Classification of Bicovariant Differential Calculi

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CLASSIFICATION OF BICOVARIANT DIFFERENTIAL CALCULI

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Abstract We show that the bicovariant first order differential calculi on a factorisable semisimple quantum group are in 1-1 correspondence with irreducible representations $V$ of the quantum group enveloping algebra. The corresponding calculus is constructed and has dimension $\dim V^2$. The differential calculi on a finite group algebra $\mathbb{C}G$ are also classified and shown to be in correspondence with pairs consisting of an irreducible representation $V$ and a continuous parameter in $\mathbb{C}^\dim V - 1$. They have dimension $\dim V$. For a classical Lie group we obtain an infinite family of non-standard calculi. General constructions for bicovariant calculi and their quantum tangent spaces are also obtained.

Keywords: one-form – differential calculus – quantum tangent space – quantum group – non-commutative geometry – quantum double

1 Introduction

One of the first steps in non-commutative geometry of the kind coming out of quantum groups is the choice of ‘first order differential calculus’ or ‘cotangent bundle’. Only once this is fixed can one begin to do gauge theory [1] or make other geometrical constructions. When the quantum space in question is a quantum group, $A$, it is natural to require that the differential calculus is covariant under left and right translations. Thus, we require $\Gamma \equiv \Omega^1(A), d : A \rightarrow \Gamma$ such that

1. $\Gamma$ is an $A$-bimodule
2. $\Gamma$ is an $A$-cobimodule, with coactions $\Delta_L : \Gamma \rightarrow A \otimes \Gamma$ and $\Delta_R : \Gamma \rightarrow \Gamma \otimes A$ bimodule maps
3. $d : A \rightarrow \Gamma$ is a bicomodule map

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4. $d(ab) = (da)b + a(db)$ for all $a, b \in A$

5. $\Gamma = \text{span} \{ adb \mid a, b \in A \}$

Here, a bicomodule is like a bimodule but with arrows reversed, i.e. a pair of commuting coactions $\Delta_L, \Delta_R$. A morphism of differential calculi means a bimodule and bicomodule map forming a commutative triangle with the $d$ maps. These are the natural axioms proposed some years ago by Woronowicz [2]. The axiom 5. here is not essential; it specifies that the calculus is irreducible, and is assumed throughout the paper. By now, several examples are known, and there is also case-by-case classification for several families of quantum groups in [3]. The class of ‘inner’ b covariant calculi has also been introduced [4]. The complete classification of the possible calculi on a general quantum group remains, however, open.

In Section 4 of the present paper we present a complete solution to this classification problem under the strict assumption of a semisimple factorisable quantum group. The standard $q$-deformed function algebras $G_q$ are essentially factorisable in the sense that they are factorisable up to suitable localisations or when working over formal power-series. In this case our algebraic result (a) constructs a calculus of dimension $(\dim V)^2$ for each irreducible representation $V$ of $U_q(g)$ and (b) indicates that these are the only ‘generic’ possibilities in the sense of extending to localisations or to working over formal power-series in the deformation parameter.

We begin in Section 2 with a clarification of the role of the quantum double in classifying calculi. This is well-known or implicit from [2] but appears to remain of current interest; see [5]. In fact, calculi correspond to subrepresentations of a given quantum-double module, a result which is somewhat different from that recently presented in [5] (these authors did not impose the optional irreducibility axiom 5 above, and hence have a more complicated result).

We reformulate the theory with these quantum double subrepresentations or ‘quantum tangent spaces’ as the starting point and then develop some preliminary general results from this point of view.

In Section 3 we apply these results to the complete classification for $A = \mathbb{C}G$ the group algebra over a finite group, as well as recovering the known classification for the function algebra $A = \mathbb{C}(G)$. We also comment on the case where $G$ is a Lie group.

Section 4 presents the main result of the paper; Section 5 concludes with a brief discussion of the open problem for the classification of higher order differential calculi (or exterior algebras). We also mention the braided version of the results in the present paper, to be presented in detail.
elsewhere[6]. We work over $\mathbb{C}$ for convenience, but all abstract Hopf-algebraic results work over any ground field or, with care, over a ring such as $\mathbb{C}[[h]]$.

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2 Role of the quantum double

In this preliminary section we clarify the role of the quantum double in classifying bicovariant differential calculi. It is needed for our main result in Section 4. This role of the quantum double should be known to experts, but we have not found a suitably explicit treatment elsewhere. Moreover, this use of the quantum double is different from the one recently presented in [5], hence it would appear necessary to emphasise it here and explain the relation with that work. The use of a braiding to describe the derivation property of partial derivatives in Proposition 2.3, the emphasis on quantum double subrepresentations, and the resulting applications at the end of the section (such as the ‘mirror’ operation) appear to be novel aspects of our formulation.

Let $H$ be a Hopf algebra non-degenerately dually paired with $A$. The Drinfeld quantum double[7] is the double cross product Hopf algebra $H\bowtie A^{op}$ built on $H \otimes A$ with the product

$$(h \otimes a)(g \otimes b) = h g_{(2)} \otimes b a_{(2)} \langle g_{(1)}, a_{(1)} \rangle \langle g_{(3)}, S a_{(3)} \rangle, \quad a, b \in A, \quad h, g \in H$$

and tensor product unit and coalgebra. This is the formulation from [8]. We use here (and throughout) the notations and conventions from [9]. Thus, $\Delta h = h_{(1)} \otimes h_{(2)}$ is the coproduct, $S$ is the antipode (which we assume for convenience to be invertible), and $\langle , \rangle$ is the pairing between $H$ and $A$. We denote the counit of any of our Hopf algebras by $\varepsilon$.

The quantum double has a formal quasitriangular structure $R = \sum_a f^a \otimes e_a$ where $\{e_a\}$ is a basis of $H$ and $\{f^a\}$ a dual basis of $A$. Although formal, this does lead to a braiding $\Psi$ among suitable representations. To make this precise, we define a representation of the quantum double of $H$ to be $H$-regular if the action of $A \subset H \bowtie A^{op}$ is given by evaluation against a (left) coaction of $H$. It is $A$-regular if the action of $H$ is given by evaluation against a (right) coaction of $A$. If $V$ is $A$-regular or $W$ is $H$-regular then $\Psi : V \otimes W \to W \otimes V$ is a well-defined operator. Thus,
\[
\Psi(v \otimes w) = \sum_a \epsilon_a v \otimes f^a(v) = \sum_a \epsilon_a v \otimes (f^a(v^{(1)}))v^{(2)} = v^{(1)} \alpha(w \otimes v^{(2)}) \text{ in the first case, where }
\]
\[v \mapsto v^{(1)} \otimes v^{(2)} \text{ (with summation understood) is the assumed coaction } V \to H \otimes V. \text{ Similarly in }
\]
The second case.

Woronowicz in [2] observed that first order bicovariant calculi are in 1-1 correspondence with Ad-invariant ‘ideals’ in \( \ker \epsilon \). More precisely, they correspond to quotients of \( \ker \epsilon \subset A \) by subspaces \( M \) which are stable under the action and coaction

\[
a \triangleright v = av, \quad \text{Ad}(v) = v_{(1)} S v_{(3)} \otimes v_{(2)}
\]
on \( v \in \ker \epsilon \). Equivalently, they correspond to quotients \( V \) to which this action and coaction descend. The corresponding calculus is \( \Gamma = V \otimes A \) with tensor product action and coaction from the left and trivial action and coaction on \( V \) from the right (here we take the left and right actions and coactions on \( A \) defined by its product and coproduct). In addition, \( da = a_{(1)} \otimes a_{(2)} - 1 \otimes a \), where \( a_{(1)} \) is projected to \( V \). This is the most general form for a bicovariant calculus up to isomorphism. The case \( V = \ker \epsilon \subset A \) is called the ‘universal’ first order calculus.

As a first step, we can write all coactions of \( A \) as actions of \( H \). Then (2) becomes

\[
a \triangleright v = av, \quad h \triangleright v = \langle h, v_{(1)} S v_{(3)} \rangle v_{(2)}
\]
and calculi correspond to quotients of which are equivariant under these actions. The quantum double is not needed to classify calculi here, but in fact these two actions do fit together to form a representation of \( A \triangleright H^{\text{op}} \), the quantum double of \( A \). This fact allows one to deduce, for example, the canonical braiding \( \sigma \) in [2] from the quantum double quasitriangular structure, which would otherwise have to be introduced by hand. This was explained in [10]. In this context, it is natural also to reformulate the bicomodules \( \Delta_L, \Delta_R \) in the axioms 2. and 3. of a bicovariant calculus as an \( H^{\text{op}} \)-bimodule by evaluation against the coactions. In principle, requiring an \( H^{\text{op}} \)-bimodule could be slightly more general when \( H \) is infinite-dimensional.

