

**Affine Orbifolds and Rational Conformal Field  
Theory Extensions of  $W_{1+\infty}$**

**Victor G. Kac  
Ivan T. Todorov**

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# AFFINE ORBIFOLDS AND RATIONAL CONFORMAL FIELD THEORY EXTENSIONS OF $W_{1+\infty}$

VICTOR G. KAC AND IVAN T. TODOROV\*

International Erwin Schrödinger  
Institute for Mathematical Physics  
and  
Department of Mathematics,  
MIT, Cambridge, MA 02139, USA

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ABSTRACT. Chiral orbifold models are defined as gauge field theories with a finite gauge group  $\Gamma$ . We start with a conformal current algebra  $\mathfrak{A}$  associated with a connected compact Lie group  $G$  and a negative definite integral invariant bilinear form on its Lie algebra. Any finite group  $\Gamma$  of inner automorphisms of  $\mathfrak{A}$  (in particular, any finite subgroup of  $G$ ) gives rise to a gauge theory with a chiral subalgebra  $\mathfrak{A}^\Gamma \subset \mathfrak{A}$  of *local observables* invariant under  $\Gamma$ . A set of positive energy  $\mathfrak{A}^\Gamma$  modules is constructed whose characters span, under some assumptions on  $\Gamma$ , a finite dimensional unitary representation of  $SL(2, \mathbb{Z})$ . We compute their *asymptotic dimensions* (thus singling out the nontrivial orbifold modules) and find explicit formulae for the modular transformations and hence, for the *fusion rules*.

As an application we construct a family of *rational conformal field theory* (RCFT) extensions of  $W_{1+\infty}$  that appear to provide a bridge between two approaches to the quantum Hall effect.

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\*On leave of absence from the Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Tsarigradsko Chaussée 72, BG-1784, Sofia, Bulgaria

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## 0. INTRODUCTION

Given a chiral conformal field theory (CFT) — i.e., a chiral algebra  $\mathfrak{A}$  and a family of positive energy  $\mathfrak{A}$ -modules (closed under “fusion”) — there are two ways of constructing other CFT with the same stress-energy tensor  $T(z)$  and associated central charge  $c$ . First, one can, in some cases, extend  $\mathfrak{A}$  by adjoining to it local primary fields. The stress energy tensor generates an RCFT for the minimal models [BPZ] corresponding to central charge  $c < 1$ . For  $c \geq 1$  one needs in addition a chiral current algebra or a  $W$ -algebra to construct an RCFT (for special rational values of  $c$ ). All RCFT extensions of the  $(c = 1)u(1)$ -current algebra have been classified in [BMT]; all local extensions of the  $su(2)$  current algebras have been described in [MST]. The second path goes in the opposite direction: one restricts  $\mathfrak{A}$  to a distinguished subalgebra of “observables” including  $T(z)$ ; we shall be concerned here with the case in which the subalgebra  $\mathfrak{A}^\Gamma$  consists of all elements of  $\mathfrak{A}$  invariant under a finite automorphism group  $\Gamma$ . The resulting CFT is called a  $\Gamma$ -orbifold. Examples of orbifolds (first in the context of a “Gaussian model” [G] [H]) have been studied in detail in [DV<sup>3</sup>] where some general properties of arbitrary orbifold models have also been pointed out. Non unitary models of  $c = 1$  have been considered in [F].

The present paper provides a systematic approach to orbifold RCFT. Our starting point is a chiral algebra  $\mathfrak{A} = \mathfrak{A}(G)$  associated with a connected compact Lie group  $G$  whose Lie algebra  $\mathfrak{g}$  is equipped with a negative definite integral invariant bilinear form. It appears as a tensor product of a lattice chiral algebra  $\mathfrak{A}(L)$  and (chiral) affine Kac-Moody algebras (corresponding to the simple components  $\mathfrak{g}^j$  of  $\mathfrak{g}$ ):

$$(0.1) \quad \mathfrak{A}(G) = \mathfrak{A}(L) \otimes \left( \bigotimes_{j=1}^s \mathfrak{A}_{k_j}(\mathfrak{g}^j) \right)$$

where  $k_j \in \mathbb{Z}_+$  is the level of the vacuum  $\hat{\mathfrak{g}}^j$ -module. The lattice  $L$  consists of all vectors  $\omega$  in the direct sum  $\mathfrak{g}^0$  of  $u(1)$ -components of the centre of  $\mathfrak{g}$  such that  $e^{2\pi i \omega} = 1$ . To each  $\omega$  of length square 2 we can associate a “charge shift” operator  $E^\omega$  providing a non-abelian extension of the Lie algebra  $\mathfrak{g}^0$ . Let  $G_c$  be the corresponding maximal compact group extension of  $G$ . Its significance stems from the fact that each finite order *inner automorphism* of  $\mathfrak{A}(G)$  is given by (the adjoint action of) an element of  $G_c$ .

An *orbifold chiral algebra* is the fixed point set  $\mathfrak{A}^\Gamma$  of a finite group of automorphisms  $\Gamma$  of a chiral algebra  $\mathfrak{A}$ . For any “non-exceptional” finite subgroup  $\Gamma \subset G_c$  we construct a finite family of  $\mathfrak{A}^\Gamma$ -modules  $V$ , which is complete in the sense that their characters transform among themselves under the modular group  $SL(2, \mathbb{Z})$ . Each  $V$  is labeled by a weight  $\Lambda$  (characterizing an  $\mathfrak{A}(G)$ -module), a conjugacy class  $\bar{b} \subset \Gamma$ , and an irreducible representation  $\sigma = \sigma^b$  of the centralizer  $\Gamma_b$  of an element  $b \in \bar{b}$  in  $\Gamma$ . It involves a choice of “phases”  $\beta(b) \in \mathfrak{g}$ , for non-exceptional conjugacy classes, satisfying the following two conditions:

$$(0.2) \quad \text{(i) } b = e^{2\pi i \beta(b)} \text{ , } \quad \text{(ii) } \beta(gbg^{-1}) = Ad_g \beta(b) \text{ for } b \in \bar{b}, g \in \Gamma \text{ .}$$

Condition (ii) implies that the centralizer of  $b$  should stabilize  $\beta$ . Two  $\beta$ 's satisfying (0.2) differ by a co root  $m$  which is also stabilized by  $\Gamma_b$ . Any such  $m$  gives rise to a 1-dimensional representation  $\sigma_m$  of  $\Gamma_b$ . The change  $\beta \rightarrow \beta + m$  can be compensated by a change in the representation  $\sigma$ :

$$(0.3) \quad V_{\Lambda, \bar{b}, \sigma}^{\beta+m} = V_{\Lambda, \bar{b}, \sigma \otimes \sigma_m^*}^\beta \text{ .}$$

Thus the family of  $\mathfrak{A}^\Gamma$ -modules is independent of the choice of  $\beta$  (allowing us to skip the superscript  $\beta$  on  $V$ ).

