the λ -family of critical models.

It makes little sense to prove the (important!) uniqueness of the solution of the integral equation for the two-point function before clarifying the freedom from the homogeneous Carleman equation.

- Clarify the combinatorics of the weight factors in the formulae expressing the N -point function in terms of the 2-point function. We already know that the non-crossing chord diagrams arise which are counted by the Catalan numbers. Identifying the combinatorics might suggest links to other models.
- The reformulation of the infinite volume limit $\theta \rightarrow \infty$ in position space is addressed in [GW13a]. We prove that the limit $\theta \rightarrow \infty$ restores the full Euclidean symmetry group and that only the diagonal $(N_1 + \dots + N_B)$ -point functions $G_{\underbrace{a_1 \dots a_1}_{N_1} | \dots | \underbrace{a_B \dots a_B}_{N_B}}$ with all N_i even contribute to position space correlation functions. These functions describe a theory with interaction but without exchange of momenta. In contrast to a free theory, clustering is violated, which corresponds to the presence of different topological sectors. This needs further investigation.
- In position space reformulation the main question concerns the analytic continuation in the Euclidean time variable to a possible Minkowskian theory. We show in [GW13a] that Osterwalder-Schrader reflection positivity of the two-point function is connected to the question whether $a \mapsto G_{aa}$ is a Stieltjes function. Settling this

question requires more knowledge on the fixed point solution G_{a0} of our master equation.

- The scaling limit in which both the size \mathcal{N} of the matrices and the noncommutativity parameter θ are sent to infinity with their ratio (4.11) kept fixed restricts the matrix model to its planar sector. The reason is that any full correlation function satisfies exactly the same equation as its restriction to the planar sector. It is an interesting and definitely much harder problem to construct the model for $\mathcal{N} \rightarrow \infty$ but finite θ . The main difficulty is to control the planar 2-point function which now solves the non-linear discrete equation (4.10). There is next a hierarchy of linear equations for higher $(N_1 + \dots + N_B)$ -point functions of fixed genus, with explicitly known solution if one $N_i \geq 3$. Assuming that the other basic functions with all $N_i \leq 2$ can be controlled there remains the resummation of the genus expansion, a problem of constructive physics. It would be very interesting to establish the limit $\theta \rightarrow \infty$ via this construction. That it agrees with the procedure based on (4.11) is plausible but by no means obvious.

A Correlation functions with two boundary components

A.1 Two cycles of odd length: Schwinger-Dyson equations

The case of $B = 2$ cycles needs distinction of several cases. For the (1+1)-point function we use $\frac{\partial(-S_{int})}{\partial\phi_{aa}} = (-V\lambda_4) \sum_{p,n \in I} \phi_{ap}\phi_{pn}\phi_{na}$ to derive

$$\begin{aligned} G_{|a|c|} &= \frac{1}{(E_a + E_a)\mathcal{Z}[0]} \left\{ \left(\phi_{cc} \frac{\partial(-S_{int})}{\partial\phi_{aa}} \right) \left[\frac{1}{V} \frac{\partial}{\partial J} \right] \right\} \mathcal{Z}[J] \Big|_{J=0} \\ &= \frac{(-\lambda_4)}{V^3(E_a + E_a)\mathcal{Z}[0]} \left\{ \frac{\partial^2((V^2W_a^1[J] + VW_a^2[J])\mathcal{Z}[J])}{\partial J_{aa}\partial J_{cc}} \right. \\ &\quad \left. + \sum_{p,n \in I} \frac{V}{E_p - E_a} \frac{\partial^2}{\partial J_{pa}\partial J_{cc}} \left(J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \right) \right\} \Big|_{J=0}. \end{aligned} \quad (\text{A.1})$$

The result after genus expansion, obtained by similar considerations as in Sec. 3.4, is

$$\begin{aligned} G_{|a|c|}^{(g)} &= \frac{(-\lambda_4)}{(E_a + E_a)} \left\{ \frac{1}{V} \sum_{n \in I} \sum_{g'+g''=g} G_{|a|c|}^{(g')} G_{|an|}^{(g'')} + \frac{3}{V} \sum_{g'+g''+1=g} G_{|a|c|}^{(g')} G_{|a|a|}^{(g'')} \right. \\ &\quad \left. + \frac{1}{V^2} \left(G_{|a|a|a|c|}^{(g-2)} + \sum_{n \in I} G_{|a|c|an|}^{(g-1)} + G_{|c|aaa|}^{(g-1)} + G_{|a|cac|}^{(g-1)} \right) \right. \\ &\quad \left. + \frac{1}{V} \sum_{p \in I} \frac{G_{|a|c|}^{(g)} - G_{|p|c|}^{(g)}}{E_p - E_a} + \frac{1}{V} \frac{G_{|ac|}^{(g)} - G_{|cc|}^{(g)}}{E_c - E_a} \right\}. \end{aligned} \quad (\text{A.2})$$

In the general case of two cycles which both have odd length we have

$$\begin{aligned} G_{|ab_1 \dots b_{2l}|c_1 \dots c_{N-2l-1}|} &= \frac{(-\lambda_4)}{V^3(E_a + E_b)\mathcal{Z}[0]} \left\{ \frac{\partial^N((V^2W_a^1[J] + VW_a^2[J])\mathcal{Z}[J])}{\partial J_{ab_1 \dots b_{2l}}^{2l+1} \partial J_{c_1 \dots c_{N-2l-1}}^{N-2l-1}} \right. \\ &\quad \left. + \sum_{p,n \in I} \frac{V}{E_p - E_a} \frac{\partial^N \left(J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \right)}{\partial J_{pb_1} \partial J_{b_1 b_2} \dots \partial J_{b_{2l-1} b_{2l}} \partial J_{b_{2l} a} \partial J_{c_1 \dots c_{N-2l-1}}^{N-2l-1}} \right\} \Big|_{J=0}, \end{aligned}$$

where $\frac{\partial^k}{\partial J_{p_1 \dots p_k}^k} := \frac{\partial^k}{\partial J_{p_1 p_2} \dots \partial J_{p_{k-1} p_k} \partial J_{p_k p_1}}$. For $l = 0$ we have to put $\frac{1}{E_a + E_{b_1}} \mapsto \frac{1}{E_a + E_a}$. The derivatives are evaluated according to the discussion preceding (3.5):

$$\begin{aligned}
& G_{|ab_1 \dots b_{2l}|c_1 \dots c_{N-2l-1}|} \\
&= \frac{(-\lambda_4)}{E_a + E_{b_1}} \left\{ \frac{1}{V} \left(G_{|ab_1 \dots b_{2l}|c_1 \dots c_{N-2l-1}|} G_{|a|a|} + \sum_{n \in I} G_{|ab_1 \dots b_{2l}|c_1 \dots c_{N-2l-1}|} G_{|an|} \right) \right. \\
&+ \frac{1}{V^2} \left(G_{|a|a|ab_1 \dots b_{2l}|c_1 \dots c_{N-2l-1}|} + G_{|aaab_1 \dots b_{2l}|c_1 \dots c_{N-2l-1}|} + \sum_{n \in I} G_{|an|ab_1 \dots b_{2l}|c_1 \dots c_{N-2l-1}|} \right. \\
&\quad \left. + \sum_{k=1}^{2l} G_{|b_1 \dots b_k ab_k \dots b_{2l}|c_1 \dots c_{N-2l-1}|} + \sum_{k=1}^{N-2l-1} G_{|c_1 \dots c_k ac_k \dots c_{N-2l-1}|ab_1 \dots b_{2l}|} \right) \\
&+ \frac{2}{V} G_{|a|ab_1 \dots b_{2l}|} G_{|a|c_1 \dots c_{N-2l-1}|}^{(g'')} - \frac{1}{V} \sum_{p \in I} \frac{G_{|pb_1 \dots b_{2l}|c_1 \dots c_{N-2l-1}|} - G_{|ab_1 \dots b_{2l}|c_1 \dots c_{N-2l-1}|}}{E_p - E_a} \\
&- \frac{1}{V} \sum_{k=1}^{2l} \frac{G_{|b_1 \dots b_k |b_{k+1} \dots b_{2l}|c_1 \dots c_{N-2l-1}|} - G_{|b_1 \dots b_k |b_{k+1} \dots b_{2l}|a|c_1 \dots c_{N-2l-1}|}}{E_{b_k} - E_a} \\
&- \sum_{j=1}^l G_{|b_1 \dots b_{2j-1}|c_1 \dots c_{N-2l-1}|} \frac{G_{|b_{2j} \dots b_{2l} b_{2j-1}|} - G_{|b_{2j} \dots b_{2l} a|}}{E_{b_{2j-1}} - E_a} \\
&- \sum_{j=1}^l G_{|b_1 \dots b_{2j}|} \frac{G_{|b_{2j+1} \dots b_{2l} b_{2j-1}|c_1 \dots c_{N-2l-1}|} - G_{|b_{2j+1} \dots b_{2l} a|c_1 \dots c_{N-2l-1}|}}{E_{b_{2j}} - E_a} \\
&\left. - \frac{1}{V} \sum_{k=1}^{N-2l-1} \frac{G_{|c_1 \dots c_k b_1 \dots b_{2l} c_k c_{k+1} \dots c_{N-2l-1}|} - G_{|c_1 \dots c_k b_1 \dots b_{2l} a c_{k+1} \dots c_{N-2l-1}|}}{E_{c_k} - E_a} \right\}. \tag{A.3}
\end{aligned}$$