**Lemma 2.1** The quantum double \( H \triangleright A^{\text{op}} \) acts on \( \ker \epsilon \subset H \) by

\[
h \triangleright x = h_{(1)} x S h_{(2)}, \quad a \triangleright x = \langle a, x_{(1)} \rangle x_{(2)} - \langle a, x \rangle 1
\]

**Proof** The quantum double has a well-known ‘Schroedinger’ representation on \( H \) [9] by the quantum adjoint action and by the coregular ‘differentiation’ representation. This induces the action stated on \( \ker \epsilon \) via the projection \( \Pi(h) = h - 1 \epsilon(h) \) as a morphism \( H \to \ker \epsilon \), i.e. it
is easy to see that it is indeed an action of the quantum double on $\ker \epsilon$ and that $H$ is an intertwiner. Also, we can identify the linear space $\ker \epsilon \subset A$ with $A/\mathbb{C}$ (the quotient by the 1-dimensional vector space spanned by the unit element); for any element in $A/\mathbb{C}$ there is a unique representative in $\ker \epsilon \subset A$. In terms of $A/\mathbb{C}$ the action in (3) is $a \triangleright v = av - \epsilon(v)a$ and $h \triangleright v = (v_{(1)}Sv_{(2)})v_{(2)}$. The action stated in the lemma is the natural right action of $A \triangleright H^{\text{op}}$ on $\ker \epsilon \subset H$ dual to this action on $A/\mathbb{C}$, viewed as a left action of the quantum double of $H$. □

We are now ready to make a further reformulation which, when the bicovariant calculus is finite-dimensional as an $A$-module, is strictly equivalent by dualizing $V$.

**Proposition 2.2** Finite-dimensional bicovariant calculi are in 1-1 correspondence with subrepresentations $L \subseteq \ker \epsilon \subset H$ of the quantum double representation in Lemma 2.1.

$$\Gamma = \text{Lin}(L, A), \quad (da)(x) = \langle x, a_{(1)} \rangle a_{(2)}$$

$$(a \cdot \gamma)(x) = a_{(2)} \gamma(a_{(1)} \triangleright x), \quad (\gamma \cdot a)(x) = \gamma(x)a$$

$$(h \cdot \gamma)(x) = \langle h_{(2)}, \gamma(h_{(1)} \triangleright x)_{(1)} \rangle \gamma(h_{(1)} \triangleright x)_{(2)}, \quad (\gamma \cdot h)(x) = \gamma(x)_{(1)} \gamma(x)_{(2)}, h)$$

for all $\gamma \in \text{Lin}(L, A)$. The vector space $L$ is called the quantum tangent space associated to the calculus.

**Proof** A quotient of $A/\mathbb{C}$ which is equivariant under the quantum double action corresponds under dualisation to a subspace of $\ker \epsilon \subset H$ which is stable under the action in Lemma 2.1. Thus, this is a dual formulation of the correspondence (3). Also, $da = a_{(1)} \otimes a_{(2)} - 1 \otimes a = a_{(1)} \otimes a_{(2)}$ when we work with $V$ as a quotient of $A/\mathbb{C}$, which leads to the form shown for $d$. It is easy to verify directly that the structures shown indeed provide a first order bicovariant differential calculus given $L$. Conversely, in the finite-dimensional case, we define $L = V^*$ where $V$ is the invariant part of $\Gamma$ under the usual correspondence in (3). In terms of the ideal $M$ which defines the bicovariant calculus via (2), the corresponding quantum double subrepresentation is $L = \{ x \in \ker \epsilon | \langle x, a \rangle = 0 \ \forall a \in M \}$. □

The correspondence in the proposition is contragradient i.e. morphisms of calculi $\Gamma_1 \to \Gamma_2$ correspond to inclusions $L_2 \hookrightarrow L_1$ of quantum double subrepresentations. Only inclusions are allowed here, corresponding to all morphisms of calculi being of the form $\Gamma_2$ a quotient of $\Gamma_1$. In the infinite-dimensional case every bicovariant calculus continues to define a subrepresentation.
$L$, and, conversely, a subrepresentation $L$ continues to provide a bicovariant first order calculus in our slightly generalised sense where an action of $H^{\text{op}}$ replaces the coactions $\Delta_L, \Delta_R$. It is this final reformulation in terms quantum tangent spaces which we will use; by definition a quantum tangent space $L$ is a subrepresentation of $\ker \epsilon \subset H$ under the action of the quantum double of $H$, and it is these which we will actually classify in the present paper. Indeed, quantum tangent spaces have many nice properties, making them an equally good starting point for differential calculus.

**Proposition 2.3** For each $x \in L$, we define the ‘braided derivation’

$$\partial_x : A \to A, \quad \partial_x(a) = (da)(x)$$

This obeys

$$\partial_x(ab) = (\partial_x a)b + \Psi(a \otimes x)^{-1}\partial_{\Psi(a \otimes x)^{-1}}b$$

where $\Psi : L \otimes A \to A \otimes L$ is the quantum double braiding between $A, L$ as quantum double modules, with inverse $\Psi^{-1}(a \otimes x)$ denoted explicitly by $\Psi(a \otimes x)^{-1} \otimes \Psi(a \otimes x)^{-2}$.

**Proof** We start with the identity

$$\partial_x(ab) = \langle x, (ab)_1(ab)_2 \rangle = \langle x_1, a_1(b_1(a_2b_2))_2 \rangle$$

$$= \langle x, a_1(b_2) + \langle a_1 \partial x, b_1(b_2) \rangle \rangle = \partial_x(a)b + a_2 \partial_{\Psi(a \otimes x)}(b)$$

based on the definition of $\partial_x$ and the action in Lemma 2.1. On the other hand, $A$ is a quantum double module by

$$h \triangleright a = \langle Sh, a_1(a_2) \rangle, \quad \psi \triangleright a = \langle S^{-1}b_1(a_2)b_2 \rangle$$

which is the conjugate (dual) of the Schrödinger representation of the quantum double on $H$. It is $H$-regular, so that the braiding $\Psi$ is well-defined. We compute it easily as

$$\Psi(x \otimes a) = \epsilon_{a \psi}a \otimes f^a \psi x = a_2 \otimes S a_1 \psi x$$

with inverse $\Psi^{-1}(a \otimes x) = a_1 \psi x \otimes a_2$, which we put into the above identity. □

Because the subspace $L$ is stable under the quantum adjoint action, it is tempting to restrict the latter to $L$ as a ‘quantum Lie bracket’ $[\cdot, \cdot] = \text{Ad} : L \otimes L \to L$. The use of Ad as ‘quantum Lie bracket’ has been discussed in [2] from this point of view and independently from another point
of view in [11] (where the ‘quantum Lie bracket structure constants’ for the $l^+ S l^-$ generators of $U_q(g)$ were computed in R-matrix form). The only content here comes from the identities

\[ [x, [y, z]] = [[x_{(1)}, y], [x_{(2)}, z]], \quad [x_{(1)}, y]x_{(2)} = xy \]  

(6)

which hold in any quantum group when $[,] = \text{Ad}$ is the left adjoint action.

**Proposition 2.4** The ‘quantum Lie bracket’ on $L$ defined by $\text{Ad}$ obeys

\[ [x, [y, z]] = [[x, y], z] + [x, [y, z]] \circ \Psi(x \otimes y), \quad [x, y] = xy - \cdot \Psi(x \otimes y) \]

where $\Psi$ is the quantum double braiding between $L$, $L$ as quantum double modules.

**Proof** We again use the formula $\Psi(x \otimes y) = e_a \cdot y \otimes f^a \cdot x$; our action of the quantum double of $H$ is regular and we still have a well-defined operator

\[ \Psi(x \otimes y) = \text{Ad}_{e_a}(y) \otimes (f^a, x_{(1)})x_{(2)} - (f^a, x)1 = [x_{(1)}, y] \otimes x_{(2)} - [x, y] \otimes 1. \]  

(7)

Then (6) can be trivially rewritten in the form stated by eliminating the coproduct in favour of $\Psi$ in these equations. \( \square \)

There are, however, some fundamental problems to be overcome before one could call this vector space $L$ with $[,]$ some kind of ‘quantum Lie algebra’. These fundamental problems have been explained in [11] and force one to the braided version [12] where these problems are resolved:

1. Although ‘enveloping algebra-like’ relations $[x, y] = xy - \cdot \Psi(x \otimes y)$ hold in $H$, we do not know that $L$ generates $H$. Even if it does, we do not know that these are the only relations in $H$. Indeed, for $U_q(g)$ they are not. So $H \neq U(L)$ as generated by $L$ and such relations.

2. Even if $H$ were to be generated in some way from $L$, we are not able to recover the coproduct of $H$ in this way. Indeed, for $U_q(g)$ the coproduct of $H$ does not have any simple form on $L$ and hence cannot be generated in some canonical way. Without this, $U(L)$ is only an algebra and not a Hopf algebra or bialgebra. Equivalently, one cannot tensor product representations of $L$ in any natural way, which makes it useless as a ‘Lie algebra’.

In the case where $H$ is quasitriangular, there is a ‘transmutation theory’ [13] which converts $H$ to a braided group. It also converts $L$ to a ‘braided-Lie algebra’ $L$. The linear maps $[, , ]$ are the same (the braided adjoint action coincides with the quantum one) but the coalgebras are
different. For the standard calculus on $G_q$, the braided coproduct takes a standard matrix form on $L$ and there is a corresponding $U(L)$ as a braided group (bialgebra in a braided category) generated from $L$. Thus, the problems 1.-2. are resolved at the price of working with the braided version of the theory. For a general Hopf algebra $H$ and general $L$, however, we do not really have a ‘quantum Lie algebra’ or braided-Lie algebra at all. Hence we prefer the term ‘quantum tangent space’ for the subspace $L$.