Knowing the character  $\chi_\Lambda$  of an  $\mathfrak{A}(G)$ -module  $V_\Lambda$  [K1] we are able to calculate the  $\mathfrak{A}^\Gamma$ -characters  $\chi_{\Lambda, \bar{b}, \sigma}$  of  $V_{\Lambda, \bar{b}, \sigma}$ . Similarly, the modular transformation properties of  $\chi_\Lambda$  [KP2] determine those of  $\chi_{\Lambda, \bar{b}, \sigma}$  and hence the orbifold fusion rules. We point out that the group factors of fusion coefficients  $N_{\bar{b}_1 \sigma_1, \bar{b}_2 \sigma_2, \bar{b}_3 \sigma_3}$  ( $\sigma_i \in \hat{\Gamma}_{b_i}$ ) of an affine orbifold differ from those of the associated Grothendieck ring (see [Lus] as well as the discussion in Sect. 4 of [DV<sup>3</sup>]) due to multipliers  $\mu(h|\Sigma\beta_i)$  which define (for  $b_1 b_2 b_3 = 1$ ) 1-dimensional representations of the intersection  $\Gamma_{b_1} \cap \Gamma_{b_2} (\ni h)$ . This difference shows up already for (finite) subgroups of  $SU(2)$ . For higher rank  $G$  it may yield a change of charge conjugation, as displayed in the examples of a 1080 element subgroup of  $SU(3)$  which admits a conjugacy class of involutive elements with a non-abelian centralizer.

We compute (in Sect. 4A) the asymptotic dimensions of orbifold characters singling out, in particular, the non-trivial orbifold modules.

If  $G$  is a simple simply-connected Lie group then the non-trivial elements of

$$(0.4) \quad Z = Z(G) \cap \Gamma \text{ ,}$$

where  $Z(G)$  is the center of  $G$ , are exceptional — they cannot be written in the form (0.2) (with  $\beta$  satisfying (ii)). Each element of  $Z$  (different from the group

unit) is associated with a fundamental weight  $\Lambda_j$  satisfying  $(\Lambda_j|\theta) = 1$  where  $\theta$  is the highest root. We associate with it (in Sect. 4) a permutation of the orbifold modules which maps, in particular, the (affine) vacuum weight  $\tilde{\Lambda}_0$  into  $\tilde{\Lambda}_j$  and thus cannot be viewed as an automorphism (“gauge transformation”) of the (vacuum) chiral algebra. Knowing the action of  $e^{2\pi i\Lambda_j}$  on  $\{V_{\Lambda\bar{b}\sigma}\}$  we can extend our treatment to all exceptional elements of a  $\Gamma \subset \text{SU}(n)$ . The treatment of Ad-exceptional elements (described in Appendix A), which are encountered in other simple Lie groups, remains however, outside the scope of the present paper.

Note an essential difference between coset models and orbifold models. For the construction of a modular invariant family of characters of coset modules it suffices to take characters of isotypic components of all (untwisted) modules of the chiral current algebra with respect to its chiral current subalgebra [KP0], [KP2], [KW], [K1]. In a sharp contrast, for an orbifold model one has to take in addition decompositions into isotypic components of twisted chiral current algebra modules which become untwisted when restricted to the orbifold chiral subalgebra.

As an application we construct a family of RCFT extensions of  $W_{1+\infty}$ -one for each value  $l(\in \mathbb{N})$  of the central charge and for each finite subgroup  $\Gamma$  of  $U(l)$ . It is designed to provide a bridge between two current attempts to understand the fractional quantum Hall effect in terms of chiral conformal algebras (see [FT] and [CTZ]).

## 1. CHIRAL ALGEBRAS ASSOCIATED WITH CONNECTED COMPACT LIE GROUPS

We shall first recall the general notion of a chiral algebra and will then introduce a class of such algebras which appear to be of paramount importance in the study of RCFT.

### 1A. Definition of a chiral algebra. Current algebras.

The mathematical concept of a vertex or chiral algebra was introduced by R. Borcherds [Bor] and later developed by a number of authors (see, e.g. [FLM], [Go], [DGM], [FZ], [LZ], [FKRW], [KR2]). The version adopted here is a specialization of [K2] to  $\mathbb{Z}$ -graded algebras (restricting from the outset attention to fields of a given conformal dimension).

Let  $V$  be a  $\mathbb{Z}_+$ -graded inner product space with a *unique vacuum state*,

$$(1.1) \quad V = \bigoplus_{n=0}^{\infty} V^{(n)}, \quad \dim V^{(0)} = 1, \quad \dim V^{(n)} < \infty;$$

the gradation defines (and can be, conversely, defined by) a distinguished hermitian operator  $L_0$  called the (chiral) *energy operator* such that

$$(1.2) \quad (L_0 - n)V^{(n)} = 0.$$

The unique (up to a phase factor) vector  $|0\rangle \in V^{(0)}$  normalized by  $\langle 0|0\rangle = 1$  is called the *vacuum*. A *chiral field*  $Y^{(s)}$  of dimension  $s$  is a power series

$$(1.3) \quad Y^{(s)}(z) = \sum_{n \in \mathbb{Z}} Y_n z^{-n-s}, \quad s \in \mathbb{Z}_+$$

with  $Y_n(= Y_n^{(s)}) \in \text{End}V$  satisfying the commutation relations (CR)

$$(1.4) \quad [Y_n, L_0] = nY_n \Leftrightarrow [L_0, Y^{(s)}(z)] = \left( z \frac{d}{dz} + s \right) Y^{(s)}(z) ,$$

$$(1.5) \quad Y_n|v_m\rangle = 0 \text{ for } v_m \in V^{(m)} , \quad n > m .$$

Equation (1.5) expresses the postulate that *the vacuum is the lowest energy state* in  $V$ . In physical terms  $V$  is the *vacuum space of finite energy states*.

A *chiral (vertex) algebra* structure on  $V$  is a linear map, called the *state-field correspondence*, from  $V^{(s)}$  to the space of fields of dimension  $s$ :  $V^{(s)} \ni v_s \rightarrow Y(v_s, z) = \sum_n Y_n(v_s)z^{-n-s}$ , defined for all  $s \in \mathbb{Z}_+$  and satisfying the following three axioms:

V1. *Vacuum axioms*: the vacuum vector corresponds to the identity operator in  $V$

$$(1.6) \quad Y(|0\rangle, z) = 1_V ;$$

the field  $Y(v_s, z)$  allows to recuperate the vector  $v_s$ :

$$(1.7) \quad \lim_{z \rightarrow 0} Y(v_s, z)|0\rangle = v_s , \quad \text{i.e. } Y_{-s}(v_s)|0\rangle = v_s \text{ and } Y_{n-s}(v_s)|0\rangle = 0 \text{ for } n > 0 .$$

V2. The *translation operator*  $L_{-1} : V \rightarrow V$  defined by  $L_{-1}v_s = Y_{-s-1}(v_s)|0\rangle$  satisfies the translation covariance condition:

$$(1.8) \quad [L_{-1}, Y(v_s, z)] = \frac{d}{dz}Y(v_s, z) .$$

V3. The chiral fields are *local*:

$$(1.9) \quad (z - w)^n [Y(v_s, z), Y(v_{s'}, w)] = 0 \text{ for } n \geq s + s' .$$

Note that the inner product is not logically necessary in this generality. It is, however, present in all CFT (being indefinite for non-unitary theories) and gives rise to a distinguished (anti-involutive) star operation ([DGM]).

