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1. Introduction

The property of ε-regularity for weak solutions plays an important role in characterizing singularities of such solutions, which enables us to obtain global smooth solutions. For harmonic maps on compact Riemann surfaces, Struwe [7] proved the ε-regularity of weak solutions $u(x,t)$ to the heat flow. Indeed, around the singular point (x,t), it occurs the concentration of energy as

$$\limsup_{r\to 0} \int_{B_r(x)} e(u(t))\, dV \geq \varepsilon,$$

where $e(u)$ is the energy density of u. Since the total energy $\int_M e(u(t))\, dV$ is monotone decreasing in time, there exist at most finitely many singularities $\{(x_i,t_i)\}_{i=1}^m$ of u.

For the Yang-Mills functional on compact 4-manifolds, Struwe [8] and Kozono, Maeda and Naito [6] gave a similar criterion and proved that there exists an $\varepsilon > 0$ such that the concentration of energy of the curvature form F_A occurs like

$$\limsup_{r\to 0} \int_{B_r(x)} |F_A(t)|^2\, dV \geq \varepsilon$$

for every singular point (x,t) of the connection $A(t)$. Making use of the gauge transformations by Uhlenbeck, they proved that there exist at most finitely many singularities of the connection $A(t,x)$ for the Yang-Mills gradient flow.

In the present paper, we show the ε-regularity of gradient flow for the Yang-Mills-Higgs heat flow on the trivial $SU(2)$-bundle over Euclidean 3-space. Let $P = \mathbb{R}^3 \times SU(2)$ be the trivial bundle over $\mathbb{R}^3$. Consider the Yang-Mills-Higgs functional: for a connection A of P and a $\mathfrak{su}(2)$-valued function Φ on $\mathbb{R}^3$, we set

$$E(A,\Phi) = \int_{\mathbb{R}^3} \left(|F_A|^2 + |d_A\Phi|^2\right) dV, \tag{1.1}$$

where d_A and F_A denote the covariant exterior differentiation and the curvature form of A, respectively. It is known that the Yang-Mills-Higgs functional has an interesting aspect on the following compactified configuration space (cf, [2]):

$$\mathcal{C} = \{(A, \Phi) : E(A, \Phi) < \infty, |\Phi(x)| \to 1 \text{ as } |x| \to \infty\}.$$

On the configuration space $\mathcal{C}$, we define $N(A, \Phi)$ as

$$N(A, \Phi) = \frac{1}{4\pi} \int_{\mathbb{R}^3} F_A \wedge d_A \Phi.$$

which is called the monopole number (or magnetic charge) for (A, Φ). Groissor [2] showed that if $(A, \Phi) \in \mathcal{C}$, then $N(A, \Phi)$ is an integer and the functional $N : \mathcal{C} \to \mathbb{Z}$ gives a path component decomposition on $\mathcal{C}$. By restricting Φ to a sufficiently large 2-sphere in $\mathbb{R}^3$, it determines a homotopy class of maps on S^2. Let S_∞ be the ideal boundary of $\mathbb{R}^3$. We can identify S_∞ with $S^2(1)$ canonically: associated to Φ, we define a map $\hat{\Phi} : S_\infty \to S^2$ by

$$\hat{\Phi}(\omega) = \lim_{r \to \infty} \frac{\Phi(r, \omega)}{|\Phi(r, \omega)|},$$

where $\Phi(r, \omega) = \Phi(x)$, $r = |x|$, $\omega = \dfrac{x}{|x|}$. Then, we have $N(A, \Phi) = -\deg(\hat{\Phi})$. Furthermore, $2N(A, \Phi)$ gives the first Chern number of some bundle over S^2.

Critical points of the functional (1.1) is called Yang-Mills-Higgs configurations. If (A, Φ) satisfies $F_A = \pm * d_A \Phi$, then (A, Φ) is a Yang-Mills-Higgs configuration. We call such (A, Φ) (anti-)monopole, and $B = F_A - *d_A \Phi$ is called Bogomolny tensor. Existence of monopoles and general Yang-Mills-Higgs configuration are given by Jaffe-Taubes [5] and Taubes [9, 10].

Consider the following heat flow associated with the Yang-Mills-Higgs functional:

$$\begin{cases} \partial_t A = -d_A^* F_A - [\Phi, d_A \Phi], \\ \partial_t \Phi = \Delta_A \Phi. \end{cases} \tag{1.2}$$

We study a property of the regularity for solutions to (1.2). In [3], Hassell showed the existence of the global smooth solutions of (1.2) under the assumption that some norm of the Bogomolny tensor of the initial data is small. We now consider the regularity of the solutions of (1.2) without smallness of the initial data. Let us call a pair $(A(t), \Phi(t))$ in the configuration space $\mathcal{C}$ *smooth* solution of (1.2), if $(A(t), \Phi(t))$ satisfies (1.2) in the classical sense. Now, we introduce the following notion:

Definition 1.1. A smooth solution $(A(t), \Phi(t))$ of (1.2) is called *extendable* if the following conditions are satisfied:

(1) For each $t \in (0, T]$, there exists a gauge transformation $g(t)$ such that $g^*(t)A(t)$ extends to a smooth connection over $S_\infty \cong S^2$.
(2) $N(A(t), \Phi(t))$ is constant for all $t \in (0, T]$.

For $\omega_0 \in S^2$, let $B_\tau(\omega_0)$ be the geodesic ball centered at ω_0 with the radius τ. Consider a smooth solution $(A(t), \Phi(t))$ of (1.2) with the following property:

$$\liminf_{r\to\infty} \int_{B_\tau(\omega_0)} r^2(|F_A(t,r,\omega)| + |d_A\Phi(t,r,\omega)|)\, d\omega \le \varepsilon_1, \tag{1.3}$$

for sufficient small τ, for all $t \in (0, T]$ and for all $\omega_0 \in S^2$. Our theorem now reads:

Theorem 1.2. *There exists a universal constant $\varepsilon_1 > 0$ such that if the smooth solution $(A(t), \Phi(t))$ of (1.2) with the following initial conditions (1.4), (1.5) and (1.6):*

(1.4)
$$|\nabla_A^n F_A(0,x)| + |\nabla_A^n d_A\Phi(0,x)| \le C|x|^{-n-2}, \quad \textit{for all } n \in \mathbb{N} \cup \{0\},$$

(1.5)
$$(A(0), \Phi(0)) \in \mathcal{C}$$

(1.6)
$$|1 - |\Phi(0,x)|^2| \le C|x|^{-1},$$

with C independent of $x \in \mathbb{R}^3$, satisfies (1.3), then $(A(t), \Phi(t))$ is extendable .

Remark 1.3. It is easy to see that the Prasad-Sommerfield monopole [5, IV.1] satisfies the assumption (1.4), (1.5) and (1.6). (See Section 8.) It is also known that for any integer N, there exists at least one monopole solution (A, Φ) with $N(A, \Phi) = N$ such that (1.4) is fulfilled for $n = 0$. (See [5, p. 109]).

Compared with the harmonic maps and the Yang-Mills functional, we shall characterized the singularity as local concentration not of the energy functional but of the L^1-norms of the curvature tensor and the first derivative of the Higgs field. From a view point of nonlinear partial differential equations, such a characterization should be done in the L^p-space whose norms is invariant under the change of scaling. Unfortunately, for the Yang-Mills-Higgs functional, the bound of norms necessarily for getting smooth solution does not coinside with that of norm defining the energy functional. This causes a lot of difficulties to obtain the global regularity for weak solutions of Yang-Mills-Higgs gradient flow.

It would be interesting to find a global weak solution for the gradient flow (1.2) for Yang-Mills-Higgs functional without any smallness on the initial data. The above theorem will be useful to get a global solution of (1.2) in weak sense, which will be discussed in a forthcoming paper.

2. Preliminaries

In this section, we prepare some fundamental estimates.

Proposition 2.1. *Let $u\colon (0,T)\times(0,\infty) \longrightarrow \mathbb{R}$ be a smooth function satisfying $u \geq 0$, $u(0) \in L^\infty(0,\infty)$ and let $f \in C^0((0,T);L^1(0,\infty))$. Suppose that u and f satisfy*

$$\partial_t u - \partial_{rr} u + \frac{c_1}{r}\partial_r u - \frac{c_2}{r^2} u \leq f,$$

where c_1 and c_2 are non-negative constants. Then for any positive ε and R, there exists a positive constant λ and C_ε such that

$$\sup_{\substack{0<t<T\\ \varepsilon<r<\infty}} |u(t,r)| \leq e^{\lambda T} \sup_{\varepsilon<r<\infty} |u(0,r)| + C_{\varepsilon,R} e^{\lambda T} T^{1/2} \sup_{0<t<T} (\|f(t)\|_{L^1} + \|u(t)\|_{L^1(\varepsilon,R)}),$$

provided $u \in C^0(0,T;L^1(\varepsilon,R))$,

Proof. Fix positive numbers $R > \varepsilon > 0$. Let ϕ be a smooth non-decreasing function on $(0,\infty)$ such that $\phi = 0$ on $(0,\varepsilon)$ and $\phi = 1$ on (R,∞). Set $\tilde{u}(t,r) = \phi(r)u(t,r)$. Then $\tilde{u}$ satisfies

$$\begin{cases} \partial_t \tilde{u} - \partial_{rr}\tilde{u} + \dfrac{c_1}{r}\partial_r \tilde{u} - \dfrac{c_2}{r^2}\tilde{u} \leq -\phi'' u - 2\phi' u + \dfrac{c_3}{r} u\phi' + f, \\ \tilde{u}(t,\varepsilon) = 0. \end{cases} \tag{2.1}$$

Here we set $g(t,r) = -\phi'' u - 2\phi' u + \dfrac{c_3}{r} u\phi'$, and then $g(t,r) = 0$ on (R,∞).