Finally, we discuss the well-known correspondence between bimodules (i.e. simultaneous modules and comodules as in the above axioms 1. and 2. for a calculus) and quantum double modules. Bicovariant bimodules are known in the mathematics literature[14] as bi-Hopf modules, and indeed correspond (in a standard way) to crossed modules[15]. The latter have been identified with quantum double modules by the author in [8]. In view of such a correspondence, it has been argued in [5] that bicovariant calculi are therefore in 1-1 correspondence with pairs $(V, \psi)$ where $V$ is a quantum double module and $\psi$ is an $A_d$-equivariant one-co cycle on $A$ with values in $V$. The corresponding calculus is

$$\Gamma = V \otimes A, \quad da = \psi(a^{(1)}) \otimes a^{(2)}$$

It is easy to see that the Leibniz rule for $d$ indeed corresponds to the cocycle condition

$$\psi(ab) = a \psi(b) + \psi(a) \epsilon(b).$$

By contrast, the classification in Proposition 2.2 tells us that this point of view, although interesting, is not so useful; not every quantum double module $V$ is needed but only the subrepresentations of one particular quantum double representation. This is because the above axiom 5. of a bicovariant calculus corresponds to $\psi$ surjective. This condition was omitted in the analysis in [5] and, as soon as it is imposed, we see that the cocycle condition forces the action of $A$ since every element of $V$ can be written as $\psi(b)$ for some $b$. Likewise, surjectivity of $\psi$ and its equivariance forces the action of $H$ on $V$ to be the image under $\psi$ of the coadjoint one. Therefore, we are forced into the setting above, with $V = L^*$ and $\psi$ the quotienting map in (3) or the adjoint of the inclusion $L \subseteq \ker \epsilon$ in Proposition 2.2. This is why not all quantum double modules are allowed. Moreover, it is not necessary to solve any cocycle conditions explicitly; these take care of themselves in the specification of the submodule inclusion into $\ker \epsilon$.

We conclude the section with some general constructions for bicovariant calculi and their quantum tangent spaces. Firstly, we recall that any element $a \in A$ which is invariant under the
adjoint coaction $\text{Ad}$ in (2) can be used to generate an ideal $A(\alpha - \epsilon(\alpha))$ to quotient $\ker \epsilon \subset A$ by. This class of bicovariant calculi can be called *inner type-I* because the exterior derivative obeys

$$\epsilon(\alpha) da = a_{(1)} \epsilon(\alpha) \otimes a_{(2)} - 1 \epsilon(\alpha) \otimes a = a_{(1)} \alpha \otimes a_{(2)} - \alpha \otimes a = a \cdot (\alpha \otimes 1) - (\alpha \otimes 1) \cdot a \quad (8)$$

projected down to the quotient. The expression on the right is shown lifted up to $A \otimes A$ as a left $A$-module by the tensor product left action and a right $A$-module by right multiplication in the second copy. A variant of this construction was introduced in [4], where we quotient by the ideal $(\ker \epsilon)(\alpha - (\epsilon(\alpha) + 1)) \subseteq \ker \epsilon \subset A$ and we have

$$da = (a_{(1)} - \epsilon(a_{(1)})) \otimes a_{(2)} = a_{(1)}(\alpha - \epsilon(\alpha)) \otimes a_{(2)} - (\alpha - \epsilon(\alpha)) \otimes a = a \cdot \omega(\alpha) - \omega(\alpha) \cdot a \quad (9)$$

in the quotient, where $\omega(\alpha) = (\alpha - \epsilon(\alpha)) \otimes 1 = (da_{(1)})sa_{(2)} \in \Gamma$. This class can be called *inner type-II*. Since $\text{Ad}$-invariant elements $\alpha$ are closed under addition and multiplication, we have whole ring of bicovariant differential calculi of either type. The standard calculi on $G_q$ were already obtained in [16] as a quotient of an inner type-II form (with $\alpha$ the $q$-trace), while [4] extended this to a ring of calculi generated by elements $a_1, \cdots, a_r \in G_q$ constructed through transmutation. Note that the cocycle point of view might suggest the more general notion of ‘coboundary’ differential calculus where $\psi$ is the Hochschild coboundary of $\nu \in V$, i.e. $\psi(\alpha) = a \psi(\nu) - \epsilon(\alpha) \nu$. However, when we again add the surjectivity of $\psi$ we are forced to $\nu = \psi(\alpha)$ for some $\alpha$ and we return to the class of inner type-II calculi from [4]. This is similar to (but different from) the discussion in [5].

**Proposition 2.5** The quantum tangent space for an *inner type-I* bicovariant differential calculi defined by any non-trivial element $\alpha \in A$ invariant under $\text{Ad}$ in (2) is the quantum double subrepresentation

$$L_\alpha = \{ x \in \ker \epsilon \subset H \mid x_{(1)} \langle x_{(2)}, \alpha \rangle = x \epsilon(\alpha) \}.$$ 

This has a canonical extension $\tilde{L}_\alpha$ where the condition on $x$ is only required to hold on evaluation against all $a \in \ker \epsilon \subset A$. Similarly, the quantum tangent space for the inner type-II case is

$$L_{\alpha,1} = \{ x \in \ker \epsilon \subset H \mid \langle x, a \alpha \rangle = \langle x, a \rangle (\epsilon(\alpha) + 1), \forall a \in \ker \epsilon \subset A \}.$$
Proof. It is convenient to first identify \( \ker \epsilon \) with \( A/\mathbb{C} \) as in the proof of Lemma 2.1. Then an inner type-I bicovariant calculus has \( V \) the quotient by the image \( A \triangleright \alpha \) in \( A/\mathbb{C} \). Hence its dual consists of the linear functionals \( x \in \ker \epsilon \subset H \) such that \( \langle x, a \triangleright \alpha \rangle = 0 \) for all \( a \), i.e. such that \( \langle x, a \alpha - a \epsilon(\alpha) \rangle = 0 \). This leads to the dual formulation; we define \( L_\alpha \) as stated and verify directly that it is stable under the quantum double action in Lemma 2.1, which is a straightforward Hopf algebra computation. The variant in which we require \( \langle x, a \alpha \rangle = \langle x, a \rangle \epsilon(\alpha) \) for all \( a \in \ker \epsilon \subset A \) is easily verified to also form a quantum double subrepresentation, and defines \( \tilde{L}_\alpha \). The type-II case is similar. \( \square \)

Note that we can define \( L_{\alpha, \lambda} \) similarly, with \( \epsilon(\alpha) + \lambda \) in place of \( \epsilon(\alpha) + 1 \) and then obtain \( \lambda \sigma a = a \cdot \omega(\alpha) - \omega(\alpha) \cdot a \) as in [4]. All non-zero \( \lambda \) are equivalent to the inner type-II construction via \( L_{\alpha, \lambda} = L_{\lambda^{-1} \alpha, 1} \), while \( L_{\alpha, 0} = \tilde{L}_\alpha \). More generally, we can restrict the condition \( \forall a \in \ker \epsilon \subset A \) in by requiring only \( a \in M \), where \( M \subseteq \ker \epsilon \subset A \) is the quotien ting ideal for any given bicovariant calculus. This gives a 1-parameter family of new calculi with quotienting ideal \( M \cdot (a - (\epsilon(\alpha) + \lambda)) \), and is the general idea behind the above constructions.

The representation-theoretic point of view suggests, however, a different type of general construction for any Hopf algebra. Namely, pick any element \( x \in \ker \epsilon \subset H \) and define \( L = (H \triangleright A^\text{op}) \triangleright x \), the image of \( x \) under the quantum double action. It evidently forms a subrepresentation of \( \ker \epsilon \) and hence by Proposition 2.2 it defines a bicovariant differential calculus. More generally, the image of any left ideal of the quantum double acting on any \( x \in \ker \epsilon \subset H \) will be a subrepresentation. An interesting special case of this idea is the following:

Proposition 2.6 Let \( c \in H \) be any non-trivial central element. There is an associated bicovariant differential calculus with

\[
L_c = \text{span}\{x_a = \langle a, \epsilon(1) \rangle c(2) - \langle a, c \rangle 1 \mid a \in \ker \epsilon \subset A\}
\]

\[
\partial_{x_a}(b) = \langle ab(k_1), c \rangle b(k_2) - \langle a, c \rangle b, \quad \Psi^{-1}(b \otimes x_a) = x_a(k_1) \otimes b(k_2)
\]

\[
[x_a, x_b] = x_b(k_2)(a S b(k_1)) b(k_3), c - x_b(a, c), \quad \Psi(x_a \otimes x_b) = x_b(k_2) \otimes x_a(S b(k_1)) b(k_3)
\]

for all \( b \in A \) in the middle line. We say that the quantum tangent space \( L_c \) or its corresponding bicovariant differential calculus is centrally generated. It has a canonical extension \( \tilde{L}_c \) spanned by \( \{x_a\} \) for all \( a \in A \).
The quantum double can also be written as $A^{op} times H$, i.e. every element can be written uniquely in the form $\sum_i a_i h_i$ with $a_i \in A$ and $h_i \in H$. A central element $c$ is precisely an element for which $h \triangleright c = \epsilon(h)c$ for all $h \in H$. Hence the image of $x = c - \epsilon(c)$ under the quantum double action reduces to the image of the action of $A$ in Lemma 2.1. Thus $\tilde{L}_c = \text{span}\{x_a = a \triangleright (c - \epsilon(c)) a \in A\}$ is a subrepresentation under the quantum double. We then restrict the allowed $\{x_a\}$ to $a \in \ker \epsilon \subset A$. It is easy to check that this still defines a quantum double subrepresentation, which is the one stated. It can sometimes coincide with $\tilde{L}_c$.

