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1 Introduction and statement of results

Let $\mathcal{O} \subset \mathbf{R}^n, n \geq 2$, be a compact set with C^∞-smooth boundary Γ and connected complement $\Omega = \mathbf{R}^n \setminus \mathcal{O}$. Denote by Δ_e the elasticity operator

$$\Delta_e v = \mu_0 \Delta v + (\lambda_0 + \mu_0)\nabla(\nabla \cdot v),$$

$v = {}^t(v_1, ..., v_n)$, where the Lamé constants λ_0 and μ_0 satisfy

$$\text{(H.1)} \qquad \mu_0 > 0, \quad n\lambda_0 + 2\mu_0 > 0.$$

Consider Δ_e in Ω with Neumann boundary conditions on Γ

$$(Bv)_i := \sum_{j=1}^n \sigma_{ij}(v)\nu_j|_\Gamma = 0, \qquad i = 1, ..., n,$$

where $\sigma_{ij}(v) = \lambda_0 \nabla \cdot v\delta_{ij} + \mu_0 \left(\partial_{x_j} v_i + \partial_{x_i} v_j\right)$ is the stress tensor, ν is the outer normal to Γ. Denote by Δ_e^N the self-adjoint realization of Δ_e in Ω with Neumann boundary conditions on Γ. It was proved in [7] that for any obstacle in odd dimensional spaces there exists an infinite sequence $\{\lambda_j\}$ of resonances associated to Δ_e^N such that $\mathrm{Im}\,\lambda_j = O(|\lambda_j|^{-\infty})$. This is due to the existence of Rayleigh surface waves mooving with a speed $c_R > 0$ strictly less than the 2 speeds in Ω, $c_1 = \sqrt{\mu_0}$, $c_2 = \sqrt{\lambda_0 + 2\mu_0}$. Moreover, for strictly convex obstacles, a large region free of resonances was obtained in [6]. This was extended in [2] for obstacles nontrapping for the Dirichlet realization, Δ_e^D, of Δ_e.

The purpose of this work is to obtain asymptotics of the counting function of the resonances associated to Δ_e^N near the real axis for a class of obstacles including the strictly convex ones. To do so, we use the following characterization of the resonances by the complex scaling method (see [4]). Fix a $\theta \in (0, \pi/2)$ and let Ω_θ be a deformation of Ω with properties described in [4] and in particular which coincides with $e^{-i\theta}\mathbf{R}^n$ outside a neighbourhood of $\mathcal{O}$. Then $z \in \mathbf{C}\setminus 0, 0 < \arg z < \theta$, is a resonance of Δ_e^N iff z^2 is an eigenvalue of the operator $-\Delta_e^N$ on the Hilbert space $L^2(\Omega_\theta)$ with domain of definition $H_B^2(\Omega_\theta) = \{u \in H^2(\Omega_\theta) : Bu = 0\}$. The multiplicity of z is the multiplicity of the corresponding eigenvalue, i.e.

$$\mathrm{mult}(z) := \mathrm{tr}\,(2\pi i)^{-1} \int_{\gamma(z)} (\Delta_e^N + \lambda^2)^{-1} 2\lambda d\lambda,$$

where $(\Delta_e^N + \lambda^2)^{-1} : L^2(\Omega_\theta) \to H_B^2(\Omega_\theta)$ is meromorphic in $\{\lambda \in \mathbf{C} : 0 < \arg \lambda < \theta\}$, $\gamma(z)$ is a small positively oriented circle centered at z and with no other poles in its interior. This enables us (see Sect.2) to express the multiplicity in terms of the Dirichlet-to-Neumann map, $\mathcal{N}(\lambda)$, defined as follows:

$$\mathcal{N}(\lambda) : H^s(\Gamma) \ni f \mapsto Bv \in H^{s-1}(\Gamma),$$

where v solves the problem

$$\begin{cases} (\Delta_e + \lambda^2)v & = & 0 & \text{in } \Omega, \\ v & = & f & \text{on } \Gamma, \\ v & - & \text{outgoing.} & \end{cases}$$

Recall that $\mathcal{N}(\lambda)$ is a meromorphic family with poles among the Dirichlet resonances. We get that if z is not a Dirichlet resonance, then

$$\mathrm{mult}(z) = \mathrm{tr}\,(2\pi i)^{-1} \int_{\gamma(z)} \mathcal{N}(\lambda)^{-1} \dot{\mathcal{N}}(\lambda) d\lambda,$$

where $\dot{a}$ denotes the first derivative $da/d\lambda$. Clearly, the formula does not change if we put $\lambda = 1/h$ and replace $\mathcal{N}(\lambda)$ by $N(h) := h\mathcal{N}(h^{-1})$. Recall next (e.g. see [6]) that in the elliptic region $\mathcal{E} = \{\zeta \in T^*\Gamma;\ c_1\|\zeta\| > 1\}$, $N(h)$ is a $h - \Psi$DO of class $L_{cl}^{1,0}(\Gamma)$ (see the appendix for the terminology) with a characteristic variety $\Sigma = \{\zeta \in T^*\Gamma;\ c_R\|\zeta\| = 1\} \subset \mathcal{E}$. The existence of such a characteristic variety is interpreted as existence of Rayleigh waves on the boundary. To get asymptotics for the counting function of the resonances generated by Σ we need the following assumptions.

(H.2) There exist some constants $C_0 \geq 1, \delta_0, k_0 > 0$ such that the elastic Dirichlet problem has no resonances in $\Lambda = \{\lambda \in \mathbf{C} : |\mathrm{Im}\,\lambda| \leq |\lambda|^{-\delta_0}, \mathrm{Re}\,\lambda \geq C_0\}$, and

$$\|\mathcal{N}(\lambda)\|_{\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))} \leq |\lambda|^{k_0}, \quad \lambda \in \Lambda. \tag{1.1}$$

Note that (H.2) is fulfilled if the obstacle is nontrapping for the Dirichlet problem (see [2]). In particular, for strictly convex obstacles it follows from the analysis in [6]. Note also that when n is odd, it suffices only to require that there are no Dirichlet resonances in some polynomial neighbourhood of the real axis, as this implies, in view of Proposition 1 in [7], that (1.1) holds in a smaller polynomial neighbourhood of the real axis. The following assumption means that $N(h)$ is invertible outside Σ and in the strictly convex case it follows from the results in [6].

(H.3) Let $\chi \in L_{cl}^{-\infty,0}(\Gamma)$ be a $h - \Psi$DO depending holomorphically on h for h^{-1} in a larger set of the same type as Λ with $\widetilde{\mathrm{WF}}(\chi)$ contained in a small neighbourhood of Σ such that $\widetilde{N}(h) := N(h) + \chi$ is elliptic in $\mathcal{E}$. Then, for $h^{-1} \in \Lambda$, the operator $\widetilde{N}(h) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is bijective with inverse of norm $O(|h|^{-k_1})$ for some constant $k_1 > 0$.

It is easy to see that the validity of (H.3) is independent of the choice of χ. See also Remark 3.2. We also need the following technical assumption.

(H.4) $n \neq 4$.

Let $\{\lambda_j\}$ be the resonances of Δ_e^N in Λ, repeated according to multiplicity. Our main result is the following theorem.

Theorem 1.1. *Under the assumptions (H.1)-(H.4), we have*

$$\sharp\{\lambda_j : |\lambda_j| \leq r\} = \tau_n c_R^{-n+1} \mathrm{Vol}\,(\Gamma) r^{n-1} + O(r^{n-2}), \quad r \to \infty,$$

where $\tau_n = (2\pi)^{-n+1} \mathrm{Vol}\{x \in \mathbf{R}^{n-1} : |x| \leq 1\}$.