In case of a real field theory we can proceed as in section 3.3 to obtain an algebraic recursion formula. For $l \geq 1$ we multiply (A.3) by $E_a + E_{b_1}$ and subtract from the resulting equation the equation with renamed indices $b_k \leftrightarrow b_{2l+1-k}$ and $c_k \leftrightarrow c_{N-2l-k}$. Taking (3.6) into account we arrive (after genus expansion) at

$$\begin{aligned}
& G_{|b_0 b_1 \dots b_{2l}|c_1 \dots c_{N-2l-1}|}^{(g)} \\
&= \frac{(-\lambda_4)}{V} \sum_{k=1}^{N-2l-1} \frac{G_{|c_1 \dots c_{k-1} b_0 b_1 \dots b_{2l} c_k c_{k+1} \dots c_{N-2l-1}|}^{(g)} - G_{|c_1 \dots c_{k-1} c_k b_1 \dots b_{2l} b_0 c_{k+1} \dots c_{N-2l-1}|}^{(g)}}{(E_{b_1} - E_{b_{2l}})(E_{b_0} - E_{c_k})} \\
&+ (-\lambda_4) \sum_{g'+g''=g} \sum_{j=1}^l \frac{G_{|b_0 b_1 \dots b_{2j-2}|c_1 \dots c_{N-2l-1}|}^{(g')} G_{|b_{2j-1} b_{2j} \dots b_{2l}|}^{(g'')} - G_{|b_{2j-1} b_1 \dots b_{2j-2}|c_1 \dots c_{N-2l-1}|}^{(g')} G_{|b_0 b_{2j} \dots b_{2l}|}^{(g'')}}{(E_{b_1} - E_{b_{2l}})(E_{b_0} - E_{b_{2j-1}})} \\
&+ (-\lambda_4) \sum_{g'+g''=g} \sum_{j=1}^l \frac{G_{|b_0 b_1 \dots b_{2j-1}|}^{(g')} G_{|b_{2j} b_{2j+1} \dots b_{2l}|c_1 \dots c_{N-2l-1}|}^{(g'')} - G_{|b_{2j} b_1 \dots b_{2j-1}|}^{(g')} G_{|b_0 b_{2j+1} \dots b_{2l}|c_1 \dots c_{N-2l-1}|}^{(g'')}}{(E_{b_1} - E_{b_{2l}})(E_{b_0} - E_{b_{2j}})} \\
&+ \frac{(-\lambda_4)}{V} \sum_{k=1}^{2l} \frac{G_{|b_0 b_1 \dots b_{k-1} |b_k b_{k+1} \dots b_{2l}|c_1 \dots c_{N-2l-1}|}^{(g-1)} - G_{|b_k b_1 \dots b_{k-1} |b_0 b_{k+1} \dots b_{2l}|c_1 \dots c_{N-2l-1}|}^{(g-1)}}{(E_{b_1} - E_{b_{2l}})(E_{b_0} - E_{b_k})}. \tag{A.4}
\end{aligned}$$

The $l = 0$ case is obtained from the symmetry $G_{|b_0|c_1 \dots c_{N-1}|} = G_{|c_1 \dots c_{N-1}|b_0|}$. The last line of (A.4) increases the genus by 1 because the external vertex connects two cycles of the

same function, from $B = 3$ to $B = 2$. The restriction to the planar ($g = 0$) sector, exact in the limit ($V \rightarrow \infty, \frac{1}{V} \sum_{p \in I}$ finite), gives:

Proposition A.1 *Given a real ϕ^4 -matrix model with injective external matrix E , the (unrenormalised) planar connected N -point functions with two cycles of odd length satisfy together with the single-cycle functions (3.10) and (3.11) the system of equations*

$$G_{|a|c|}^{(0)} = \frac{\lambda_4}{V} \sum_{p \in I} \frac{G_{|p|c|}^{(0)} G_{|aa|}^{(0)} - G_{|a|c|}^{(0)} G_{|pa|}^{(0)}}{E_p - E_a} + \frac{\lambda_4}{V} G_{|aa|}^{(0)} \frac{G_{|cc|}^{(0)} - G_{|ca|}^{(0)}}{E_c - E_a}, \quad (\text{A.5})$$

$$\begin{aligned} & G_{|b_0 b_1 \dots b_{2l} | c_1 \dots c_{N-2l-1} |}^{(0)} \\ &= \frac{(-\lambda_4)}{V} \sum_{k=1}^{N-2l-1} \frac{G_{|c_1 \dots c_{k-1} b_0 b_1 \dots b_{2l} c_k c_{k+1} \dots c_{N-2l-1} |}^{(0)} - G_{|c_1 \dots c_{k-1} c_k b_1 \dots b_{2l} b_0 c_{k+1} \dots c_{N-2l-1} |}^{(0)}}{(E_{b_1} - E_{b_{2l}})(E_{b_0} - E_{c_k})} \\ &+ (-\lambda_4) \sum_{j=1}^l \frac{G_{|b_0 b_1 \dots b_{2j-2} | c_1 \dots c_{N-2l-1} |}^{(0)} G_{|b_{2j-1} b_{2j} \dots b_{2l} |}^{(0)} - G_{|b_{2j-1} b_1 \dots b_{2j-2} | c_1 \dots c_{N-2l-1} |}^{(0)} G_{|b_0 b_{2j} \dots b_{2l} |}^{(0)}}{(E_{b_1} - E_{b_{2l}})(E_{b_0} - E_{b_{2j-1}})} \\ &+ (-\lambda_4) \sum_{j=1}^l \frac{G_{|b_0 b_1 \dots b_{2j-1} |}^{(0)} G_{|b_{2j} b_{2j+1} \dots b_{2l} | c_1 \dots c_{N-2l-1} |}^{(0)} - G_{|b_{2j} b_1 \dots b_{2j-1} |}^{(0)} G_{|b_0 b_{2j+1} \dots b_{2l} | c_1 \dots c_{N-2l-1} |}^{(0)}}{(E_{b_1} - E_{b_{2l}})(E_{b_0} - E_{b_{2j}})}. \end{aligned} \quad (\text{A.6})$$

In the limit ($V \rightarrow \infty, \frac{1}{V} \sum_{p \in I}$ finite), the full function and the genus-0 function satisfy the same equation: $\lim_{\frac{1}{V} \sum_{p \in I} \text{finite}}^{V \rightarrow \infty} G_{|ab_1 \dots b_{2l} | c_1 \dots c_{N-2l-1} |}^{(0)} = \lim_{\frac{1}{V} \sum_{p \in I} \text{finite}}^{V \rightarrow \infty} G_{|ab_1 \dots b_{2l} | c_1 \dots c_{N-2l-1} |}$.

Proof. Equation (A.5) follows from the ($g = 0$)-case of (A.2) by elimination of $\sum_{n \in I} G_{|an|}^{(0)}$ via (3.10). \square

A.2 Two cycles of even length: Schwinger-Dyson equations

The situation is different in the case of two cycles which both have even length because the action of the two J -cycles on \mathcal{Z} decomposes into its action on $\log \mathcal{Z}$ and the separate action of each one cycle on one of the factors $\log \mathcal{Z}$ in $\frac{1}{2}(\log \mathcal{Z})^2$. This means

$$\begin{aligned} & V G_{|ab_1 \dots b_{2l-1} | c_1 \dots c_{N-2l} |} + V^2 G_{|ab_1 \dots b_{2l-1} |} G_{|c_1 \dots c_{N-2l} |} \\ &= V \frac{(-\lambda_4)}{V^3 (E_a + E_b) \mathcal{Z}[0]} \left\{ \frac{\partial^N ((V^2 W_a^1[J] + V W_a^2[J]) \mathcal{Z}[J])}{\partial J_{ab_1 \dots b_{2l-1}}^{2l} \partial J_{c_1 \dots c_{N-2l}}^{N-2l}} \right. \\ &\quad \left. + V \sum_{p, n \in I} \frac{V}{E_p - E_a} \frac{\partial^N \left(J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \right)}{\partial J_{pb_1} \partial J_{b_1 b_2} \dots \partial J_{b_{2l-2} b_{2l-1}} \partial J_{b_{2l-1} a} \partial J_{c_1 \dots c_{N-2l}}^{N-2l}} \right\}_{J=0}. \end{aligned}$$