Consider the equation

$$\begin{cases} \partial_t U - \partial_{rr} U + \dfrac{c_1}{r}\partial_r U - \dfrac{c_2}{r^2} U = f + g, \\ U(t,\varepsilon) = 0, U(t,\infty) = 0, \\ U(0,r) = \tilde{u}(0,r). \end{cases} \tag{2.2}$$

By a comparison theorem, we have

$$\tilde{u}(t,r) \leq U(t,r) \quad \text{for } x \geq \varepsilon. \tag{2.3}$$

Set $Hu = -\partial_{rr} u + \dfrac{c_1}{r}\partial_r u - \dfrac{c_2}{r^2} u$. Since $U(t,\varepsilon) = 0$, we have

$$\begin{aligned} \langle HU, U\rangle_{L^2} &= -\int_\varepsilon^\infty \partial_{rr} U\, U\, dr + c_1 \int_\varepsilon^\infty r^{-1}\partial_r U\, U\, dr - c_2 \int_\varepsilon^\infty r^{-2}|U|^2\, dr \\ &\geq \frac{1}{2}\int_\varepsilon^\infty |\partial_r U|^2\, dr - (c_2 + \frac{c_1^2}{2}) \int_\varepsilon^\infty r^{-2}|U|^2\, dr \\ &\geq -(c_2 + \frac{c_1^2}{2}) \int_\varepsilon^\infty r^{-2}|U|^2\, dr. \end{aligned}$$

Taking $\lambda > (c_2 + \frac{c_1^2}{2})\frac{1}{\varepsilon^2}$, we get

$$\langle (H+\lambda)U, U\rangle_{L^2} > 0. \tag{2.4}$$

The solution of (2.2) is written as

$$U(t,r) = e^{-tH}u_0 + \int_0^t e^{-(t-s)H}(f(s)+g(s))\,ds. \tag{2.5}$$

Use fundamental estimates for semi-group e^{-tH} and (2.4), and we have

$$\begin{aligned}
\|U(t,\cdot)\|_{L^\infty(\varepsilon,\infty)} &\le e^{\lambda t}\|u_0\|_{L^\infty(\varepsilon,\infty)} + \int_0^t \|e^{(t-s)H}(f(s)+g(s))\|_{L^\infty}\,ds \\
&\le e^{\lambda t}\|u_0\|_{L^\infty(\varepsilon,\infty)} + \int_0^t (t-s)^{-1/2}e^{\lambda(t-s)}\|f(s)+g(s)\|_{L^1}\,ds \\
&\le e^{\lambda T}\|u_0\|_{L^\infty(\varepsilon,\infty)} + 2e^{\lambda T}T^{1/2}\sup_{0<t<T}\|f(t)+g(t)\|_{L^1},
\end{aligned}$$

since $f(t,\cdot)\in L^1(0,\infty)$ by the assumption. Moreover, we have

$$\|g(t)\|_{L^1} = \|g(t)\|_{L^1(\varepsilon,R)},$$

since $\operatorname{supp} g \subset [\varepsilon, R]$. Thus we get

$$\|U(t,\cdot)\|_{L^\infty(\varepsilon,\infty)} \le e^{\lambda t}\|u_0\|_{L^\infty(\varepsilon,\infty)} + CT^{1/2}e^{\lambda T}\sup_{0<t<T}(\|f(t)\|_{L^1} + \|u(t)\|_{L^1(\varepsilon,R)}),$$

and by (2.3),

$$\|u(t,\cdot)\|_{L^\infty(\varepsilon,\infty)} \le e^{\lambda T}\|u_0\|_{L^\infty(\varepsilon,\infty)} + CT^{1/2}e^{\lambda T}\sup_{0<t<T}(\|f(t)\|_{L^1} + \|u(t)\|_{L^1(\varepsilon,R)}).$$

This completes the proof. □

Proposition 2.2. *Let $B_\tau(\omega)$ be the geodesic ball centered at ω with the radius τ. For $u\in W^{1,2}(B_\tau(\omega))$, $u\ge 0$, we have*

$$\int_{B_\tau(\omega)}|u|^3\,d\omega \le C\left(\int_{B_\tau(\omega)}|u|\,d\omega\right)\left(\int_{B_\tau(\omega)}|\nabla_\omega u|^2\,d\omega + \tau^{-2}\int_{B_\tau(\omega)}|u|^2\,d\omega\right),$$

where $d\omega$ and ∇_ω denote the standard volume form and the differentiation in the direction of S^2, respectively.

Proof. First we prove

$$\int_{B_\tau(\omega)} |u|^3 \, d\omega \le C \left(\int_{B_\tau(\omega)} |u| \, d\omega \right) \left(\int_{B_\tau(\omega)} |\nabla_\omega u|^2 \, d\omega \right), \tag{2.6}$$

for $u \in W_0^{1,2}(B_\tau(\omega))$. By Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} \|u\|_{L^3} &\le C \|u\|_{L^2}^{2/3} \|\nabla_\omega u\|_{L^2}^{1/3} \\ &\le C \left(\|u\|_{L^1}^{1/4} \|u\|_{L^3}^{3/4} \right)^{2/3} \|\nabla_\omega u\|_{L^2}^{1/3} \\ &= C \|u\|_{L^1}^{1/6} \|u\|_{L^3}^{1/2} \|\nabla_\omega u\|_{L^2}^{1/3}. \end{aligned} \tag{2.7}$$

Dividing the both sides of (2.7) by $\|u\|_{L^3}^{1/2}$, we have (2.6).

To obtain Proposition 2.2, we have

$$\int_{B_\tau} |u|^3 \, d\omega \le C \int_{B_\tau} |u - \bar{u}|^3 \, d\omega + C \int_{B_\tau} |\bar{u}|^3 \, d\omega, \tag{2.8}$$

where $\bar{u}$ is the average of u on $B_\tau(\omega)$. The second term of the right hand side of (2.8) is dominated by

$$\int_{B_\tau} |\bar{u}|^3 \, d\omega = |B_\tau|^{-2} \left(\int_{B_\tau} u \, d\omega \right)^3 \le |B_\tau|^{-1/2} \left(\int_{B_\tau} |u|^2 \, d\omega \right)^{3/2}. \tag{2.9}$$

By (2.7), (2.8) and (2.9), we have

$$\begin{aligned} \int_{B_\tau} |u|^3 \, d\omega &\le C \left[\int_{B_\tau} |u - \bar{u}|^2 \, d\omega \left(\int_{B_\tau} |\nabla_\omega u|^2 \, d\omega \right)^{1/2} + |B_\tau|^{-1/2} \left(\int_{B_\tau} |u|^2 \, d\omega \right)^{3/2} \right] \\ &\le C \left(\int_{B_\tau} |u|^2 \, d\omega \right) \left[\left(\int_{B_\tau} |\nabla_\omega u|^2 \, d\omega \right)^{1/2} + |B_\tau|^{-1/2} \left(\int_{B_\tau} |u|^2 \, d\omega \right)^{1/2} \right] \\ &\le C \left(\int_{B_\tau} |u| \, d\omega \right)^{1/2} \left(\int_{B_\tau} |u|^3 \, d\omega \right)^{1/2} \\ &\quad \times \left[\left(\int_{B_\tau} |\nabla_\omega u|^2 \, d\omega \right)^{1/2} + |B_\tau|^{-1/2} \left(\int_{B_\tau} |u|^2 \, d\omega \right)^{1/2} \right]. \end{aligned} \tag{2.10}$$

Dividing (2.10) by $\left(\int_{B_\tau} |u|^3 \, d\omega \right)^{1/2}$, we get the claim. □

3. Fundamental Properties of Smooth Solutions

We give fundamental properties for the smooth solution $(A(t), \Phi(t))$ of (1.2). First we give the energy formula:

Proposition 3.1. *Let $(A(t), \Phi(t))$ be a smooth solution of (1.2) on $(0,T]\times\mathbb{R}^3$. Then, we have*

$$\frac{\partial}{\partial t}E(A(t),\Phi(t)) = -2\int_{\mathbb{R}^3} |\partial_t A(t)|^2 + |\partial_t\Phi(t)|^2\, dV.$$

In particular, the energy is non-increasing in time along the solution of (1.2).

This proposition is due to [3, Proposition 4.3]. Taking the standard coordinates of $\mathbb{R}^3$, we define

$$F_A = \sum_{i<j} F_{ij}dx^i\wedge dx^j, \quad F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j],$$
$$d_A\Phi = \nabla_i\Phi dx^i, \qquad \nabla_i\Phi = \partial_i\Phi + [A_i,\Phi],$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$. Moreover Ψ denotes the 6-vectors obtained by putting the components of F_A and $d_A\Phi$, that is,

$$\Psi = \begin{pmatrix} F_A \\ *d_A\Phi\end{pmatrix}.$$

Proposition 3.2. *Let $(A(t), \Phi(t))$ be a smooth solution of (1.2). Then we have*

$$\partial_t F_{ij} = -\nabla_A^*\nabla_A F_{ij} - 2[F_{ik}, F_{jk}] - 2[\nabla_i\Phi, \nabla_j\Phi] + [[F_{ij},\Phi],\Phi], \tag{3.1}$$

$$\partial_t\nabla_i\Phi = -\nabla_A^*\nabla_A\nabla_i\Phi - 2[\nabla_k\Phi, F_{ik}] + [[\nabla_i\Phi,\Phi],\Phi], \tag{3.2}$$

$$\partial_t|\Phi|^2 = \Delta|\Phi|^2 - 2|d_A\Phi|^2, \tag{3.3}$$

$$\partial_t|F_A|^2 \le \Delta|F_A|^2 - 2|\nabla_A F_A|^2 + C|F_A|^3 + C|F_A|^2|d_A\Phi|, \tag{3.4}$$

$$\partial_t|\nabla_A\Phi|^2 \le \Delta|d_A\Phi|^2 - 2|\nabla_A^2\Phi|^2 + C|d_A\Phi|^2|F_A|, \tag{3.5}$$

$$\partial_t|\Psi|^2 \le \Delta|\Psi|^2 - 2|\nabla_A\Psi|^2 + C|\Psi|^3, \tag{3.6}$$

$$\begin{aligned}\partial_t\nabla_A^n\Psi &= -\nabla_A^*\nabla_A\nabla_A^n\Psi + \sum_{i=0}^n \nabla_A^i\Psi * \nabla_A^{n-i}\Psi \\ &\quad + \sum_{i=0}^n[\Phi, \nabla_A^i\Psi*\nabla_A^{n-i}\Psi] + [[\nabla_A^n\Psi,\Phi],\Phi],\end{aligned} \tag{3.7}$$

$$\partial_t|\nabla_A^n\Psi|^2 \le \Delta|\nabla_A^n\Psi|^2 - |\nabla_A^{n+1}\Psi|^2 + C|\nabla_A^n\Psi|\sum_{i=0}^n|\nabla_A^i\Psi|\,|\nabla_A^{n-i}\Psi|. \tag{3.8}$$

Here, $A*B$ denotes some linear combination of tensor products of components of A and B, and ∇_A^* denotes the formal adjoint operator of ∇_A with respect to the standard L^2-inner product.

Proof. For any $\mathfrak{su}(2)$-valued tensor field T, a, b and c, we have

$$\Delta|T|^2 = 2(\nabla_A^*\nabla_A T, T) + 2|\nabla_A T|^2,$$
$$(\nabla_i\nabla_j - \nabla_j\nabla_i)T = [F_{ij}, T],$$
$$(a, [b, c]) = (b, [c, a]),$$

where $(\cdot,\cdot)$ denotes the pointwise inner product for $\mathfrak{su}(2)$-valued tensor fields. Using these relations, we obtain (3.3). Taking the inner product between Φ and the second equation of (1.2) in the direction of fiber, we have

$$(\partial_t\Phi, \Phi) = (\Delta_A\Phi, \Phi) = \frac{1}{2}\Delta|\Phi|^2 - |\nabla_A\Phi|^2,$$

which shows (3.3).