The calculation of $\partial x_a$ and $\Psi^{-1}$ from Proposition 2.3 is trivial. For the quantum Lie bracket in Proposition 2.4, we note first the Ad-invariance identity

$$c_{(1)} \otimes h_{(1)} c_{(2)} S h_{(2)} = (S h_{(1)}) c_{(1)} h_{(2)} \otimes c_{(2)}, \quad \forall h \in H$$

which holds for any central element $c$. Then

$$\text{Ad}_h(x_a) = \langle a, c_{(1)} \rangle \text{Ad}_h(c_{(2)}) - \epsilon(h) \langle a, c \rangle = \langle a, (S h_{(1)}) c_{(1)} h_{(2)} \rangle c_{(2)} - \epsilon(h) \langle a, c \rangle = x_{a_{(2)} h_{(1)}} \langle h, (S a_{(1)}) a_{(2)} \rangle$$

where the last two terms cancel because $c$ is central. In other words, the map $A \rightarrow H$ sending $a \mapsto (a \otimes \text{id}) \Delta c$ intertwines the quantum adjoint action and the quantum coadjoint action; likewise for its projection to ker $\epsilon$, which is the map $a \mapsto x_a$. Using this observation, we have the quantum Lie bracket

$$[x_a, x_b] = x_{b_{(2)} a_{(1)}} \langle x_a, (S b_{(1)}) b_{(2)} \rangle = x_{b_{(2)} a_{(1)}} \langle a, c_{(1)} \rangle c_{(2)} \langle S b_{(1)} b_{(2)} \rangle - x_b \langle a, c \rangle$$

giving the result as stated. Likewise, the braiding in Proposition 2.4 comes out as

$$\Psi(x_a \otimes x_b) = \langle a, c_{(1)} \rangle \text{Ad}_{c_{(2)}}(x_b) \otimes c_{(3)} - \langle a, c_{(1)} \rangle \text{Ad}_{c_{(2)}}(x_a) \otimes 1 = x_{b_{(2)} a_{(1)}} \langle a(Sb_{(1)}) b_{(3)}, c_{(1)} \rangle - x_{b_{(2)} a_{(1)}} \langle a(Sb_{(1)}) b_{(3)}, c \rangle = x_{b_{(2)} a_{(1)}} - x_{a(Sb_{(1)}) b_{(3)}},$$

again using the intertwining property for the map $a \mapsto x_a$. \hfill \Box

The centrally generated calculi are dual in a certain sense to inner type-I calculi. Thus, the quantum tangent space for the inner calculus in Proposition 2.5 can be viewed as the kernel under differentiation for a suitable (right-handed) calculus on $H$, taken along the direction of $a - \epsilon(a)$. By contrast, the quantum tangent space for a centrally generated calculus can be viewed as the projection to ker $\epsilon \subset H$ of the image of $c$ under differentiation along all possible
a \in \ker \epsilon \subseteq A$, for a suitable (left-handed) calculus on $H$. Also, for factorisable quantum groups (see the next section) there is a correspondence between central elements and elements invariant under a right handed $\Ad_R$ coaction, via the quantum Killing form, see [4]. So centrally generated calculi and (a right-handed version of) inner calculi are in correspondence in this case, although quite different in character.

A centrally generated calculus typically has many quotients, i.e. its quantum tangent space $L_c$ itself has further subrepresentations. For example, we can restrict the allowed $\{x_a\}$ to $a \in M_R$ whenever $M_R$ is a right ideal in $\ker \epsilon \subseteq A$ stable under the right coaction $\Ad_R(a) = a_{(2)} \otimes (Sa_{(1)})a_{(3)}$. This is because the relation $\hbar a = \langle h, (Sa_{(1)})a_{(3)} \rangle a_{(2)}$ holds for the quantum double when acting on a central element. Moreover, comparing with (2), we see that every non-trivial central element $c$ defines a ‘mirror’ operation from the moduli space of right-handed bicontractive calculi to the moduli space of left-handed bicontractive calculi. It sends a calculus defined by quotienting the universal one by a right ideal $M_R$ according to a (right-handed version of) (2) to the calculus with quantum tangent space $\{x_a| a \in M_R\} \subseteq L_c$. The latter calculus is itself a quotient of the universal one, namely by the (left) ideal

$$\overline{M_R} = \{a \in \ker \epsilon | \langle ma, c \rangle = 0 \ \forall m \in M_R \}.$$  \hspace{1cm} (11)

This is our ‘mirror’ operation at the level of the quotienting ideals. The same operation with left-right interchanged takes us from left-handed to right handed calculi, and there is a canonical inclusion $M_R \subseteq \overline{M_R}$. The calculus with quantum tangent space $L_c$ in Proposition 2.6 is the mirror image of the zero differential calculus, and vice-versa. In addition, the zero differential calculus is the mirror image of the universal differential calculus.

Also clear from this point of view, if $L_1, L_2$ are subrepresentations of the quantum double then $L_1 \cap L_2$ is also. We denote its calculus by $\Gamma_1 \cdot \Gamma_2$; it is a quotient of both $\Gamma_1, \Gamma_2$. $L_1 + L_2$ is also a subrepresentation and we denote its calculus $\Gamma_1 * \Gamma_2$; it has $\Gamma_1, \Gamma_2$ as quotients. If $L_1 \cap L_2 = \{0\}$ then the resulting calculus is the obvious direct product calculus. We say that a differential calculus is coirreducible if its corresponding quantum tangent space $L$ is irreducible as a quantum double representation. This implies that the calculus has no proper quotient calculus. Note that this should not be confused with irreducibility for the calculus (no proper subcalculus) which is automatically true for all calculi in the paper as a consequence of axiom 5. in the definition of a bicontractive calculus. Moreover, in nice cases (where the quantum double
is semisimple) one has only to decompose

$$\ker \epsilon = L_1 \oplus L_2 \oplus \cdots \quad (12)$$

into irreducibles in order to classify coirreducible calculi; each distinct irreducible in the decomposition corresponds to an isolated coirreducible calculus and each irreducible with multiplicity typically corresponds to a continuous family of calculi given by a parameter describing the embeddings of the irreducible into its multiple copies in $\ker \epsilon$. Moreover, we see in this situation that the universal differential calculus, which corresponds to $L = \ker \epsilon$, can be built up as a direct product of coirreducible calculi.

3 Calculi on finite groups and enveloping algebras

In this section we apply the formulation of the classification problem in the preceding section to the elementary cases of finite groups and enveloping algebras. The result in the case $A = \mathbb{C}(G)$ (the algebra of functions on a finite group) is already known by other means, but recovered now from Proposition 2.2. But we also give the dual case $A = \mathbb{C}G$ (the group algebra). It turns out to be more similar to the quantum group case in the following section. The classification for Lie groups and enveloping algebras remains open, but we make some remarks.

Proposition 3.1 Let $A = \mathbb{C}(G)$ where $G$ is a finite group. It is known in this case[17] that the coirreducible biconvariant differential calculi are in 1-1 correspondence with the non-trivial conjugacy classes $C \subset G$. We recover this result from the above approach as corresponding to

$$L = \text{span} \{ x_g \equiv g - \epsilon | g \in C \}, \quad \partial_{x_g} a = a(g \cdot ( )) - a, \quad \Psi^{-1}(a \otimes x_g) = x_g \otimes a(g \cdot ( ))$$

$$[x_g, x_h] = x_{ghg^{-1}} - x_h, \quad \Psi(x_g \otimes x_h) = x_{ghg^{-1}} \otimes x_g$$

where $\epsilon$ is the group identity element.