The global prefactor V arises from $\frac{\partial}{\partial J_{ab}} \exp(\frac{V}{2} \langle J, J \rangle_E) = \frac{V J_{ba}}{E_a + E_b} \exp(\frac{V}{2} \langle J, J \rangle_E)$. In addition to the discussion of (3.5) there is now the possibility that the derivatives with respect to $J_{ab_1 \dots b_{2l-1}}^{2l}$ and $J_{c_1 \dots c_{N-2l}}^{N-2l}$ act separately on one of the functions $V^2 W_a^1[J] + V W_a^2[J]$ and $\mathcal{Z}[J]$. The analogous consideration applies to the last line. In this way all terms which

constitute $G_{|c_1 \dots c_{N-2l}|}$ times the rhs of equation (3.5) for $G_{|ab_1 \dots b_{2l-1}|}$ are generated, and these cancel with the lhs. For $(N = 4, l = 1)$ we thus obtain after genus expansion

$$\begin{aligned}
G_{|ab|cd}^{(g)} &= \frac{(-\lambda_4)}{E_a + E_b} \left\{ \frac{1}{V} \sum_{g'+g''+1=g} G_{|ab|cd}^{(g')} G_{|a|a|}^{(g'')} + \frac{1}{V} \sum_{g'+g''=g} \sum_{n \in I} G_{|ab|cd}^{(g')} G_{|an|}^{(g'')} \right. \\
&+ \frac{1}{V^2} G_{|a|a|ab|cd}^{(g-2)} + \frac{1}{V} \sum_{g'+g''+1=g} G_{|a|a|cd|}^{(g')} G_{|ab|}^{(g'')} + \frac{1}{V^2} \sum_{n \in I} G_{|an|ab|cd}^{(g-1)} \\
&+ \frac{1}{V} \sum_{g'+g''=g} \sum_{n \in I} G_{|an|cd|}^{(g')} G_{|ab|}^{(g'')} + \frac{1}{V^2} (G_{|aaab|cd}^{(g-1)} + G_{|baba|cd}^{(g-1)} + G_{|cacd|ab}^{(g-1)} + G_{|cdad|ab}^{(g-1)}) \\
&+ \frac{1}{V} \sum_{g'+g''=g} (G_{|cacd|}^{(g')} G_{|ab|}^{(g'')} + G_{|cdad|}^{(g')} G_{|ab|}^{(g'')}) - \frac{1}{V} \sum_{p \in I} \frac{G_{|pb|cd}^{(g)} - G_{|ab|cd}^{(g)}}{E_p - E_a} \\
&\left. - \frac{1}{V} \frac{G_{|b|b|cd}^{(g-1)} - G_{|b|a|cd}^{(g-1)}}{E_b - E_a} - \frac{1}{V} \frac{G_{|cbcd}^{(g)} - G_{|cbad}^{(g)}}{E_c - E_a} - \frac{1}{V} \frac{G_{|cdbd}^{(g)} - G_{|cdba}^{(g)}}{E_d - E_a} \right\}. \tag{A.7}
\end{aligned}$$

For $N \geq 6$ we can achieve $l \geq 2$ by symmetry, and then the invariance of the real theory under orientation reversal (3.6) allows us to derive the purely algebraic solution

$$\begin{aligned}
&G_{|ab_1 \dots b_{2l-1}|c_1 \dots c_{N-2l}|}^{(g)} \\
&= (-\lambda_4) \sum_{j=1}^{l-1} \frac{G_{|b_1 \dots b_{2j-1}a|c_1 \dots c_{N-2l}|}^{(g)} G_{|b_{2j}b_{2j+1} \dots b_{2l-1}|}^{(g)} - G_{|b_1 \dots b_{2j-1}b_{2j}|c_1 \dots c_{N-2l}|}^{(g)} G_{|ab_{2j+1} \dots b_{2l-1}|}^{(g)}}{(E_{b_1} - E_{b_{2l-1}})(E_a - E_{b_{2j}})} \\
&+ (-\lambda_4) \sum_{j=1}^{l-1} \frac{G_{|b_1 \dots b_{2j-1}a|}^{(g)} G_{|b_{2j}b_{2j+1} \dots b_{2l-1}|c_1 \dots c_{N-2l}|}^{(g)} - G_{|b_1 \dots b_{2j-1}b_{2j}|}^{(g)} G_{|ab_{2j+1} \dots b_{2l-1}|c_1 \dots c_{N-2l}|}^{(g)}}{(E_{b_1} - E_{b_{2l-1}})(E_a - E_{b_{2j}})} \\
&+ \frac{(-\lambda_4)}{V} \sum_{k=1}^{N-2l} \frac{G_{|c_1 \dots c_{k-1}ab_1 \dots b_{2l-1}c_k c_{k+1} \dots c_{N-2l}|}^{(g)} - G_{|c_1 \dots c_{k-1}c_k b_1 \dots b_{2l-1}ac_{k+1} \dots c_{N-2l}|}^{(g)}}{(E_{b_1} - E_{b_{2l-1}})(E_a - E_{c_k})} \\
&+ \frac{(-\lambda_4)}{V} \sum_{k=1}^{2l-1} \frac{G_{|b_1 \dots b_{k-1}a|b_k b_{k+1} \dots b_{2l-1}|c_1 \dots c_{N-2l}|}^{(g-1)} - G_{|b_1 \dots b_{k-1}b_k|ab_{k+1} \dots b_{2l-1}|c_1 \dots c_{N-2l}|}^{(g-1)}}{(E_{b_1} - E_{b_{2l-1}})(E_a - E_{b_k})} \Big\}. \tag{A.8}
\end{aligned}$$

The restriction to the planar ($g = 0$) sector gives:

Proposition A.2 *Given a real ϕ^4 -matrix model with injective external matrix E , the (unrenormalised) planar connected N -point functions with two cycles of even length satisfy together with the single-cycle functions (3.10) and (3.11) the system of equations*

$$\begin{aligned}
G_{|ab|cd}^{(0)} &= \frac{(-\lambda_4)}{V} G_{|ab|}^{(0)} G_{|ab|}^{(0)} \left(\sum_{n \in I} G_{|an|cd}^{(0)} + G_{|cacd|}^{(0)} + G_{|cdad|}^{(0)} \right) \\
&+ \frac{\lambda_4}{V} \sum_{p \in I} \frac{G_{|pb|cd}^{(g)} G_{|ab|}^{(0)} - G_{|ab|cd}^{(g)} G_{|pb|}^{(0)}}{E_p - E_a} + \frac{\lambda_4}{V} \frac{G_{|cbcd}^{(0)} - G_{|cbad}^{(0)}}{E_c - E_a} + \frac{\lambda_4}{V} \frac{G_{|cdbd}^{(0)} - G_{|cdba}^{(0)}}{E_d - E_a} \Big\}, \tag{A.9}
\end{aligned}$$

$$\begin{aligned}
& G_{|ab_1 \dots b_{2l-1}|c_1 \dots c_{N-2l}|}^{(0)} \\
& (-\lambda_4) \sum_{j=1}^{l-1} \frac{G_{|b_1 \dots b_{2j-1}a|c_1 \dots c_{N-2l}|}^{(0)} G_{|b_{2j}b_{2j+1} \dots b_{2l-1}|}^{(0)} - G_{|b_1 \dots b_{2j-1}b_{2j}|c_1 \dots c_{N-2l}|}^{(0)} G_{|ab_{2j+1} \dots b_{2l-1}|}^{(0)}}{(E_{b_1} - E_{b_{2l-1}})(E_a - E_{b_{2j}})} \\
& + (-\lambda_4) \sum_{j=1}^{l-1} \frac{G_{|b_1 \dots b_{2j-1}a|}^{(0)} G_{|b_{2j}b_{2j+1} \dots b_{2l-1}|c_1 \dots c_{N-2l}|}^{(0)} - G_{|b_1 \dots b_{2j-1}b_{2j}|}^{(0)} G_{|ab_{2j+1} \dots b_{2l-1}|c_1 \dots c_{N-2l}|}^{(0)}}{(E_{b_1} - E_{b_{2l-1}})(E_a - E_{b_{2j}})} \\
& + \frac{(-\lambda_4)}{V} \sum_{k=1}^{N-2l} \frac{G_{|c_1 \dots c_{k-1}ab_1 \dots b_{2l-1}c_k c_{k+1} \dots c_{N-2l}|}^{(0)} - G_{|c_1 \dots c_{k-1}c_k b_1 \dots b_{2l-1}ac_{k+1} \dots c_{N-2l}|}^{(0)}}{(E_{b_1} - E_{b_{2l-1}})(E_a - E_{c_k})}. \tag{A.10}
\end{aligned}$$

We have $\lim_{\frac{1}{V} \sum_{p \in I} \text{finite} \rightarrow \infty} G_{|ab_1 \dots b_{2l-1}|c_1 \dots c_{N-2l}|}^{(0)} = \lim_{\frac{1}{V} \sum_{p \in I} \text{finite} \rightarrow \infty} G_{|ab_1 \dots b_{2l-1}|c_1 \dots c_{N-2l}|}$ in the scaling limit.

Proof. Equation (A.9) follows from the ($g = 0$)-case of (A.7) by elimination of $\sum_{n \in I} G_{|an|}^{(0)}$ via (3.10). \square

A.3 Remarks on functions with more than two cycles

Although we are not going to work out the details, it is clear how to write down the Schwinger-Dyson equation for any planar B -cycle N -point function and how to achieve its solution. By analogy with the (1+1)-point function and the (2+2)-point function we expect that the solution for the $(N_1 + \dots + N_B)$ -point functions with all $N_i \leq 2$ requires a case by case Carleman solution for known input data. If at least one $N_j \geq 3$, then the invariance under orientation reversal leads to an explicit algebraic formula for the $(N_1 + \dots + N_B)$ -point function.

A.4 The $B = 2$ sector of noncommutative ϕ_4^4 -theory

We solve the basic equations (A.5) for $G_{|a|c|}^{(0)}$ and (A.9) for $G_{|ab|cd|}^{(0)}$ for the parameters (4.3b) of the noncommutative ϕ_4^4 -model in matrix representation. Higher correlation functions are algebraic. The equations only depend on the 1-norms (4.12) of the indices so that $\sum_{p \in \mathbb{N}_{\mathcal{N}}^2} \mapsto \sum_{|p|=0}^{\mathcal{N}} (|p| + 1)$. We absorb the mass dimension and the volume factor

$$G_{ab_1 \dots b_k|c_1 \dots c_{N-k-1}} := \mu^{3N-4} (V\mu^4) G_{|ab_1 \dots b_k|c_1 \dots c_{N-k-1}|}^{(0)}. \tag{A.11}$$

Conversely, (A.11) suggests that if $G_{ab_1 \dots b_k|c_1 \dots c_{N-k}}$ has a limit for $V \rightarrow \infty$ (and that is the case as shown in the sequel), then $G_{|ab_1 \dots b_k|c_1 \dots c_{N-k-1}|}^{(0)}$ is scaled to zero. We give in section 4.7 some arguments why the sector $B \geq 2$ is nonetheless interesting.