Using the Bianch indentity, we have

$$\begin{aligned} &d_{A_j}\partial_t A_i = \partial_t F_{ji}, \\ &d_A(\nabla_j F_{ji}) = \nabla_k(\nabla_j F_{ki} - \nabla_i F_{kj}) + 2[F_{ik}, F_{jk}] \\ &= -\nabla_A^*\nabla_A F_{ji} + 2[F_{ik}, F_{jk}], \\ &d_A[\Phi, \nabla_i\Phi] = -2[\nabla_i\Phi, \nabla_j\Phi] + [[F_{ij}, \Phi], \Phi]. \end{aligned} \tag{3.9}$$

By (3.9) and the first equation of (1.2), we have (3.1). A similar calculation to the second equation of (1.2) yields (3.2). (3.4) and (3.5) are obtained easily from (3.1) and (3.2). From (3.4) and (3.5), we get (3.6).

Let us show (3.7) by induction. The case $n = 0$ is shown in (3.6). Assume (3.7) is true for n and we have

$$\begin{aligned} &\partial_t\nabla_A^{n+1}\Psi = \nabla_A\partial_t\nabla_A^n\Psi - [d_A^*F_A, \nabla_A^n\Psi] - [[\Phi, d_A\Phi], \nabla_A^n\Psi] \\ &= \nabla_A\left(-\nabla_A^*\nabla_A\nabla_A^n\Psi + \sum_{i=0}^{n}\nabla_A^i\Psi * \nabla_A^{n-i}\Psi + \sum_{i=0}^{n}[\Phi, \nabla_A^i\Phi * \nabla_A^{n-i}\Psi] + [[\nabla_A^n\Psi, \Phi], \Phi]\right) \\ &- [d_A^*F_A, \nabla_A^n\Psi] - [[\Phi, d_A\Phi], \nabla_A^n\Psi] \\ &= -\nabla_A^*\nabla_A\nabla_A^{n+1}\Psi + \sum_{i=0}^{n+1}(\nabla_A^i\Psi * \nabla_A^{n+1-i}\Psi + [\Phi, \nabla_A^i\Phi * \nabla_A^{n+1-i}\Psi]) + [[\nabla_A^{n+1}\Psi, \Phi], \Phi], \end{aligned}$$

which implies that (3.7) is true for $n+1$. So, we have (3.8) by (3.7). □

By applying the maximum principle for (3.3), $\sup_{(t,x)}|\Phi|$ is bounded by its initial value.

Proposition 3.3. *Let $(A(t), \Phi(t))$ be a smooth solution of (1.2) satisfying (1.5). Then we have the following uniform bounds for $t \in (0, T]$:*

$$\sup_{0<t<T} \|\Psi(t)\|_{L^\infty} \leq C \max(\|\Psi_0\|_{L^2}^2, \|\Psi_0\|_{L^\infty}),$$
$$\sup_{0<t<T} \|\nabla^n \Psi\|_{L^\infty} \leq C_n \quad (n \in \mathbb{N}),$$

where C_n depends on (A_0, Φ_0) and $n \in \mathbb{N} \cup \{0\}$.

For the proof, see [3, Proposition 5.2, 5.3].

Proposition 3.4. *Let $(A(t), \Phi(t))$ be a smooth solution of (1.2). Assume that*

$$|1 - |\Phi(0, r, \omega)|^2| \leq C_0 r^{-1},$$
$$|d_A \Phi(t, r, \omega)|^2 \leq C_1 r^{-\alpha}, \text{ for } \alpha > 3, r \geq 1,$$
$$|d_A \Phi(t, r, \omega)| \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3).$$

Then we have

$$|1 - |\Phi(t, r, \omega)|^2| \leq C_2 r^{-1},$$

which yields

$$|1 - |\Phi(t, r, \omega)|\,| \leq C r^{-1}.$$

Here constant C depends on C_0 and C_1.

Proof. By setting $w := |\Phi|^2 - 1 - C_0 r^{-1}$, w satisfies

$$\partial_t w - \Delta w = -2|d_A \Phi|^2 \leq 0.$$

Therefore, we have

$$w(t, r, \omega) \leq \max_{(r,\omega)} w(0, r, \omega) \leq 0,$$

which implies

$$|\Phi(t, r, \omega)|^2 - 1 \leq C_0 r^{-1}. \tag{3.10}$$

To show the Proposition, we need to show the bound from below:

$$|\Phi(t, r, \omega)|^2 - 1 \geq -C_2 r^{-1},$$

with some constant C_2 independent of r and ω.

Let us take a bounded continuos function v on $\mathbb{R}^3$ as

$$v := \begin{cases} C_3 & \text{for } r \leq 1, \\ C_3 r^{-\alpha} & \text{for } r \geq 1, \end{cases}$$

where $\alpha > 3$ and $C_3 = 2\max(C_1, \max_{r\le 1} |d_A\Phi|^2)$. Choose f as

$$f(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{v(x)}{|x-y|}\,dy \quad (\ge 0),$$

which solves $-\Delta f = v$. Since v is bounded near the origin and since $v(x) = O(|x|^{-\alpha})$ with $\alpha > 3$, it is easy to verify that $f \in L^\infty(\mathbb{R}^3)$ with decay like

$$|f(x)| \le C_4|x|^{-1} \text{ for all } |x| \ge 1. \tag{3.11}$$

Now set $u := |\Phi|^2 - 1 + C_2 r^{-1} + f$ and we have by (3.3) and the definition of v that

$$\begin{aligned}\partial_t u - \Delta u &= -2|d_A\Phi|^2 - \Delta f\\ &= -2|d_A\Phi|^2 + v \ge 0.\end{aligned}$$

Hence the maximum principle yields

$$\min_{(r,\omega)} u(0,t,\omega) \le u(t,x) \text{ for all } (t,x) \in (0,\infty)\times\mathbb{R}^3. \tag{3.12}$$

Since $f \ge 0$, we have by assumption,

$$\min_{(r,\omega)} u(0,r,\omega) \ge (C_2 - C_0)r^{-1}$$

and (3.12) yields

$$|\Phi(t,x)|^2 - 1 \ge f(x) - C_0 r^{-1} \ge -C_0 r^{-1}. \tag{3.13}$$

Now the desired estimate follows from (3.11).

By (3.13) and

$$|1-|\Phi||\le |1-|\Phi||\,|1+|\Phi|| = |1-|\Phi|^2|,$$

we get the conclusion. □

Proposition 3.5. *Let $(A(t),\Phi(t))$ be a smooth solution of (1.2). Assume (1.4), then we have*

$$|\nabla_A^n \Psi(t,r,\omega)| \le C r^{-(n+1)} \tag{3.14}$$

for $t \in (0,T]$.

Proof. First we show (3.14) for $n = 0$.

From (3.6) and Proposition 3.3, we have

$$\partial_t|\Psi| \le \Delta|\Psi| + C|\Psi|.$$

Set $w = e^{Ct}|\Phi(t)| - Cr^{-1}$, then w satisfies

$$\partial_t w - \Delta w \le 0. \tag{3.15}$$

Applying the maximum principle for (3.15), we have

$$w(t,x) \le \max_{x\in\mathbb{R}^3} w(0,x).$$

On the other hand, by Proposition 3.3, we have

$$|\Psi(0)| \le \begin{cases} Cr^{-2} & \text{for } r \ge 1, \\ C & \text{for } r < 1. \end{cases}$$

Hence we have

$$w(0) = |\Psi(0)| - Cr^{-1} \le \begin{cases} C(r^{-2} - r^{-1}) & \text{for } r \ge 1, \\ C(1 - r^{-1}) & \text{for } r < 1, \end{cases}$$

and

$$w(0,x) \le 0.$$

Therefore we have

$$w(t) = |\Psi(t)| - Cr^{-1} \le 0.$$

To prove (3.14) for general n, we assume (3.14) is true for $m \le n-1$. Set $u_n = r^{n+1}|\nabla_A^n \Psi|$. By (3.8), we have

$$\begin{aligned} \partial_t u_n \le{}& \Delta u_n - \frac{2(n+1)}{r}\partial_r u_n + \frac{(n+1)(n+2)}{r^2}u_n \\ &+ C|\Psi|u_n + C\sum_{i=1}^{n-1}(r^{i+\frac{1}{2}}|\nabla_A^i\Psi|)(r^{n-i+\frac{1}{2}}|\nabla_A^{n-i}\Psi|). \end{aligned}$$

Using Proposition 3.3 and the assumption of the induction, we have

$$\partial_t u_n \le \Delta u_n - \frac{C_1}{r}\partial_r u_n + \frac{C_2}{r^2}u_n + C_3 u_n + g_n, \tag{3.16}$$

where $g_n = C\sum_{i=1}^{n-1}(r^{i+\frac{1}{2}}|\nabla_A^i\Psi|)(r^{n-i+\frac{1}{2}}|\nabla_A^{n-i}\Psi|)$.

For any $0 < \varepsilon < R$, let $\phi(r)$ be a non-decreasing function which satisfies $\phi = 0$ on $[0,\varepsilon]$, $\phi = 1$ on $[R,\infty)$. Set $f = u\phi$, then, by (3.16), we have

$$\partial_t f \le \Delta f - \frac{C_1}{r}\partial_r f + \frac{C_2}{r^2}f + C_3 f + g_n + G, \tag{3.17}$$

where $G = C(\phi'' u_n + \phi'\partial_r u_n + \frac{1}{r}\phi' u_n)$. Here we note that $\operatorname{supp} G \subset [\varepsilon, R]$, $g_n + G$ is bounded and $f(\varepsilon,\omega) = 0$.

Let v be the solution for

$$\begin{cases} -\Delta v = g_n + G, \\ v|_{|x|=\varepsilon} = 0, \end{cases}$$

then v is bounded on $\mathbb{R}^3$. Applying the maximum principle for (3.17), we have

$$f(t,x) - v(x) \le \max_{x\in\mathbb{R}^3} e^{Ct}(f(0,x) - v(x)) \le C, \text{ for } t \in (0,T].$$

Hence we have

$$r^{n+1}|\nabla_A^n \Psi| \le C.$$

Therefore we have (3.14) by induction. □

4. Estimates

Throughout this section, we assume that $(A(t), \Phi(t))$ is a smooth and $W^{1,2}(\mathbb{R}^3)$-solution of (1.2) with the initial value $(A_0, \Phi_0) \in \mathcal{C}$ on $(0, T) \times \mathbb{R}^3$, $T < \infty$. Taking the polar coordinates on $\mathbb{R}^3$, we denote $\mathbb{R}^3 \ni x = (r, \omega) \in (0, \infty) \times S^2$.