Proof Here $H = \mathbb{C}G$ and we classify all irreducible subspaces $L \subseteq \ker \epsilon \subset \mathbb{C}G$ which are stable under the adjoint action and the action of $\mathbb{C}(G)$ in Lemma 2.1. The algebra $A = \mathbb{C}(G)$ is commutative and elements of the form $x_g = g - \epsilon$ are a basis of simultaneous eigenfunctions for its action on $\ker \epsilon$, since $a \cdot x_g = a(g)g - a(\epsilon)\epsilon - a(g)\epsilon + a(\epsilon)\epsilon = a(g)x_g$ for any $g \in G$. By choosing $a$ a Kronecker delta function we see that if $L$ contains a linear combination involving $x_g$ then it contains $x_g$ itself. Hence $L = \text{span} \{ x_g | g \in C \}$ for some subset $C \subset G$ not containing
\(e\). This is the content of stability under the \(A\) part of the quantum double action. The content of stability under the \(H\) part of the quantum double action (the adjoint action of \(G\) extended linearly) is therefore that \(C\) should be a union of non-trivial conjugacy classes. The irreducible \(L\) then correspond precisely to the non-trivial conjugacy classes. The corresponding braided derivations are \(\partial_{x,g} a = \langle x_g, a_{(1)} \rangle a_{(2)} = a(g( ) - a(\epsilon( ))\) and the ‘quantum Lie bracket’ is \([x_g, x_h] = \text{Ad}_g(x_h) - x_h = x_{g\{h^{-1}} - x_h\) as stated. Likewise, we compute \(\Psi\) from (5) and (7) in the form stated. One also has \(\ker \epsilon = \oplus_{C \neq \{e\}} L_C\), corresponding to the decomposition of \(G - \{\epsilon\}\) into non-trivial conjugacy classes, i.e. the universal calculus as a direct product of the coirreducibles.

These calculi are all ‘non-classical’ in the sense that the braiding needed for the derivation property is non-trivial (when \(G\) is non-Abelian). They are in fact a variant of the familiar \(q\)-derivative, with \(q\) being replaced by a group element taken from the conjugacy class. The non-classical nature also appears as non-commutativity of the calculus in the sense \(a b \neq (db)a\) for some \(a, b\). The calculus is inner type-I with \(\alpha\) the characteristic function of \(C \cup \{\epsilon\}\), and inner type-II with \(\alpha\) the characteristic function of \(C\). We can also apply our formalism to \(A = \odot G\).

If \(G\) is Abelian we have \(\odot G = \odot (\hat{G})\) and return to the preceding example applied to the dual group. But when \(G\) is non-Abelian, the algebra \(A\) is non-commutative and we are really doing ‘non-commutative geometry’.

**Proposition 3.2** Let \(A = \odot G\) where \(G\) is a finite group. The coirreducible bicovariant differential calculi are in 1-1 correspondence with pairs \(V, \lambda\), where \(V\) is a non-trivial irreducible (right) representation of \(G\) and \(\lambda \in P(V^*)\). The corresponding calculus has dimension \(\dim V\) and

\[
L = \text{span}\{v \in \langle v \lambda( ) - \langle v, \lambda\rangle 1 \mid v \in V\}, \quad \partial_{x,v}(g) = (\langle v \lambda( ) - \langle v, \lambda\rangle)g, \quad \Psi^{-1}(v \otimes g) = g \otimes v \lambda(g) \quad \text{and trivial ‘quantum Lie bracket’}.
\]

**Proof** Here \(H = \odot (G)\) is commutative. Hence the adjoint action in Lemma 2.1 is trivial (as is the bracket \([, , ]\) and its associated braiding). We therefore need only to classify irreducible subspaces \(L \subseteq \ker \epsilon \subset \odot G\) under the action of \(\odot G^\mathfrak{op}\). This action is \(h v x = x(h( )) - x(h)1\) for all \(x \in \ker \epsilon\), which is the standard projection \(\Pi\) to \(\ker \epsilon\) of the right regular representation of \(G\) on \(\odot G\) by multiplication from the left in the group. The Peter-Weyl decomposition \(\odot G \cong \odot \oplus_{V \neq \{e\}} V \otimes V^*\) projected via the projection \(\Pi\) is an isomorphism \(\oplus_{V \neq \{e\}} V \otimes V^* \cong \ker \epsilon\),
giving the decomposition of this into irreducibles. In the Peter-Weyl decomposition, the element \( v \otimes \lambda \) maps to the function \( \langle v_\lambda(\cdot), \lambda \rangle \in \mathbb{C}(G) \), giving the form of \( L \) shown. We need \( \lambda \neq 0 \) and we identify all \( \lambda \) which are related by a phase since these give the same \( L \), i.e. the continuous parameter is \( \lambda \in \mathbb{P}(V^*) = \mathbb{C}^{\dim V-1} \). The braided-derivation is \( \partial_{x_v} g = \langle x_v, g \rangle g \) on group-like elements of \( \mathbb{C}G \), which gives the form shown. The group-like elements are simultaneous eigenfunctions for all the braided-derivations. The braiding is easily computed as \( \Psi^{-1}(g \otimes x_v) = g \circ x_v \otimes g = \langle v_\lambda(x_v), \lambda \rangle \otimes g - \langle v_\lambda g, \lambda \rangle \otimes g = x_{v_\lambda g} \otimes g \) \( \square \)

Note that a basis of \( V^* \) specifies \( \dim V \) isomorphic copies of \( V \) in the Peter-Weyl decomposition. However, we need here not only the multiplicities but the actual corresponding subspaces \( L \). We obtain a subspace isomorphic to \( V \) for every non-trivial linear combination (modulo an overall scale) of the basis elements, i.e. a continuous family of calculi parametrized by the projective space \( \mathbb{P}(V^*) \) for each irreducible representation \( V \). Also, since irreducible representations of \( G \) correspond to characters, one can recast this result in terms of these. For a given character \( \chi \) we identify \( V^*_\chi \) as the quotient of \( \mathbb{C}G \) where \( [\lambda] = [\lambda'] \) if \( \chi(g\lambda) = \chi(g\lambda') \) for all \( g \). Then co-irreducible calculi are in 1-1 correspondence with pairs \( \chi, [\lambda] \) according to

\[
L = \text{span}\{x_g \equiv \chi(g)(\lambda) - \chi(g\lambda)1 | g \in G\}, \quad \partial_{x_g} h = (\chi(g_gh\lambda) - \chi(g\lambda))h, \quad \Psi^{-1}(h \otimes x_g) = x_{g_gh} \otimes h
\]

where \( g, h \in G \). Here \( V_\chi \) is the vector space spanned by \( \chi(\cdot) \lambda g \) as \( g \) runs over \( G \) and is an irreducible (right) representation of \( G \) acting by left multiplication in the argument of \( \chi \). From this, it is clear that these calculi on \( \mathbb{C}G \) are all centrally generated by \( c = \chi(g \lambda) \).

Finally, we consider the differential calculi on a classical Lie group co-ordinate ring \( A = \mathbb{C}(G) \). Here \( \mathbb{C}(G) \) denotes an algebraic model of the functions on \( G \) constructed as a Hopf algebra non-degenerately paired to the enveloping algebra \( U(g) \).

**Proposition 3.3** Let \( g \) be a Lie algebra. For each natural number \( n \) there is a bicovariant differential calculus with \( L = g + gg + \cdots g^n \), the subspace of degree \( \leq n \) and \( n \geq 1 \). For example, for \( n = 2 \):

\[
L = \text{span}\{\xi, \eta \xi | \xi, \eta, \zeta \in g\}, \quad \partial_{\xi} = -\tilde{\xi}, \quad \partial_{\eta \xi} = \tilde{\zeta} \tilde{\eta}
\]

\[
\Psi^{-1}(a \otimes \xi) = \xi \otimes a, \quad \Psi^{-1}(a \otimes \eta \zeta) = \eta \zeta \otimes a - \zeta \otimes \eta(a) - \eta \otimes \tilde{\zeta}(a)
\]

\[
[\xi, x] = \xi x - x \xi, \quad [\eta \zeta, x] = \eta \zeta x - \eta x \zeta - \zeta x \eta + x \zeta \eta
\]

\[
\Psi(\xi \otimes x) = x \otimes \xi, \quad \Psi(\xi \eta \otimes x) = [\xi, x] \otimes \eta + [\eta, x] \otimes \xi + x \otimes \xi \eta
\]

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for all $x \in L$. Here $\xi$ is the right-invariant vector-field associated to $\xi \in g$.

**Proof** Note that the degree of a given element in $U(g)$ is not well-defined but the subspace spanned by products of up to $n$ elements is. We show only that such a subspace $L^{(n)}$ forms a quantum double subrepresentation. To see that it is closed under the adjoint action of $U(g)$ it suffices to see that it is closed under the action of each $\xi \in g$. This action is by commutator in $U(g)$. Hence assuming the result for $L^{(n-1)}$ and the Leibniz rule for commutators, we obtain the result for $L^{(n)}$ by induction. The other part of the quantum double action (that of $A = C(G)$) is given by evaluation against the left coaction $\beta = \Delta - 1 \otimes \id$. Then $\beta(\xi x) = \Delta(\xi x) - \xi x \otimes 1 = (\xi \otimes 1)\Delta x + (1 \otimes \xi)\Delta x - \xi x \otimes 1 = (\xi \otimes 1)\beta(x) + (1 \otimes \xi)\beta(x) + x \otimes \xi \in U(g) \otimes L^{(n)}$ as $\beta(x) \in U(g) \otimes L^{(n-1)}$ and $n \geq 1$. Here $x \in L^{(n-1)}$ and we proceed by induction. The explicit computations for $L^{(2)}$ are immediate from the form of the coproduct on $\eta \zeta$ in the formulae above. Here, $\partial(\xi) = \langle \xi, a(1) a(2) \rangle = \frac{d}{dt} |_{t=0} a(e^{t\xi}(a)) = -\zeta(a)$ for $a \in C(G)$ (this is given explicitly by the matrix representation of $g$ used in defining the pairing between $U(g)$ and $C(G)$, i.e. it is actually algebraic.) $\square$

We see that it is possible to view higher order differential operators as if they are ‘first order vector fields’ – but braided. A second order operator, for example, is clearly not a derivation in the usual sense but it is a braided-derivation for suitable $\Psi$. For example, one could compute its ‘flow’ as a corresponding braided-exponential. This opens up the possibility of a ‘geometrical’ picture for the evolution of quantum systems generated by second or higher order Hamiltonians, to be given in detail elsewhere.