We shall be concerned with (orbifolds of) *chiral current algebras* described below. Let  $G$  be a compact Lie group of the form  $G = G^0 \times G^1 \times \cdots \times G^s$  where  $G^0 = U(1)^r$ , and  $G^j, j = 1, \dots, s$ , are simple simply-connected groups. (Every compact Lie group can be viewed as a product of the above form factored by a finite central subgroup). Let  $\mathfrak{g}^j$  denote the Lie algebra of  $G^j (j = 0, \dots, s)$  and let  $L = \{\omega \in \mathfrak{g}^0 | \exp 2\pi i\omega = 1\}$ . We assume that  $\mathfrak{g}$  is equipped with a symmetric integral negative definite invariant bilinear form. A bilinear form on  $\mathfrak{g}$  is called *integral* if the length square of any  $a \in i\mathfrak{g}^j (j = 1, \dots, s)$  such that  $\exp 2\pi i\omega = 1$  (resp. of any  $\omega \in L$ ) is an even integer (respectively an integer). When restricted to a simple  $\mathfrak{g}^j$ , the integrality property means that the bilinear form is equal to  $k_j(v|v')$ , where  $k_j \in \mathbb{N}$  will be identified with the level of the affine Kac-Moody algebra  $\hat{\mathfrak{g}}^j$  and

$$(v|v') = \frac{1}{2g_j} \text{tr}_{\mathfrak{g}^j} (ad_v ad_{v'})$$

( $g_j^\vee$  is the dual Coxeter number of  $\mathfrak{g}^j$ ; recall that with such a normalization  $(\alpha|\alpha) = 2$  for long roots  $\alpha$ ).

In what follows we let also  $k_0 = 1$ .

*Remark 1.1.* Admitting lattice vectors  $\alpha$  with odd square lengths requires, as it will become clear shortly, extending the  $\mathbb{Z}_+$  gradation of the vacuum space (1.1) to a  $\frac{1}{2}\mathbb{Z}_+$  gradation. In physical terms it amounts to admitting locally anti-commuting (Fermi) fields of half-integer conformal dimensions in the chiral algebra. Such fields do not describe local observables (in the strict sense of the word) and could alternatively be incorporated in the positive energy representations of a chiral Bose algebra corresponding to an even integral lattice. A way to get rid of Fermi fields is to go to a double cover of the group  $G$ , which makes the lattice  $L$  even.

Given the above data one can construct a chiral algebra

$$(1.10) \quad \mathfrak{A}(G) = \mathfrak{A}(L) \otimes \left( \otimes_{j=1}^s \mathfrak{A}_{k_j}(\mathfrak{g}^j) \right) ,$$

called an *affine (or current) chiral algebra* as follows.

For each  $\mathfrak{g}^j (j = 0, 1, \dots, s)$  consider its affinization [K1]:

$$\hat{\mathfrak{g}}^j = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{R}} \mathfrak{g}^j + \mathbb{C}K_j .$$

It is a  $\mathbb{Z}$ -graded algebra, the energy operator  $L_0$  acting on it as  $-t \frac{d}{dt}$ .

Let  $V_0(\mathfrak{g}_j, k_j)$  denote the unique irreducible  $\hat{\mathfrak{g}}^j$ -module which admits a non-zero vector  $|0\rangle$  such that  $(\mathbb{C}[t] \otimes \mathfrak{g}^j)|0\rangle = 0$  and  $K_j|0\rangle = k_j|0\rangle$ . Given an element  $v \in \mathfrak{g}^j$  and  $n \in \mathbb{Z}$  we let  $v_n$  denote the operator on  $V_0(\mathfrak{g}^j, k_j)$  corresponding to  $t^n \otimes v$ . Let  $v(z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$  be the *current* corresponding to  $v$ .

Then the chiral algebra structure  $\mathfrak{A}_{k_j}(\mathfrak{g}^j)$  on the vacuum space  $V_0(\mathfrak{g}^j, k_j)$  is defined for each  $j = 1, \dots, s$  by the following state-field correspondence:

$$Y(v_{-i_1-1}^1 \dots v_{-i_n-1}^n |0\rangle_j, z) =: \partial^{i_1} v^1(z) \dots \partial^{i_n} v^n(z) : / i_1! \dots i_n!$$

(with appropriately defined normal products, [K2]).

The vacuum space  $V$  is given by

$$(1.11) \quad V = V(L) \otimes \otimes_{j=1}^s V_0(\mathfrak{g}^j, k_j) .$$

In the next section we describe the first factor in (1.11) and the corresponding chiral algebra structure  $\mathfrak{A}(L)$  (cf. [K2], Sect. 5.4).

### 1B. Lattice vertex algebras.

Let  $\mathbb{C}_\varepsilon[L]$  be the twisted group algebra of the lattice  $L$  with basis  $e^\omega (\omega \in L)$  and multiplication rules

$$(1.12) \quad e^\omega e^{\omega'} = \varepsilon(\omega, \omega') e^{\omega+\omega'} , \quad \omega, \omega' \in L ,$$

where  $\varepsilon(\omega_1, \omega_2)$  is a  $\pm 1$ -valued cocycle:

$$(1.13a) \quad \varepsilon(\omega, 0) = \varepsilon(0, \omega) = 1 ,$$

$$(1.13b) \quad \varepsilon(\omega, \omega') \varepsilon(\omega + \omega', \omega'') = \varepsilon(\omega, \omega' + \omega'') \varepsilon(\omega', \omega'') .$$

(Equation (1.13a) means that  $e^0 = 1$  and equation (1.13b) is equivalent to associativity.)

Let  $S = V_0(\mathfrak{g}^0, 1)$ . This is the symmetric algebra over the positive energy subspace

$$(1.14) \quad \hat{\mathfrak{g}}^{0(+)} = \bigoplus_{n < 0} \mathbb{C} t^n \mathfrak{g}^0 .$$

(Here and below we omit the tensor product sign between  $t^n$  and  $\mathfrak{g}$ ). The space  $V(L)$  is then defined as the tensor product

$$(1.15) \quad V(L) = S \otimes \mathbb{C}_\varepsilon[L] .$$

It is an infinite direct sum (over the lattice) of irreducible positive energy  $\hat{\mathfrak{g}}^0$  modules with  $k_0 = 1$ :

$$(1.16) \quad V(L) = \bigoplus_{\omega \in L} S \otimes e^\omega .$$

The corresponding ground state vectors are  $1 \otimes e^\omega$ ; in particular, the  $V(L)$  vacuum is  $|0\rangle = 1 \otimes 1$ . The chiral subalgebra  $\mathfrak{A}(S \otimes 1)$  is generated by currents. The ground state vector  $|\omega\rangle \equiv 1 \otimes e^\omega$  of each term in (1.16) is characterized by being an eigenvector of  $\hat{\mathfrak{g}}^{0(-)} = \bigoplus_{n \geq 0} \mathbb{C} t^n \mathfrak{g}^0$ :

$$(1.17) \quad (v_0 - (v|\omega))|\omega\rangle = 0 = v_n|\omega\rangle, n = 1, 2, \dots$$