Proposition 4.1. *If $(A(0), \Phi(0))$ satisfies (1.4) and (1.5), then we have*

$$\Psi(t) \in W^{m,2}(\mathbb{R}^3) \cap W^{m,\infty}(\mathbb{R}^3),$$

for all $m \in \mathbb{N} \cup \{0\}$.

Proof. For $m = 0$, we see for Propositions 3.1 and 3.3, $\Psi \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.

Suppose Proposition 4.1 is true for m. Then we have by (3.8) that

$$\partial_t |\nabla_A^{m+1}\Psi| \le \Delta |\nabla_A^{m+1}\Psi| + F(|\Psi|, |\nabla_A \Psi|, \cdots, |\nabla_A^{m+1}\Psi|),$$

where F denotes the polynomial for $|\Psi|, \cdots, |\nabla_A^{m+1}\Psi|$. By the assumption of the induction, there is a function $f_m = f_m(t,x) \in L^\infty(\mathbb{R}^3 \times (0,T))$ such that $|F| \le f_m$ for all $(t,x) \in (0,T) \times \mathbb{R}^3$. Using the maximum principle,

$$|\nabla_A^{m+1}\Psi| \le |u(t,x)|, \tag{4.1}$$

where $u(t,x)$ is the solution of the heat equation

$$\begin{cases} \partial_t u - \Delta u = f \\ u(x,0) = |\nabla_A^{m+1}\Psi(0,x)|. \end{cases} \tag{4.2}$$

Note that if $(A(0), \Phi(0))$ satisfies (1.4) and (1.5). We have

$$\int_{\mathbb{R}^3} |\nabla_A^{m+1}\Phi(0)|^2 \, dV \le \int_{|x|\le 1} |\nabla_A^{m+1}\Phi(0)|^2 \, dV + C \int_1^\infty r^{-2m-4+2} \, dr \le C,$$

therefore $\nabla_A^{m+1}\Psi(0) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. (4.1) and (4.2) yield that

$$\nabla_A^{m+1}\Psi(t) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3).$$

This completes the proof. □

Proposition 4.2. *If $(A(0), \Phi(0))$ satisfies (1.4) and (1.5), then we have*

$$\int_{S^2} r^2 |\Psi(t,r,\omega)|\, d\omega$$

is uniformly bounded on $(0,T) \times (0,\infty)$. The bound depends only on T and $E(0)$.

Proof. Set

$$h(t,r) := \int_{S^2} r^2 |\Psi(t,r,\omega)|\, d\omega.$$

Then we have

$$\partial_t h - \partial_{rr} h + \frac{4}{r}\partial_r h - \frac{6}{r^2} h \le C r^2 \int_{S^2} |\Psi|^2\, d\omega. \tag{4.3}$$

Note that $h \in L^\infty((0,T)\times(0,\infty))$, $\|h(0)\|_{L^1(\varepsilon,R)} \le CR^3$, and

$$\int_0^\infty \int_{S^2} r^2 |\Psi|^2\, d\omega dr = E(t).$$

Applying Proposition 2.1 to (4.3), we have

$$\sup_{(t,r)} \int_{S^2} r^2 |\Psi(t,r,\omega)|\, d\omega \le C,$$

where the constant depends on T and $E(0)$. $\square$

For a positive $\tau > 0$, we take a smooth non-negative function η on S^2 satisfying

$$\eta = \begin{cases} 1 \text{ on } & B_{\tau/2}(\omega_0) \\ 0 \text{ outside } & B_\tau(\omega_0) \end{cases}$$

and $|\nabla_\omega \eta| \le C/\tau$. In the following, we assume the following condition for the solution (A, Φ).

Condition 4.3. There exists a universal constant $\varepsilon_1 > 0$ such that the solution (A,Φ) satisfies

$$\sup_{(t,r)} \int_{S^2} r^2 |\Psi(t,r,\omega)|\, \eta^2\, d\omega < \varepsilon_1.$$

Proposition 4.4. *If (A,Φ) satisfies the condition 4.3, then we have*

$$\frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\Psi|^2 \eta^2\, dV + C\int_{\mathbb{R}^3} |\nabla_A \Psi|^2 \eta^2\, dV \le C,$$

where the constant $C = C(E(0), \tau)$ is independent of r.

Proof. Taking the L^2-inner product to (3.7) with $\Psi\eta^2$, for $n=0$, we have

$$\frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\Psi|^2\eta^2\, dV = -\int_{\mathbb{R}^3} \langle \nabla_A^*\nabla_A\Psi, \Psi\rangle \eta^2\, dV + C\int_{\mathbb{R}^3} \langle \Psi * \Psi, \Psi\rangle \eta^2\, dV - C\int_{\mathbb{R}^3} |[\Phi,\Psi]|^2\eta^2\, dV.$$

In particular, we have

$$\frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\Psi|^2\eta^2\, dV \le -\int_{\mathbb{R}^3} \langle \nabla_A^*\nabla_A\Psi, \Psi\rangle \eta^2\, dV + C\int_{\mathbb{R}^3} |\Psi|^3\eta^2\, dV. \tag{4.4}$$

Using $|r^{-1}\nabla_{A_\omega}\Phi| \le |\nabla_A\Phi|$ and elementary calculations, we get

$$\begin{aligned}
&-\int_{\mathbb{R}^3} \langle \nabla_A^*\nabla_A\Psi, \Psi\rangle \eta^2\, dV\\
&= -\int_{\mathbb{R}^3} |\nabla_A\Psi|^2\eta^2\, dV + \int_{\mathbb{R}^3} \langle \nabla_A\Psi, \Psi\rangle \nabla\eta\,\eta\, dV\\
&\le -\int_{\mathbb{R}^3} |\nabla_A\Psi|^2\eta^2\, dV + \int_{\mathbb{R}^3} |\nabla_A\Psi|\,|\Psi|\,|\nabla\eta|\,\eta\, dV\\
&\le -\int_{\mathbb{R}^3} |\nabla_A\Psi|^2\eta^2\, dV + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_A\Psi|^2\eta^2\, dV\\
&+ 4\int_{\mathbb{R}^3} r^{-2}|\Psi|^2|\nabla_\omega\eta|^2\, dV.
\end{aligned} \tag{4.5}$$

In the above integration by parts, the surface integrand at infinity vanishes because

$$\left|\int_{S_r} \langle \Psi, \nabla_A\Psi\rangle\, dS_r\right| \le C\int_{S^2} r^2|\Psi|\,|\nabla_A\Psi|\, d\omega \le Cr^{2-1-2} = Cr^{-1},$$

by Proposition 3.5. Combining (4.4) with (4.5), we have

$$\begin{aligned}
&\frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\Psi|^2\eta^2\, dV + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\Psi|^2\eta^2\, dV\\
&\le 4\int_{\mathbb{R}^3} r^{-2}|\Psi|^2|\nabla_\omega\eta|^2\, dV + C\int_{\mathbb{R}^3} r^2|\Psi|^3\eta^2\, dV.
\end{aligned} \tag{4.6}$$

Applying Proposition 2.2, we have

(4.7)

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\Psi|^3\eta^2\, dV\\
&\le C_1\int_0^\infty r^2 \left(\int_{S^2} |\Psi|\,\eta^2\, d\omega\right)\left(\int_{S^2} |\nabla_\omega|\Psi||^2\eta^2\, d\omega + \tau^{-2}\int_{S^2} |\Psi|^2\eta^2\, d\omega\right) dr\\
&\le C_1 \sup_{(t,r)} \left(\int_{S^2} r^2|\Psi|\,\eta^2\, d\omega\right)\left(\int_{\mathbb{R}^3} (r^{-2}|\nabla_\omega|\Psi||^2)\eta^2\, dV + \tau^{-2}\int_{\mathbb{R}^3} r^{-2}|\Psi|^2\eta^2\, dV\right)\\
&\le C_1 \sup_{(t,r)} \left(\int_{S^2} r^2|\Psi|\,\eta^2\, d\omega\right)\left(\int_{\mathbb{R}^3} |\nabla_A\Psi|^2\eta^2\, dV + \tau^{-2}\int_{\mathbb{R}^3} r^{-2}|\Psi|^2\eta^2\, dV\right).
\end{aligned}$$

If

$$\sup_{(t,r)} \int_{S^2} r^2 |\Psi|\, \eta^2 \, d\omega < \frac{1}{2C_1},$$

then, by Proposition 4.2, (4.6) and (4.7), we have

$$\frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\Psi|^2 \eta^2 \, dV + C \int_{\mathbb{R}^3} |\nabla_A \Psi|^2 \eta^2 \, dV \leq C(1+\tau^{-2}) \int_{\mathbb{R}^3} r^{-2} |\Psi|^2 \eta^2 \, dV.$$

On the other hand, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} r^{-2} |\Psi|^2 \, dV = \int_0^1 \int_{S^2} |\Psi|^2 \, d\omega dr + \int_{|x|\geq 1} |\Psi|^2 \, dV \\
&\leq C \sup_{(t,r)} |\Psi|^2 + \int_{\mathbb{R}^3} |\Psi|^2 \, dV,
\end{aligned}$$

which completes the proof. □

Proposition 4.5. *For $n \geq 1$, we have*

$$\begin{aligned}
&\frac{1}{2}\partial_t \int_{\mathbb{R}^3} r^{2n} |\nabla_A^n \Psi|^2 \eta^2 \, dV + C \int_{\mathbb{R}^3} r^{2n} |\nabla_A^{n+1} \Psi|^2 \eta^2 \, dV \\
&\leq C(1+\tau^{-2}) \int_{\mathbb{R}^3} r^{2n-2} |\nabla_A^n \Psi|^2 \, dV + C \sup_{(t,r)} \left(\int_{S^2} r^4 |\Psi|^2 \eta^2 \, d\omega \right) \int_{\mathbb{R}^3} r^{2n-2} |\nabla_A^n \Psi|^2 \eta^2 \, dV \\
&+ C \sum_{i=1}^{n-1} \left[\sup_{(t,r)} \left(\int_{S^2} r^{2i+4} |\nabla_A^i \Psi|^2 \eta^2 \, d\omega \right) \right. \\
&\left. \times \left(\int_{\mathbb{R}^3} r^{2i} |\nabla_A^{i+1} \Psi|^2 \eta^2 \, dV + \tau^{-2} \int_{\mathbb{R}^3} r^{2(i-1)} |\nabla_A^i \Psi|^2 \eta^2 \, dV \right) \right],
\end{aligned}$$

where constant C depends only on n.