On the other hand, we do not attempt to classify all bicovariant calculi here. This would appear to be an interesting problem in the classical theory of enveloping algebras: find all subspaces $L$ which are stable under the adjoint action and under the left coaction $\beta = \Delta - 1 \otimes \id$. Moreover, the $L^{(n)}$ are of course not coirreducible. Instead, we have a filtration

$$
g = L^{(1)} \subset L^{(2)} \subset L^{(3)} \cdots \subset L^{(\infty)} = \ker \epsilon,
$$

where $g = L^{(1)}$ corresponds to the classical differential calculus on $C(G)$. At the level of bicovariant calculi we have a sequence of quotients of the universal one (of all finite degree invariant differential operators) eventually quotienting down to the standard one.

There are certainly bicovariant calculi other than the $L^{(n)}$. For example, if $g \otimes g$ has an Ad-invariant element $t = t_i \otimes t^i$ (e.g. if $g$ is semisimple) then $L = g \oplus \mathbb{C}$ spanned by $g$ and the central
element $c = t_it^i$ corresponds to a b covariant differential calculus in between those corresponding to $L^{(1)}$ and $L^{(2)}$. In the semisimple case, $g$ can be viewed as being centrally generated according to Proposition 2.6 by the quadratic Casimir, and $g \oplus \mathbb{C}$ is its canonical extension. (Equivalently, the mirror operation (11) in this case turns the zero differential calculus into the classical one, and vice-versa.) The elements of $g$ act as ordinary vector fields, while $c$ acts as a second order operator viewed as a braided-vector field. The quantum Lie bracket restricted to $g$ is its usual Lie bracket. The other cases and the braiding are

$$[\xi, c] = 0, \quad [c, c] = 0, \quad [c, \xi] = [t_i; t^i, \xi], \quad \Psi(\xi \otimes \eta) = \eta \otimes \xi$$

$$\Psi(\xi \otimes c) = c \otimes \xi, \quad \Psi(c \otimes \xi) = \xi \otimes c + [t_i; \xi] \otimes t^i - t^i \otimes [t_i; \xi], \quad \Psi(c \otimes c) = c \otimes c. \quad (14)$$

If $g$ under the adjoint action is isotypical (as for $sl_2$) then $[c, \xi]$ here is fixed multiple of $\xi$. The simplest case $L = sl_2 \oplus \mathbb{C}$ corresponds to the non-standard 4-dimensional differential calculus on $SU(2)$ which has been studied in [17] as the $q \to 1$ limit of the known 4-dimensional calculus on the quantum group $SU_q(2)$ in [2]. Similarly, $L^{(n-1)} \oplus \mathbb{C}$ corresponds to a natural calculus in between the calculi corresponding to $L^{(n-1)}$ and $L^{(n)}$, whenever we have a degree $n$ central element. Intermediate calculi are generally what arise when we take the limit of quantum group differential calculi (these will be classified in the next section), i.e. this is a general feature.

Put another way, we will see from the classification in the next section that the standard dim $g$-dimensional calculus on a simple Lie group $G$ violates the ‘principle of $q$-deformisability’; only certain extensions of ordinary vector fields on a Lie group by higher order vector fields can deform to calculi on $G_q$.

## 4 Calculi on factorisable quantum groups

In this section we present our main result, which is a classification of the b covariant calculi for a factorisable semisimple quantum group. We then discuss the application of the result to the standard quantum groups $G_q$, which these are essentially factorisable.

We recall that a ‘strict quantum group’ or quasitriangular Hopf algebra is factorisable[18] if $R_{21}R$ viewed as a map $Q : A \to H$ by $Q(a) = (a \otimes \text{id})(R_{21}R)$ is an isomorphism. This is the strongest form; one may also demand separately that the map is injective or surjective. We also consider, by definition, that a quantum group is semisimple if there is a Peter-Weyl decomposition

$$\oplus_V V^* \otimes V \cong A \quad (15)$$
provided by the matrix elements of the inequivalent finite-dimensional irreducible representations \( V \) of \( H \). This is broadly equivalent to other notions of semisimplicity, and is at any rate the condition that we suppose in this section. If \( V \) is such a representation, with basis \( \{ e_i \} \) and dual basis \( \{ f^i \} \), we define the matrix elements \( \rho^j_i \in A \) by \( h \triangleright e_i = e_j \rho^j_i(h) \) and the above map by \( f^i \otimes e_j \mapsto \rho^j_i \).

**Lemma 4.1** Let \( H \) be a factorisable quantum group with dual \( A \). The map \( Q \) identifies \( \ker \epsilon \subset A \) and \( \ker \epsilon \subset H \). Under this identification, the action of the quantum double in Lemma 2.1 becomes the action on \( \ker \epsilon \subset A \) given by

\[
h \triangleright a = a_{(2)} \langle h, (Sa_{(1)})a_{(3)} \rangle, \quad b \triangleright a = \langle b, R^{(1)} R^{(2)} \rangle \langle a_{(1)}, R^{(2)} \rangle \langle a_{(3)}, R^{(1)} \rangle a_{(2)} - \langle b, Q(a) \rangle 1
\]

for all \( h \in H \) and \( b \in A \). Here, \( R^j \equiv R^{(1)} \otimes R^{(2)} \) denotes a second copy of the quasitriangular structure \( R \).

**Proof** It is immediate from the county property of the quasitriangular structure \( R \) that \( Q(1) = 1 \). Hence \( Q(\ker \epsilon) = \ker \epsilon \subset H \). Moreover, we know from \( \mathrm{Ad} \)-invariance of the quantum Killing form that \( \mathrm{Ad}_b \circ Q(a) = Q(\mathrm{Ad}_b a) \) where \( \mathrm{Ad}_b \) is the left quantum coadjoint action as stated for \( h \triangleright a \) in the lemma, and \( \mathrm{Ad} \) is the quantum adjoint action used for \( x \) in Lemma 2.1. For a proof see [11] or the text[9]. The new part concerns the other action:

\[
b \triangleright Q(a) = \langle b, Q(a)_{(1)} \rangle Q(a)_{(2)} - \langle b, Q(a) \rangle 1
\]

\[
= \langle a, R^{(2)} R^{(1)} \rangle \langle b, R^{(1)} R^{(2)} \rangle \langle a_{(1)}, R^{(1)} \rangle R^{(2)} a_{(2)} - \langle b, Q(a) \rangle Q(1)
\]

\[
= \langle a, R^{(2)} R^{(2)} R^{(1)} R^{(1)} \rangle \langle b, R^{(1)} R^{(2)} \rangle \langle a_{(1)}, R^{(1)} \rangle R^{(2)} Q(a_{(2)}) - \langle b, Q(a) \rangle Q(1)
\]

\[
= \langle a_{(1)}, R^{(2)} \rangle \langle a_{(2)}, R^{(2)} \rangle \langle b, R^{(1)} R^{(2)} Q(a_{(2)}) - \langle b, Q(a) \rangle Q(1) = Q(b \triangleright a)
\]

for all \( a \in \ker \epsilon \subset A \) and \( b \in A \), where \( R^{(1)}_1 \otimes R^{(2)}_1, \ldots, R^{(1)}_4 \otimes R^{(2)}_4 \) are four copies of \( R \). The first equality is the action of \( A \) in Lemma 2.1. The second puts in the formula for \( Q \). The third is the coproduct property of the quasitriangular structure and finally we recognise the required result in terms of the action \( b \triangleright a \) stated. Hence \( Q \) intertwines the stated action of the quantum double with the action in Lemma 2.1. Note that this computation also works at the level of a coaction of \( H \) rather than an action by \( b \in A \) (i.e. the action of the quantum double remains \( A \)-regular). □

So the possible quantum tangent spaces \( L \) are in 1-1 correspondence with subrepresentations of \( \ker \epsilon \subset A \) under this action of the quantum double. This action looks more complicated than
before. However, there is a well-known isomorphism in the factorisable case of the quantum double with $H \bowtie H$. The latter is $H \otimes H$ as an algebra and has a coalgebra which is a twisting of the tensor product one. The map $\theta$ to $H \bowtie H$ is\cite{18}

$$\theta(h \otimes a) = h_{(1)}R^{(2)}(\otimes) h_{(2)}R^{(1)}(\otimes)R^{(2)}, a)$$

(16)

The full details of the isomorphism and an explicit formula for $\theta^{-1}$ are in the author’s text\cite{9}.