To display the full chiral algebra  $\mathfrak{A}(L)$  it remains to recall the *Frenkel-Kac construction* for the charged fields [FK]:

$$(1.18) \quad Y(e^\omega, z) = e^\omega e^{\varphi_+(z, \omega)} z^{\omega_0} e^{\varphi_-(z, \omega)}$$

where

$$(1.19) \quad \varphi_\pm(z, \omega) = \pm \sum_{n=1}^{\infty} \omega_{\mp n} \frac{z^{\pm n}}{n} .$$

$Y(e^\omega, z)$  is a *primary field* with respect to the current subalgebra  $\mathfrak{A}(S \otimes 1)$ :

$$(1.20) \quad [v(z), Y(e^\omega, w)] = (v|\omega)\delta(z-w)Y(e^\omega, w) .$$

Let  $p(\omega) \in \mathbb{Z}/2\mathbb{Z}$  denote the parity of  $(\omega|\omega)$ ,  $\omega \in L$ . The fields  $Y(e^\omega, z)$  and  $Y(e^{\omega'}, z)$  are local if and only if

$$(1.21) \quad \varepsilon(\omega, \omega') = (-1)^{p(\omega)p(\omega') + (\omega|\omega')} \varepsilon(\omega', \omega) .$$

The state-field correspondence for the chiral algebra  $\mathfrak{A}(L)$  is defined by

$$(1.22) \quad Y(v_{-i_1-1}^1 \dots v_{-i_n-1}^n \otimes e^\omega, z) =: \partial^{i_1} v^1(z) \dots \partial^{i_n} v^n(z) Y(e^\omega, z) : / i_1! \dots i_n! .$$

Note that a 2-cocycle  $\varepsilon(\omega, \omega')$  satisfying (1.21) always exists and the chiral algebra  $\mathfrak{A}(L)$  is independent of the choice of this cocycle.

The conformal properties of  $Y(e^\omega, z)$  are given by

$$(1.23) \quad [T(z), Y(e^\omega, w)] = \delta(z-w) \frac{\partial}{\partial w} Y(e^\omega, w) + \frac{|\omega|^2}{2} Y(e^\omega, w) \frac{\partial}{\partial w} \delta(z-w), \quad |\omega|^2 \equiv (\omega|\omega).$$

Here  $T$  is the *stress energy tensor*

$$(1.24) \quad T(z) = \sum_n L_n z^{-n-2} \quad ([v_m, L_n] = m v_{m+n})$$

expressed in terms of an orthonormal basis  $v^i(z)$  of  $u(1)$  currents,  $(v^i|v^j) = \delta_{ij}$ , by the Sugawara formula

$$(1.25) \quad T(z) = \frac{1}{2} \sum_{i=1}^r : (v^i(z))^2 : ,$$

where the normal product can be thought as a limit

$$\lim_{z_1, z_2 \rightarrow z} \left\{ v^i(z_1) v^i(z_2) - \frac{1}{z_{12}^2} \right\}$$

(see comment following Eq. (1.36) below). The fusion rules for the  $\mathfrak{A}(L)$  vertex operators have the form

$$(1.26) \quad \lim_{z \rightarrow w} \{ (z-w)^{-(\omega|\omega')} Y(e^\omega, z) Y(e^{\omega'}, w) \} = \varepsilon(\omega, \omega') Y(e^{\omega+\omega'}, w);$$

the operator product expansion for oppositely charged fields can be written in more detail as

$$(1.27) \quad z_{12}^{|\omega|^2} Y(e^\omega, z_1) Y(e^{-\omega}, z_2) =: \exp \left\{ \int_{z_2}^{z_1} \omega(z) dz \right\} : , \quad z_{12} = z_1 - z_2 .$$

(The normal ordered exponential is defined in such a way that the  $n$ th term of its Taylor expansion is an integral over a single quasiprimary field of dimension  $n$  — cf. [FST]).

### 1C. Current chiral algebras associated to simple Lie algebras.

The CR between two currents  $Y(t^{-1}v^i, z_i)$ ,  $i = 1, 2$ , for two arbitrary elements  $v^1$  and  $v^2$  of  $\hat{\mathfrak{g}}$  are given by

$$(1.28) \quad [Y(t^{-1}v^1, z_1), Y(t^{-1}v^2, z_2)] Y(t^{-1}[v^1, v^2], z_2) \delta(z_{12}) - (v^1|v^2) \delta'(z_{12}) .$$

Here and further  $z_{12} = z_1 - z_2$ . We shall write down for later reference these relations for the Chevalley-Cartan basis of a simple component  $\mathfrak{g}^j$  of  $\mathfrak{g}$ . We shall set

$$(1.29) \quad (v_1|v_2)_k = k_j (v_1|v_2) \text{ for } v_1, v_2 \in \mathfrak{g}^j .$$

Restricting attention to a simple component we skip the index  $j$  on  $\mathfrak{g}$ . We choose a Cartan subalgebra  $\mathfrak{h}$  in  $\mathfrak{g}$  and a basis  $\alpha_i, i = 1, \dots, l$ , of *simple roots* in its dual thus introducing a standard partial order in  $\mathfrak{h}^*$ , which from now on we shall identify with  $\mathfrak{h}$  using the bilinear form  $(\cdot|\cdot)$ .

To each positive root  $\alpha > 0$  we associate a certain current representing the corresponding coroot  $\alpha^\vee$ :

$$(1.30) \quad H^\alpha(z) = \sum_n H_n^\alpha z^{-n-1}, H_0^\alpha = \alpha^\vee := \frac{2\alpha}{|\alpha|^2}, |\alpha|^2 = (\alpha|\alpha).$$

We shall use the positive integer *marks*  $a_i$  (and  $a_i^\vee$ ) of the Dynkin diagram of  $\mathfrak{g}$  which enter the expression for the *highest root*

$$(1.31) \quad \theta = \sum_{i=1}^l a_i \alpha_i = \sum_{i=1}^l a_i^\vee \alpha_i^\vee = \theta^\vee$$

(see [K1], Chap. 4, Tables). Their ratio relates the *Cartan matrix*  $a_{ij}$  of  $\mathfrak{g}$  to the symmetric Gram matrix of the coroots,

$$(\alpha_i^\vee | \alpha_j^\vee) = a_{ij} \frac{a_i}{a_j} \quad (a_{ij} = (\alpha_i^\vee | \alpha_j^\vee))$$

while the sum of check marks of the extended Dynkin diagram gives the dual Coxeter number

$$(1.32) \quad g^\vee = 1 + a_1^\vee + \dots + a_l^\vee \quad \left( \text{tr}(ad_{v_1} ad_{v_2}) = 2g^\vee (v_1 | v_2) \right).$$

The set of indices  $(j \in) J$  for which the exponentials  $e^{2\pi i \Lambda_j}$  of the corresponding fundamental weights  $\Lambda_j$  generate the center  $Z(G)$  of the simply connected group  $G$  with the Lie algebra  $\mathfrak{g}$  is given by

$$(1.33) \quad J = \{j = 1, \dots, l \mid a_j = 1\}.$$

Let  $E^\alpha$  be a raising or a lowering operator, depending on the sign of  $\alpha$ . Then the current CR (1.28) assume the form:

$$(1.34) \quad \begin{aligned} [H^\alpha(z_1), E^\beta(z_2)] &= (\alpha^\vee | \beta) E^\beta(z_2) \delta(z_{12}) \quad (\alpha, \beta \text{ roots}), \\ [H^\alpha(z_1), H^\beta(z_2)] &= k(\alpha^\vee | \beta^\vee) \delta'(z_{12}), \\ [E^{\alpha_i}(z_1), E^{-\alpha_j}(z_2)] &= 0 \text{ for } i \neq j, \\ [E^\alpha(z_1), E^{-\alpha}(z_2)] &= H^\alpha(z_2) \delta(z_{12}) - \frac{2k}{|\alpha|^2} \delta'(z_{12}). \end{aligned}$$

