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ON THE POINT SPECTRUM OF DIFFIRENCE SCHRÖDINGER OPERATORS

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Abstract. One considers the equation $\psi\,(x+h)+\psi\,(x-h)+v\,(x)\,\psi\,(x)=E\psi\,(x)$, where v is an almost everywhere finite periodic function, and h is a positive number. It is proved that this equation has no solutions from $L_2(\mathbb{R})$. This implies in particular that the spectrum of Harper operator appears to be singular continuous in all the cases where its geometrical structure was investigated.

1 Introduction

In this note we consider the equation

$$H\psi\,(x)=E\psi\,(x),\quad x\in\mathbb{R}, \tag{1.1}$$

where

$$(H\,\psi)\,(x)=\psi\,(x+h)+\psi\,(x-h)+v\,(x)\psi\,(x), \tag{1.2}$$

h is a positive number, and v is an almost everywhere finite measurable function periodic with a period h_0. This function can be complex valued. We prove

Theorem 1.1. Equation (1.1) has no solutions from $L_2(\mathbb{R})$.

The central point in the proof is related to the notion of Bloch solutions. For an ordinary differential equation with periodic coefficients, one calls its solution ψ a Bloch solution if it is invariant up to a constant factor with respect to the translation by the period:

$$\psi\,(x+h_0)=u\,\psi\,(x),\quad x\in\mathbb{R}. \tag{1.3}$$

For equation (1.1) ψ is called a Bloch solution if it satisfies (1.3) with a coefficient u depending h-periodically on x,

$$u\,(x+h)=u\,(x).$$

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This definition is natural since the set of solutions of equation (1.1) is a two-dimensional modul over the ring of h-periodic functions. The idea of the proof is to show that if there is an $L_2(\mathbb{R})$-solution of (1.1), then this equation has also a Bloch solution belonging to $L_2(\mathbb{R})$, and to check that this is impossible. The same idea leads to an immediate proof of the analogous theorem for the one-dimensional differential equation with periodic coefficients. In the case under considiration the coefficient u from the definition of Bloch solutions depends on x, and the proof becomes a little more complicated.

In the case where v is, for example, a bounded real-valued function, the statement of the theorem can be easily derived from the direct-integral decomposition of the problem in terms of the corresponding problems on the invariant lattices, and from the known theorems on the structure of the discrete spectrum of the ergodic operators, see [PF]. But even in this case the present proof, probably, is not completely useless since it is quite direct and elementary.

In section 2, we characterize the set of solutions of equation (1.3). The theorem 1.1 is proved in section 3.

In the sequel, for brevity, we often omit in the standard way the words "almost everywhere". In particular, instead of writing "$a(x) = b(x)$ for a.e. $x \in X$, we write simply $a(x) = b(x)$ $x \in X$", instead of saying "an a.e. finite function" , we say "a finite function", and so on.

2 Set of solutions of the equation $H\psi = E\psi$

To prove the theorem we have to prepare several auxiliary results.

2.1 The structure of the set of solutions.

Here, we slightly generalize two statements from [BF3].

Let $\psi(x)$ and $\tilde{\psi}(x)$ satisfy equation (1.1). We call the expression

$$\left(\psi(x), \tilde{\psi}(x)\right) = \psi(x+h)\tilde{\psi}(x) - \psi(x)\tilde{\psi}(x+h)$$

the *wronskian* of the solutions ψ and $\tilde{\psi}$.

Lemma 2.1 The wronskian of ψ and $\tilde{\psi}$ is an h-periodic function.

Proof.

$$\psi(x)(\tilde{\psi}(x+h) + \tilde{\psi}(x-h)) = (E - v(x))\psi(x)\tilde{\psi}(x) = \tilde{\psi}(x)(\psi(x+h) + \psi(x-h)).$$

□

Proposition 2.2 Let

$$(\psi(x), \tilde{\psi}(x)) = 1, \quad x \in \mathbb{R}. \tag{2.1}$$

Then any function g satisfying (1.1) can be represented in the form

$$g(x) = (g(x),\, \tilde{\psi}(x))\,\psi(x) + (\psi(x),\, g(x))\,\tilde{\psi}(x), \quad x \in \mathbb{R}.$$

Proof. Let

$$M(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}, \quad G(x) = \begin{pmatrix} g(x) \\ g(x-h) \end{pmatrix},$$

$$\Psi(x) = \begin{pmatrix} \psi(x) & \tilde{\psi}(x) \\ \psi(x-h) & \tilde{\psi}(x-h) \end{pmatrix}.$$