To prove the theorem we first show that $N(h)$ in $\mathcal{E}$ minus a small neighbourhood of its boundary can be extended to an $h - \Psi\mathrm{DO}$, $P(h) \in L_{cl}^{1,0}(\Gamma)$, which is selfadjoint for real h, with a principal symbol having one eigenvalue vanishing on Σ, negative in $\mathcal{B} := \{\zeta \in T^*\Gamma;\ c_R\|\zeta\| < 1\}$, positive in $\{\zeta \in T^*\Gamma;\ c_R\|\zeta\| > 1\}$, and all the other eigenvalues positive on $T^*\Gamma$. We further show that the eigenvalues $\{\mu_j(h)\}$, repeated according to multiplicity, of $P(h)$ near 0 are increasing functions of h, for h small enough, so we can define an infinite sequence $\{\widetilde{\lambda}_j\} \subset \mathbf{R}^+$ by $\mu_j(\widetilde{\lambda}_j^{-1}) = 0$. Thus, modulo some constant, the number of $\{\widetilde{\lambda}_j : \widetilde{\lambda}_j \leq r\}$ is equal to the number of the eigenvalues of $P(r^{-1})$ in $(-\infty, 0]$ which, according to well known semi-classical asymptotics, is

$$\left(\frac{r}{2\pi}\right)^{n-1} \mathrm{Vol}(\mathcal{B}) + O(r^{n-2}).$$

The final step in the proof is to show that there exists a bijection between $\{\widetilde{\lambda}_j\}$ and the resonances $\{\lambda_j\}$.

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2 Trace integrals for $\mathcal{N}$

The purpose of this section is to prove the following

Proposition 2.1. *Let* $\gamma \subset \mathbf{C}_\theta := \{\lambda \in \mathbf{C} : |\arg\lambda| < \theta\}$ *be a closed positively oriented* C^1 *curve without self intersections which avoids the resonances. If there are no Dirichlet resonances on* γ *and in its interior, then*

$$\mathrm{tr}\,(2\pi i)^{-1} \int_\gamma \mathcal{N}(\lambda)^{-1} \dot{\mathcal{N}}(\lambda) d\lambda$$

is equal to the number of the Neumann resonances inside γ.

Proof. Let $\Omega \subset \mathbf{C}$ be an open domain and let H_1, H_2 be two Hilbert spaces. A meromorphic function $B(\lambda) : \Omega \to \mathcal{L}(H_1, H_2)$ will be said to be a family with finite rank singularities in Ω if near every pole $\lambda_0 \in \Omega$, $B(\lambda)$ has a Laurent expansion

$$B(\lambda) = \widetilde{B}(\lambda) + \sum_{j=1}^{k} P_j (\lambda - \lambda_0)^{-j}, \tag{2.1}$$

with P_j of finite rank and $\widetilde{B}(\lambda)$ holomorphic at λ_0. As $\mathcal{N}(\lambda)$ (resp. $\mathcal{N}(\lambda)^{-1}$) can be expressed in terms of the Dirichlet (resp. Neumann) resolvent, $\mathcal{N}(\lambda)$ (resp. $\mathcal{N}(\lambda)^{-1}$) is a family with finite rank singularities in $\mathbf{C}_\theta$ with poles among the Dirichlet (resp. Neumann) resonances. We need the following technical lemma.

Lemma 2.2. *Let $B(\lambda)$ be as above and let $A(\lambda) : \Omega \to \mathcal{L}(H_2, H_1)$ be a holomorphic function. Let also $\gamma : S^1 \to \Omega$ be a C^1 curve avoiding the poles. Then the operators*

$$(2\pi i)^{-1} \int_\gamma A(\lambda) B(\lambda) d\lambda : H_1 \to H_1 \quad \textit{and} \quad (2\pi i)^{-1} \int_\gamma B(\lambda) A(\lambda) d\lambda : H_2 \to H_2$$

are of trace class and have the same trace.

Proof. We may replace γ by a union of closed loops around the poles inside γ, so we may assume that γ is already a small closed loop around a pole λ_0, where (2.1) holds. Then, we can replace $B(\lambda)$ by its singular part $\sum_{j=1}^{k} P_j (\lambda - \lambda_0)^{-j}$ in both the integrals above, and we are reduced to the case when $B(\lambda)$ is of trace class. In this case, however, the desired conclusion is immediate as we can put the traces inside the integrals and use then the cyclicity of the trace. $\square$

For $\lambda \in C_\theta$ we can solve the inhomogeneous Dirichlet problem

$$\begin{cases} (\Delta_e + \lambda^2) u &= f \in L^2(\Omega_\theta), \\ u|_\Gamma &= g \in H^{3/2}(\Gamma), \\ u &- \text{outgoing}, \end{cases}$$

by

$$u = G^D(\lambda) f + K^D(\lambda) g,$$

where $G^D : L^2(\Omega_\theta) \to H^2(\Omega_\theta)$ is the Dirichlet-Green operator, and $K^D : H^{3/2}(\Gamma) \to H^2(\Omega_\theta)$ is the Dirichlet-Poisson operator. Now the Dirichlet-Neumann operator $\mathcal{N} : H^{3/2}(\Gamma) \to H^{1/2}(\Gamma)$ is given by

$$\mathcal{N}(\lambda) = B K^D(\lambda).$$

Hence,

$$\dot{\mathcal{N}}(\lambda) = B \dot{K}^D(\lambda) = -B G^D(\lambda) K^D(\lambda) 2\lambda. \tag{2.2}$$

On the other hand, it is easy to see that

$$(\Delta_e^N + \lambda^2)^{-1} = G^D(\lambda) - K^D(\lambda) \mathcal{N}(\lambda)^{-1} B G^D(\lambda). \tag{2.3}$$

Since G^D is holomorphic inside γ, by (2.2), (2.3) and Lemma 2.2, we obtain

$$\mathrm{tr}\,(2\pi i)^{-1}\int_\gamma (\Delta_e^N+\lambda^2)^{-1}2\lambda d\lambda$$

$$=-\mathrm{tr}\,(2\pi i)^{-1}\int_\gamma K^D(\lambda)\mathcal{N}(\lambda)^{-1}BG^D(\lambda)2\lambda d\lambda$$

$$=-\mathrm{tr}\,(2\pi i)^{-1}\int_\gamma \mathcal{N}(\lambda)^{-1}BG^D(\lambda)K^D(\lambda)2\lambda d\lambda$$

$$=\mathrm{tr}\,(2\pi i)^{-1}\int_\gamma \mathcal{N}(\lambda)^{-1}\dot{\mathcal{N}}(\lambda)d\lambda.$$

□

3 Study of N in the elliptic region

As we are going to use the semi-classical calculus (see the appendix for the terminology and notations) it will be more convenient to work with the semi-classical parameter $h=1/\lambda$ which will vary in $L:=\{h\in\mathbf{C}:|\mathrm{Im}\,h|\le|h|^{2+\delta_0},|h|\le h_0,\mathrm{Re}\,h>0\}$ when λ varies in Λ. Let $\chi\in L_{cl}^{-\infty,0}(\Gamma)$ be an $h-\Psi$DO depending holomorphically on h such that $\widetilde{\mathrm{WF}}(I-\chi)\subset\mathcal{E}$. Then it follows from [6] that $(I-\chi)N, N(I-\chi)\in L_{cl}^{1,0}(\Gamma)$ with symbols having for every choice of the local coordinates a common asymptotic expansion

$$\sum_{k=0}^{\infty}h^k n_k(x,\xi) \tag{3.1}$$

in the complement of $\widetilde{\mathrm{WF}}(\chi)$, where $n_k=O(|\xi|^{1-k}), |\xi|\gg c_1^{-1}$, and extends holomorphically in ξ to a complex neighbourhood of $\mathcal{E}$ such that $(x,\lambda\xi)$ belongs to the neighbourhood whenever (x,ξ) belongs to it and $\lambda\ge1$. Furthermore, it follows from the Green formula for the elastic Laplacian that for real h, we have

$$(I-\chi)(N-N^*),\,(N-N^*)(I-\chi)\in L^{-\infty,-\infty}(\Gamma). \tag{3.2}$$

In particular, this implies that $n_0(x,\xi)$ is a Hermitian matrix. Let $a_1(x,\xi)\le a_2(x,\xi)\le...\le a_n(x,\xi)$ be its eigenvalues. Then we know (see [1],[8]) that $a_1(x,\xi)=\tilde{a}_1(x,\xi)(c_R|\xi|-1)$ with $\tilde{a}_1$ smooth and $\tilde{a}_1,a_2>0$ everywhere in $\mathcal{E}$.