Recalling $G_{aa} := \mu^2 G_{|aa|}^{(0)}$, $V = (\frac{\theta}{4})^2$, $\lambda_4 = Z^2 \lambda$ and $E_{\underline{m}}$ from (4.3b), multiplication of (A.5) by $\frac{\mu^6 V}{Z(1+\mathcal{Y})G_{aa}}$ thus leads to

$$\frac{Z^{-1}}{(1+\mathcal{Y})} \frac{G_{a|c}}{G_{aa}} = \lambda \left(\frac{4}{\theta \mu^2 (1+\mathcal{Y})} \right)^2 \sum_{|p|=0}^{\mathcal{N}} (|p| + 1) \frac{G_{p|c} - \frac{G_{a|c} G_{pa}}{G_{aa}}}{p - a} + \frac{\lambda}{(1+\mathcal{Y})^2} \frac{G_{cc} - G_{ac}}{c - a}, \tag{A.12}$$

where p and $|p|$ are related by (4.12). Now the thermodynamic limit (4.11) to continuous variables $a, c, p \in [0, \Lambda^2]$ exists. Under the assumption that $G_{a|c}$ is Hölder-continuous, we can move $\frac{G_{a|c}}{G_{aa}} G_{pa}$ to the lhs of (A.12), and taking (4.54) at $b_1 = a$ into account we arrive at the following Carleman equation for the planar (1+1)-point function:

$$\left(\frac{a}{a} + \frac{1 + \lambda\pi a \mathcal{H}_a^\Lambda[G_{\bullet 0}]}{aG_{a0}}\right)(aG_{a|c}) - \lambda\pi \mathcal{H}_a^\Lambda[\bullet G_{\bullet|c}] = \frac{\lambda}{(1 + \mathcal{Y})^2} \frac{G_{cc} - G_{ac}}{c - a}. \quad (\text{A.13})$$

We let $\vartheta(a) := \vartheta_a(a)$ and stress that the (non-)presence of the subscript at ϑ will be important in the following calculation. The Carleman formula (4.24) and rational fraction expansion give the solution of (A.13) as

$$\begin{aligned} G_{a|c} &= \frac{\sin \vartheta(a)}{(1 + \mathcal{Y})^2 \pi a} \left(\frac{G_{cc} - G_{ac}}{c - a} \cos(\vartheta(a)) + e^{\mathcal{H}_a^\Lambda[\vartheta]} \mathcal{H}_a^\Lambda \left[e^{-\mathcal{H}_a^\Lambda[\vartheta]} \sin(\vartheta(\bullet)) \frac{G_{cc} - G_{\bullet c}}{c - \bullet} \right] \right) \\ &= \frac{\sin \vartheta(a)}{(1 + \mathcal{Y})^2 \pi a (c - a)} \left((G_{cc} - G_{ac}) \cos(\vartheta(a)) + e^{\mathcal{H}_a^\Lambda[\vartheta]} \mathcal{H}_a^\Lambda \left[e^{-\mathcal{H}_a^\Lambda[\vartheta]} \sin(\vartheta(\bullet)) (G_{cc} - G_{\bullet c}) \right] \right. \\ &\quad \left. - e^{\mathcal{H}_a^\Lambda[\vartheta]} \mathcal{H}_c^\Lambda \left[e^{-\mathcal{H}_c^\Lambda[\vartheta]} \sin(\vartheta(\bullet)) (G_{cc} - G_{\bullet c}) \right] \right) \\ &= \frac{\sin \vartheta(a) e^{\mathcal{H}_a^\Lambda[\vartheta]}}{(1 + \mathcal{Y})^2 \pi a (c - a)} \left(G_{cc} \cos(\vartheta(c)) e^{-\mathcal{H}_c^\Lambda[\vartheta]} - G_{ac} \cos(\vartheta(a)) e^{-\mathcal{H}_a^\Lambda[\vartheta]} \right. \\ &\quad \left. - \mathcal{H}_a^\Lambda \left[e^{-\mathcal{H}_a^\Lambda[\vartheta]} \sin(\vartheta(\bullet)) G_{\bullet c} \right] + \mathcal{H}_c^\Lambda \left[e^{-\mathcal{H}_c^\Lambda[\vartheta]} \sin(\vartheta(\bullet)) G_{\bullet c} \right] \right). \quad (\text{A.14}) \end{aligned}$$

The last identity follows from (4.25a). We see no possibility at the moment to further simplify this expression. A non-trivial consistency check is the symmetry $G_{a|c} = G_{c|a}$. In perturbation theory up to $\mathcal{O}(\lambda^2)$ we confirm the symmetry and the agreement with the Feynman graph expansion, see Appendix B.

Higher $(N_1 + N_2)$ -point functions with N_i odd are obtained from (A.6) for $E_a - E_b \mapsto Z^2 \mu^2 (1 + \mathcal{Y})(a - b)$ and $\lambda_4 = Z^2 \lambda$. In the simplest case ($N = 4, l = 1$) the solution in dimensionless functions (A.11) reads

$$\begin{aligned} G_{a_1 a_2 a_3 | c} &= \frac{\lambda}{(1 + \mathcal{Y})^2} \frac{G_{a_1 | c} G_{a_3 a_2} - G_{a_2 | c} G_{a_3 a_1}}{(a_2 - a_3)(a_2 - a_1)} + \frac{\lambda}{(1 + \mathcal{Y})^2} \frac{G_{a_2 a_1} G_{a_3 | c} - G_{a_2 a_3} G_{a_1 | c}}{(a_2 - a_3)(a_3 - a_1)} \\ &\quad + \frac{\lambda}{(1 + \mathcal{Y})^2} \frac{G_{a_1 a_2 a_3 c} - G_{c a_2 a_3 a_1}}{(a_2 - a_3)(c - a_1)}. \quad (\text{A.15}) \end{aligned}$$

The basic function for $B = 2$ cycles of even length is $G_{|ab|cd}^{(0)}$ determined by the solution of (A.9). In the parametrisation (4.3b) and after taking the limit (4.11) to continuous matrix indices (4.12), this equation reads in terms of (A.11)

$$\begin{aligned} \left(\frac{b}{a} + \frac{1 + \lambda\pi a \mathcal{H}_a^\Lambda[G_{\bullet 0}]}{aG_{a0}}\right)(aG_{ab|cd}) - \lambda\pi \mathcal{H}_a^\Lambda[\bullet G_{\bullet b|cd}] \\ = -Z(1 + \mathcal{Y})G_{ab} \left(\lambda I_{a|cd} + \frac{\lambda}{(1 + \mathcal{Y})^2} (G_{acdc} + G_{adcd}) \right) \\ + \frac{\lambda}{(1 + \mathcal{Y})^2} \frac{G_{dcbc} - G_{dcba}}{c - a} + \frac{\lambda}{(1 + \mathcal{Y})^2} \frac{G_{cdbd} - G_{cdba}}{d - a}, \quad (\text{A.16a}) \end{aligned}$$

$$I_{a|cd} := \int_0^{\Lambda^2} q dq G_{aq|cd}. \quad (\text{A.16b})$$

We regard $I_{a|cd}$ as a function which is independent of $G_{ab|cd}$ so that we can apply the Carleman formula (4.24). By linearity we first treat the auxiliary problem

$$\begin{aligned} \left(\frac{b_1}{a} + \frac{1 + \lambda \pi a \mathcal{H}_a^\Lambda[G_{\bullet 0}]}{a G_{a0}} \right) (a F_{ab_1|c_1 \dots c_N}) - \lambda \pi \mathcal{H}_a^\Lambda[\bullet F_{\bullet b_1|c_1 \dots c_N}] \\ = \frac{\lambda}{(1 + \mathcal{Y})^2} \frac{G_{c_2 \dots c_N c_1} - G_{c_2 \dots c_N a}}{c_1 - a}. \end{aligned} \quad (\text{A.17})$$

Comparing with (4.55) we notice that both sides of $G_{ab_1 c_1 \dots c_N} = \sum_{l=1}^{\frac{N}{2}} G_{b_1 c_1 \dots c_{2l-1}} F_{ab_1|c_{2l-1} \dots c_N}$ satisfy the same Carleman equation. We can thus compute the functions F by iteration:

$$\begin{aligned} F_{ab_1|c_1 c_2} &= \frac{G_{ab_1 c_1 c_2}}{G_{b_1 c_1}}, \\ F_{ab_1|c_1 c_2 c_3 c_4} &= \frac{G_{ab_1 c_1 c_2 c_3 c_4} - G_{b_1 c_1 c_2 c_3} F_{ab_1 c_3 c_4}}{G_{b_1 c_1}} = \frac{G_{ab_1 c_1 c_2 c_3 c_4} G_{b_1 c_3} - G_{b_1 c_1 c_2 c_3} G_{ab_1 c_3 c_4}}{G_{b_1 c_1} G_{b_1 c_3}}. \end{aligned} \quad (\text{A.18})$$