Obviously,

$$\Psi(x+h) = M(x)\,\Psi(x), \quad G(x+h) = M(x)\,G(x).$$

Moreover, due to (2.1) $\det \Psi = 1$. Therefore, one can write $G(x+h) = \Psi(x+h)\,\Psi^{-1}(x)\,G(x)$. This implies that the vector $\Psi^{-1}\,G$ is an h-periodic function. This leads to the representation $g = p\,\psi + \tilde{p}\,\tilde{\psi}$ where p and $\tilde{p}$ are two h-periodic functions. Using this representation and taking into account (2.1), one can easily check that $(g,\, \tilde{\psi}) = p$, and $(\psi,\, g) = \tilde{p}$, which proves the lemma. $\square$

Note that this proposition is a part of the proof that the set of solutions of equation (1.1) is a two-dimensional modul over the ring of h-periodic functions.

2.2 Relations between L_2-solutions.

Assume that two functions ψ and $\tilde{\psi}$ belong to $L_2(\mathbb{R})$ and satisfy equation (1.1). Study relations between them.

Lemma 2.3. The wronskian of ψ and $\tilde{\psi}$ equals to zero.

Proof. By Lemma 2.1, the function $(\psi_1(x), \psi_2(x))$ is h-periodic. But, since $\psi_1 \in L_2(\mathbb{R})$ and $\psi_2 \in L_2(\mathbb{R})$, one has

$$\int_{\mathbb{R}} |(\psi_1(x), \psi_2(x))|\; dx < \infty.$$

This implies the statement of the lemma. $\square$

Proposition 2.4. Let ψ and $\tilde{\psi}$ satisfy equation (1.1) and belong to $L_2(\mathbb{R})$. Let

$$\delta = \{ x \in \mathbb{R} : \psi(x) \neq 0 \},$$

$$\Delta = \cup_{k \in \mathbb{Z}} \{ x \in \mathbb{R} : x - hk \in \delta \}$$

There exists an h-periodic finite function $a(x)$, defined on Δ, such that

$$\tilde{\psi}(x) = a(x)\psi(x), \quad x \in \Delta. \tag{2.2}$$

Proof. Let

$$S(x) = \begin{cases} 0, & \text{when } 0 \leq x < h, \\ 1, & \text{when } h \leq x < 2h, \end{cases} \qquad C(x) = \begin{cases} 1, & \text{when } 0 \leq x < h, \\ 0, & \text{when } h \leq x < 2h. \end{cases}$$

By means of the formulae $S(x \pm h) = -S(x \mp h) + (E - v(x))\,S(x)$ define $S(x)$ for other $x \in \mathbb{R}$. By construction, S satisfies equation (1.1). Define in the same way C. Obviously,

$$(S(x), C(x)) = 1. \tag{2.3}$$

Using proposition 2.2, one can write

$$\psi(x) = \alpha(x)\,C(x) + \beta(x)\,S(x), \tag{2.4}$$

$$\tilde{\psi}(x) = \tilde{\alpha}(x)\,C(x) + \tilde{\beta}(x)\,S(x). \tag{2.5}$$

This is clear that α, β, $\tilde{\alpha}$ and $\tilde{\beta}$ are h-periodic finite functions. By Lemma 2.1, the wronskian $(\psi(x), \tilde{\psi}(x)) = 0$. Substituting in the formula for this wronskian representations (2.4) and (2.5) one gets

$$\alpha(x)\tilde{\beta}(x) - \tilde{\alpha}(x)\beta(x) = 0. \tag{2.6}$$

On the other hand, for all $x \in \delta$,

$$|\alpha(x)|^2 + |\beta(x)|^2 \neq 0. \tag{2.7}$$

The h-periodicity of α and β implies that (2.7) remains true on Δ. Therefore (2.6) and (2.7) show that

$$\begin{pmatrix} \tilde{\alpha}(x) \\ \tilde{\beta}(x) \end{pmatrix} = a(x) \begin{pmatrix} \alpha(x) \\ \beta(x) \end{pmatrix}, \quad x \in \Delta, \tag{2.8}$$

where the function a is defined and finite on the set Δ. Obviously, a is h-periodic. Formulae (2.4), (2.5) and (2.8) lead to the representation (2.2). □

3. Absence of the L_2-solutions

3.1 Bloch solution.

Proposition 3.1. If equation (1.1) has a solution from $L_2(\mathbb{R})$, then it also has a solution $\Psi \in L_2(\mathbb{R})$ satisfying the relation

$$\Psi\,(x + h_0) = u\,(x)\,\Psi\,(x), \tag{3.1}$$

with an h-periodic function u being finite and non-zero for a.e. $x \in \mathbb{R}$.