Next we shall construct a selfadjoint (for real h) operator $P\in L_{cl}^{1,0}(\Gamma)$ which concides (mod $L^{-\infty,-\infty}(\Gamma)$) with N in $|\xi|>c^{-1}$, where we can choose c with $c_1-c>0$ arbitrarily small, and which is elliptic away from Σ. The only difficulty in doing so is to extend $n_0(x,\xi)$ to the whole $T^*\Gamma$. Let $c_0\in(c_R,c_1)$ and put $\Sigma_0=\{(x,\xi)\in T^*\Gamma:c_0|\xi|=1\}$. For any $\rho\in\Sigma_0$, let $\gamma(\rho)\in\mathbf{P}^{n-1}$ be the point corresponding to the eigenspace of $n_0(\rho)$ associated with the (unique) negative eigenvalue $a_1(\rho)$. Obviously, $\gamma:\Sigma_0\to\mathbf{P}^{n-1}$ is continuous. Now, according to [3], under the assumption (H.4), γ has a continuous extension $\tilde{\gamma}:B_0\to\mathbf{P}^{n-1}$, where $B_0=\{(x,\xi)\in T^*\Gamma:c_0|\xi|\le1\}$. We will think of $\tilde{\gamma}(\rho)$ as a 1-dimensional subspace of $\mathbf{C}^n$. Extend a_1 to a continuous function $a_1(\rho)<0$ on B_0. Then it is clear that we can extend

n_0 to a continuous function $\widetilde{n}_0$ on B_0 with values in the Hermitian matrices such that $\widetilde{n}_0(\rho)$ maps $\widetilde{\gamma}(\rho)$ (and hence also $\widetilde{\gamma}(\rho)^-$) into itself, such that $\widetilde{n}_0(\rho) = a_1(\rho)I$ on $\widetilde{\gamma}$ and $\widetilde{n}_0(\rho) > 0$ on $\widetilde{\gamma}(\rho)^-$. This means that we have found a continuous extension $\widetilde{n}_0$ of n_0 from $T^*\Gamma \setminus B_0$ to the whole $T^*\Gamma$ such that $\widetilde{n}_0$ has one eigenvalue < 0 in B_0 while the other eigenvalues are > 0. After decreasing c_0 arbitrarily little and regularizing, we may assume that $\widetilde{n}_0$ is C^∞. The lower symbols are much easier to handle. Thus we get an $h - \Psi$DO, $P \in L^{1,0}_{cl}(\Gamma)$ with leading symbol $\widetilde{n}_0$, depending holomorphicaly on h with the following properties:

$$P^* = P \qquad \text{for} \quad h \quad \text{real}, \tag{3.3}$$

$$(I - \chi)(N - P),\ (N - P)(I - \chi) \in L^{-\infty,-\infty}(\Gamma). \tag{3.4}$$

In what follows we will use the notations n_0 and a_1 for the principal symbol of P and its first eigenvalue, respectively. By well known results on the semi-classical eigenvalue assymptotics, if $\varepsilon_0 > 0$ is small enough, then for real h, the number of the eigenvalues of P in $(-\infty, \varepsilon_0]$ is

$$(2\pi h)^{-n+1} \left(\mathrm{Vol}\left(\{(x,\xi) \in T^*\Gamma : a_1(x,\xi) \le \varepsilon_0\}\right) + O(h)\right). \tag{3.5}$$

Remark 3.1 Since $N(h)$ is an elliptic $h - \Psi$DO near $|\xi| = \infty$, we see that $N(h) : H^{1/2} \to H^{-1/2}$ is a Fredholm operator and in particular $N(h)$ is of constant index for $h \in L$. Moreover $N(h)$ depends holomorphically on $h \in L$ and is invertible when h^{-1} is not a resonance, so $N(h)$ is of index 0 for all h. Therefore, whenever $N(h)$ has a bounded left or right inverse, that inverse is two sided. This remark also applies to the operator $\widetilde{N}(h)$ introduced in the assumption (H.3).

Remark 3.2 Assuming (H.1), (H.2) and that h_0 in the definition of L is small enough, the following assumption is equivalent to (H.3), where $\chi_0 \in L^{0,0}_{cl}$ is elliptic near Σ and with $\widetilde{\mathrm{WF}}(\chi_0)$ contained in a small neighbourhood of Σ:

(H.3$'$) For $h \in L, u \in H^{1/2}$, we have

$$\|u\|_{H^{1/2}} \le C|h|^{-k_1} \left(\|N(h)u\|_{H^{-1/2}} + \|\chi_0 u\|_{H^{1/2}}\right).$$

The proof is easy and we will only indicate how to get (H.3) from (H.3$'$). In $\mathcal{E}$ we can construct a microlocal parametrix for $\widetilde{N}(h)$ and it follows that if $\chi \in L^{0,0}_{cl}$ and $\widetilde{\mathrm{WF}}(\chi) \subset \mathcal{E}$, then for any $N_0 > 0$:

$$\|\chi u\|_{H^{1/2}} \le C_{N_0} \left(\|\widetilde{N}(h)u\|_{H^{-1/2}} + |h|^{N_0}\|u\|_{H^{1/2}}\right). \tag{3.6}$$

Here we use for simplicity the natural h-dependent Sobolev norms discussed in the appendix. Assume in addition that $\widetilde{\mathrm{WF}}(I - \chi)$ is compact and disjoint from Σ and even disjoint from $\widetilde{\mathrm{WF}}(\chi_0)$ in (H.3$'$). Applying (H.3$'$) to $(I - \chi)u$ we get

$$\begin{aligned} \|(I-\chi)u\|_{H^{1/2}} \le C|h|^{-k_1}(&\|(I-\chi)N(h)u\|_{H^{-1/2}} \\ &+\|[N(h),\chi]u\|_{H^{-1/2}} + \|\chi_0 u\|_{H^{1/2}}). \end{aligned} \tag{3.7}$$

Here

$$\|(I-\chi)N(h)u\|_{H^{-1/2}} = \|(I-\chi)\widetilde{N}(h)u\|_{H^{-1/2}} + O(|h|^{\infty})\|u\|_{H^{1/2}}$$

and (using a new χ) $\|[N(h),\chi]u\|_{H^{-1/2}}$ can be estimated by the RHS of (3.6). Using this in (3.7) and adding (3.6) and (3.7), we get

$$\|u\|_{H^{1/2}} \leq C|h|^{-k_1}\left(\|\widetilde{N}(h)u\|_{H^{-1/2}} + |h|^{N_0}\|u\|_{H^{1/2}}\right),$$

so with $N_0 > k_1$ and h_0 sufficiently small, we finaly deduce

$$\|u\|_{H^{1/2}} \leq 2C|h|^{-k_1}\|\widetilde{N}(h)u\|_{H^{-1/2}}.$$

Then $\widetilde{N}(h)$ has a left inverse which, according to Remark 3.1, is also a right inverse and (H.3) follows.