Since G_{ab} is differentiable, these functions extend to coinciding indices c_i so that the inhomogeneity $\frac{\lambda}{(1+\mathcal{Y})^2} \frac{G_{dabc} - G_{dcba}}{c-a} + \frac{\lambda}{(1+\mathcal{Y})^2} \frac{G_{cdbd} - G_{cdba}}{d-a}$ in (A.16a) gives rise to the contribution $F_{ab|cdcb} + F_{ab|dcdb}$ in $G_{ab|cd}$. The inhomogeneity proportional to $Z(1 + \mathcal{Y})G_{ab}$ needs to be treated by the Carleman formula (4.24):

$$\begin{aligned} G_{ab|cd} &= F_{ab|cdcb} + F_{ab|dcdb} \\ &\quad - \frac{\sin \vartheta_b(a)}{\lambda \pi a} \left\{ Z(1 + \mathcal{Y}) G_{ab} \left(\lambda I_{a|cd} + \frac{\lambda}{(1 + \mathcal{Y})^2} (G_{acdc} + G_{adcd}) \right) \cos \vartheta_b(a) \right. \\ &\quad \left. + e^{\mathcal{H}_a^\Lambda[\vartheta_b]} \mathcal{H}_a^\Lambda \left[e^{-\mathcal{H}_a^\Lambda[\vartheta_b]} \sin \vartheta_b(\bullet) Z(1 + \mathcal{Y}) G_{\bullet b} \left(\lambda I_{\bullet|cd} + \frac{\lambda}{(1 + \mathcal{Y})^2} (G_{\bullet cdc} + G_{\bullet dcd}) \right) \right] \right\} \\ &= F_{ab|cdcb} + F_{ab|dcdb} \\ &\quad - \frac{\sin \vartheta_b(a)}{\lambda \pi a} \cos \vartheta_b(a) G_{ab} Z(1 + \mathcal{Y}) \left(\lambda I_{a|cd} + \frac{\lambda}{(1 + \mathcal{Y})^2} (G_{acdc} + G_{adcd}) \right) \\ &\quad - G_{ab} Z(1 + \mathcal{Y}) \mathcal{H}_a^\Lambda \left[\frac{\sin^2 \vartheta_b(\bullet)}{\lambda \pi \bullet} \left(\lambda I_{\bullet|cd} + \frac{\lambda}{(1 + \mathcal{Y})^2} (G_{\bullet cdc} + G_{\bullet dcd}) \right) \right], \end{aligned} \quad (\text{A.19})$$

where (4.29) has been used. We multiply by λb and integrate over $b =: q$ to obtain an equation for $I_{a|cd} := \int_0^{\Lambda^2} q dq G_{aq|cd}$:

$$\begin{aligned} X_{a|cd} \left\{ \frac{Z^{-1}}{1 + \mathcal{Y}} + \frac{1}{\pi a} \int_0^{\Lambda^2} q dq \sin \vartheta_q(a) \cos \vartheta_q(a) G_{aq} \right\} + \mathcal{H}_a^\Lambda \left[\frac{X_{\bullet|cd}}{\pi \bullet} \int_0^{\Lambda^2} q dq \sin^2 \vartheta_q(\bullet) G_{aq} \right] \\ = \lambda \int_0^{\Lambda^2} q dq (F_{aq|cdcq} + F_{aq|dcdq}) + \frac{\lambda}{(1 + \mathcal{Y})^2} (G_{acdc} + G_{adcd}), \end{aligned} \quad (\text{A.20a})$$

$$X_{a|cd} := Z(1 + \mathcal{Y}) \left(\lambda I_{a|cd} + \frac{\lambda}{(1 + \mathcal{Y})^2} (G_{acdc} + G_{adcd}) \right). \quad (\text{A.20b})$$

Using $\frac{q}{\lambda\pi a} = \cot \vartheta_q(a) - \cot \vartheta_0(a)$ and inserting Z^{-1} from (4.17), we have

$$\begin{aligned} & \frac{Z^{-1}}{1+\mathcal{Y}} + \frac{1}{\pi a} \int_0^{\Lambda^2} q dq \sin \vartheta_q(a) \cos \vartheta_q(a) G_{aq} \\ &= 1 + \lambda \int_0^{\Lambda^2} dq (G_{aq} - G_{0q}) - \lambda \int_0^{\Lambda^2} dq \frac{G_{aq} \sin \vartheta_q(a) \cos (\vartheta_q(a) - \vartheta_0(a))}{\sin \vartheta_0(a)}. \end{aligned} \quad (\text{A.20c})$$

All integrals in this equation, and also $\int_0^{\Lambda^2} q dq \sin^2 \vartheta_q(b) G_{aq}$ and $\int_0^{\Lambda^2} q dq (F_{aq|cdcq} + F_{aq|dcdq})$ exist for $\Lambda \rightarrow \infty$, even in perturbation theory. Therefore, (A.20a) is again a Carleman singular integral equation (4.23) for $X_{\bullet|cd}$ where the input functions h and f only depend on the known data G_{ab} and $\vartheta_b(a)$ and exist in the limit $\Lambda \rightarrow \infty$. Its solution is given by (4.24). Inserted into (A.19) we thus get the (in principle explicit) solution

$$G_{ab|cd} = F_{ab|cdcb} + F_{ab|dadb} - \frac{\sin \vartheta_b(a)}{\lambda\pi a} \cos \vartheta_b(a) G_{ab} X_{a|cd} - G_{ab} \mathcal{H}_a^\Lambda \left[\frac{\sin^2 \vartheta_b(\bullet)}{\lambda\pi\bullet} X_{\bullet|cd} \right]. \quad (\text{A.21})$$

Higher (N_1+N_2) -point functions with N_i even are obtained from (A.10) for $E_a - E_b \mapsto Z^2 \mu^2 (1+\mathcal{Y})(a-b)$ and $\lambda_4 = Z^2 \lambda$. In the simplest case ($N=6, l=2$) the solution in dimensionless functions (A.11) reads

$$\begin{aligned} G_{ab_1b_2b_3|c_1c_2} &= \frac{\lambda}{(1+\mathcal{Y})^2} \frac{G_{b_1a} G_{b_2b_3|c_1c_2} - G_{b_1b_2} G_{ab_3|c_1c_2} + G_{b_1a|c_1c_2} G_{b_2b_3} - G_{b_1b_2|c_1c_2} G_{ab_3}}{(b_2-a)(b_1-b_3)} \\ &+ \frac{\lambda}{(1+\mathcal{Y})^2} \left(\frac{G_{ab_1b_2b_3c_1c_2} - G_{c_1b_1b_2b_3ac_2}}{(c_1-a)(b_1-b_3)} + \frac{G_{c_1ab_1b_2b_3c_2} - G_{c_1c_2b_1b_2b_3a}}{(c_2-a)(b_1-b_3)} \right). \end{aligned} \quad (\text{A.22})$$

B Perturbative expansion

It is interesting to expand our solutions for the correlation functions into power series in λ . These series should reproduce the expansion of the partition function into ribbon graphs, and indeed this agreement was for us a non-trivial consistency check of our equations. The Feynman rules are:

- $\frac{1}{1+(1+\mathcal{Y})(a+b)}$ for a (fat) line separating faces with indices a, b
- $(-Z^2\lambda)$ for a (ribbon) vertex with four outgoing ribbons and index conservation at every corner,
- $(1+\mathcal{Y})^2 \int_0^{\Lambda^2} pdp$ for every closed face with index p .

These rules lead to a (for $\Lambda \rightarrow \infty$) divergent one-particle irreducible (1PI) two-point function Γ_{ab} which therefore needs renormalisation through subtraction of the constant and linear Taylor terms. There is no subtraction for the four-point function: As result of the vanishing β -function, the divergences in the ribbon graphs of the 1PI four-point function for $\Lambda \rightarrow \infty$ cancel exactly with the divergence of the wavefunction renormalisation Z .

All of the following results are given for the limit $\Lambda \rightarrow \infty$. Starting point is the master equation (4.48) for G_{b_0} which yields $G_{a_0} = \frac{1}{a+1} + \mathcal{O}(\lambda)$. To next order we thus have

$$G_{a_0} = \frac{1}{1+a} - \lambda \frac{\log(1+a)}{(1+a)} + \mathcal{O}(\lambda^2).$$

For the Hilbert transform we have

$$\lambda \pi \mathcal{H}_a^\infty[G_{\bullet_0}] = -\lambda \frac{\log(a)}{1+a} + \mathcal{O}(\lambda^2).$$

This yields for (4.26b) the expansions

$$\begin{aligned} \vartheta_b(a) &= \lambda \frac{a\pi}{1+a+b} - \lambda^2 \frac{a\pi((1+a)\log(1+a) - a\log a)}{(1+a+b)^2} + \mathcal{O}(\lambda^3), \\ \mathcal{H}_a^\infty[\vartheta_b] - \mathcal{H}_0^\infty[\vartheta_0] &= \lambda \frac{-(1+b)\log(1+b) - a\log a}{1+a+b} + \mathcal{O}(\lambda^2), \end{aligned} \quad (\text{B.1})$$

which inserted into (4.29) give

$$G_{ab} = \frac{1}{1+a+b} - \lambda \frac{(1+a)\log(1+a) + (1+b)\log(1+b)}{(1+a+b)^2} + \mathcal{O}(\lambda^2) \quad (\text{B.2})$$

for the 2-point function. This result is reproduced by renormalised one-loop ribbon graphs if we take the following expansion of \mathcal{Y} (see (4.30)) and $Z(1+\mathcal{Y}) = e^{\mathcal{H}_0^\infty[\vartheta_0]}$ (see (4.32)) into account (strictly speaking, Z is only defined for finite Λ):

$$\mathcal{Y} = \lambda + \mathcal{O}(\lambda^2), \quad Z(1+\mathcal{Y}) = 1 + \mathcal{O}(\lambda). \quad (\text{B.3})$$

It is interesting to compare this with the perturbative solution of the true master equation (4.36). This leads to $\mathcal{T}_a = 1 + a + \mathcal{O}(\lambda)$ and then in next order

$$\mathcal{T}_a = 1 + a + \lambda \left((1+a)\log(1+a) + a\log a \right) + \mathcal{O}(\lambda^2) = \frac{\lambda\pi a}{\tan \vartheta_0(a)}, \quad (\text{B.4})$$

in agreement with (B.1). We see this as good indication that in the limit $\Lambda \rightarrow \infty$ the master equations (4.36) and (4.48) are equivalent.