**Proposition 4.2** The action in Lemma 4.1 of the quantum double, in the form $H \bowtie H$ acting on $\ker \epsilon \subset A$, takes the form

$$(h \otimes 1)\triangleright a = \langle Sh, a_{(1)}a_{(2)} \rangle - 1\langle Sh, a \rangle, \quad (1 \otimes g)\triangleright a = a_{(1)}\langle g, a_{(2)} \rangle - 1\langle g, a \rangle$$

for all $h, g \in H$ and $a \in \ker \epsilon \subset A$.

**Proof** To find the action of $H \bowtie H$ we need the explicit inversion formula for $\theta$ in [9]. Then $(h \otimes 1)\triangleright a = \theta^{-1}(h \otimes 1)\triangleright a$ etc. can be computed, and one obtains the result stated in the proposition. Once these actions have been obtained, however, it is enough (and rather easier) to verify that pull back along $\theta$ indeed recovers the action of $H \bowtie A^\text{op}$ in Lemma 4.1. Thus,

$$\theta(h \otimes 1)\triangleright a = (h_{(1)} \otimes h_{(2)})\triangleright a = (h_{(1)} \otimes 1)\triangleright a_{(1)}\langle h_{(2)}, a_{(2)} \rangle - (h_{(1)} \otimes 1)\triangleright 1\langle h_{(2)}, a \rangle$$

$$= \langle Sh_{(1)}, a_{(1)}a_{(2)} \rangle - \langle Sh_{(1)}, a_{(1)} \rangle 1\langle h_{(2)}, a_{(2)} \rangle = \langle h, (Sa_{(1)})a_{(2)} \rangle \triangleright a_{(2)}$$

$$\theta(1 \otimes b)\triangleright a = (R^{(2)} \otimes R^{(1)})\triangleright a(\otimes)R^{(1)}R^{(2)}, b)$$

$$= (R^{(2)} \otimes 1)\triangleright a_{(1)}\langle R^{(1)}, a_{(2)} \rangle \langle R^{(1)}R^{(2)}, b \rangle - (R^{(2)} \otimes 1)\triangleright 1\langle R^{(1)}, a \rangle \langle R^{(1)}R^{(2)}, b \rangle$$

$$= \langle SR^{(2)}a_{(1)}a_{(2)}(\otimes)R^{(1)}, a_{(2)} \rangle \langle R^{(1)}R^{(2)}, b \rangle - \langle SR^{(2)}a_{(1)}a_{(2)} \rangle 1\langle R^{(1)}, a_{(2)} \rangle \langle R^{(1)}R^{(2)}, b \rangle$$

$$= a_{(1)}\langle R^{(2)}, a_{(2)} \rangle \langle R^{(1)}, a_{(2)} \rangle \langle R^{(1)}R^{(2)}, b \rangle - \langle Q(a), b \rangle 1$$

as required. We used the form of $\theta$, the actions as stated in the proposition and, in the last line, the antipode property $(S \otimes \text{id})R^{-1} = R$ of a quasitriangular structure. Our notation is $R^{(1)} \otimes R^{(2)} = R^{-1}$. \Box

So, quantum tangent spaces $L$ are in correspondence with subrepresentations of $\ker \epsilon$ under this action of $H \otimes H$. We can now obtain our main result.

**Theorem 4.3** Let $H$ be a factorisable quantum group with dual $A$, and suppose that the Peter-Weyl decomposition (15) holds. Then the finite dimensional bicovariant coirreducible calculi on
A are in 1-1 correspondence with the non-trivial finite-dimensional irreducible representations \( V \) of \( H \). The corresponding calculus has dimension \( (\dim V)^2 \) and

\[
L = \text{span}\{ x^i_j \equiv Q(\rho^i_j - 1\delta^i_j) \mid i, j = 1, \ldots, \dim V \}
\]

\[
\partial x^i_j(a) = Q(\rho^i_j \otimes a(2))a(1) - \delta^i_j a, \quad \Psi^{-1}(a \otimes x^i_j) = x^a_b \otimes a(2) \mathcal{R}(a(1) \otimes \rho^b_a)\mathcal{R}(\rho^b_j \otimes a(2))
\]

\[
[x^i_j, x^k_l] = x^a_b Q(\rho^i_j \otimes (S\rho^b_a)\rho^k_l) - x^k_l \delta^i_j
\]

\[
\Psi(x^i_j \otimes x^k_l) = x^m_n \otimes x^a_b \mathcal{R}((S\rho^m_n)\rho^n_d \otimes \rho^i_j)\mathcal{R}(\rho^k_j \otimes (S\rho^d_c)\rho^l_i)
\]

where we also regard the quantum Killing form and quasitriangular structure as functionals

\[
Q, \mathcal{R} : A \otimes A \to \mathbb{C}
\]

**Proof** We first separate off the trivial representation in (15), so \( A \cong \mathbb{C} \oplus (\oplus_{V \neq \mathbb{C}} V^* \otimes V) \) where the sum is over non-trivial \( V \). The projection \( \Pi(a) = a - 1\epsilon(a) \) from \( A \to \ker \epsilon \) establishes an isomorphism

\[
\ker \epsilon \cong \oplus_{V \neq \mathbb{C}} V^* \otimes V.
\]

This is because \( \Pi \) and the projection to \( \oplus_{V \neq \mathbb{C}} V^* \otimes V \) have the same kernel, namely the span of the identity element in \( A \). By Proposition 4.2, we therefore have an isomorphism of \( H \otimes H \) modules, where the second \( H \) acts on \( V \) as in the Peter-Weyl decomposition (the given irreducible representation \( V \)) and the first copy of \( H \) acts on \( V^* \) by the conjugate representation \( h \mapsto f(S_h( ) \, \, ) \) for \( f \in V^* \). Next, as \( H \otimes H \) modules, these \( V^* \otimes V \) are distinct and irreducible. Hence they are precisely the choices for irreducible subrepresentations of \( \ker \epsilon \subset A \).

The explicit formula for the braided-derivations and their requisite braiding are easily computed from the formulae in Proposition 2.3. From the proof of Lemma 4.1 we have

\[
(\Delta - \text{id} \otimes 1)x^i_j = \mathcal{R}^{(1)}\mathcal{R}^{(2)}((\rho^i_j - \delta^i_j)_{(1)}, \mathcal{R}^{(2)})((\rho^i_j - \delta^i_j)_{(2)}, \mathcal{R}^{(1)}) \otimes Q((\rho^i_j - \delta^i_j)_{(3)}) - Q(\rho^i_j - \delta^i_j) \otimes 1
\]

\[
= \mathcal{R}^{(1)}\mathcal{R}^{(2)} \otimes (\rho^i_a, \mathcal{R}^{(2)})(\rho^b_j, \mathcal{R}^{(1)})Q(\rho^a_b) - Q(\rho^i_j) \otimes 1
\]

\[
= \mathcal{R}^{(1)}\mathcal{R}^{(2)} \otimes (\rho^i_a, \mathcal{R}^{(2)})(\rho^b_j, \mathcal{R}^{(1)})x^a_b
\]

Evaluation against this is the action of \( A \) in Lemma 4.1, which is the action needed to compute the braiding. Thus, \( \Psi^{-1}(a \otimes x^i_j) = a(2) \otimes (a(1), \mathcal{R}^{(1)}\mathcal{R}^{(2)})(\rho^i_a, \mathcal{R}^{(2)})(\rho^b_j, \mathcal{R}^{(1)})x^a_b \), which can be written in the form shown where \( \mathcal{R} \) is regarded as a functional on \( A \otimes A \). The quantum Lie bracket and its braiding from Proposition 2.4 are also easily computed and follow the same
lines as in [11][19], except that we are not tied to any particular representation $V$ or any fixed $R$-matrix; we include the proofs only for completeness in our present conventions. Thus, by Ad-invariance of $Q$ we have

$$[x^i_j, x^k_l] = Q[Q(\rho^i_j - \delta^i_j) \rho^k_l, (\rho^k_l - \delta^k_l)] = Q(\rho^a_b)(Q(\rho^i_j), (S\rho^a_k)\rho^b_l) - \delta^i_j Q(\rho^k_l) = x^a_b(Q(\rho^i_j), (S\rho^a_k)\rho^b_l) - \delta^i_j x^k_l,$$

which we write in the form stated where $Q = R_21 \mathcal{R}$ is regarded as a functional on $A \otimes A$. Here $\triangleright$ is the quantum coadjoint action of $H$ in Lemma 4.1. Finally, using the above result for $\Delta x^i_j$ and Ad-invariance of $Q$, we have

$$\Psi(x^i_j \otimes x^k_l) = [x^i_j(1), x^k_l] \otimes x^i_j(2) - [x^i_j, x^k_l] \otimes 1 = Q[R^{(1)} \mathcal{R}^{(2)} \triangleright (\rho^k_l - \delta^k_l)] \otimes (\rho^i_j, \mathcal{R}^{(2)})(\rho^k_l, \mathcal{R}^{(1)})x^a_b$$