Proof. Taking the L^2-inner product to (3.7) with $r^{2n} \nabla_A^n \Psi \eta^2$, by a similar calculation with the previous proof, we have

$$\begin{aligned}
&\frac{1}{2}\partial_t \int_{\mathbb{R}^3} r^{2n} |\nabla_A^n \Psi|^2 \eta^2 \, dV \leq - \int_{\mathbb{R}^3} r^{2n} \langle \nabla_A^* \nabla_A \nabla_A^n \Psi, \nabla_A^n \Psi \rangle \eta^2 \, dV \\
&+ C \sum_{i=0}^{n} \int_{\mathbb{R}^3} r^{2n} |\nabla_A^i \Psi|\, |\nabla_A^{n-i} \Psi|\, |\nabla_A^n \Psi|\, \eta^2 \, dV.
\end{aligned} \tag{4.8}$$

Moreover, we have

$$
\begin{aligned}
&-\int_{\mathbb{R}^3} r^{2n}\langle \nabla_A^*\nabla_A\nabla_A^n\Psi, \nabla_A^n\Psi\rangle\eta^2\, dV \\
&\le -\int_{\mathbb{R}^3} r^{2n}|\nabla_A^{n+1}\Psi|^2\eta^2\, dV + \frac{1}{2}\int_{\mathbb{R}^3} r^{2n}|\nabla_A^{n+1}\Psi|^2\eta^2\, dV \\
&+ 4\int_{\mathbb{R}^3} r^{2n-2}|\nabla_A^n\Psi|^2|\nabla_\omega\eta|^2\, dV + C_n\int_{\mathbb{R}^3} r^{2n-2}|\nabla_A^n\Psi|^2\eta^2\, dV.
\end{aligned}
\tag{4.9}
$$

Using Proposition 3.5, we have

$$
\left|\int_{S_r} r^{2n}\langle \nabla_A^n\Psi, \nabla_A^{n+1}\Psi\rangle\, dS_r\right| \le C\int_{S^2} r^{2n}|\nabla_A^n\Psi|\,|\nabla_A^{n+1}\Psi|\, d\omega \le Cr^{2n-n-1-n-2} = Cr^{-1},
$$

and therefore the boundary term of the integral by part is vanish. By (4.8) and (4.9), we get

$$
\begin{aligned}
&\frac{1}{2}\partial_t\int_{\mathbb{R}^3} r^{2n}|\nabla_A^n\Psi|^2\eta^2\, dV + \frac{1}{2}\int_{\mathbb{R}^3} r^{2n}|\nabla_A^{n+1}\Psi|^2\eta^2\, dV \\
&\le C\int_{\mathbb{R}^3} r^{2n-2}|\nabla_A^n\Psi|^2|\nabla_\omega\eta|^2\, dV + C\int_{\mathbb{R}^3} r^{2n-2}|\nabla_A^n\Psi|^2\eta^2\, dV \\
&+ C\int_{\mathbb{R}^3} r^{2n}|\nabla_A^n\Psi|^2|\Psi|\,\eta^2\, dV + C\sum_{i=0}^{n-1}\int_{\mathbb{R}^3} r^{2n}|\nabla_A^i\Psi|\,|\nabla_A^{n-i}\Psi|\,|\nabla_A^n\Psi|\,\eta^2\, dV.
\end{aligned}
\tag{4.10}
$$

By a direct calculation, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} r^{2n}|\nabla_A^i\Psi|\,|\nabla_A^{n-i}\Psi|\,|\nabla_A^n\Psi|\,\eta^2\, dV \le \int_{\mathbb{R}^3} r^{2n-2}|\nabla_A^n\Psi|^2\eta^2\, dV \\
&+ \int_{\mathbb{R}^3} r^{4i+2}|\nabla_A^i\Psi|^4\eta^2\, dV + \int_{\mathbb{R}^3} r^{4(n-i)+2}|\nabla_A^{n-i}\Psi|^4\eta^2\, dV,
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^3} r^{4i+2}|\nabla_A^i\Psi|^4\eta^2\, dV = \int_0^\infty r^{4i+4}\int_{S^2} |\nabla_A^i\Psi|^4\eta^2\, d\omega dr \\
&\le C\int_0^\infty r^{4i+4}\left(\int_{S^2} |\nabla_A^i\Psi|^2\eta^2\, d\omega\right) \\
&\times \left(\int_{S^2} |\nabla_\omega|\nabla_A^i\Psi||^2\eta^2\, d\omega + \tau^{-2}\int_{S^2} |\nabla_A^i\Psi|^2\eta^2\, d\omega\right) dr \\
&\le C\sup_{(t,r)}\left(\int_{S^2} r^{2i+4}|\nabla_A^i\Psi|^2\eta^2\, d\omega\right) \\
&\times \left(\int_{\mathbb{R}^3} r^{2i}|\nabla_A^{i+1}\Psi|^2\eta^2\, dV + \tau^{-2}\int_{\mathbb{R}^3} r^{2(i-1)}|\nabla_A^i\Psi|^2\eta^2\, dV\right).
\end{aligned}
\tag{4.12}
$$

Combining (4.11) with (4.12), we get

$$\begin{aligned}
&\sum_{i=1}^{n-1}\int_{\mathbb{R}^3} r^{2n}|\nabla_A^i\Psi|\,|\nabla_A^{n-i}\Psi|\,|\nabla_A^n\Psi|\,\eta^2\,dV\\
&\le (n-2)\int_{\mathbb{R}^3} r^{2n-2}|\nabla_A^n\Psi|^2\eta^2\,dV + C\sum_{i=1}^{n-1}\left[\sup_{(t,r)}\left(\int_{S^2} r^{2i+4}|\nabla_A^i\Psi|^2\eta^2\,d\omega\right)\right.\\
&\left.\times\left(\int_{\mathbb{R}^3} r^{2i}|\nabla_A^{i+1}\Psi|^2\eta^2\,dV + \tau^{-2}\int_{\mathbb{R}^3} r^{2(i-1)}|\nabla_A^i\Psi|^2\eta^2\,dV\right)\right].
\end{aligned}\tag{4.13}$$

Using the Sobolev and Schwartz inequalities, we have

$$\begin{aligned}
&\int_{B_\tau}|\Psi|\,|\nabla_A^n\Psi|^2\eta^2\,d\omega \le \left(\int_{B_\tau}|\Psi|^2\eta^2\,d\omega\right)^{1/2}\left(\int_{B_\tau}|\nabla_A^n\Psi|^4\eta^2\,d\omega\right)^{1/2}\\
&\le \left(\int_{B_\tau}|\Psi|^2\eta^2\,d\omega\right)^{1/2}\left(\int_{B_\tau}|\nabla_A^n\Psi|^2\eta^2\,d\omega\right)^{1/2}\\
&\times\left(\int_{B_\tau}|\nabla_\omega|\nabla_A^n\Psi||^2\eta^2\,d\omega + \tau^{-2}\int_{B_\tau}|\nabla_A^n\Psi|^2\eta^2\,d\omega\right)^{1/2}\\
&\le \frac{\varepsilon}{2}\int_{B_\tau}|\nabla_\omega|\nabla_A^n\Psi||^2\eta^2\,d\omega + \frac{\varepsilon}{2}\tau^{-2}\int_{B_\tau}|\nabla_A^n\Psi|^2\eta^2\,d\omega\\
&+\frac{1}{2\varepsilon}\left(\int_{B_\tau}|\Psi|^2\eta^2\,d\omega\right)\left(\int_{B_\tau}|\nabla_A^n\Psi|^2\eta^2\,d\omega\right).
\end{aligned}$$

Therefore, taking $\varepsilon = 2r^{-2}\varepsilon'$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} r^{2n}|\Psi|\,|\nabla_A^n\Psi|^2\eta^2\,dV\\
&\le \varepsilon'\int_{\mathbb{R}^3} r^{2n}|\nabla_A^{n+1}\Psi|^2\eta^2\,dV + C\tau^{-2}\int_{\mathbb{R}^3} r^{2n-2}|\nabla_A^n\Psi|^2\eta^2\,dV\\
&+ C\sup_{(t,r)}\left(\int_{S^2} r^4|\Psi|^2\eta^2\,d\omega\right)\int_{\mathbb{R}^3} r^{2n-2}|\nabla_A^n\Psi|^2\eta^2\,dV.
\end{aligned}\tag{4.14}$$

By (4.10), (4.13) and (4.14), we get Proposition 4.5. □

Set

$$A_n := \frac{1}{2}\partial_{rr} - \frac{n+2}{r}\partial_r + \frac{(n+2)(n+3)}{r^2},$$

for $n \ge 0$.

Proposition 4.6. *If (A, Φ) satisfies Condition 4.3, then, we have*

$$\frac{1}{2}\partial_t\int_{S^2} r^4|\Psi|^2\eta^2\,d\omega - A_0\int_{S^2} r^4|\Psi|^2\eta^2\,d\omega \le C\tau^{-2}\int_{\operatorname{supp}\eta} r^2|\Psi|^2\,d\omega.$$

Proof. For the sake of simplicity, set

$$h_n := r^{n+2}|\nabla_A^n \Psi|,$$
$$g_n := \int_{S^2} h_n^2 \eta^2 \, d\omega = \int_{S^2} r^{2n+4}|\nabla_A^n \Psi|^2 \eta^2 \, d\omega.$$

Since

$$\Delta|\Psi| = r^{-2}\partial_{rr} h_0 - 4r^{-3}\partial_r h_0 + 6r^{-4} h_0 + r^{-4}\Delta_{S^2} h_0,$$

by (3.6), we have

$$\partial_t h_0 \leq \partial_{rr} h_0 - \frac{4}{r}\partial_r h_0 + 6h_0 + r^{-2}\Delta_{S^2} h_0 + C h_0 |\Psi|. \tag{4.15}$$

Multiplying (4.15) by $h_0\eta^2$ and then integrating over S^2, we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t g_0 - A_0 g_0 + \int_{S^2} \left(|\partial_r h_0|^2 + r^{-2}|\partial_\omega h_0|^2\right)\eta^2 \, d\omega \\ &\leq 2r^{-2}\int_{S^2} |\partial_\omega h_0|\,|h_0|\,|\partial_\omega \eta|\,|\eta| \, d\omega + C\int_{S^2} h_0^2 |\Psi| \, d\omega. \end{aligned} \tag{4.16}$$

On the other hand, we have

$$\begin{aligned} &\int_{S^2} h_0^2 |\Psi| \eta^2 \, d\omega = \int_{S^2} r^4 |\Psi|^3 \eta^2 \, d\omega \\ &\leq C_1 \left(\int_{S^2} r^2 |\Psi| \eta^2 \, d\omega\right)\left(\int_{S^2} r^{-2}(r^4|\partial_\omega \Psi|^2\eta^2)\, d\omega + \tau^{-2}\int_{S^2} r^2|\Psi|^2\eta^2 \, d\omega\right). \end{aligned} \tag{4.17}$$