From (3.13) we then obtain the 4-point function $G_{abcd} =: -G_{ab}G_{bc}G_{cd}G_{da}\Gamma_{abcd}$ up to one loop, with the 1PI contribution

$$\begin{aligned} \Gamma_{abcd} &= \lambda \left(1 - \lambda \frac{a - (1+a)\log(1+a) - c + (1+c)\log(1+c)}{a-c} \right. \\ &\quad \left. - \lambda \frac{b - (1+b)\log(1+b) - d + (1+d)\log(1+d)}{b-d} \right) + \mathcal{O}(\lambda^3) \end{aligned} \quad (\text{B.5})$$

that agrees with the ribbon graph calculation. For the 6-point function, the result of (3.14) can be arranged as

$$G_{abcdef} = G_{ab}G_{bc}G_{cd}G_{de}G_{ef}G_{fa} \left(\Gamma_{abcd}G_{ad}\Gamma_{defa} + \Gamma_{bcde}G_{be}\Gamma_{efab} + \Gamma_{cdef}G_{cf}\Gamma_{fabc} - \Gamma_{abcdef} \right),$$

$$\begin{aligned}
\Gamma_{abcdef} &= (-\lambda)^3 \frac{(a-c)(1+e) \log(1+e) + (c-e)(1+a) \log(1+a) + (e-a)(1+c) \log(1+c)}{(a-c)(c-e)(e-a)} \\
&\quad + (-\lambda)^3 \frac{(b-d)(1+f) \log(1+f) + (d-f)(1+b) \log(1+b) + (f-b)(1+d) \log(1+d)}{(b-d)(d-f)(f-b)} \\
&\quad + \mathcal{O}(\lambda^4), \tag{B.6}
\end{aligned}$$

also in agreement with the ribbon graph calculation.

For the (1+1)-point function (A.14) we need

$$\begin{aligned}
\mathcal{H}_a^\Lambda[\vartheta] - \mathcal{H}_c^\Lambda[\vartheta] &= \lambda \left(\frac{c \log c + c \log 2}{1+2c} - \frac{a \log a + a \log 2}{1+2a} \right) + \mathcal{O}(\lambda^2), \\
\mathcal{H}_b^\Lambda [e^{\mathcal{H}_a^\Lambda[\vartheta] - \mathcal{H}_c^\Lambda[\vartheta]} \sin \vartheta(\bullet) G_{\bullet c}] &= \lambda \left(\frac{(1+c) \log(1+c)}{(1+2c)(1+b+c)} + \frac{\log 2}{(1+2c)(1+2b)} \right. \\
&\quad \left. - \frac{b \log b}{(1+b+c)(1+2b)} \right) + \mathcal{O}(\lambda^2).
\end{aligned}$$

Now (A.14) gives after straightforward but lengthy computation

$$\begin{aligned}
G_{a|c} &= -G_{aa} G_{cc} (\Gamma_{aacc} G_{ac} - \Gamma_{a|c}), \\
\Gamma_{a|c} &= \lambda^2 \left(\frac{\frac{(1+c) \log(1+c)}{1+c+c} - \frac{(1+a) \log(1+a)}{1+a+a}}{(c-a)} - \frac{\log(2)}{(1+a+a)(1+c+c)} \right) + \mathcal{O}(\lambda^3), \tag{B.7}
\end{aligned}$$

where $\Gamma_{a|c}$ agrees with the Feynman graph computation

$$\Gamma_{a|c} = \int_0^\infty p dp \frac{(-\lambda)^2}{(p+a+1)(p+p+1)(p+c+1)} + \mathcal{O}(\lambda^3).$$

For the (2+2)-point function (A.21) we first have to provide some intermediate results. From (A.18) and (B.6) we have

$$F_{ab|cdcb} = G_{ab} G_{cd} G_{dc} G_{ba} \left(\Gamma_{abcd} G_{ad} G_{bc} \Gamma_{dcba} + \Gamma_{bcde} G_{bc} G_{bc} \Gamma_{cbab} - G_{bc} \Gamma_{abcdcb} \right).$$

Next we compute $X_{a|cd}$ from (A.20a). Since $\mathcal{H}_a^\Lambda \left[\frac{X_{\bullet|cd}}{\pi \bullet} \int_0^{\Lambda^2} q dq \sin^2 \vartheta_q(\bullet) G_{aq} \right] = \mathcal{O}(\lambda^4)$, the equations (A.20a+A.20c) are to $\mathcal{O}(\lambda^3)$ purely algebraic with solution

$$\begin{aligned}
X_{a|cd} &= G_{cd}^2 \left\{ -G_{ac}^2 \Gamma_{acdc} \left(\lambda - \lambda^2 (1 - \log(1+a)) \right) - \Gamma_{adcd} G_{ad}^2 \left(\lambda - \lambda^2 (1 - \log(1+a)) \right) \right. \\
&\quad \left. - G_{ad} J_{aac} - G_{ac} J_{aad} - J_{aadd} - J_{aacc} \right\} + \mathcal{O}(\lambda^4), \tag{B.8a}
\end{aligned}$$

$$J_{aac} := \int_0^\infty p dp \frac{(-\lambda^3)}{(1+a+p)^2(1+c+p)} = (-\lambda)^3 \frac{a-c + (1+c)(\log(1+c) - \log(1+a))}{(a-c)^2}, \tag{B.8b}$$

$$\begin{aligned}
J_{aacc} &:= \int_0^\infty p dp \frac{(-\lambda^3)}{(1+a+p)^2(1+c+p)^2} \\
&= (-\lambda)^3 \frac{2c - 2a + (2+a+c) \log(1+a) - (2+a+c) \log(1+c)}{(a-c)^3}. \tag{B.8c}
\end{aligned}$$

From (B.8a) we obtain after some rearrangements

$$\begin{aligned}
& \mathcal{H}_a^\infty \left[\frac{\sin^2 \vartheta_b(\bullet)}{\lambda \pi^\bullet} X_{\bullet|cd} \right] \\
&= (-\lambda)^3 G_{cd}^2 G_{ab}^2 \left(-\frac{(1+c)\log(1+c) + a\log(a)}{(1+a+c)^2} + \frac{(1+b)\log(1+b) - (1+c)\log(1+c)}{(1+a+c)(b-c)} \right. \\
&\quad \left. - \frac{(1+d)\log(1+d) + a\log(a)}{(1+a+d)^2} + \frac{(1+b)\log(1+b) - (1+d)\log(1+d)}{(1+a+d)(b-d)} \right) \\
&\quad - G_{cd}^2 G_{ab} \left(J_{ccb} G_{ca} + J_{ddb} G_{da} + J_{bcc} + J_{bdd} \right) + \mathcal{O}(\lambda^4) . \tag{B.8d}
\end{aligned}$$

These results and $\Gamma_{abcdcb} = -J_{cca} - J_{bdd}$ give for (A.21)

$$\begin{aligned}
G_{ab|cd} &= G_{ab} G_{cd} G_{dc} G_{ba} \left(\Gamma_{abcd} G_{ad} G_{bc} \Gamma_{dcba} + \Gamma_{abdc} G_{ac} G_{bd} \Gamma_{cdba} + \Gamma_{bcdc} G_{bc} G_{bc} \Gamma_{cbab} \right. \\
&\quad \left. + \Gamma_{bdcd} G_{bd} G_{bd} \Gamma_{dbab} + \Gamma_{acdc} G_{ac} G_{ac} \Gamma_{caba} + \Gamma_{adcd} G_{ad} G_{ad} \Gamma_{daba} + \Gamma_{ab|cd} \right) , \\
\Gamma_{ab|cd} &= G_{ad} J_{aac} + G_{bd} J_{bbc} + G_{ac} J_{aad} + G_{bc} J_{bbd} + G_{cb} J_{cca} + G_{ca} J_{ccb} + G_{db} J_{dda} + G_{da} J_{ddb} \\
&\quad + J_{aacc} + J_{aadd} + J_{bbcc} + J_{bbdd} + \mathcal{O}(\lambda^4) . \tag{B.9}
\end{aligned}$$

Acknowledgements

We would like to thank the Erwin-Schrödinger-Institute for Mathematical Physics in Vienna and the Collaborative Research Centre “Groups, Geometry and Actions” (SFB 878) in Münster for financing several mutual visits. We also thank Alan Carey, Christian Krattenthaler and Vincent Rivasseau for helpful discussions. We are grateful to the referee for suggestions of improvements and most importantly for the remarks concerning the type of summability of the genus expansion.