Proof. Let $\psi_0 \in L_2(\mathbb{R})$ be a solution of equation (1.1), and let

$$\Delta_0 = \cup_{k\in\mathbb{Z}}\{\, x \in \mathbb{R} \,:\, x - hk \in \delta_0 \,\}, \quad \text{where} \quad \delta_0 = \{\, x \in \mathbb{R} \,:\, \psi_0(x) \neq 0 \}\,.$$

Put

$$\Delta_l = \{\, x \in \mathbb{R} \,:\, x - lh_0 \in \Delta_0 \,\}\,, \quad l \in \mathbb{Z}.$$

Note that for any $l \in \mathbb{Z}$

$$x \in \Delta_l \Longrightarrow x + nh \in \Delta_l \quad \forall n \in \mathbb{Z}. \tag{3.2}$$

Fix $q \in \mathbb{R}$ so that $0 < q < 1$. Define the functions

$$\psi_l(x) = q^{|l|}\,\psi_0(x - lh_0), \quad l \in \mathbb{Z}.$$

All the functions ψ_l satisfy equation (1.1) and belong to $L_2(\mathbb{R})$. Moreover, Proposition 2.4 implies

Lemma 3.2. If $\psi \in L_2(\mathbb{R})$ and satisfies (1.1), then for any $l \in \mathbb{Z}$

$$\psi\,(x) = a_l(x)\,\psi_l(x), \quad x \in \Delta_l,$$

where a_l is a finite h-periodic function.

To construct the solution Ψ, it is covenient to introduce one more family of sets. Let $\mathcal{M}_0 = \Delta_0$, and

$$\mathcal{M}_{l+1} = \Delta_{l+1} \setminus \left(\bigcup_{j=-l}^{j=l} \mathcal{M}_j\right), \quad \mathcal{M}_{-l-1} = \Delta_{-l-1} \setminus \left(\bigcup_{j=-l}^{j=l+1} \mathcal{M}_j\right)$$

for all $l \in \mathbb{N} \cup \{0\}$. Note that by definition for all $l \in \mathbb{Z}$

$$\mathcal{M}_l \subset \Delta_l. \tag{3.3}$$

Furthemore, it can be easily seen that

$$\mathcal{M}_l \cap \mathcal{M}_k = \emptyset \quad \text{if} \quad k \neq l, \tag{3.4}$$

and

$$\bigcup_{j=-l}^{j=l} \mathcal{M}_j = \bigcup_{j=-l}^{j=l} \Delta_j, \quad \bigcup_{j=-l}^{j=l+1} \mathcal{M}_j = \bigcup_{j=-l}^{j=l+1} \Delta_j. \tag{3.5}$$

This together with the definition of $\mathcal{M}_l$ and (3.2) shows that for any $l \in \mathbb{Z}$

$$x \in \mathcal{M}_l \Longrightarrow x + nh \in \mathcal{M}_l, \quad n \in \mathbb{Z}. \tag{3.6}$$

Let

$$\mathcal{M} = \bigcup_{l \in \mathbb{Z}} \mathcal{M}_l. \tag{3.7}$$

Note that, due to (3.5), this is equivalent to

$$\mathcal{M} = \bigcup_{l \in \mathbb{Z}} \Delta_l. \tag{3.8}$$

Therefore the definitions of Δ_l and (3.2) imply that

$$x \in \mathcal{M} \Longrightarrow x + mh_0 + nh \in \mathcal{M} \quad n, m \in \mathbb{Z}. \tag{3.9}$$

It is convenient to intriduce one more set,

$$\mathcal{N} = \mathbb{R} \setminus \mathcal{M}.$$

In view of (3.9),

$$x \in \mathcal{N} \Longrightarrow x + mh_0 + nh \in \mathcal{N} \quad n, m \in \mathbb{Z}. \tag{3.10}$$

Remark that all the above sets are measurable.

Define $\Psi(x)$ by the formula

$$\Psi(x) = \sum_{l \in \mathbb{Z}} \chi_{\mathcal{M}_l}(x)\, \psi_l(x), \tag{3.11}$$

where $\chi_{\mathcal{M}_l}$ is the characteristic function of the set $\mathcal{M}_l$. Note that, by definition, $\Psi(x) = 0$ for all $x \in \mathcal{N}$, and, due to (3.4), $\Psi(x) = \psi_l(x)$ when $x \in \mathcal{M}_l$.