4 Positivity of $\dot{P}$

In this section we will study P for real h only. Notice first that if h-derivatives are denoted by points,

$$\dot{P} \in L^{1,1}(\Gamma) \quad \text{with principal symbol} \quad h^{-1}\nu(n_0), \tag{4.1}$$

where $\nu = \xi \cdot \nabla_\xi$. We are going to study $\nu(n_0)$ in a neighbourhood of Σ. Recall that the first eigenvalue $a := a_1$ of n_0 vanishes on Σ, and moreover it is easy to see that $\nu(a) > 0$ there. Let $\pi(x,\xi)$ be the spectral projection associated to a. Then there exists a smooth matrix-valued function $q(x,\xi)$, $q > 0$, $[q,\pi] = 0$, such that

$$n_0 = a\pi + (I-\pi)q(I-\pi). \tag{4.2}$$

Hence

$$\begin{aligned}\nu(n_0) &= \nu(a)\pi + a\nu(\pi) - \nu(\pi)q(I-\pi)\\ &-(I-\pi)q\nu(\pi) + (I-\pi)\nu(q)(I-\pi).\end{aligned} \tag{4.3}$$

Differentiating the identity $\pi^2 = \pi$, we get

$$\nu(\pi) = \pi\nu(\pi) + \nu(\pi)\pi.$$

Applying π to the left and to the right yields $\pi\nu(\pi)\pi = 0$, so

$$\nu(\pi) = (I-\pi)\nu(\pi)\pi + \pi\nu(\pi)(I-\pi). \tag{4.4}$$

Using this in (4.3), we get

$$\begin{aligned}\nu(n_0) &= \pi\nu(a)\pi + (I-\pi)(a\nu(\pi) - q\nu(\pi))\pi\\ &+\pi(a\nu(\pi) - \nu(\pi)q)(I-\pi)\\ &+(I-\pi)\nu(q)(I-\pi).\end{aligned} \tag{4.5}$$

Since $\nu(a) > 0$, it follows that in sense of Hermitian matrices

$$\nu(n_0) \geq C^{-1}\pi - C(I - \pi)$$

for some constant $C > 0$, and using that $q > 0$ we obtain

$$\nu(n_0) \geq C^{-1}I - Cn_0^2, \tag{4.6}$$

with a new constant $C > 0$. Outside a neighbourhood of Σ we do not know the sign of $\nu(n_0)$ any more, but we can here use that n_0 is elliptic, and (4.6) can be globalized to

$$\nu(n_0) \geq C^{-1}\langle\xi\rangle I - C\langle\xi\rangle^{-1}n_0^2, \tag{4.7}$$

where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. Let $\langle hD\rangle = (1 - h^2\Delta)^{1/2}$, Δ being the Laplace operator on Γ, so that $\langle hD\rangle$ has the principal symbol $\langle\xi\rangle$ and is selfadjoint for real h. Then, combining (4.1) and (4.7), we get

$$h\dot{P} + CP\langle hD\rangle^{-1}P \geq C^{-1}\langle hD\rangle. \tag{4.8}$$

In the remainder of this section we will derive from (4.8) that the eigenvalues of P near 0 are increasing functions of h. In the next section we will use (4.8) to show the invertibility of $P(h)$ when $\operatorname{Im} h \neq 0$.

From now on we equip the Sobolev spaces H^s with the h-dependent norm $\|u\|_{H^s} = \|\langle hD\rangle^s u\|_{L^2}$. Let $\mu_1(h) \leq \mu_2(h) \leq \ldots$ be the eigenvalues of $P(h)$ repeated according to multiplicity. The domain of P is H^1 and from the fact that $\dot{P} = O(h^{-1}) : H^1 \to H^0$, it follows that if μ_k is close to 0 (so that $k \sim h^{-n+1}$), then $\mu_k(h)$ is a locally Lipschitz function in h whose a.e. defined derivative satisfies

$$\frac{d\mu_k(h)}{dh} = O(h^{-1}). \tag{4.9}$$

We also want a lower bound of the same type. Assume for $h = h_1$ (small) that $\mu_k(h_1)$ is of multiplicity m and let $F(h_1)$ be the corresponding m-dimensional spectral subspace. For h close to h_1, we have precisely m eigenvalues close to $\mu_k(h_1)$ and we let $F(h)$ be the corresponding spectral subspace. Let $h_2 > h_1$ be close to h_1 and let $e(h_2) \in F(h_2)$ be a normalized eigenvector with associated eigenvalue $\mu_k(h_2)$. We then extend it to $e(h)$ for $h \in [h_1, h_2]$ as the unique smooth function of h in $F(h)$ with $\dot{e}(h) \in F(h)^{-}$. Trivially $e(h_1)$ will be an eigenvector of $P(h_1)$ with eigenvalue $\mu_k(h_1)$. We have

$$h\frac{d}{dh}\langle P(h)e(h), e(h)\rangle = \langle h\dot{P}(h)e(h), e(h)\rangle. \tag{4.10}$$

Restricting the attention to the eigenvalues in $[-\delta, \delta]$ for $\delta > 0$ small enough, we have $\|P(h)e(h)\| \leq \delta$, and (4.8) gives

$$\begin{aligned}\langle h\dot{P}(h)e(h), e(h)\rangle &\geq C^{-1}\langle\langle hD\rangle e(h), e(h)\rangle \\ &-C\langle\langle hD\rangle^{-1}P(h)e(h), P(h)e(h)\rangle \\ &\geq C^{-1} - C\delta^2 \geq (2C)^{-1}.\end{aligned} \tag{4.11}$$

Using (4.11) in (4.10) and integrating between h_1 and h_2, we get

$$\mu_k(h_2) - \mu_k(h_1) \geq (2C)^{-1} \int_{h_1}^{h_2} h^{-1} dh. \tag{4.12}$$

By (4.9) and (4.12), we conclude that the a.e. defined derivative of $\mu_k(h) \in [-\delta, \delta]$, with $\delta > 0$ small enough, satisfies

$$C^{-1} \leq h \frac{d\mu_k(h)}{dh} \leq C, \tag{4.13}$$

with some constant $C > 0$.

Fix a $h_0 > 0$ small enough. Since $\mu_k(h)$ decreases when h decreases, as soon as $\mu_k(h) \in [-\delta, \delta]$, there exists an infinite sequence $h_{k_0} \geq h_{k_0+1} \geq \ldots$ of values in $(0, h_0]$ defined by $\mu_k(h_k) = 0$. Clearly, h_k are precisely the values of h in $(0, h_0]$ for which $P(h)$ is not invertible. Let $p > n$. We have $|\mu_k(h)| \leq h^p$ ($k \sim h^{1-n}$) for h in some interval $I_{k,p}$ containing h_k, of length $|I_{k,p}| \sim h^{p+1}$. If $0 < \tilde{h} \leq h_0$, then $(\tilde{h}/2, \tilde{h}]$ can intersect at most $O(\tilde{h}^{-n+1})$ of the intervals $I_{k,p}$ and we conclude that the union of all such $I_{k,p}$ is a union of at most $O(\tilde{h}^{-n+1})$ disjoint intervals $J_{k,p}$, where each $J_{k,p}$ is of length at most $O(\tilde{h}^{p-n+2})$. Varying $\tilde{h}$, we get

Proposition 4.1. *The inverse $P(h)^{-1} : L^2 \to L^2$ exists and is of norm $O(h^{-p})$ for $h \in (0, h_0] \backslash \Omega_p$, where Ω_p is a union of disjoint closed intervals $J_{1,p}, J_{2,p}, \ldots$ with $|J_{k,p}| = O(h^{p+2-n})$ for $h \in J_{k,p}$. Moreover, the number of such intervals that intersect $[h/2, h]$, for $0 < h \leq h_0$, is at most $O(h^{1-n})$.*

5 Trace integrals for P

We will now work with $h \in L$. Assuming h_0 sufficiently small we will first prove the following

Lemma 5.1. *If $\operatorname{Im} h \neq 0$, the inverse $P(h)^{-1} : H^s \to H^{s+1}$ exists and*

$$\|P(h)^{-1}\|_{\mathcal{L}(H^s, H^{s+1})} \leq C_s \frac{\operatorname{Re} h}{|\operatorname{Im} h|}.$$

Proof. Without loss of generality we may suppose that $\operatorname{Im} h > 0$. We have with $h_1 = \operatorname{Re} h$:

$$P(h) = P(h_1) + i \operatorname{Im} h \dot{P}(h_1) + r(h),$$

where

$$\langle h_1 D \rangle^{-1/2} r(h) \langle h_1 D \rangle^{-1/2} = O\left(\left(h_1^{-1} \operatorname{Im} h \right)^2 \right) : L^2 \to L^2.$$