References

- [AR94] A. Abdesselam and V. Rivasseau, “Trees, forests and jungles: A botanical garden for cluster expansions,” In: *Constructive physics*, pp. 7–36, Lect. Notes Phys. **446**, Springer-Verlag (1994) [hep-th/9409094].
- [Aiz81] M. Aizenman, “Proof of the triviality of ϕ_d^4 field theory and some mean field features of Ising models for $d > 4$,” Phys. Rev. Lett. **47** (1981) 1–4.
- [Avr69] C. Avramescu, “Sur l’existence des solutions convergentes des systèmes d’équations différentielles non linéaires,” Ann. Mat. Pura Appl. **81** (1969) 147–168.
- [BFSS97] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A conjecture,” Phys. Rev. D **55** (1997) 5112–5128 [hep-th/9610043].
- [Bax71] R. J. Baxter, “Eight-vertex model in lattice statistics,” Phys. Rev. Lett. **26** (1971) 832–833.
- [Bax80] R. J. Baxter, “Hard hexagons: exact solution,” J. Phys. A: Math. Gen. **13** (1980) L61–L70.
- [BGI02] C. Becchi, S. Giusto and C. Imbimbo, “The Wilson-Polchinski renormalization group equation in the planar limit,” Nucl. Phys. B **633** (2002) 250–270 [hep-th/0202155].

- [BGI03] C. Becchi, S. Giusto and C. Imbimbo, “The renormalization of noncommutative field theories in the limit of large noncommutativity,” Nucl. Phys. B **664** (2003) 371–399 [hep-th/0304159].
- [BPZ84] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, “Infinite conformal symmetry in two-dimensional quantum field theory,” Nucl. Phys. B **241** (1984) 333–380.
- [Bet31] H. Bethe, “Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette,” Z. Phys. **71** (1931) 205–226.
- [BK90] E. Brezin and V. A. Kazakov, “Exactly solvable field theories of closed strings,” Phys. Lett. B **236** (1990) 144–150.
- [BF09] R. Brunetti and K. Fredenhagen, “Quantum field theory on curved backgrounds,” Lect. Notes Phys. **786** (2009) 129–155 [arXiv:0901.2063 [gr-qc]].
- [BK87] D. C. Brydges and T. Kennedy, “Mayer expansions and the Hamilton-Jacobi equation,” J. Stat. Phys. **48** (1987) 19–49.
- [Car22] T. Carleman, “Sur la résolution de certaines équations intégrales,” Arkiv for Mat., Astron. och Fysik **16** (1922), 19pp.
- [CC96] A. H. Chamseddine and A. Connes, “The spectral action principle,” Commun. Math. Phys. **186** (1997) 731–750 [hep-th/9606001].
- [CCMX11] F. Cianciaruso, V. Colao, G. Marino and H.-K. Xu, “A compactness result for differentiable functions with an application to boundary value problems,” Ann. Mat. Pura Appl. (2011), DOI 10.1007/s10231-011-0230-1.
- [Con94] A. Connes, *Noncommutative geometry*, Academic Press, San Diego (1994).
- [Con96] A. Connes, “Gravity coupled with matter and foundation of noncommutative geometry,” Commun. Math. Phys. **182** (1996) 155–176 [hep-th/9603053].
- [Con08] A. Connes, “On the spectral characterization of manifolds,” arXiv:0810.2088 [math.OA].
- [DGZ95] P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, “2D gravity and random matrices,” Phys. Rept. **254** (1995) 1–133 [hep-th/9306153].
- [DR07] M. Disertori and V. Rivasseau, “Two and three loops beta function of non commutative ϕ_4^4 theory,” Eur. Phys. J. C **50** (2007) 661–671 [hep-th/0610224].
- [DGMR07] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, “Vanishing of beta function of non commutative ϕ_4^4 theory to all orders,” Phys. Lett. B **649** (2007) 95–102 [hep-th/0612251].
- [DFR95] S. Doplicher, K. Fredenhagen and J. E. Roberts, “The Quantum structure of space-time at the Planck scale and quantum fields,” Commun. Math. Phys. **172** (1995) 187–220 [hep-th/0303037].
- [DS90] M. R. Douglas and S. H. Shenker, “Strings in less than one dimension,” Nucl. Phys. B **335** (1990) 635–654.
- [Fad95] L. D. Faddeev, “Instructive history of the quantum inverse scattering method,” Acta Appl. Math. **39** (1995) 69–84.
- [FMRS85] J. Feldman, J. Magnen, V. Rivasseau and R. Seneor, “Massive Gross-Neveu model: A rigorous perturbative construction,” Phys. Rev. Lett. **54** (1985) 1479–1481.
- [FO76] J. S. Feldman and K. Osterwalder, “The Wightman axioms and the mass gap for weakly coupled ϕ_3^4 quantum field theories,” Annals Phys. **97** (1976) 80–135.
- [Fil96] T. Filk, “Divergencies in a field theory on quantum space,” Phys. Lett. B **376** (1996) 53–58.

- [FQS84] D. Friedan, Z.-a. Qiu and S. H. Shenker, “Conformal invariance, unitarity and two-dimensional critical exponents,” *Phys. Rev. Lett.* **52** (1984) 1575–1578.
- [Frö82] J. Fröhlich, “On the triviality of $\lambda\phi_d^4$ theories and the approach to the critical point in $d \geq 4$ dimensions,” *Nucl. Phys. B* **200** (1982) 281–296.
- [GK85] K. Gawedzki and A. Kupiainen, “Gross-Neveu model through convergent perturbation expansions,” *Commun. Math. Phys.* **102** (1985) 1–30.
- [GGISV03] V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker and J. C. Várilly, “Moyal planes are spectral triples,” *Commun. Math. Phys.* **246** (2004) 569–623.
- [GW11] V. Gayral and R. Wulkenhaar, “Spectral geometry of the Moyal plane with harmonic propagation,” arXiv:1108.2184 [math.OA], to appear in *J. Noncommut. Geom.*
- [Gin70] J. Ginibre, “General formulation of Griffiths’ inequalities,” *Comm. Math. Phys.* **16** (1970) 310–328.
- [GJ68-72] J. Glimm and A. M. Jaffe, “The $\lambda\phi_2^4$ quantum field theory without cut-offs, I.” *Phys. Rev.* **176** (1968) 1945–1951.
 II. The field operators and the approximate vacuum, *Ann. Math.* **91** (1970) 362–401.
 III. The physical vacuum, *Acta Math.* **125** (1970) 203–267.
 IV. Perturbations of the Hamiltonian,” *J. Math. Phys.* **13** (1972) 1568–1584.
- [GJ73] J. Glimm and A. M. Jaffe, “Positivity of the ϕ_3^4 Hamiltonian,” *Fortsch. Phys.* **21** (1973) 327–376.
- [GJS74] J. Glimm, A. M. Jaffe and T. Spencer, “The Wightman axioms and particle structure in the $P(\phi)_2$ quantum field model,” *Ann. of Math.* **100** (1974) 585–632.
- [GJ87] J. Glimm and A. M. Jaffe, *Quantum physics. A functional integral point of view*, Springer-Verlag, New York (1987).
- [GV88] J. M. Gracia-Bondía and J. C. Várilly, “Algebras of distributions suitable for phase space quantum mechanics. I.,” *J. Math. Phys.* **29** (1988) 869–879.
 II. “Topologies on the Moyal algebra,” *J. Math. Phys.* **29** (1988) 880–887.
- [GR94] I.S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press (1994).
- [GM90] D. J. Gross and A. A. Migdal, “Nonperturbative two-dimensional quantum gravity,” *Phys. Rev. Lett.* **64** (1990) 127–130.
- [GN74] D. J. Gross and A. Neveu, “Dynamical symmetry breaking in asymptotically free field theories,” *Phys. Rev. D* **10** (1974) 3235–3253.
- [GW73] D. J. Gross and F. Wilczek, “Ultraviolet behavior of nonabelian gauge theories,” *Phys. Rev. Lett.* **30** (1973) 1343–1346.
- [GS06] H. Grosse and H. Steinacker, “Renormalization of the noncommutative ϕ^3 -model through the Kontsevich model,” *Nucl. Phys. B* **746** (2006) 202–226 [hep-th/0512203].
- [GS08] H. Grosse and H. Steinacker, “Exact renormalization of a noncommutative ϕ^3 model in 6 dimensions,” *Adv. Theor. Math. Phys.* **12** (2008) 605–639 [hep-th/0607235].
- [GW04] H. Grosse and R. Wulkenhaar, “The β -function in duality-covariant noncommutative ϕ^4 -theory,” *Eur. Phys. J. C* **35** (2004) 277–282 [hep-th/0402093].
- [GW05a] H. Grosse and R. Wulkenhaar, “Power-counting theorem for non-local matrix models and renormalisation,” *Commun. Math. Phys.* **254** (2005) 91–127 [hep-th/0305066].
- [GW05b] H. Grosse and R. Wulkenhaar, “Renormalisation of ϕ^4 -theory on noncommutative \mathbb{R}^4 in the matrix base,” *Commun. Math. Phys.* **256** (2005) 305–374 [hep-