Combining (4.16) with (4.17), we set

$$\begin{aligned} &\frac{1}{2}\partial_t g_0 - A_0 g_0 + \int_{S^2}\left(|\partial_r h_0|^2 + r^{-2}|\partial_\omega h_0|^2\right)\eta^2 \, d\omega \\ &\leq \frac{1}{2}\int_{S^2} r^{-2}|\partial_\omega h_0|^2\eta^2 \, d\omega + 2\int_{S^2} r^2|\Psi|^2|\partial_\omega \eta|^2 \, d\omega \\ &+ C_1\left(\int_{S^2} r^2|\Psi|\eta^2 \, d\omega\right)\left(\int_{S^2} r^{-2}|\partial_\omega h_0|^2\eta^2 \, d\omega + \tau^{-2}\int_{S^2} r^2|\Psi|^2\eta^2 \, d\omega\right). \end{aligned}$$

If

$$\sup_{(t,r)} \int_{S^2} r^2 |\Psi| \, \eta^2 \, d\omega < \frac{1}{2C_1}$$

then we have

$$\frac{1}{2}\partial_t g_0 - A_0 g_0 \leq C\tau^{-2}\int_{\operatorname{supp}\eta} r^2|\Psi|^2 \, d\omega,$$

which implies Proposition 4.6. $\square$

Proposition 4.7. *For $n \geq 1$, we have*

$$\begin{aligned}
&\frac{1}{2}\partial_t \int_{S^2} r^{2n+4}|\nabla_A^n \Psi|^2\eta^2\, d\omega - A_n \int_{S^2} r^{2n+4}|\nabla_A^n\Psi|^2\eta^2\, d\omega \\
&\leq C\tau^{-2}\int_{S^2} r^{2n+2}|\nabla_A^n\Psi|^2\eta^2\, d\omega + C\sup_{(t,r)}\left(\int_{S^2} r^4|\Psi|^2\eta^2\, d\omega\right)\int_{S^2} r^{2n+2}|\nabla_A^n\Psi|^2\eta^2\, d\omega \\
&+ C\sum_{i=1}^{n-1}\sup_{(t,r)}\left(\int_{S^2} r^{2i+4}|\nabla_A^i\Psi|^2\eta^2\, d\omega\right) \\
&\times\left(\int_{S^2} r^{2(i+1)+2}|\nabla_A^{i+1}\Psi|^2\eta^2\, d\omega + \tau^{-2}\int_{S^2} r^{2i+2}|\nabla_A^i\Psi|^2\eta^2\, d\omega\right),
\end{aligned}$$

where constants C are depending only on n.

Proof. By a similar calculation in the proof of Proposition 4.6, we have

$$\begin{aligned}
\partial_t h_n &\leq \partial_{rr}h_n - \frac{2(n+2)}{r}\partial_r h_n + \frac{(n+2)(n+3)}{r^2}h_n + r^{-2}\Delta_{S^2}h_n \\
&+ Ch_n|\Psi| + C\sum_{i=1}^{n-1} r^{n+2}|\nabla_A^i\Psi|\,|\nabla_A^{n-i}\Psi|.
\end{aligned} \tag{4.18}$$

Multiplying (4.18) by $h_n\eta^2$ and then integrating over S^2, we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t g_n - A_n g_n + \int_{S^2}\left(|\partial_r h_n|^2 + r^{-2}|\partial_\omega h_n|^2\right)\eta^2\, d\omega \\
&\leq 2r^{-2}\int_{S^2}|\partial_\omega h_n|\,|h_n|\,|\partial_\omega\eta|\,|\eta|\, d\omega + C\int_{S^2} h_n^2|\Psi|\, d\omega \\
&+ C\sum_{i=0}^{n-1} r^{n+2}\int_{S^2}|\nabla_A^i\Psi|\,|\nabla_A^{n-i}\Psi|\,\eta^2\, d\omega.
\end{aligned} \tag{4.19}$$

By the Sobolev and the Hölder inequalities, we have

$$\begin{aligned}
&\int_{S^2} h_n^2|\Psi|\,\eta^2\, d\omega = \int_{S^2} r^{2(n+2)}|\nabla_A^n\Psi|^2|\Psi|\,\eta^2\, d\omega \\
&\leq \varepsilon\int_{S^2} r^{2n+4}r^{-2}|\partial_\omega|\nabla_A^{n+1}\Psi||^2\eta\, d\omega + C\tau^{-2}\int_{S^2} r^{2n+2}|\nabla_A^n\Psi|^2\eta^2\, d\omega \\
&+ C\sup_{(t,r)}\left(\int_{S^2} r^4|\Psi|^2\eta^2\, d\omega\right)\int_{S^2} r^{2n+2}|\nabla_A^n\Psi|^2\eta^2\, d\omega \\
&\leq \varepsilon\int_{S^2} r^{-2}|\partial_\omega h_n|^2\eta\, d\omega + C\tau^{-2}\int_{S^2} r^{-2}h_n^2\eta^2\, d\omega \\
&+ C\sup_{(t,r)}\left(\int_{S^2} h_0^2\eta^2\, d\omega\right)\int_{S^2} r^{-2}h_n^2\eta^2\, d\omega,
\end{aligned} \tag{4.20}$$

and

$$
\begin{aligned}
&\sum_{i=1}^{n-1} \int_{S^2} r^{n+2} |\nabla_A^i \Psi|\, |\nabla_A^{n-i}\Psi|\, h_n \eta^2 \, d\omega \\
&= \sum_{i=1}^{n-1} \int_{S^2} r^{2n+4} |\nabla_A^i \Psi|\, |\nabla_A^{n-i}\Psi|\, |\nabla_A^n \Psi|\, \eta^2 \, d\omega \\
&\le C \sum_{i=1}^{n-1} \left[\left(\int_{S^2} r^{2i+4} |\nabla_A^i \Psi|^2 \eta^2 \, d\omega \right) \right. \\
&\left. \times \left(\int_{S^2} r^{2i+4} |\nabla_A^{i+1}\Psi|^2 \eta^2 \, d\omega + \tau^{-2} \int_{S^2} r^{2i+2} |\nabla_A^i \Psi|^2 \eta^2 \, d\omega \right) \right] \\
&+ C \int_{S^2} r^{2n+2} |\nabla_A^n \Psi|^2 \eta^2 \, d\omega \\
&= C \sum_{i=1}^{n-1} \left(\int_{S^2} h_i^2 \eta^2 \, d\omega \right) \left(\int_{S^2} r^{-2} h_{i+1}^2 \eta^2 \, d\omega + \tau^{-2} \int_{S^2} r^{-2} h_i^2 \eta^2 \, d\omega \right) \\
&+ C \int_{S^2} r^{-2} h_n^2 \eta^2 \, d\omega.
\end{aligned} \tag{4.21}
$$

From (4.19), (4.20) and (4.21), we obtain

$$
\begin{aligned}
&\frac{1}{2} \partial_t g_n - A_n g_n \le C \tau^{-2} \int_{S^2} r^{-2} h_n^2 \eta^2 \, d\omega \\
&+ C \sup_{(t,r)} \left(\int_{S^2} h_0^2 \eta^2 \, d\omega \right) \left(\int_{S^2} r^{-2} h_n^2 \eta^2 \, d\omega \right) \\
&+ C \sum_{i=1}^{n-1} \sup_{(t,r)} \left(\int_{S^2} h_i^2 \eta^2 \, d\omega \right) \left(\int_{S^2} r^{-2} h_{i+1}^2 \eta^2 \, d\omega + \tau^{-2} \int_{S^2} r^{-2} h_i^2 \eta^2 \, d\omega \right),
\end{aligned}
$$

which yields the desired result. □

In the following, we impose the following assumption on the initial value:

$$
|\nabla_A^n F_A(0,x)| + |\nabla_A^n d_A \Phi(0,x)| \le C |x|^{-n-2}, \qquad \text{for all } n \in \mathbb{N} \cup \{0\}, \tag{4.22}
$$

$$
(A(0), \Phi(0)) \in \mathcal{C}, \tag{4.23}
$$

$$
|1 - |\Phi(0,r,\omega)|^2| \le C r^{-1}. \tag{4.24}
$$

We remark that these assumptions are equivalent to (1.4), (1.5) and (1.6), respectively.

Proposition 4.8. *Let $(A(t),\Phi(t))$ be a smooth solution of (1.2) with the initial value $(A(0),\Phi(0))$. Assume that $(A(0),\Phi(0))$ satisfies (4.22) and (4.23). If (A,Φ) satisfies Condition 4.3, then, for any $n\geq 0$,*

$$\sup_{(t,r)}\int_{B_{\tau/2}(\omega_0)} r^{2n+4}|\nabla_A^n\Psi|^2\,d\omega$$

is finite.

Proof. First we show that

$$\sup_{(t,r)}\int_{S^2} r^4|\Psi|^2\eta^2\,d\omega\leq C. \tag{4.25}$$

By Proposition 4.6,

$$g_0(t,r)=\int_{S^2} r^4|\Psi|^2\eta^2\,d\omega$$

satisfies

$$\frac{1}{2}\partial_t g_0 - A_0 g_0\leq C\tau^{-2}\int_{S^2} r^2|\Psi|^2\,d\omega. \tag{4.26}$$

We verify the assumption of Proposition 2.1 for (4.26). Since $|\Psi(0)|\leq Cr^{-2}$, we have

$$g_0(0,r)=\int_{S^2} r^4|\Psi(0)|^2\eta^2\,d\omega\leq C\,\mathrm{vol}(S^2). \tag{4.27}$$

By Proposition 3.1, we have

$$\int_0^\infty\int_{S^2} r^2|\Psi|^2\,d\omega dr=\int_{\mathbb{R}^3}|\Psi|^2\,dV\leq E(0).$$

Since $\|\Psi(t)\|_{L^\infty}\leq C$, we have

$$\|g_0(t)\|_{L^1(\varepsilon,R)}=\int_\varepsilon^R\int_{S^2} r^4|\Psi|^2\eta^2\,d\omega dr\leq CR^5.$$

Applying (4.26) to Proposition 2.1, we have (4.25).