Let $u \in H^{1/2}$. Then

$$\operatorname{Im} \langle P(h)u,\, u \rangle = \operatorname{Im} h \langle \dot{P}(h_1)u, u \rangle + \operatorname{Im} \langle \langle h_1 D \rangle^{-1/2} r(h)u,\, \langle h_1 D \rangle^{1/2} u \rangle,$$

and using (4.8) and the estimate on r, we get

$$\operatorname{Im}\langle P(h)u,\, u\rangle + C\|\langle h_1 D\rangle^{-1/2} P(h_1)u\|^2$$

$$\geq (Ch_1)^{-1}\operatorname{Im} h\langle\langle h_1 D\rangle u,\, u\rangle - O\left(\left(h_1^{-1}\operatorname{Im} h\right)^2\right)\|\langle h_1 D\rangle^{1/2} u\|^2.$$

Assuming h_0 sufficiently small, the last term can be absorbed, and we obtain

$$\operatorname{Im}\langle P(h)u,\, u\rangle + C\|\langle h_1 D\rangle^{-1/2} P(h_1)u\|^2$$

$$\geq (2Ch_1)^{-1}\operatorname{Im} h\|\langle h_1 D\rangle^{1/2} u\|^2. \tag{5.1}$$

Here

$$\|\langle h_1 D\rangle^{-1/2} P(h_1)u\| \leq \|\langle h_1 D\rangle^{-1/2} P(h)u\| + O\left(h_1^{-1}\operatorname{Im} h\right)\|\langle h_1 D\rangle^{1/2} u\|,$$

and hence from (5.1):

$$(2Ch_1)^{-1}\operatorname{Im} h\|\langle h_1 D\rangle^{1/2} u\|^2 \leq \|\langle h_1 D\rangle^{-1/2} P(h)u\|\|\langle h_1 D\rangle^{1/2} u\|$$

$$+2\|\langle h_1 D\rangle^{-1/2} P(h)u\|^2 + O\left(\left(h_1^{-1}\operatorname{Im} h\right)^2\right)\|\langle h_1 D\rangle^{1/2} u\|^2.$$

The last term can be absorbed as before and we get

$$(3Ch_1)^{-1}\operatorname{Im} h\|\langle h_1 D\rangle^{1/2} u\|^2 \leq 2\|\langle h_1 D\rangle^{-1/2} P(h)u\|^2$$

$$+(6Ch_1)^{-1}\operatorname{Im} h\|\langle h_1 D\rangle^{1/2} u\|^2 + 3Ch_1(2\operatorname{Im} h)^{-1}\|\langle h_1 D\rangle^{-1/2} P(h)u\|^2.$$

This gives with a new constant $C > 0$:

$$\|\langle h_1 D\rangle^{1/2} u\| \leq Ch_1(\operatorname{Im} h)^{-1}\|\langle h_1 D\rangle^{-1/2} P(h)u\|. \tag{5.2}$$

In other words, $P(h) : H^{1/2} \to H^{-1/2}$ has a bounded left inverse of norm $O(h_1/|\operatorname{Im} h|)$. Since $P(\bar{h})^* = P(h)$, the adjoint of the left inverse of $P(\bar{h})$ is a right inverse of $P(h)$, so we see that $P(h) : H^{1/2} \to H^{-1/2}$ has a two sided inverse of norm $O(h_1/|\operatorname{Im} h|)$.

To prove the lemma for any s we will use that P is elliptic outside Σ. Let $\chi \in L^{-\infty,0}$ have its $\widetilde{\mathrm{WF}}$ in a small neighbourhood of Σ so that $M = P + \chi$ is elliptic. It is easy to see that

$$P^{-1} = M^{-1} - M^{-1}(P-M)M^{-1} + M^{-1}(P-M)P^{-1}(P-M)M^{-1}. \tag{5.3}$$

Here $M^{-1} \in L^{-1,0}$ is $O(1) : H^s \to H^{s+1}$ and $P - M$ is $O(1) : H^{-s} \to H^s, \forall s$. Using this in (5.3) together with the fact that the lemma holds for $s = -1/2$, we obtain the lemma in general. $\square$

Let $J_{k,p}$ be one of the intervals in Ω_p given in Proposition 4.1. Let $\gamma_{k,p}$ be the piecewise smooth simple positively oriented loop given by the four segments: $\operatorname{Re} h \in J_{k,p}$, $\operatorname{Im} h = \pm(\operatorname{Re} h)^{p+1}$ and $\operatorname{Re} h \in \partial J_{k,p}$, $|\operatorname{Im} h| \leq (\operatorname{Re} h)^{p+1}$.

Proposition 5.2. *For every $h \in \gamma_{k,p}$, the inverse $P(h)^{-1} : H^s \to H^{s+1}$ exists and*

$$\|P(h)^{-1}\|_{\mathcal{L}(H^s,H^{s+1})} \leq C_s(\operatorname{Re} h)^{-p}, \quad h \in \gamma_{k,p}.$$

Proof. Let $h_1 \in (0, h_0] \setminus \Omega_p$, so that by Proposition 4.1,

$$\|P(h_1)^{-1}\|_{\mathcal{L}(L^2,L^2)} \leq C h_1^{-p}.$$

By the same argument as in the end of the proof of Lemma 5.1 we derive from this

$$\|P(h_1)^{-1}\|_{\mathcal{L}(H^s,H^{s+1})} \leq C_s h_1^{-p}. \tag{5.4}$$

For $|h - h_1| \leq h_1/2$, write

$$P(h) = P(h_1)(I + P(h_1)^{-1}(P(h) - P(h_1))).$$

Here

$$P(h) - P(h_1) = O(1)h_1^{-1}|h - h_1| : H^1 \to L^2,$$

so if $|h - h_1| \ll h_1^{p+1}$, we get from (5.4) with $s = 0$, that $P(h) : H^1 \to L^2$ is invertible with inverse $O(|h|^{-p})$. As before this extends to $H^{s+1} \to H^s$ for all s. In particular, if we let $\operatorname{Re} h = h_1$, then

$$\|P(h)^{-1}\|_{\mathcal{L}(H^s,H^{s+1})} \leq C_s h_1^{-p}$$

first for $|\operatorname{Im} h| \ll h_1^{p+1}$, and then also, by Lemma 5.1, when $|\operatorname{Im} h| \geq C^{-1} h_1^{p+1}$. This completes the proof of Proposition 5.2. $\square$

If $h_k \in (0, h_0], k \geq k_0$, is defined by $\mu_k(h_k) = 0$, we define the multiplicity of h_k to be the multiplicity of μ_k as eigenvalue. In the remainder of this section we will prove the following

Proposition 5.3. *Let $\gamma \subset L$ be a closed positively oriented C^1 curve without self intersections which avoids the points h_k. Then*

$$\operatorname{tr}(2\pi i)^{-1} \int_\gamma P(h)^{-1} \dot{P}(h) dh$$

is equal to the number of h_k inside γ.

Proof. Fix an h_k and assume that the multiplicity of h_k is m so that $h_{k-m_1} = h_{k-m_1+1} = \ldots = h_{k+m_2}$ where $m_1 \geq 0, m_2 \geq 0, m = m_1 + m_2 + 1$. For h close to h_k, let $F(h)$ be the spectral subspace corresponding to the eigenvalues $\mu_{k-m_1}(h), \ldots, \mu_{k+m_2}(h)$. Then $\dim F(h) = m$ and $F(h)$ is also well defined for h in a small complex neighbourhood of h_k and depends holomorphically on h. Let $e_1(h), \ldots, e_m(h)$ be a basis in $F(h)$ which depends holomorphically on h and which is orthonormal for real h. Define $R_+(h) : L^2 \to \mathbf{C}^m$ by $R_+(h)u(j) = \langle u, e_j(\bar{h})\rangle$ and put $R_-(h) = R_+(\bar{h})^* : \mathbf{C}^m \to L^2$. Then

$$\mathcal{P}(h) := \begin{pmatrix} P(h) & R_-(h) \\ R_+(h) & 0 \end{pmatrix} : H^1 \times \mathbf{C}^m \to L^2 \times \mathbf{C}^m$$

depends holomorphically on h and has the inverse

$$\mathcal{E}(h) := \begin{pmatrix} E(h) & E_+(h) \\ E_-(h) & E_{-+}(h) \end{pmatrix} : L^2 \times \mathbf{C}^m \to H^1 \times \mathbf{C}^m,$$