The affine chiral algebra  $\mathfrak{A}_k(\mathfrak{g})$  contains the *Sugawara stress energy tensor* (see e.g. [K2] Sect. 5.7.):

$$(1.35) \quad \begin{aligned} T(z) &= \frac{1}{2h} \left\{ \sum_{\alpha > 0} \frac{(\alpha|\alpha)}{2} : (E^\alpha(z) E^{-\alpha}(z) + E^{-\alpha}(z) E^\alpha(z)) : \right. \\ &\quad \left. + \sum_{i=1}^l : H_i(z) H^i(z) : \right\}, \quad h = k + g^\vee. \end{aligned}$$

Here  $H^i$  and  $H_i$  correspond to dual bases in the Cartan subalgebra:

$$(1.36) \quad H^i = \alpha_i^\vee, \quad H_i = \Lambda_i, \quad (\alpha_i^\vee | \Lambda_j) = \delta_{ij}.$$

The normal product  $::$  can be defined by either subtracting the singular in  $z_{12}$  part of an ordinary product  $J_a(z_1)J^a(z_2)$  or by ordering the frequency parts of the currents (inequivalent definitions of the normal product used in [FST] and [K2] yield the same expression for the stress energy tensor). Equations (1.34) (1.35) imply the Virasoro CR

$$(1.37) \quad [T(z_1), T(z_2)] = \delta(z_{12})\partial_2 T(z_2) + 2T(z_2)\partial_2 \delta(z_{12}) + \frac{c}{12}\partial_2^3 \delta(z_{12}) \left( \partial_2 \equiv \frac{\partial}{\partial z_2} \right)$$

where the Virasoro central charge exceeds the rank  $l$  of  $\mathfrak{g}$ . Denoting by  $d(\mathfrak{g})$  the dimension of  $\mathfrak{g}$ , we have

$$(1.38) \quad c = c_k(\mathfrak{g}) = \frac{k}{h}d(\mathfrak{g}) \geq l.$$

The positive integer  $h$  entering (1.35) and (1.38) (the sum of the level and the dual Coxeter number) is called the *height*. The last inequality in (1.38) follows from the fact, that  $T$  can be split into a sum of two commuting terms, the stress tensor  $T_H$  of the Cartan subalgebra and a remainder  $T_R$ :

$$(1.39) \quad T = T_H + T_R, \quad T_H(z) = \frac{1}{2k} \sum_{i=1}^l : H_i(z) H^i(z) :$$

We find, as a consequence of (1.34), (1.35) and (1.36)

$$(1.40a) \quad [T_H(z_1), H^i(z_2)] = \partial_2(\delta(z_{12})H^i(z_2)) = [T(z_1), H^i(z_2)]$$

and hence

$$(1.40b) \quad [T_R(z_1), H^i(z_2)] = 0 = [T_R(z_1), T_H(z_2)]$$

For a level  $k > 1$  simply laced (A-D-E) simple Lie algebra the RCFT with stress energy tensor  $T_R$  correspond to (generalized)  $G/H$  parafermions — see [KP0] and [Gep]. For a simply laced level 1  $\hat{\mathfrak{g}}$  we have  $c_1(\hat{\mathfrak{g}}) = l$  and hence  $T_R = 0$ .

Note that the lattice chiral algebra  $\mathfrak{A}(L)$  could also contain a level 1 simply laced current subalgebra. In fact, each even (integral) lattice  $L_r$  has a sublattice  $W_{r-\nu} \oplus L_\nu \subset L_r$  of the same dimension  $r$ . Here  $W_{r-\nu}$  is the root lattice of a direct sum of A-D-E (simple) Lie algebras, generated by vectors of length square 2, and  $L_\nu$  is its orthogonal complement (with no vector of length square 2), so that  $L/(W_{r-\nu} \oplus L_\nu)$  is a finite abelian group, the glue group.

The stress energy tensor  $T(z)$  of the chiral algebra  $\mathfrak{A}(G)$  is defined as the sum of the stress energy tensors of the factors of  $\mathfrak{A}(G)$ .

## 2. TWISTED MODULES OF A CURRENT CHIRAL ALGEBRA

### 2A. Positive energy irreducible $\mathfrak{A}(G)$ -modules.

Let  $\mathfrak{A}(G) = \mathfrak{A}(L) \otimes (\otimes_{j=1}^s \mathfrak{A}_{k_j}(\mathfrak{g}^j))$  be a current chiral algebra. Its positive energy irreducible modules are tensor products of such modules for each factor.

Let  $L^* = \{\mu \in \mathfrak{g}_0 \mid (\mu|\omega) \in \mathbb{Z} \text{ for all } \omega \in L\}$  be the dual lattice. It is easy to see that the positive energy irreducible modules over  $\mathfrak{A}(L)$  are labeled by the elements of the finite abelian group  $L^*/L$  as follows. Extend the cocycle  $\varepsilon(\omega_1, \omega_2)$  to  $L^*$  in such a way that (1.13) holds for  $\omega, \omega' \in L$  and  $\omega'' \in L^*$ . We choose a vector  $\mu$  of a coset of  $L^* \bmod L$ , and let

$$(2.1) \quad V_\mu(L) = \sum_{\omega \in \mu + L} S \otimes e^\omega .$$

Then Eqs. (1.18), (1.19) and (1.22) define an irreducible positive energy module over  $\mathfrak{A}(L)$ .

As a consequence of the Sugawara formula (1.25), the ground state energy  $\Delta(\mu)$  of the module  $V_\mu(L)$  is given by

$$(2.2) \quad \Delta(\mu) = \frac{(\mu|\mu)}{2}, \text{ if } \mu \text{ is a minimal length vector in } \mu + L .$$

Let  $\mathfrak{g}$  be the Lie algebra of a simple connected compact Lie group and let  $k$  be a non-negative integer. Then the integrable positive energy irreducible modules over  $\hat{\mathfrak{g}}$  of level  $k$  are labeled by the highest weight  $\Lambda$  of  $\mathfrak{g}$  in the lowest energy subspace (which is a finite-dimensional irreducible  $\mathfrak{g}$ -module). We denote these modules by  $V_\Lambda(\mathfrak{g}, k)$ . Recall that  $\Lambda$  then satisfies the integrability condition ([K1], Chap. 12):

$$(2.3) \quad (\Lambda|\alpha_i^\vee) \in \mathbb{Z}_+ \text{ for } i = 1, \dots, l, \quad (\Lambda|\theta) \leq k .$$

Each of the  $\hat{\mathfrak{g}}$ -modules  $V_\Lambda(\mathfrak{g}, k)$  extends to a  $\mathfrak{A}_k(\mathfrak{g})$ -module and all positive energy irreducible  $\mathfrak{A}_k(\mathfrak{g})$ -modules are obtained in this way [FZ].