Check that Ψ satisfies equation (1.1). Consider at first the case when $x \in \mathcal{M}$. There is an $l \in \mathbb{Z}$ such that $x \in \mathcal{M}_l$. By (3.6), $x \pm h \in \mathcal{M}_l$. Therefore $(H\Psi)(x) = (H\psi_l)(x) = E\psi_l(x) = E\Psi(x)$. If $x \in \mathcal{N}$, then $x \pm h \in \mathcal{N}$, and $\Psi(x-h) = \Psi(x) = \Psi(x+h) = 0$. So, again $(H\Psi)(x) = E\Psi(x)$.

The solution Ψ belongs to $L_2(\mathbb{R})$:

$$\|\Psi\|_{L_2}^2 = \sum_{l \in \mathbb{Z}} \int_{\mathcal{M}_l} |\psi_l(x)|^2\, dx \leq \sum_{l \in \mathbb{Z}} \|\psi_l\|_{L_2}^2 = \sum_{l \in \mathbb{Z}} q^{2|l|}\, \|\psi_0\|_{L_2}^2 < \infty.$$

Now prove

Lemma 3.3. If $\psi \in L_2(\mathbb{R})$, and $H\psi = E\psi$, then

$$\chi_{\mathcal{M}}\psi(x) = a(x)\Psi(x), \quad x \in \mathbb{R},$$

where a is a finite h-periodic function.

Proof. Using Lemma 3.2, one can write

$$\chi_{\mathcal{M}}(x)\, \psi(x) = \sum_{l \in \mathbb{Z}} \chi_{\mathcal{M}_l}(x)\, \psi(x) = \sum_{l \in \mathbb{Z}} \chi_{\mathcal{M}_l}(x)\, a_l(x)\, \psi_l(x)$$

with some finite h-periodic functions a_l. Due to (3.4), this formula can be rewritten in the form

$$\chi_{\mathcal{M}}(x)\ \psi(x) = \sum_{l \in \mathbb{Z}} \chi_{\mathcal{M}_l}(x)\, a_l(x) \sum_{l \in \mathbb{Z}} \chi_{\mathcal{M}_l}(x)\, \psi_l(x) = a(x)\, \Psi(x).$$

The function $a(x) = \sum_{l \in \mathbb{Z}} \chi_{\mathcal{M}_l}(x)\, a_l(x)$ is finite and, by (3.6), h-periodic.

□

Now one can easily finish the proof of the proposition. Since $\Psi = 0$ outside $\mathcal{M}$, and since $\mathcal{M}$ is invariant with respect to the h_0-translations, one has by Lemma 3.3

$$\Psi(x + h_0) = a(x)\,\Psi(x), \quad x \in \mathbb{R}, \tag{3.12}$$

with an finite h-periodic function a. Discuss the set where $a(x) = 0$.

Denote by $\mathcal{D}$ the set where $\Psi(x) \neq 0$. By (3.12), $\Psi(x+h_0) = \Psi(x) = 0$, $x \notin \mathcal{D}$. Thus, for all these x one can change a by letting $a(x) = 1$. Now, to finish the proof, it suffices to show that $a(x) \neq 0$ for a.e. $x \in \mathcal{D}$. In the same way as we have proved (3.12) one can show that $\Psi(x - h_0) = \tilde{b}(x)\,\Psi(x)$, where $\tilde{b}$ is a finite h-periodic function. Therefore,

$$\Psi(x) = b(x)\Psi(x + h_0), \quad x \in \mathbb{R},$$

where $b(x) = \tilde{b}(x + h_0)$. This formula and formula (3.12) imply that

$$a(x)\,b(x) = 1, \quad x \in \mathcal{D}.$$

Since both a and b are finite, this finishes the proof.

□

Corollary 3.4. For the Bloch solution Ψ described in Proposition 3.1, one has

$$\mathrm{mes}\,\{x \in [0, h_0] \,:\, \Psi(x) \neq 0\} > 0.$$

Proof. Suppose that $\Psi(x) = 0$ on the interval $[0, h_0)$. Then, since $u(x) \neq 0$ and $1/u(x) \neq 0$ for a.e. x, formula (3.1) implies that $\Psi = 0$. But this is impossible, because $\Psi(x) = \psi_0(x)$ for all x where $\psi_0(x) \neq 0$. □

3.2 Proof of the theorem.

Now we can easily prove the theorem. Suppose equation (1.1) has a solution from $L_2(\mathbb{R})$. Then it has also the Bloch solution Ψ described in the proposition. Show that this is impossible.

Define a measure by the formula $\nu(\sigma) = \int_\sigma |\Psi(x)|^2\, dx$, where σ is a measurable set. Note that by Corollary 3.2 $\nu([0, h_0]) > 0$.