which is also holomorphic in h. Moreover, $-E_{-+}(h)$ is simply $P(h)|_{F(h)}$, expressed in the basis $e_1(h), ..., e_m(h)$. In particular, the eigenvalues of $-E_{-+}(h)$ are $\mu_{k-m_1}(h), ..., \mu_{k+m_2}(h)$. Each of these eigenvalues is $\sim (h-h_k)/h_k$ both in sign and size (for h real). Therefore

$$\det E_{-+}(h) = C_k(1 + O_k((h-h_k)))(h-h_k)^m, \quad C_k \neq 0. \tag{5.5}$$

Let γ be a sufficiently small positively oriented circle centered at h_k and consider

$$I_\gamma := \mathrm{tr}\,(2\pi i)^{-1} \int_\gamma P(h)^{-1} \dot{P}(h) dh.$$

Note that it follows from the formula

$$P(h)^{-1} = E(h) - E_+(h) E_{-+}^{-1}(h) E_-(h),$$

where $E(h)$ is holomorphic inside γ and E_+, E_- are of finite rank, that $P(h)^{-1}$ is a Fredholm family, so I_γ is well defined and

$$I_\gamma = -(2\pi i)^{-1} \int_\gamma \mathrm{tr}\,\left(E_+(h) E_{-+}(h)^{-1} E_-(h) \dot{P}(h)\right) dh. \tag{5.6}$$

From the relation

$$\dot{\mathcal{E}}(h) = -\mathcal{E}(h) \dot{\mathcal{P}}(h) \mathcal{E}(h),$$

we get

$$-\dot{E}_{-+} = E_- \dot{P} E_+ + E_{-+} \dot{R}_+ E_+ + E_- \dot{R}_- E_{-+}. \tag{5.7}$$

By (5.6), (5.7) and the cyclicity of the trace, we obtain

$$\begin{aligned} I_\gamma &= -(2\pi i)^{-1} \int_\gamma \mathrm{tr}\,\left(E_{-+}^{-1} E_- \dot{P} E_+\right) dh \\ &= (2\pi i)^{-1} \int_\gamma \mathrm{tr}\,\left(E_{-+}^{-1} \dot{E}_{-+}\right) dh + (2\pi i)^{-1} \int_\gamma \mathrm{tr}\,\left(\dot{R}_+ E_+\right) dh \\ &\quad + (2\pi i)^{-1} \int_\gamma \mathrm{tr}\,\left(E_{-+}^{-1} E_- \dot{R}_- E_{-+}\right) dh. \end{aligned} \tag{5.8}$$

Here $\dot{R}_+ E_+$ is holomorphic inside γ, so the corresponding integral vanishes. The same holds for the last integral in (5.8) since

$$\mathrm{tr}\,\left(E_{-+}^{-1} E_- \dot{R}_- E_{-+}\right) = \mathrm{tr}\,\left(E_- \dot{R}_-\right).$$

We then have in view of (5.5),

$$\begin{aligned} I_\gamma &= (2\pi i)^{-1} \int_\gamma \mathrm{tr}\,\left(E_{-+}^{-1} \dot{E}_{-+}\right) dh \\ &= (2\pi i)^{-1} \int_\gamma d(\log \det E_{-+}(h)) = m, \end{aligned} \tag{5.9}$$

which is the desired result for this simple curve γ. It is now immediate to extend this for a general curve γ. $\square$

6 Relationship between the trace integrals for P and N

In this section we will compare the trace integrals that we have already studied in the preceding sections. Before doing so, however, we will prove the following

Proposition 6.1. *For every $h \in \gamma_{k,p}$, the inverse $N(h)^{-1} : H^s \to H^{s+1}$ exists and*

$$\|N(h)^{-1}\|_{\mathcal{L}(H^s,H^{s+1})} \leq C_s(\operatorname{Re} h)^{-p}, \quad h \in \gamma_{k,p}.$$

Proof. In view of Proposition 5.2, it suffices to take h varying in a subset of $\{h \in L : |h| \leq h_0\}$ where $P(h)^{-1}$ exists and is of norm $O_s(|h|^{-p})$ in $\mathcal{L}(H^s, H^{s+1})$, with $h_0 = h_0(p) > 0$ to be taken small enough. Let $\chi_1, \chi_2 \in L^{0,0}$ with $\widetilde{\mathrm{WF}}(\chi_1) \cap \widetilde{\mathrm{WF}}(\chi_2) = \emptyset$, $\widetilde{\mathrm{WF}}(\chi_1)$ or $\widetilde{\mathrm{WF}}(\chi_2)$ is disjoint from Σ. Let M be an elliptic $h - \Psi$DO whose symbol coincides (modulo $S^{-\infty,-\infty}$) with that of P outside some sufficiently small neighbourhood of Σ. Then from (5.3) we get

$$\chi_1 P^{-1}\chi_2 = O(|h|^s) : H^{-s} \to H^s, \quad \forall s. \tag{6.1}$$

We can treat $\widetilde{N}^{-1}$ in the same way, $\widetilde{N}$ being defined in (H.3). Let $\chi_1, \chi_2 \in L^{0,0}$ with $\widetilde{\mathrm{WF}}(\chi_1) \cap \widetilde{\mathrm{WF}}(\chi_2) = \emptyset$, $\widetilde{\mathrm{WF}}(\chi_1)$ or $\widetilde{\mathrm{WF}}(\chi_2)$ is disjoint from $\{(x,\xi) \in T^*\Gamma : |\xi| \leq c_1^{-1}\}$. Choose M as above, so that the symbols of M and $\widetilde{N}$, both operators having well defined symbols in $|\xi| > c_1^{-1}$ mod $S^{-\infty,-\infty}$, coincide mod $S^{-\infty,-\infty}$ in $|\xi| > c^{-1}$, where $c < c_1$ is sufficiently close to c_1. This means that if $\chi \in L^{-\infty,0}$ and $\widetilde{\mathrm{WF}}(I-\chi) \cap \{|\xi| \leq c^{-1}\} = \emptyset$, then

$$(M - \widetilde{N})(I - \chi),\ (I - \chi)(M - \widetilde{N}) \in L^{-\infty,-\infty}. \tag{6.2}$$

Replacing P by $\widetilde{N}$ in (5.3) we then get

$$\chi_1 \widetilde{N}^{-1}\chi_2 = O(|h|^s) : H^{-s} \to H^s, \quad \forall s. \tag{6.3}$$

Let $\chi \in L^{-\infty,0}$ with $\widetilde{\mathrm{WF}}(\chi)$ contained in a small neighbourhood of Σ. Now we are going to show that the operator

$$R = \widetilde{N}^{-1}(I - \chi) + P^{-1}\chi$$

is an approximative right inverse of N. We have

$$NR = I + (N - \widetilde{N})\widetilde{N}^{-1}(I - \chi) + (N - P)P^{-1}\chi.$$

Here $(N-\widetilde{N})\widetilde{N}^{-1}(I-\chi) \in L^{-\infty,-\infty}$ by (6.3), assuming of course that the symbols of N and $\widetilde{N}$ coincide outside a sufficiently small neighbourhood of Σ in the elliptic region. Let $\widetilde{\chi} \in L^{-\infty,0}$ with $\widetilde{\mathrm{WF}}(I-\widetilde{\chi}) \cap \{|\xi| \leq c^{-1}\} = \emptyset$, $\widetilde{\mathrm{WF}}(\widetilde{\chi}) \cap \widetilde{\mathrm{WF}}(\chi) = \emptyset$, so that $(N-P)(I-\widetilde{\chi}) \in L^{-\infty,-\infty}$. Then, in view of (6.1),

$$(N - P)P^{-1}\chi = (N - P)(I - \widetilde{\chi})P^{-1}\chi + (N - P)\widetilde{\chi}P^{-1}\chi$$

$$= O(|h|^s) : H^{-s} \to H^s, \quad \forall k.$$

By inversion of a Neumann series we then get a right inverse which, according to Remark 3.1, is a two sided inverse and the desired result follows. $\square$