As a consequence of Eq. (1.35), the ground state energy (= conformal dimension)  $\Delta(\Lambda)$  of the module  $V_\Lambda(\mathfrak{g}, k)$  is given by:

$$(2.4) \quad \Delta(\Lambda) = \frac{(\Lambda + 2\rho|\Lambda)}{2h}, \text{ where } h = k + g^\vee, 2\rho = \sum_{\alpha > 0} \alpha .$$

### 2B. $\mathbb{Z}_N$ -twisted current chiral algebra modules.

Let  $G^0$  be the connected compact Lie group whose maximal torus is  $U(1)^r = \mathbb{R}^r/L$ , i.e.  $L$  is the coroot lattice of  $G^0$ . ( $G^0$  contains the torus  $U(1)^r$  but can, in general, be larger due to the presence of  $\omega$ 's in  $L$  of length square 2; the semi-simple part of  $G^0$  is a product of simply laced compact simple Lie groups). Let

$$(2.5) \quad G_c = G^0 \times G^1 \times \dots \times G^s ,$$

the corresponding decomposition of Lie algebras being

$$(2.6) \quad \mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^s .$$

Let  $Z^j \subset G^j$  denote the center of  $G^j, j = 1, \dots, s$ , and let  $Z^0 = L^*/L$  ( $Z^0$  is central subgroup of  $G^0$ ). The following finite subgroup of  $G_c$  will play an important role in the sequel:

$$(2.7) \quad Z(G_c) = Z^0 \times Z^1 \times \cdots \times Z^s .$$

Recall (see (1.33)) that the center of a simple connected simply connected compact Lie group consists of 1 and the elements

$$(2.8) \quad \exp 2\pi i \Lambda_j , \text{ where } j \in J .$$

Recall that if  $Y(v_1, z) = \sum_{n \in \mathbb{Z}} Y_n(v_1) z^{-n-1}$  is a field of conformal dimension 1 of a chiral algebra, then  $Y_0(v_1)$  is a derivation of  $\mathfrak{A}$  and  $\exp Y_0(v_1)$  converge in any positive energy  $\mathfrak{A}$ -module (see e.g. [K2], Sect. 4.9.). Since such derivations of the chiral algebra  $\mathfrak{A}(G)$  form the Lie algebra  $\mathfrak{g}_{\mathbb{C}}$  (the complexification of  $\mathfrak{g}$ ), the group  $G_c$  acts on  $\mathfrak{A}(G)$  by automorphisms, and moreover, acts on each positive energy  $\mathfrak{A}(G)$ -module  $U$  in a consistent way (i.e.  $g(au) = g(a)g(u)$  for  $g \in G_c, a \in \mathfrak{A}(G), u \in U$ ) preserving the Hilbert metric.

It follows from the usual properties of the Casimir operator that the stress energy tensor  $T(z)$  is a  $G_c$ -invariant observable:

$$(2.9) \quad T(z) \in \mathfrak{A}(G)^{G_c} .$$

Now we recall briefly the notion of a twisted module  $U$  over a chiral algebra  $\mathfrak{A}$ . Let  $b$  be an automorphism of order  $N$  of  $\mathfrak{A}$ ; then we get a  $\mathbb{Z}/N\mathbb{Z}$ -grading

$$\mathfrak{A} = \bigoplus_{m \in \mathbb{Z}/N\mathbb{Z}} \mathfrak{A}_m ,$$

where  $\mathfrak{A}_m$  is the  $\exp 2\pi im/N$  eigenspace of  $b$ . A  $b$ -twisted  $\mathfrak{A}$ -module  $U$  is a linear map  $a \mapsto \pi(a)$  from  $\mathfrak{A}$  to the space of fields with values in  $\text{End } U$  such that the twisted Borcherds identity holds (see e.g. [KR2]), in particular all the CR are preserved and

$$(2.10) \quad e^{2\pi i L_0} \pi(a) e^{-2\pi i L_0} = (-1)^{p(a)} e^{\frac{2\pi im}{N}} \pi(a) \text{ for } a \in \mathfrak{A}_m .$$

If  $\mathfrak{A} = \mathfrak{A}_0$ , we get a usual (untwisted)  $\mathfrak{A}$ -module.

Returning to  $\mathfrak{A}(G)$ , fix  $\beta \in i\mathfrak{g}$  such that the corresponding element  $b = \exp 2\pi i \beta \in G_c$  has finite order  $N$  and choose a Cartan subalgebra of  $\mathfrak{g}$  containing  $i\beta$ .

Given a positive energy representation  $\pi$  of  $\mathfrak{A}(G)$  in a vector space  $U$ , we construct a  $b$ -twisted representation  $\pi_\beta$  in  $U$  as follows. First, due to the decomposition (1.10) of  $\mathfrak{A}(G)$  and the corresponding decomposition  $U = \bigotimes_{j=0}^s U^j$ ,

it suffices to construct the  $b_j$ -twisted representation  $\pi_{\beta^j}$  in  $U^j$  for each  $j$ , where  $\beta = \sum_j \beta^j$  is the decomposition (2.6) and  $b_j = \exp 2\pi i \beta^j$ .

Next, for a positive energy representation  $\pi$  of  $\mathfrak{A}(\mathfrak{g})$  we let

$$(2.11) \quad \pi_{\beta}(E^{\alpha}(z)) = \pi(E^{\alpha}(z))z^{-(\alpha|\beta)} = \sum_{n \in \mathbb{Z}} E_{n+(\alpha|\beta)}^{\alpha} z^{-n-(\alpha|\beta)-1},$$

and extend to the whole  $\mathfrak{A}(\mathfrak{g})$  using the twisted Borchers identity.

In order to preserve CR we should have

$$(2.12) \quad \pi_{\beta}(H^{\alpha}(z)) = \pi(H^{\alpha}(z)) - \frac{k(\alpha^{\vee}|\beta)}{z}.$$

Similarly, for a positive energy representation  $\pi$  of  $\mathfrak{A}(L)$  we let

$$(2.13) \quad \pi_{\beta}(Y(e^{\omega}, z)) = \pi(Y(e^{\omega}, z))z^{-(\omega|\beta)},$$

$$(2.14) \quad \pi_{\beta}(\omega(z)) = \pi(\omega(z)) - \frac{(\omega|\beta)}{z},$$

and extend to  $\mathfrak{A}(L)$  using the twisted Borchers identity.

The constructed  $b$ -twisted  $\mathfrak{A}(G)$ -module will be denoted by  $U^{(\beta)}$ .

Going to the stress tensor, which is a sum of a torus part,  $T_L$  (1.25), and a contribution of type (1.35), (1.39) for each simple factor in  $G$ , we shall see that only the Cartan part

$$(2.15) \quad T_{\mathfrak{h}} = T_L + T_H, \quad T_L = \frac{1}{2} \sum_{i=1}^r :v_i(z)^z:, \quad T_H = \frac{1}{2k} \sum_{j=1}^l :H_j H^j:(z)$$

changes following (2.12), (2.14) while the remainder  $T_R$  in (1.39) is left unchanged.