$$= Q(\rho^d_a) \otimes x^a_b(R^{(1)} \mathcal{R}^{(2)}(1) \mathcal{R}^{(2)}(2), (S\rho^k_l)c_d) \rho^i_j(1) \rho^a_a, \mathcal{R}^{(2)}(2)) \rho^k_l, \mathcal{R}^{(1)})x^a_b - \delta^k_l \otimes x^i_j$$

$$= x^d \otimes x^a_b(R^{(1)} \mathcal{R}^{(2)}(2), (S\rho^k_l)c_d) \rho^i_j(1) \rho^a_a, \mathcal{R}^{(2)}(2)) \rho^k_l, \mathcal{R}^{(1)})x^a_b,$$

which we write in the form stated. Note that both of the expressions $Q(\rho^i_j \otimes (S\rho^k_l)c_d) \rho^i_j(1)$ and $R((S\rho^k_l)c_d) \rho^i_j \mathcal{R}(1) \mathcal{R}(2)$ can be expanded out as four-fold products of the matrices $R = (\rho \otimes \rho)\mathcal{R}$, its inverse and $\tilde{R} = (\rho \otimes \rho \circ S)\mathcal{R}$. This step and the resulting R-matrix formulae are identical in form to the computation of the quantum Lie algebra ‘structure constants’ in [11] and the computation of the quadratic relations of the braided matrices in [19] (the matrix denoted $\Psi'$ there), respectively. Hence we omit the proofs and note only that, after rearranging the R-matrices, one has the same form as for a quantum or braided-Lie algebra of matrix type, namely

$$R_{21}[x_1, Rx_2] = x_2 Q - Q x_2, \quad R_{21}\Psi(x_1 \otimes Rx_2) = x_2 R_{21} \otimes x_1 R$$

(18)

where the numerical suffices denote positions in a matrix tensor product and $Q = R_{21} \mathcal{R}$. The relation between (18), braided matrices $u = x + id$ and the quantum double is explained further in [10] (where the quantum double braiding $\Psi$ is denoted $\tilde{\mathcal{R}}$). On the other hand, now (18) applies to any irreducible representation $V$ of $H$ and not some fundamental basic representation, which need not exist.

Let us note that if $\mathcal{R}$ is a quasitriangular structure in a quantum group then so is $\mathcal{R}^{-1}_{21}$. Thus all results involving a quasitriangular Hopf algebra have a ‘conjugate’ one in which this conjugate $\mathcal{R}^{-1}_{21}$ is used instead of $\mathcal{R}$. This conjugation is also intimately tied to the $*$-operation
or complex conjugation in many systems [20]. In the above theorem, we see that for every $V$ we have equally well the conjugate

$$L = \text{span}\{x^i_j \equiv Q(\rho^i_j - i\delta^i_j) | i, j = 1, \ldots, \dim V\}$$

(19)

where $Q(a) = (a \otimes \text{id})(R^{-1}R^*)$. Here $L$ is isomorphic to $L$ but the isomorphism (which is $Q \circ Q^{-1}$ restricted to $L$) is non-trivial. This fits also with the general point of view of quasi-structures on inhomogeneous quantum groups [20] where the tensor product of unitaries is unitary only up to a non-trivial isomorphism.

These results can be applied formally to the standard quantum groups $H = U_q(g)$ with dual $A = G_q$ associated to complex semisimple Lie algebras, provided we work over formal power-series $\mathbb{C}[[\hbar]]$ and introduce suitable logarithms for some of the $G_q$ generators, etc. Or, if we want to work algebraically over $\mathbb{C}$ (with generic $q$), we need to localise and introduce roots of some of the generators of $G_q$ and use the algebraic form of $U_q(g)$ where $q^\hbar$ etc. is regarded as a single generator. This is clear from the standard cases such $SU_q(2)$: In standard notations the value of $Q$ on the generators is

$$Q\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = \left(\begin{array}{cc} q^H & q^{-\frac{1}{2}}(q - q^{-1})q^\frac{\hbar}{2}X_- \\ q^{-\frac{1}{2}}(q - q^{-1})q^\frac{\hbar}{2}X_+ & q^{-1}C - q^{-2}q^H \end{array}\right)$$

(20)

where $C = q^{H-1} + q^{-H+1} + (q - q^{-1})^2X_+X_-$ is the $q$-quadratic Casimir. According to [18], the standard quantum groups are all factorisable modulo such formal extensions. Likewise, the Peter-Weyl decomposition (15) holds formally for the standard semisimple $g$. This is because the category of finite-dimensional representations in the classical and quantum cases are generically equivalent, and the assumption holds in some form for the classical case. Note also that the entries in (20) projected to $\ker \epsilon$ span a 4-dimensional $L$ associated to the spin $1/2$ representation, and has the structure in Theorem 4.3 without any powerseries. Indeed, the quantum double of $U_q(su_2)$ is known to be a $q$-deformation of the Lorentz group and hence the lowest possible generic representation is the 4-dimensional one on $q$-Minkowski space. In this simplest case, $L$ is the same subspace $L$. The latter also coincides with $L_C$ from Proposition 2.6 with $C$ the $q$-quadratic Casimir above, and is a subspace of $L_{\alpha,1}$ from Proposition 2.5 with $\alpha = (qa + q^{-1}d)/q^{-2}(q^3 - 1)(q - 1)$ the normalised $q$-trace.

Therefore we should understand Theorem 4.3 not as a complete algebraic classification for a given version of each given $G_q$ (this is a much harder problem and has been recently addressed in some cases [3]), but as a classification of those calculi which are ‘generic’ in the sense that
they extend to the various localisations and square-roots of the generators etc. needed for exact factorisability. In other words, there are natural calculi, corresponding to $L$ (or $L$) for each finite-dimensional irreducible representation $V$, and these are the only ones modulo ‘pathological’ possibilities for particular $q$ for particular versions of particular $G$. 

For the $A, B, C, D$ series we have a natural ‘fundamental representations’ $V$ and in this case it should be clear that the calculus corresponding to $L$ is the one found by Jurco[16] by other means. We therefore have a new construction for this and the result that its slight generalisation to other irreducible representations exhausts all the generic first order bicovariant differential calculi on the standard semisimple quantisations.

5 Concluding remarks

We conclude with some remarks about further work. Firstly, the bicovariant calculi studied here are ‘first order’. They play the role of 1-forms. It remains to construct and classify all possible higher order calculi or ‘exterior algebras’. One canonical construction is to take the tensor algebra on the first order calculus $\Gamma$ and quotient with the aid of a ‘skew-symmetrizer’ built from the quantum double braiding $\Psi$, see[2]. In this case the exterior algebra is a super-Hopf algebra[21]. On the other hand, even when $\Gamma$ is the classical calculus, the canonical exterior algebra is not the classical one. One must quotient it further. The classification of exterior algebras therefore remains open even after we have classified the first order calculi.

Secondly, all of the results in Section 2 about first order calculi on quantum groups have an analogue for braided groups. Braided groups are needed to include $q$-deformations $\mathbb{R}^n_q$ and $\mathbb{R}^{1,3}_q$ etc., with their additive (braided) coproduct. The classification of differential calculi on such objects would therefore seem to be the starting point for some form of $q$-geometry based on $\mathbb{R}^n$. Our result in this direction is a negative but rather unexpected one: generically there is only one coirreducible braided-bicovariant differential calculus on $\mathbb{R}^{1,3}_q$ (say), and it is infinite-dimensional. Its braided tangent space $L$ consists (in a suitable completion) of a $q$-deformation of the space of solutions of the massless Klein-Gordon equation projected to the functions vanishing at the origin. Briefly, (details will be presented elsewhere) the sketch is as follows. Let $B$ be a braided group in a braided category generated by ‘background quantum group’ $H$ as its category of modules. We define a braided-bicovariant calculus $\Gamma$ in the obvious way and proceed in a similar manner to Section 2. The role of the quantum double is now played by the author’s
double-bosonisation’ $B^* \triangleright H \triangleleft B$ quantum group[22]. This acts on $\ker \epsilon \subset B$ and the possible braided tangent spaces $L$ are in 1-1 correspondence with sub-representations of $\ker \epsilon$. When $B = \mathbb{R}^1$ it is known from [23] that the double-bosonisation is the $q$-conformal group and the action on $B$ is a $q$-deformation of its action on $\mathbb{R}^n$. Classically, however, this representation has one irreducible subrepresentation, which is the space of solutions of the massless Klein-Gordon equation.

The braided version of the theory may also help to solve the above-mentioned problem of exterior algebras on quantum groups, at least in the case of strict (quasitriangular) quantum groups. This is because the braided groups corresponding under transmutation to strict quantum groups are always braided-commutative in a certain sense[24], i.e. closer to the classical situation. Using this braided-commutativity one may reasonably expect a natural exterior algebra $q$-deforming the classical one. Such a result would be the ‘skew’ analogue of the situation in Section 2, where we explained that the braided version of the ‘quantum Lie bracket’ is better behaved for constructing a braided enveloping algebra. This remains a direction for further work.

References