It follows from Propositions 6.1 and 5.2, and the analysis above, that

$$N^{-1} = \widetilde{N}^{-1}(I - \chi) + P^{-1}\chi + K, \tag{6.4}$$

$$P^{-1} = M^{-1}(I - \chi) + P^{-1}\chi + K', \tag{6.5}$$

with $K, K' = O(|h|^s) : H^{-s} \to H^s$, $\forall s \in \mathbf{R}, h \in \gamma_{k,p}$. By (6.4) and (6.5) we have with a fixed $\widetilde{h} \in \gamma_{k,p}$:

$$\begin{aligned} &\operatorname{tr} (2\pi i)^{-1} \int_{\gamma_{k,p}} N(h)^{-1} \dot{N}(h) dh \\ &= \operatorname{tr} (2\pi i)^{-1} \int_{\gamma_{k,p}} P(h)^{-1} \chi \dot{N}(h) dh + O_p(|\widetilde{h}|^\infty) \\ &= \operatorname{tr} (2\pi i)^{-1} \int_{\gamma_{k,p}} P(h)^{-1} \chi \dot{P}(h) dh + O_p(|\widetilde{h}|^\infty) \\ &= \operatorname{tr} (2\pi i)^{-1} \int_{\gamma_{k,p}} P(h)^{-1} \dot{P}(h) dh + O_p(|\widetilde{h}|^\infty). \end{aligned} \tag{6.6}$$

Here we have used that we can take χ holomorphic in h, so that the contributions from the first term in the right hand sides of (6.4) and (6.5) vanish. In view of Propositions 2.1 and 5.3, (6.6) tels us that there are exactly as many inverse resonaces inside $\gamma_{k,p}$ as there are points h_k in $J_{k,p}$. It is also clear that there are no inverse resonances outside the union of $\gamma_{k,p}$. In view of Proposition 4.1 we get with a suitable choice of $h(p)$:

Proposition 6.2. *For every $p > n$, there is a bijection l_p from the set of h_k in $(0, h(p)]$, counted with multiplicity, into the set of the inverse resonances in $\{h \in L : |h| \leq h(p)\}$, counted with multiplicity, such that*

$$|l_p(h) - h| \leq C_p h^{p+2-n}.$$

By skillfully patching together different l_p we get

Proposition 6.3. *If $h_0 > 0$ is suitably chosen, there is a bijection l from the set of h_k in $(0, h_0]$, counted with multiplicity, into the set of the inverse resonances in $\{h \in L : |h| \leq h_0\}$, counted with multiplicity, such that for every $p > 0$:*

$$|l(h) - h| \leq C_p h^p.$$

Define $\widetilde{\lambda}_k := h_k^{-1}$. We have

Proposition 6.4. *If $C_0 > 0$ is suitably chosen, there is a bijection l from the set of $\widetilde{\lambda}_k$ in $[C_0, +\infty)$, counted with multiplicity, into the set of the resonances in $\{\lambda \in \Lambda, |\lambda| \geq C_0\}$, counted with multiplicity, such that for every $p > 0$:*

$$|l(\lambda) - \lambda| \leq C_p \lambda^{-p}.$$

It is now clear how to finish the proof of Theorem 1.1 as indicated at the end of the introduction.

Appendix

Let Γ be a compact manifold without boundary of dimension $n-1$, that we equip with a smooth Riemannian metric, and let $h \in (0, h_0], h_0 > 0$. Define the Sobolev spaces H^s with h-dependent norm: $u \in H^s \Leftrightarrow \langle hD \rangle^s u \in L^2$. Here $\langle hD \rangle := (1 - h^2\Delta)^{1/2}$, where Δ is the Laplace-Beltrami operator on Γ. We say that $A \in L^{-\infty,-\infty}$ if $A = O(h^s) : H^{-s} \to H^s, \forall s$.

Let $U \subset \mathbf{R}^{n-1}$ be an open domain. We let $S^{m,k}(U \times \mathbf{R}^{n-1})$ be the space of functions $a(x, \xi; h)$ on $U \times \mathbf{R}^{n-1} \times (0, h_0]$ which are smooth in (x, ξ) and such that for every compact $K \subset U$ and all $\alpha, \beta \in \mathbf{N}^{n-1}$:

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{K,\alpha,\beta} h^{-k} \langle \xi \rangle^{m-|\beta|}, \quad \forall (x, \xi) \in K \times \mathbf{R}^{n-1}.$$

If $a_j \in S^{m_j,k_j}, j \in \{0, 1, 2, ...\}, m_j \searrow -\infty, k_j \searrow -\infty$, then as usual we define $a \in S^{m_0,k_0}$ given by $a \sim \sum_{j=0}^{\infty} a_j$, and a is unique up to $S^{-\infty,-\infty} = \cap_{m,k} S^{m,k}$. We define $S_{cl}^{m,k} \subset S^{m,k}$ as the subspace of all a which have an asymptotic expansion

$$a(x, \xi; h) \sim \sum_{j=0}^{\infty} h^{j-k} a_j(x, \xi),$$

where $a_j \in S_{1,0}^{m-j}(U \times \mathbf{R}^{n-1})$ are independent of h. Recall that $S_{\rho,\delta}^m(U \times \mathbf{R}^{n-1})$ is the space of all functions $a(x, \xi) \in C^\infty(U \times \mathbf{R}^{n-1})$ such that for every compact $K \subset U$ and all $\alpha, \beta \in \mathbf{N}^{n-1}$:

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{K,\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}, \quad \forall (x, \xi) \in K \times \mathbf{R}^{n-1}.$$

The spaces $S_{cl}^{m,k}$ have a natural regularity in h. We can and will choose the asymptotic sums $a(x, \xi; h)$ such that $(h\partial_h)^l a \in S_{cl}^{m,k}, \forall l$, with the asymptotic expansion

$$(h\partial_h)^l a \sim \sum_{j=0}^{\infty} h^{j-k} (j-k)^l a_j(x, \xi).$$

Denote by $L^{m,k}, L_{cl}^{m,k}$ the corresponding spaces of $h - \Psi$DO which up to $L^{-\infty,-\infty}$ are given in local coordinates by $a(x, hD_x; h)$ with $a \in S^{m,k}$ and $S_{cl}^{m,k}$ respectively. Now we are going to introduce the notion of wave front, $\widetilde{\mathrm{WF}}(A)$, of an operator $A \in L^{m,k}(\Gamma)$. Let $\widetilde{T^*\Gamma}$ be the compactification of $T^*\Gamma$ by adding $\{(x,\infty\xi) : (x,\xi) \in S^*\Gamma\} \simeq S^*\Gamma$ as a natural boundary ($\widetilde{T^*\Gamma}$ is then homeomorphic to $\{(x,\xi) \in T^*\Gamma : |\xi| \leq 1\}$). If $\rho \in T^*\Gamma$, we say that $\rho \notin \widetilde{\mathrm{WF}}(A)$ if the symbol of A, for some choice of local coordinates near the projection of ρ, is of class $S^{-\infty,-\infty}$ near ρ. If $\rho = (x_0, \infty\xi_0) \in \widetilde{T^*\Gamma} \setminus T^*\Gamma$, we say that $\rho \notin \widetilde{\mathrm{WF}}(A)$ if the symbol of A is of class $S^{-\infty,-\infty}$ in $\{(x,\xi) : x \in \mathrm{neigh}(x_0),\ \xi/|\xi| \in \mathrm{neigh}(\xi_0),\ |\xi| \geq C\}$ for some neighbourhoods of x_0 and ξ_0, and for some constant $C > 0$. It follows from the definition that $\widetilde{\mathrm{WF}}(A)$ is closed, $\widetilde{\mathrm{WF}}(AB) \subset \widetilde{\mathrm{WF}}(A) \cap \widetilde{\mathrm{WF}}(B)$, $\widetilde{\mathrm{WF}}(A) = \emptyset \Rightarrow A \in L^{-\infty,-\infty}$. Moreover, if $A \in L^{m,k}_{cl}$, then $\widetilde{\mathrm{WF}}(h\partial_h A) \subset \widetilde{\mathrm{WF}}(A)$.