**Proposition 2.1.** *If we set*

$$(2.16a) \quad \tilde{T}_{\mathfrak{h}} = T_{\mathfrak{h}} - \frac{1}{2}\beta(z) + \frac{1}{2z^2}(\beta|\beta)_k, \quad \tilde{T}_R = T_R$$

*implying for the Laurent modes of  $\tilde{T}$*

$$(2.16b) \quad \tilde{L}_n = L_n - \beta_n + \frac{1}{2}(\beta|\beta)\delta_{n0},$$

*where  $(\beta|\beta)_k = k|\beta|^2$  for each simple component of (2.6), then  $\tilde{T}$  and  $\tilde{J}$  satisfy the same CR as  $T$  and  $J$  ( $J$  standing for any of the  $\mathfrak{g}$ -currents,  $H^{\alpha}, E^{\alpha}, v^i$ ) e.g.*

$$(2.17) \quad [\tilde{L}_n, \tilde{J}(z)] = \frac{d}{dz} \left( z^{n+1} \tilde{J}(z) \right).$$

*Proof.* It is straightforward to verify that the commutator of  $\tilde{L}_n$  with  $\tilde{E}^{\alpha} \equiv \pi_{\beta}(E^{\alpha})$  (2.11) reproduces (2.17). One further uses the fact that  $\pi_{\beta}$  defines a Lie algebra homomorphism on the currents, preserving their CR. The constant term in  $\tilde{L}_0$  is obtained by computing  $[\tilde{L}_1, \tilde{L}_{-1}]$ .  $\square$

### 3. TWISTED CHARACTERS AND MODULAR TRANSFORMATIONS

The *complete character* of a positive energy  $\mathfrak{A}(G)$ -module  $V$  is defined on the product of the upper half plane  $\tau$  and the group  $G$  as follows:

$$(3.1a) \quad \chi_V(\tau, z, u) = e^{2\pi i(k, u)} \text{tr}_V \left( q^{L_0 - \frac{c_k}{24}} e^{2\pi i z} \right) .$$

Here

$$(3.1b) \quad q = e^{2\pi i \tau} (|q| < 1), z \in i\mathfrak{g}, (k, u) = u_0 + \sum_{j=1}^s k_j u_j ,$$

$u_j$  are auxiliary (complex) parameters,  $L_0$  is the chiral energy operator (1.2), (1.4) (the zero mode of the stress energy tensor (1.24)),  $c_k$  is the Virasoro central charge (cf. (1.38)):

$$(3.1c) \quad c_k = r + \sum_{j=1}^s c_{k_j}(\mathfrak{g}^j) .$$

If  $V$  is irreducible then  $\chi_V$  splits into a product of Kac-Moody and lattice characters; we reproduce their expressions and transformation properties separately. This will allow us to write down the general orbifold characters.

#### 3A. Kac-Moody and lattice characters.

Let now  $G$  be a connected simply connected compact Lie group with a simple Lie algebra  $\mathfrak{g}$ . We shall use the following notation:  $M^*$  is the weight lattice dual to the co root lattice  $M$ ; the set of level  $k$  dominant weights is [K1]

$$(3.2) \quad P_+^k = \left\{ \Lambda \in M^* \mid (\Lambda | \alpha_i^\vee) \geq 0, i = 1, \dots, l; (\Lambda | \theta^\vee) \leq k \right\} ;$$

$Q \subset M^*$  is the root lattice; the quotient  $M^*/kM$  is a finite abelian group of order  $|M^*/kM| = k^l |M^*/M|$ . The values of  $|M^*/M|$  may be found e.g. in [KW] (in the simply laced case  $|M^*/M|$  is the determinant of the Cartan matrix). The Kac-Moody character  $\chi_\Lambda(\tau, z, u) \equiv \chi_{V_\Lambda(\mathfrak{g}, k)}(\tau, z, u)$  can be expressed in terms of classical  $\Theta$  functions of weight  $l/2$  and certain almost holomorphic modular forms  $c_\lambda^\Lambda$ , the *string functions*, of opposite weight ([K1], Eq. (12.7.12)):

$$(3.3a) \quad \chi_\Lambda(\tau, z, u) = \sum_{\substack{\lambda \in M^*/kM \\ \Lambda - \lambda \in Q}} c_\lambda^\Lambda(\tau) \Theta_{\lambda k}^M(\tau, z, u) ,$$

$$(3.3b) \quad \Theta_{\lambda k}^M(\tau, z, u) = e^{2\pi i k u} \sum_{\gamma \in M + \frac{\lambda}{k}} q^{\frac{k}{2}(\gamma|\gamma)} e^{2\pi i k(\gamma|z)} .$$

We assume here that  $iz$  is an element of  $\mathfrak{g}$  and choose a Cartan subalgebra containing  $iz$ .

The modular transformation law for  $\Theta$  is given by (see Theorem 13.5 of [K1]):

$$(3.4) \quad \begin{aligned} S &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \Theta_{\lambda k}^M(\tau, z, u) \rightarrow \Theta_{\lambda k}^M\left(-\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{(z|z)}{2\tau}\right) \\ &= (-i\tau)^{l/2} |M^*/kM|^{-1/2} \sum_{\lambda' \in M^*/kM} e^{-2\pi i \frac{(\lambda|\lambda')}{k}} \Theta_{\lambda' k}^M(\tau, z, u), \end{aligned}$$

$$(3.5) \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \Theta_{\lambda k}^M(\tau, z, u) \rightarrow \Theta_{\lambda k}^M(\tau + 1, z, u) = e^{i\pi \frac{(\lambda|\lambda)}{k}} \Theta_{\lambda k}^M(\tau, z, u).$$

The characters  $\chi_\Lambda$  span a finite dimensional representation of  $SL(2, \mathbb{Z})$  as well (see [KP] or Theorem 13.8 of [K1]):

$$(3.6) \quad \chi_\Lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{(z|z)}{2\tau}\right) = \sum_{\Lambda' \in P_+^k} S_{\Lambda\Lambda'} \chi_{\Lambda'}(\tau, z, u);$$

here the  $S_{\Lambda\Lambda'}$  are given by the Kac-Peterson formula:

$$(3.7) \quad S_{\Lambda\Lambda'} = i^{|\Delta_+|} |M^*/kM|^{-1/2} \sum_{w \in W(\mathfrak{g})} \varepsilon(w) \exp\left\{-2\pi i \frac{(\Lambda + \rho|w(\Lambda' + \rho))}{h}\right\},$$

where  $|\Delta_+|$  is the number of positive roots,  $W(\mathfrak{g})$  is the Weyl group of  $\mathfrak{g}$ ,  $\varepsilon(w) = \pm 1$  according to the parity of  $w$ ,  $2\rho$  and  $h$  are defined in (2.4),

$$(3.8) \quad \chi_\Lambda(\tau + 1, z, u) = e^{2\pi i m_\Lambda} \chi_\Lambda(\tau, z, u),$$

$$(3.9) \quad m_\Lambda = \Delta(\Lambda) - \frac{c_k(\mathfrak{g})}{24}$$

where  $\Delta(\Lambda)$  is the conformal dimension (2.4),  $c_k(\mathfrak{g})$  is the Virasoro central charge (1.38).