By (3.1) one has

$$\int_0^\infty |\Psi(x)|\, dx = \sum_{l=0}^{\infty} \int_0^{h_0} f_l(x)\, d\nu(x), \tag{3.13}$$

where $f_0(x) = 1$, and

$$f_l(x) = |u(x)\,u(x + h_0) \ldots u(x + (l-1)h_0)|^2, \quad l \in \mathbb{N}.$$

Since $\int_0^\infty |\Psi|^2\, dx < \infty$, and $f_l(x) \geq 0$ for all $x \in \mathbb{R}$ and $l \in \mathbb{N}$, formula (3.13) implies that for any fixed $C > 0$

$$\nu\,\{x \in [0, h_0] \,:\, f_l(x) > C\} \to 0 \quad \text{as} \quad l \to \infty. \tag{3.14}$$

Since u in (3.1) is (a.e.) non-zero, one can consider the functions

$$g_l(x) = |u(x - h_0)\, u(x - 2h_0) \ldots u(x - lh_0)|^{-2}, \quad l \in \mathbb{N}.$$

In the same way as we have obtained (3.14), one can easily prove that the inequality $\int_{-\infty}^{0} |\Psi|^2\, dx < \infty$ implies that for any fixed $C > 0$

$$\nu\, \{x \in [0, h_0]\, :\, g_l(x) > C\} \to 0 \quad \text{as} \quad l \to \infty. \tag{3.15}$$

Show that (3.14) and (3.15) can not be valid simultaneously. Note that

$$g_l(x) = \frac{1}{f_l(x - lh_0)}, \quad l \in \mathbb{N}.$$

Therefore $\nu\, \{x \in [0, h_0]\, :\, g_l(x) > C\} = \nu\, \left\{x \in [0, h_0]\, :\, f_l(x - lh_0) < C^{-1}\right\}$. Since f_l is h-periodic, one has also

$$\nu\, \{x \in [0, h_0]\, :\, g_l(x) > C\} = \nu\, \left\{x \in [0, h_0]\, :\, f_l(x + jh - lh_0) < C^{-1}\right\}, \quad j \in \mathbb{Z}. \tag{3.16}$$

To finish the proof we have to consider the cases $h/h_0 \in \mathbb{Q}$ and $h/h_0 \notin \mathbb{Q}$ separately.

Let $h/h_0 \in \mathbb{Q}$. Then one can write $h/h_0 = m/n$ for some natural m and n. Since $h_0 m = hn$, by (3.16)

$$\nu\, \{x \in [0, h_0]\, :\, g_{l\cdot m}(x) > C\} = \nu\, \{[0, h_0]\} - \nu\, \left\{x \in [0, h_0]\, :\, f_{l\cdot m}(x) \geq C^{-1}\right\}$$

for any $l \in \mathbb{N}$. Direct l in this formula to infinity. By (3.14) and (3.15), we get in result that $0 = \nu\, \{[0, h_0]\}$ which is impossible. This completes the proof of the theorem in the case where $h/h_0 \in \mathbb{Q}$.

Assume that $h/h_0 \notin \mathbb{Q}$. Then for any $\epsilon > 0$ one can choose an increasing sequence of natural numbers l_k so that

$$\inf_{j \in \mathbb{Z}} |jh - l_k h_0| < \epsilon.$$

Fix k. Choose $j \in \mathbb{N}$ so that $|\epsilon_0| < \epsilon$, where $\epsilon_0 = jh - l_k h_0$. Relation (3.16) implies

$$\nu\, \{x \in [0, h_0]\, :\, g_{l_k}(x) > C\} = \nu\, \left\{x \in [0, h_0]\, :\, f_{l_k}(x + \epsilon_0) < C^{-1}\right\} =$$

$$= \nu\, \left\{x \in [\epsilon_0, h_0 + \epsilon_0]\, :\, f_{l_k}(x) < C^{-1}\right\} \geq \nu\, \left\{x \in [\epsilon, h_0 - \epsilon]\, :\, f_{l_k}(x) < C^{-1}\right\} =$$

$$= \nu([\epsilon, h_0 - \epsilon]) - \nu\, \{x \in [\epsilon, h_0 - \epsilon]\, :\, f_{l_k}(x) \geq C\}.$$

Directing in this formula k to ∞ one obtains $0 \geq \nu\, \{[\epsilon, h_0 - \epsilon]\}$, and, since ϵ is arbitrary, $0 \geq \nu\, \{[0, h_0]\}$, which is impossible. This completes the proof of the theorem. $\square$

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