On the other hand, by Proposition 4.6 and Proposition 4.7, we have

(4.28)
$$\int_{\mathbb{R}^3}|\Psi(t)|^2\eta^2\,dV + C\int_0^t\int_{\mathbb{R}^3}|\nabla_A\Psi|^2\eta^2\,dV\,dt\leq E(0)+Ct,$$

(4.29)
$$\begin{aligned}\int_{\mathbb{R}^3} r^2|\nabla_A\Psi(t)|^2\eta^2\,dV + C\int_0^t\int_{\mathbb{R}^3} r^2|\nabla_A^2\Psi|^2\eta^2\,dV\,dt&\leq C\int_0^t\int_{\mathbb{R}^3}|\nabla_A\Psi|^2\eta^2\,dV\,dt\\ + C\sup_{(t,r)}\left(\int_{S^2} r^4|\Psi|^2\eta^2\,d\omega\right)\int_0^t\int_{\mathbb{R}^3}|\nabla_A\Psi|^2\eta^2\,dV\,dt&+\int_{\mathbb{R}^3} r^2|\nabla_A\Psi(0)|^2\eta^2\,dV.\end{aligned}$$

Since $|\nabla_A \Psi(0)|^2 \le Cr^{-6}$, we get

$$\int_{\mathbb{R}^3} r^2 |\nabla_A \Psi(0)|^2 \eta^2 \, dV \le \int_{|x|\le 1} |\nabla_A \Psi(0)|^2 + C \int_1^\infty r^{-2} \, dr \le C.$$

Combining (4.28) with (4.29), we have

$$\begin{aligned} &\sup_t \int_{\mathbb{R}^3} |\Psi(t)|^2 \eta^2 \, dV \le C, \\ &\sup_t \int_{\mathbb{R}^3} r^2 |\nabla_A \Psi(t)|^2 \eta^2 \, dV \le C, \\ &\int_0^t \int_{\mathbb{R}^3} |\nabla_A \Psi(t)|^2 \eta^2 \, dV \, dt \le C, \\ &\int_0^t \int_{\mathbb{R}^3} r^2 |\nabla_A^2 \Psi(t)|^2 \eta^2 \, dV \, dt \le C. \end{aligned} \tag{4.30}$$

Here, we assume that

$$\sup_{(t,r)} \int_{S^2} r^{2m+4} |\nabla_A^m \Psi(t)|^2 \eta^2 \, d\omega \le C, \tag{4.31}$$

$$\sup_t \int_{\mathbb{R}^3} r^{2m} |\nabla_A^m \Psi(t)|^2 \eta^2 \, dV \le C, \tag{4.32}$$

$$\int_0^t \int_{\mathbb{R}^3} r^{2m} |\nabla_A^{m+1} \Psi(t)|^2 \eta^2 \, dV \, dt \le C, \tag{4.33}$$

for all $m \le n-1$. By Proposition 4.7,

$$g_n(t,r) = \int_{S^2} r^{2n+4} |\nabla_A^n \Psi|^2 \eta^2 \, d\omega$$

satisfies

$$\begin{aligned} &\frac{1}{2}\partial_t g_n - A_n g_n \le C(1+\tau^{-2}) \int_{S^2} r^{2n+2} |\nabla_A^n \Psi|^2 \eta^2 \, d\omega \\ &+ C \sum_{i=1}^{n-1} \sup_{(t,r)} \left(\int_{S^2} r^{2i+4} |\nabla_A^i \Psi|^2 \eta^2 \, d\omega \right) \\ &\times \left(\int_{S^2} r^{2(i+1)+2} |\nabla_A^{i+1} \Psi|^2 \eta^2 \, d\omega + \tau^{-2} \int_{S^2} r^2 |\nabla_A^i \Psi|^2 \eta^2 \, d\omega \right). \end{aligned}$$

By (4.31), we have

$$\frac{1}{2}\partial_t g_n - A_n g_n \le C \sum_{i=1}^{n} \int_{S^2} r^{2i+2} |\nabla_A^i \Psi|^2 \eta^2 \, d\omega. \tag{4.34}$$

Since $|\nabla_A^n \Psi(0)| \leq C r^{-(n+2)}$, $\|\nabla_A^n \Psi(t)\|_{L^\infty} \leq C$, we get

$$|g_n(0,r)| \leq \int_{S^2} r^{2n+4} |\nabla_A^n \Psi(0)|^2 \, d\omega \leq C \operatorname{vol}(S^2),$$
$$\|g_n(t)\|_{L^1(\varepsilon,R)} = \int_\varepsilon^R \int_{S^2} r^{2n+4} |\nabla_A^n \Psi(t)|^2 \eta^2 \, d\omega dr \leq C R^{2n+5}.$$

By (4.32), we have

$$\int_0^\infty \int_{S^2} r^{2i+2} |\nabla_A^i \Psi|^2 \eta^2 \, d\omega dr = \int_{\mathbb{R}^3} r^{2i} |\nabla_A^i \Psi|^2 \eta^2 \, dV \leq C.$$

Applying (4.34) to Proposition 2.1, we have (4.31) for $m = n$.

Moreover, by Proposition 4.5, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} r^{2n} |\nabla_A^n \Psi(t)|^2 \eta^2 \, dV + C \int_0^t \int_{\mathbb{R}^3} r^{2n} |\nabla_A^{n+1} \Psi|^2 \eta^2 \, dV dt \\ &\leq C \sum_{i=1}^n \int_0^t \int_{\mathbb{R}^3} r^{2i-2} |\nabla_A^i \Psi|^2 \eta^2 \, dV dt + \int_{\mathbb{R}^3} r^{2n} |\nabla_A^n \Psi(0)|^2 \eta^2 \, dV. \end{aligned} \tag{4.35}$$

Since $|\nabla_A^n \Psi(0)| \leq C r^{-n-2}$, we have

(4.36)
$$\int_{\mathbb{R}^3} r^{2n} |\nabla_A^n \Psi(0)|^2 \eta^2 \, dV \leq \int_{|x| \leq 1} |\nabla_A^n \Psi(0)|^2 \, dV + C \int_1^\infty r^{2n+2-2n-4} \, dr \leq C.$$

Combining (4.35) with (4.36), we get (4.32) and (4.33). This completes the proof by the induction. $\square$

Theorem 4.9. *Let $(A(t), \Phi(t))$ be a smooth solution of (1.2) with the initial value $(A(0), \Phi(0))$. Assume that $(A(0), \Phi(0))$ satisfies (4.22) and (4.23). If (A, Φ) satisfies Condition 4.3, then, for any $n \geq 0$,*

$$\sup_{\substack{(t,r) \\ \omega \in B_{\tau/2}(\omega_0)}} r^{n+2} |\nabla_A^n \Psi(t, r, \omega)| \leq C.$$

Proof. By Proposition 4.8, for any $n \geq 0$,

$$r^{n+2} \nabla_A^n \Psi \in L^2(B_{\tau/2}(\omega_0)), \quad \text{for } (t,r) \in (0,T) \times (0,\infty).$$

In particular, we have

$$r^n \nabla_{A_\omega}^2 \nabla_A^{n-2} \Psi \in L^2(B_{\tau/2}(\omega_0)), \quad \text{for } (t,r) \in (0,T) \times (0,\infty).$$

Using, for any $\mathfrak{g}$-valued form Ω, $|\partial_\omega |\Omega|| \leq |\nabla_{A_\omega} \Omega|$, we get

$$r^n \partial_\omega^2 |\nabla_A^{n-2} \Psi| \in L^2(B_{\tau/2}(\omega_0)), \quad \text{for } (t,r) \in (0,T) \times (0,\infty).$$

Note that $W^{n,2}(B_{\tau/2}(\omega_0)) \subset W^{n-2,\infty}(B_{\tau/2}(\omega_0))$ for all $n \geq 2$, we have

$$r^n \nabla_A^{n-2} \Psi \in C^0((0,\infty) \times B_{\tau/2}(\omega_0)), \qquad \text{for } t \in (0,T).$$

□

5. Existence of Gauge Transformations

In this section, we show the existence of an exponential gauge for a smooth solution $(A(t), \Phi(t))$ of (1.2).

Theorem 5.1. *Let $(A(t), \Phi(t))$ be a smooth solution of (1.2) with the initial value $(A(0), \Phi(0))$. Assume that $(A(0), \Phi(0))$ satisfies (4.22), (4.23) and (4.24). If (A, Φ) satisfies Condition 4.3 then we have*

$$\sup_{(t,r,\omega)} |1 - |\Phi(t,r,\omega)|^2| \leq C r^{-1}, \tag{5.1}$$

$$\sup_{\substack{(t,r) \\ \omega \in B_{\tau/2}(\omega_0)}} |F_A(t,r,\omega)| \leq C r^{-2}, \tag{5.2}$$

$$\sup_{\substack{(t,r) \\ \omega \in B_{\tau/2}(\omega_0)}} |d_A \Phi(t,r,\omega)| \leq C r^{-2}. \tag{5.3}$$

Proof. By Theorem 4.9, we recall

$$\sup_{\substack{(t,r) \\ \omega \in B_{\tau/2}(\omega_0)}} |\Psi(t,r,\omega)| \leq C r^{-2}. \tag{5.4}$$

By the definition of Ψ, (5.2) and (5.3) follows from (5.4).

From (5.3) we have $|d_A \Phi(t,r,\omega)|^2 \leq C r^{-4}$. Combining (5.3) with Proposition 3.4, we get (5.1). □

Theorem 5.2. *Under the same assumptions of Theorem 5.1, if (A, Φ) satisfies Condition 4.3, for all $\omega_0 \in S^2$, then there exists a gauge transformation $g(t,r,\omega)$ such that $\tilde{A} := g^* A$ satisfies*

$$\tilde{A}_r = \sum_{i=1}^{3} x_i \tilde{A}_i = 0.$$

By Theorem 5.1, we can apply the result of Jaffe-Taubes [5, p. 37], (cf. [11]) and obtain Theorem 5.2. However we remark that such a g is not unique. Those two transformations g_1 and g_2 differ only by a radially constant gauge transformation h $(\dfrac{\partial h}{\partial r} = 0)$.