Now we are going to extend the above notions to complex h. More precisely we will work in the domain

$$\{h \in \mathbf{C} : 0 < |h| \leq h_0,\ |\mathrm{Im}\, h| \leq |h|^{1+\delta_0}\}, \tag{A.1}$$

where $\delta_0 > 0$ and $h_0 > 0$ is small enough. Clearly, we may assume that $\arg h \in (-\pi/2, \pi/2)$. Then $\langle hD\rangle^s$ is a well defined operator and we can still define the h-dependent Sobolev spaces. Moreover, the norms $\|\langle hD\rangle^s u\|_{L^2}, \|\langle |h|D\rangle^s u\|_{L^2}, \|\langle \mathrm{Re}\, h\, D\rangle^s u\|_{L^2}$ are equivalent, uniformly with respect to h, for every fixed $s \in \mathbf{R}$. It also makes sense to speak about operators of class $L^{-\infty,-\infty}$, when h varies in the set (A.1). Now consider an $A \in L^{m,k}_{cl}$ and assume that there exists an $\varepsilon_0 > 0$ such that for every choice of local coordinates in Γ, the full symbol of A becomes

$$a(x,\xi;h) \sim \sum_{j=0}^{\infty} h^{j-k} a_j(x,\xi), \tag{A.2}$$

where the function $r \to a_j(x, r\xi)$ has a holomorphic extension to $1/2 \leq |r| \leq 2, |\arg r| \leq \varepsilon_0$, for $x \in U, \xi \in \mathbf{R}^{n-1}, |\xi| \geq C_{j,U} > 0$. Moreover, we assume that this extension, which we can also denote by $a_j(x, r\xi)$, is of class $S^{m-j}_{1,0}$ in $U \times \{\xi \in \mathbf{R}^{n-1} : |\xi| \geq C_{j,U}\}$. For $|\xi| \leq 2C_{j,U}$, let $\tilde{a}_j(x,\xi)$ denote an almost analytic extension so that $\bar\partial_\xi \tilde{a}_j(x,\xi)$ is $O(|\mathrm{Im}\,\xi|^\infty)$. Then for $(1+\varepsilon)C_{j,U} \leq |\xi| \leq (2-\varepsilon)C_{j,U}$, and r as above, we have

$$\tilde{a}_j(x, r\xi) - a_j(x, r\xi) = O(|\mathrm{Im}\, r|^\infty). \tag{A.3}$$

Pasting together $\tilde{a}_j(x, r\xi)$ and $a_j(x, r\xi)$, by means of a cutoff $\chi(x, r\xi)$, we get an extension $\hat{a}_j(x, r\xi)$ of $a_j(x, r\xi)$ to $x \in U, \xi \in \mathbf{R}^{n-1}, 1/2 \leq |r| \leq 2, |\arg r| \leq \varepsilon_0$, such that $\hat{a}_j(\cdot, r\cdot)$ is a C^∞ function of r with values in the class $S^{m-j}_{1,0}$, with

$$\partial_{\bar r} \hat{a}_j(x, r\xi) = \begin{cases} O(|\mathrm{Im}\, r|^\infty), \\ 0 \quad \text{for} \quad |\xi| \geq 2C_{j,U}, \end{cases}$$

so that $\partial_{\bar r} \hat{a}_j(x, r\xi) = O(|\mathrm{Im}\, r|^\infty)$ in $S^{-\infty}_{1,0}$. As an extension of $a_j(x, hD), 0 < h \leq h_0$, we now take $\hat{a}_j(x, hD)$ for h in the set (A.1). We have

$$\partial_{\bar h} \hat{a}_j(x, h\xi) = h_1 \frac{\partial}{\partial(\bar h/h_1)} \hat{a}_j(x, (h/h_1)\, h_1\xi),$$

where we let $h_1 > 0$ be h-independent, so that

$$\partial_{\bar h} \hat{a}_j(x, h\xi) = h_1 O((h_1^{-1}|\mathrm{Im}\, h|)^\infty)$$

in the sense of the functions $f(x, h_1\xi), f \in S^{-\infty}$. Because of the shape of the domain (A.1), we then get

$$\partial_{\bar{h}}\hat{a}_j(x, h\xi) = r_j(x, |h|\xi; h), \quad r_j \in S^{-\infty,-\infty},$$

and similarly for $\partial_h^k\partial_{\bar{h}}\hat{a}_j$. It is now clear that we can extend $A \in L_{cl}^{m,k}$ for h in the set (A.1) in such a way that $\partial_h^k\partial_{\bar{h}}A \in L^{-\infty,-\infty}$. By solving a $\bar{\partial}$-problem we will next see that we can, after modifying A by an operator in $L^{-\infty,-\infty}$, extend A holomorphically in the domain (A.1). We may assume that we have the extension of A above in the slightly bigger domain $|\mathrm{Im}\, h| \leq 2(\mathrm{Re}\, h)^{1+\delta_0}, 0 < \mathrm{Re}\, h \leq 2h_0$. Let χ be a C^∞ function on this bigger domain which vanishes near the boundary and is equal to 1 on (A.1). We may further assume that $\partial_h^k\chi = O(|h|^{-p(k)}), \forall k$. Then

$$\partial_h^k(\chi\bar{\partial}_h A) = O(|h|^\infty) : H^{-s} \to H^s, \forall (k, s).$$

Next we try to solve the equation

$$\frac{\partial R}{\partial \bar{h}} = \chi\frac{\partial A}{\partial \bar{h}}$$

in a small sector around $(0, 2h_0]$. Put $h = e^z$. Then we want to solve

$$\frac{\partial R}{\partial \bar{z}} = \chi\frac{\partial A}{\partial \bar{z}} \tag{A.4}$$

in a half band

$$|\mathrm{Im}\, z| < C_0, 0 < \mathrm{Re}\, z < C_1, \tag{A.5}$$

in which the right hand side has its support. Also,

$$\partial_z^k(\chi\bar{\partial}_z A) = O((e^{\mathrm{Re}\, z})^\infty) : H^{-s} \to H^s, \forall (k, s). \tag{A.6}$$

Now let $\frac{e^{-(z-w)^2}}{\pi(z-w)}$ be a convenient fundamental solution of $\partial_{\bar{z}}$ and we choose R to be the corresponding solution of (A.4):

$$R(z) = \int \frac{e^{-(z-w)^2}}{\pi(z-w)}\left(\chi\frac{\partial A}{\partial \bar{z}}\right)(w)L(dw), \tag{A.7}$$

where $L(dw)$ denotes the Lebesgue measure. Then in (A.5) we have

$$\partial_z^k R = O((e^{\mathrm{Re}\, z})^\infty) : H^{-s} \to H^s, \forall (k, s),$$

and going back to the h-variable, we get

$$\frac{\partial R}{\partial \bar{h}} = \chi\frac{\partial A}{\partial \bar{h}}, \tag{A.8}$$

with

$$\partial_h^k R = O(|h|^\infty) : H^{-s} \to H^s, \forall (k, s),$$

in the domain (A.1). Summing up, we have proved

Proposition A.1. *Let $A \in L^{m,k}_{cl}(\Gamma)$ satisfy the assumptions around (A.2). Then, if $h_0 > 0$ is small enough in (A.1), we can find $R(h)$, for h in the domain (A.1), such that*

$$\partial_h^k R = O(|h|^\infty) : H^{-s} \to H^s, \ \forall (k,s),$$

and such that $\widetilde{A}(h) = A(h) + R(h)$ extends holomorphically to the domain (A.1). Moreover, we have $\widetilde{A}(h) = B(\operatorname{Re} h; h/|h|) = C(|h|; h/|h|)$, where $B, C \in L^{m,k}_{cl}(\Gamma)$ uniformly in $h/|h|$.

Remark A.2. It is easy to see that if $\rho \notin \widetilde{\mathrm{WF}}(A)$, then $\rho \notin \widetilde{\mathrm{WF}}(B), \rho \notin \widetilde{\mathrm{WF}}(C)$.

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