- th/0401128].
- [GW05c] H. Grosse and R. Wulkenhaar, “Renormalization of ϕ^4 -theory on noncommutative \mathbb{R}^4 to all orders,” *Lett. Math. Phys.* **71** (2005) 13–26 [hep-th/0403232].
 - [GW09] H. Grosse and R. Wulkenhaar, “Progress in solving a noncommutative quantum field theory in four dimensions,” arXiv:0909.1389 [hep-th].
 - [GW12] H. Grosse and R. Wulkenhaar, “8D-spectral triple on 4D-Moyal space and the vacuum of noncommutative gauge theory,” *J. Geom. Phys.* **62** (2012) 1583–1599 [arXiv:0709.0095 [hep-th]].
 - [GW13a] H. Grosse and R. Wulkenhaar, “Solvable limits of a 4D noncommutative QFT,” arXiv:1306.2816 [math-ph].
 - [GW13b] H. Grosse and R. Wulkenhaar, “Numerical investigation of an exactly solvable noncommutative quantum field theory in four dimensions,” in preparation.
 - [GRS75] F. Guerra, L. Rosen and B. Simon, “The $P(\phi)_2$ Euclidean quantum field theory as classical statistical mechanics,” *Ann. of Math.* **101** (1975) 111–189 and *Ann. of Math.* **101** (1975) 191–259.
 - [GMRV06] R. Gurau, J. Magnen, V. Rivasseau and F. Vignes-Tourneret, “Renormalization of non-commutative ϕ_4^4 field theory in x space,” *Commun. Math. Phys.* **267** (2006) 515–542 [hep-th/0512271].
 - [Haa92] R. Haag, *Local quantum physics: Fields, particles, algebras*, Springer-Verlag, Berlin (1992).
 - [HK64] R. Haag and D. Kastler, “An algebraic approach to quantum field theory,” *J. Math. Phys.* **5** (1964) 848–861.
 - [Hag67] C. R. Hagen, “The Thirring model,” *Nuovo Cim. B* **51** (1967) 169–186.
 - [IKKT97] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A large- N reduced model as superstring,” *Nucl. Phys. B* **498** (1997) 467–491 [hep-th/9612115].
 - [Isi25] E. Ising, “Beitrag zur Theorie des Ferromagnetismus,” *Z. Phys.* **31** (1925) 253–258.
 - [Jaf00] A. M. Jaffe, “Constructive quantum field theory,” in: *Mathematical physics 2000*, pp. 111–127, eds. A. Fokas et al, Imperial College Press, London (2000).
 - [JW00] A. M. Jaffe and E. Witten, “Quantum Yang-Mills theory,” in: *The millenium prize problems*, pp. 129–152, eds. J. Carlson et al, Amer. Math. Soc., Providence (2006).
 - [Jim90] M. Jimbo (ed.), *Yang-Baxter equation in integrable systems*, World Scientific (1990).
 - [Joh61] K. Johnson, “Solution of the equations for the Green’s functions of a two-dimensional relativistic field theory,” *Nuovo Cim.* **20** (1961) 773–790.
 - [Kac49] M. Kac, “On distributions of certain Wiener functionals,” *Trans. Amer. Math. Soc.* **65** (1949) 1–13.
 - [Kla67] B. Klaiber, “The Thirring model,” in: *Boulder 1967 lectures in theoretical physics, Vol. Xa – Quantum theory and statistical physics*, pp. 141–176, New York (1968).
 - [Kon82] M. Kontsevich, “Intersection theory on the moduli space of curves and the matrix Airy function,” *Commun. Math. Phys.* **147** (1992) 1–23.
 - [Kor79] V. E. Korepin, “Direct calculation of the S-matrix in the massive Thirring model,” *Theor. Math. Phys.* **41** (1979) 953–967 [*Teor. Mat. Fiz.* **41** (1979) 169–189].
 - [KW41] H. A. Kramers and G. H. Wannier, “Statistics of the two-dimensional ferromagnet. I+II,” *Phys. Rev.* **60** (1941) 252–262, 263–276.
 - [LAK54] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, “On the removal of infinities in quantum electrodynamics,” (in russ.) *Dokl. Akad. Nauk SSSR* **95** (1954) 497–500.

- “Asymptotic expression of the electron Green function in quantum electrodynamics,” (in russ.) Dokl. Akad. Nauk SSSR **95** (1954) 773–776.
- “An asymptotic expression for the photon Green function in quantum electrodynamics,” (in russ.) Dokl. Akad. Nauk SSSR **95** (1954) 1117–1120.
- [LS02] E. Langmann and R. J. Szabo, “Duality in scalar field theory on noncommutative phase spaces,” Phys. Lett. B **533** (2002) 168–177 [hep-th/0202039].
- [LSZ04] E. Langmann, R. J. Szabo and K. Zarembo, “Exact solution of quantum field theory on noncommutative phase spaces,” JHEP **0401** (2004) 017 [hep-th/0308043].
- [Lie67] E. H. Lieb, “Residual entropy of square ice,” Phys. Rev. **162** (1967) 162–172.
- [MR08] J. Magnen and V. Rivasseau, “Constructive ϕ^4 field theory without tears,” Annales Henri Poincaré **9** (2008) 403–424 [arXiv:0706.2457 [math-ph]].
- [MVS00] S. Minwalla, M. Van Raamsdonk and N. Seiberg, “Noncommutative perturbative dynamics,” JHEP **0002** (2000) 020 [hep-th/9912072].
- [Mus65] N. I. Muskhelishvili, *Singuläre Integralgleichungen*, Akademie-Verlag Berlin (1965).
- [Nel71] E. Nelson, “Quantum fields and Markoff fields,” in: *Partial differential equations*, pp. 413–420, ed. D. C. Spencer, Amer. Math. Soc., Providence (1973).
- [OS73] K. Osterwalder and R. Schrader, “Axioms for Euclidean Green’s functions,” Commun. Math. Phys. **31** (1973) 83–112.
- [OS75] K. Osterwalder and R. Schrader, “Axioms for Euclidean Green’s functions II,” Commun. Math. Phys. **42** (1975) 281–305.
- [Ons44] L. Onsager, “Crystal statistics. I. A two-dimensional model with an order-disorder transition,” Phys. Rev. **65** (1944) 117–149.
- [Pol73] H. D. Politzer, “Reliable perturbative results for strong interactions?,” Phys. Rev. Lett. **30** (1973) 1346–1349.
- [Rie93] M. A. Rieffel, “Deformation quantization for actions of \mathbb{R}^d ,” Memoirs AMS **506** (1993) 1–96.
- [Riv83] V. Rivasseau, “Construction and Borel summability of planar four-dimensional Euclidean field theory,” Commun. Math. Phys. **95** (1984) 445–486.
- [Riv91] V. Rivasseau, *From perturbative to constructive renormalization*, Princeton University Press (1991).
- [RVW06] V. Rivasseau, F. Vignes-Tourneret and R. Wulkenhaar, “Renormalisation of non-commutative ϕ^4 -theory by multi-scale analysis,” Commun. Math. Phys. **262** (2006) 565–594 [hep-th/0501036].
- [Riv07a] V. Rivasseau, “Non-commutative renormalization” In: *Quantum spaces (Séminaire Poincaré X)*, pp 19–109, eds. B. Duplantier and V. Rivasseau, Birkhäuser Verlag Basel (2007) [arXiv:0705.0705 [hep-th]].
- [Riv07b] V. Rivasseau, “Constructive matrix theory,” JHEP **0709** (2007) 008 [arXiv:0706.1224 [hep-th]].
- [RW12] V. Rivasseau and Z. Wang, “Constructive renormalization for Φ_2^4 theory with loop vertex expansion,” J. Math. Phys. **53** (2012) 042302 [arXiv:1104.3443 [math-ph]].
- [Sch59] J. Schwinger, “Euclidean quantum electrodynamics,” Phys. Rev. **115** (1959) 721–731.
- [Sch62] J. S. Schwinger, “Gauge invariance and mass. II,” Phys. Rev. **128** (1962) 2425–2429.
- [SW64] R. F. Streater and A. S. Wightman, *PCT, spin and statistics, and all that*, Benjamin, New York (1964).

- [Sym64] K. Symanzik, *A modified model of Euclidean quantum field theory*, Courant Institute of Mathematical Sciences, New York University, Report IMM-NYU 327 (1964).
- [Thi58] W. E. Thirring, “A soluble relativistic field theory?,” *Annals Phys.* **3** (1958) 91–112.
- [tHo74] G. ’t Hooft, “A planar diagram theory for strong interactions,” *Nucl. Phys. B* **72** (1974) 461–473.
- [tHo82] G. ’t Hooft, “Rigorous construction of planar diagram field theories in four-dimensional Euclidean space,” *Commun. Math. Phys.* **88** (1983) 1–25.
- [Tri57] F. G. Tricomi, *Integral equations*, Interscience, New York (1957).
- [Wan12] Z. Wang, “Constructive renormalization of 2-dimensional Grosse-Wulkenhaar model,” arXiv:1205.0196 [hep-th].
- [WZ71] J. Wess and B. Zumino, “Consequences of anomalous Ward identities,” *Phys. Lett. B* **37** (1971) 95–97.
- [Wig56] A. S. Wightman, “Quantum field theory in terms of vacuum expectation values,” *Phys. Rev.* **101** (1956) 860–866.
- [WG64] A. S. Wightman and L. Gårding, “Fields as operator-valued distributions in quantum field theory,” *Ark. Fys.* **28** (1964) 129–184.
- [WK74] K. G. Wilson and J. B. Kogut, “The renormalization group and the ϵ -expansion,” *Phys. Rept.* **12** (1974) 75–200.
- [Wit84] E. Witten, “Nonabelian bosonization in two dimensions,” *Commun. Math. Phys.* **92** (1984) 455–472.
- [Wit91] E. Witten, “Two-dimensional gravity and intersection theory on moduli space,” *Surveys Diff. Geom.* **1** (1991) 243–310.