6. Connections at Infinity

Throughout of this section, we assume that connections A satisfy the exponential gauge conditions. Take the orthonormal basis $\{dr, r\sin\theta d\phi, rd\theta\}$ on the cotangent space $\mathbb{R}^3$. In polar coordinate, we can represent the condition A as $A(r,\phi,\theta) = A_r dr + A_\phi(r\sin\theta d\phi) + A_\theta(rd\theta)$. Since we take the exponential gauge, we may assume $A_r = 0$.

$$F(r,\phi,\theta) = F_{\phi\theta}(R,\phi,\theta)(r\sin\theta d\phi)\wedge(rd\theta)$$
$$+ F_{r\theta}(R,\phi,\theta)dr\wedge(rd\theta) + F_{r\phi}(R,\phi,\theta)dr\wedge(r\sin\theta d\phi),$$
$$d_A\Phi = (d_A\Phi)_r dr + (d_A\Phi)_\theta(rd\theta) + (d_A\Phi)_\phi(r\sin\theta)d\phi.$$

Set $A^R(\phi,\theta) := A(R,\phi,\theta)$ and $\Phi^R(\phi,\theta) = \Phi(R,\phi,\theta)$. So, we get

$$F_{A^R}(\phi,\theta) = F_{\phi\theta}(R,\phi,\theta)(R\sin\theta d\phi)\wedge(Rd\theta),$$
$$d_{A^R}\Phi^R = (d_A\Phi)^R = (d_A\Phi)_\theta(Rd\theta) + (d_A\Phi)_\phi(R\sin\theta)d\phi.$$

Therefore this implies

$$|F_{A^r}(\phi,\theta)| = |F_{\phi\theta}(r,\phi,\theta)(r\sin\theta d\phi)\wedge(rd\theta)| \le r^2|F_{\phi\theta}(r,\phi,\theta)|,$$
$$|F_{A^r}(\phi,\theta) - F(r,\phi,\theta)| = |F_{r\theta}(r,\phi,\theta)dr\wedge(rd\theta) - F_{r\phi}(r,\phi,\theta)dr\wedge(r\sin\theta d\phi)|$$
$$\le r|F(r,\phi,\theta)|,$$
$$|(d_A\Phi)^r(\phi,\theta)| = |(d_A\Phi)_\theta(rd\theta) + (d_A\Phi)_\phi(r\sin\theta)d\phi| \le r|d_A\Phi(r,\phi,\theta)|,$$

Thus, we have

Proposition 6.1. *Under the gauge condition $A_r = \sum x^i A_i = 0$, we have*

$$|F_{A^r}(\phi,\theta)| \le r^2|F(r,\phi,\theta)| \le r^2|\Psi(r,\phi,\theta)|,$$
$$|(d_A\Phi)^r(\phi,\theta)| \le r|d_A\Phi(r,\phi,\theta)| = r|*d_A\Phi(r,\phi,\theta)| \le r|\Psi(r,\phi,\theta)|.$$

Using Proposition 6.1, we have

Theorem 6.2. *Let $(\tilde{A}(t),\Phi(t))$ be a smooth solution of (1.2) with the initial value $(\tilde{A}(0),\Phi(0))$. Assume that $(\tilde{A}(0),\Phi(0))$ satisfies (4.22), (4.23) and (4.24). Moreover, we assume that there exists gauge transformations $g(t,r,\omega)$ such that $A := g^*\tilde{A}$ satisfies $A_r = 0$. Then there exists a universal constant $\varepsilon_1 > 0$ such that if*

$$\sup_{(t,r)} \int_{B_\tau(\omega_0)} r^2|\Psi(t,r,\omega)|\,d\omega < \varepsilon_1,$$

then there exist a smooth $\mathfrak{su}(2)$-valued function Φ^∞ on $B_{\tau/2}(\omega_0)$ and a connection A^∞ over $B_{\tau/2}(\omega_0)$ such that

$$\Phi^r \to \Phi^\infty,$$
$$A^r \to A^\infty,$$

in $C^\infty(B_{\tau/2}(\omega_0))$.

Proof. Using the polar coordinate (r, ω) in $\mathbb{R}^3$, we write $A = A_r dr + A_\omega d\omega$. By the assumption $A_r = 0$, it is easy to show that

$$F_{r\omega} = \partial_r A_\omega - \partial_\omega A_r + [A_r, A_\omega] = \partial_r A_\omega.$$

(See Uhlenbeck [11]). Thus, we obtain

$$\frac{\partial}{\partial r}(rA_\omega(x)) = A_\omega + rF_{r\omega}(x). \tag{6.1}$$

Integrating (6.1) over (r_1, r_2), we have

$$\begin{aligned} |r_2 A_\omega(r_2, \omega) - r_1 A_\omega(r_1, \omega)| &= \left| \int_{r_1}^{r_2} \frac{\partial}{\partial \tau}(\tau A_\omega(\tau, \omega))\, d\tau \right| \\ \leq \int_{r_1}^{r_2} \tau |F_{r\omega}(\tau, \omega)|\, d\tau &+ \int_{r_1}^{r_2} |A_\omega(\tau, \omega)|\, d\tau. \end{aligned} \tag{6.2}$$

Using $A_r = 0$, we have

$$\begin{aligned} &|A_\omega(\tau)| \leq \tau^{-1}|A(\tau)| \leq |F(\tau)| \\ &\tau |F_{r\omega}(\tau)| \leq |F(\tau)|, \end{aligned} \tag{6.3}$$

and

$$rA_\omega(r, \omega) = (A^r)_\omega(\omega). \tag{6.4}$$

Combining (6.2) (6.3), (6.4) and the finiteness of the energy, we obtain

$$\begin{aligned} |A^{r_2}(\omega) - A^{r_1}(\omega)| \leq C \int_{r_1}^{r_2} |F(\tau, \omega)|\, d\tau \leq C \left(\int_{r_1}^{r_2} \tau^{-2}\, d\tau \right)^{1/2} \\ \left(\int_{r_1}^{r_2} \tau^2 |F(\tau, \omega)|^2\, d\tau \right)^{1/2} \leq C \left(\frac{1}{r_1} - \frac{1}{r_2} \right)^{1/2}. \end{aligned}$$

Thus, the sequence $\{A^r(\omega)\}$ is a Cauchy sequence.

In general, we have

$$\partial_\omega^n \partial_r (rA_\omega) = \partial_\omega^n A_\omega + \partial_\omega^n (rF_{r\omega}),$$

which yields

$$|r_2^{n+1} \partial_\omega^n A(r_2, \omega) - r_1^{n+1} \partial_\omega^n A(r_1, \omega)| C \leq \int_{r_1}^{r_2} \tau^{n+1} |\partial^n F|\, d\tau. \tag{6.5}$$

The right hand side of (6.5) is bounded from above in terms of

$$\sup_{r_1 < r < r_2} |\tau^{i+1} \partial^i A|, \text{ and } \int_{r_1}^{r_2} \tau^{2j+2} |\nabla_A^j F_A|^2\, d\tau,$$

for $i = 0, \cdots, n-1$, $j = 0, \cdots, n$. Therefore, by (4.32), we have

$$|\partial^n A^{r_2} - \partial^n A^{r_1}| \le C_n \left(\frac{1}{r_1} - \frac{1}{r_2} \right)^{1/2}, \text{ for all } n \in \mathbb{N}.$$

Hence, there exists A^∞ such that $A^r \to A^\infty$ in C^∞-topology.

Similarly, we obtain the convergence of Φ^r. (cf. [1, pp. 2491-2492], [5, II. 4]).

Obviously, A^∞ and Φ^∞ are smooth with respect to t. □

7. Proof of Main Theorem

Let $(A(t), \Phi(t))$ be a smooth solution of (1.2) with the initial value $(A(0), \Phi(0))$. Assume that that $(A(0), \Phi(0))$ satisfies (4.22), (4.23) and (4.24). We furthermore assume that

$$\sup_{(t,r)} \int_{B_\tau(\omega_0)} r^2 |\Psi(t, r, \omega)| \, d\omega < \varepsilon_1.$$

By Theorem 5.1, for any $n \ge 0$, we have (5.1), (5.2) and (5.3). By Theorem 5.2, there exists gauge transformations $g(t, r, \omega)$ such that $\tilde{A}(t, r, \omega) := g(t, r, \omega)^* A(t, r, \omega)$ satisfies

$$\tilde{A}_r = \sum_{i=1}^{3} x_i \tilde{A}_i = 0.$$

Hence, by Theorem 6.2, there exists a smooth $\mathfrak{su}(2)$-valued function Φ^∞ on $B_{\tau/2}(\omega_0)$ and a connection A^∞ over $B_{\tau/2}(\omega_0)$ such that

$$\Phi^r \to \Phi^\infty,$$
$$\tilde{A}^r \to A^\infty,$$

in $C^\infty(B_{\tau/2}(\omega_0))$.

Let us summerize the above arguments, if the initial value $(A(0), \Phi(0))$ satisfies

$$|\nabla_A^n F_A(0, x)| + |\nabla_A^n d_A \Phi(0, x)| \le C|x|^{-n-2}, \quad \text{for all } n \in \mathbb{N} \cup \{0\},$$
$$(A(0), \Phi(0)) \in \mathcal{C}$$

and if

$$\sup_{(t,r)} \int_{B_\tau(\omega_0)} r^2 |\Psi(t, r, \omega)| \, d\omega < \varepsilon_1 \quad \text{for any } \omega_0 \in S^2,$$

then the solution $(A(t), \Phi(t))$ is extendable on $(0, T]$. Hence we get the main theorem.

If A_∞ exists, by Proposition 6.1, then we have

$$|(d_A \Phi)^r| \le r|\Psi| \le Cr^{-1}.$$

Therefore, we have $d_{A^\infty} \Phi^\infty = 0$. Recall that $|\Phi^\infty| = 1$. Then the connection $A^\infty(t, \omega)$ is reduced to the $U(1)$-bundle over S^2.

8. The Prasad-Sommerfield Monopole

We show that the Prasad-Sommerfield monopole satisfies the condition (1.4), (1.5) and (1.6) in Section 1.

The Prasad-Sommerfield monopole has the monopole number $N = \pm 1$:

$$
\begin{cases} \Phi(x) = \mp \left(\dfrac{1}{\tanh r} - \dfrac{1}{r} \right) \vec{n} \cdot \vec{e}, \\ A(x) = \left(\dfrac{1}{\sinh r} - \dfrac{1}{r} \right) (\vec{n} \times \vec{e}) \cdot d\vec{x}. \end{cases} \tag{8.1}
$$

where $r = |x|$, $\vec{n} = \vec{x}/r$, $e_j = \dfrac{i}{2}\sigma_j$, σ_j are Pauli matrices.

It is easy to see that

$$
|1 - |\Phi(x)|\,| = \frac{1}{r} - 2e^{-2r} + O(e^{-3r}),
$$

which shows (1.6), (cf. [5, IV.1, pp. 104-105]).) Remark that the energy of the monopole (8.1) is equal to 4π, and we get (1.5).

Let us show (1.4) for $n = 0$. Since $F_A = dA + [A, A]$, and we have by (1.16) in [5, IV.1, p. 105]

$$
\begin{aligned} |F_A(x)| &\le Cr^{-2}, \\ |d_A\Phi(x)| &\le Cr^{-2}, \\ |A(x)| &\le Cr^{-1}. \end{aligned} \tag{8.2}
$$

For general n, we have

$$
|\nabla_A^n F_A| \le |\partial_r^n F_A| + C \sum_{i=0}^{n-1} |\partial_r^i A|\, |\partial_r^{n-i-1} A| \le Cr^{-n-2}.
$$

For $|\nabla_A^n d_A \Phi|$, we have

$$
|\nabla_A^n d_A\Phi| \le |\partial_r^n d_A\Phi| + C \sum_{i=0}^{n-1} |\partial_r^i A|\, |\partial_r^{n-i-1} d_A\Phi| \le Cr^{-n-2}.
$$

This show that the Prasad-Sommerfield monopole (8.1) satisfies (1.4), (1.5) and (1.6).

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