In the special case of  $\mathfrak{g} = \mathfrak{su}(2)$  we have

$$(3.10a) \quad S_{\lambda\lambda'} = \sqrt{2/h} \sin \pi \frac{(\lambda+1)(\lambda'+1)}{h}, \quad h = k+2,$$

$$(3.10b) \quad m_\lambda = \frac{\lambda(\lambda+2)}{4h} - \frac{c}{24}, \quad c = c_k(\mathfrak{su}(2)) = \frac{3k}{h}.$$

Note that for a simply laced affine algebra at level 1 (so that  $c = l$ ) there is only one non-zero string function, which is a negative power of the Dedekind  $\eta$ -function:  $c_\Lambda^\lambda(\tau)|_{k=1} = (\eta(\tau))^{-l}$ . Recall the transformation properties of the  $\eta$ -function:

$$(3.11) \quad \eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2} \eta(\tau), \quad \eta(\tau + 1) = e^{\pi i/12} \eta(\tau).$$

The matrix  $S$  simplifies in this case as it coincides with that for the lattice characters (see (3.14) below).

It is clear from the construction that the lattice character  $\chi_\mu$  of the module  $V_\mu(L)$  is given by

$$(3.12) \quad \chi_\mu(\tau, z, u) = (\eta(\tau))^{-r} \Theta_{\mu 1}^L(\tau, z, u) .$$

Here, as before,  $z$  is an element of  $\mathfrak{g}^0$  and we choose a Cartan subalgebra containing  $z$ . (The expression (3.12) has, of course, the same form as the level 1 simply laced Kac-Moody character; it coincides with (3.3), (3.11) for  $L = M, r = l$ ). The modular transformation law for  $\chi_\mu$  can be read off (3.4), (3.5) and (3.11) (the expression for  $S$  in the counterpart of (3.6) being simpler than (3.7)):

$$(3.13) \quad \chi_\mu \left( -\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{1}{2\tau}|z|^2 \right) = \sum_{\mu' \in L^*/L} S_{\mu\mu'} \chi_\mu(\tau, z, u)$$

where

$$(3.14) \quad S_{\mu\mu'} = |L^*/L|^{-1/2} e^{-2\pi i(\mu|\mu')} ;$$

$$(3.15)$$

$$\chi_\mu(\tau + 1, z, u) = e^{2\pi i m_\mu} \chi_\mu(\tau, z, u) , \quad m_\mu = \Delta(\mu) - \frac{r}{24} , \quad \Delta(\mu) = \frac{1}{2}|\mu|^2 .$$

As mentioned above, an irreducible positive energy  $\mathfrak{A}(G)$ -module  $V$  is the tensor product of the  $\mathfrak{A}(L)$ -module  $V_\mu(L)$  and  $\mathfrak{A}(\mathfrak{g}^j)$ -modules  $V_{\Lambda^j}(\mathfrak{g}^j)$ . Hence positive energy irreducible  $\mathfrak{A}(G)$ -modules are parameterized by the set

$$P_+^k = (L^*/L) \times P_+^{k_1} \times \cdots \times P_+^{k_s} .$$

We let  $\mu = \Lambda^0$ , call  $\Lambda = \sum_{j=0}^s \Lambda^j$  the *highest weight* of  $V$ , and write  $V = V_\Lambda$ . The character of  $V_\Lambda$ ,  $\Lambda \in P_+^k$ , is the product

$$(3.16) \quad \chi_\Lambda(\tau, z, u) = \prod_{j=0}^s \chi_{\Lambda^j}(\tau, z^j, u^j) .$$

### 3B. Modular transformations of twisted characters.

Recall that the affine chiral algebra  $\mathfrak{A}(G)$  is defined by the data consisting of a compact group  $G$  and an invariant bilinear form on its Lie algebra  $\mathfrak{g}$ . This invariant bilinear form looks as follows:

$$(3.17a) \quad (x|y)_k \equiv \sum_{j=0}^r k_j (x^j|y^j) .$$

We will also use the normalized invariant bilinear form

$$(3.17b) \quad (x|y) = \sum_{j=0}^r (x^j|y^j) .$$

Let now  $\beta \in i\mathfrak{g}$  be such that  $b = \exp 2\pi i\beta \in G$  has finite order and choose a Cartan subalgebra of  $\mathfrak{g}$  containing  $i\beta$ . It follows from (2.16b) and (3.1) that the value of the character of the  $b$ -twisted  $\mathfrak{A}(G)$ -module  $V_\Lambda^{(\beta)}$  at  $e^{2\pi i\alpha} \in G$  is given by the following formula:

$$(3.18) \quad \begin{aligned} \chi_\Lambda^{\alpha,\beta}(\tau) &\equiv \text{tr}_{V_\Lambda} q^{L_0 - \beta + \frac{1}{2}(\beta|\beta)_k - \frac{c_k}{24}} e^{2\pi i\alpha} \\ &= e^{\pi i(\alpha|\beta)_k} \chi_\Lambda \left( \tau, \alpha - \tau\beta, -\frac{1}{2}(\alpha - \tau\beta|\beta) \right). \end{aligned}$$

Each factor in (3.18) can be written in a similar form for the Kac-Moody and the lattice case (assuming that  $\alpha$  and  $\beta$  lie in the same Cartan subalgebra):

$$(3.19) \quad \chi_\Lambda^{\alpha,\beta}(\tau) = \sum_{\substack{\lambda \in M^*/kM \\ \Lambda - \lambda \in Q}} c_\lambda^\Lambda(\tau) \Theta_{\lambda k}^{\alpha,\beta}(\tau)$$

$$(3.20) \quad \chi_\mu^{\alpha,\beta}(\tau) = [\eta(\tau)]^{-r} \Theta_{\mu 1}^{\alpha,\beta}(\tau)$$

where in both cases

$$(3.21) \quad \Theta_{\lambda k}^{\alpha,\beta}(\tau) = e^{i\pi k(\alpha|\beta)} \Theta_{\lambda k}^M \left( \tau, \alpha - \beta\tau, \frac{1}{2}(\beta\tau - \alpha|\beta) \right) = \sum_{\gamma \in M + \frac{\lambda}{k}} q^{\frac{k}{2}|\gamma - \beta|^2} e^{2\pi i k(\gamma|\alpha)}$$

(We can read off the lattice  $\Theta$ -function from (3.21) setting  $M = Q = L$ ,  $\lambda = \mu$ ,  $k = 1$ ).

The modular transformation law for twisted characters is deduced from the known transformation properties of Kac-Moody and lattice characters (3.6–3.9) and (3.13–3.15) using the following lemma (cf. [KP2] and [K1]).

**Lemma 3.1.** *Let the finite set of functions  $\{F_i(\tau, z, u), i \in I\}$  be closed under modular transformations:*

$$(3.22) \quad F_i \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, u - \frac{c|z|^2}{2(c\tau + d)} \right) = \sum_{j \in I} A_{ij} F_j(\tau, z, u), \quad A_{ij} \in \mathbb{C},$$

for  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ . Define

$$(3.23) \quad F_i^{\alpha,\beta}(\tau) = F_i \left( \tau, \alpha - \tau\beta, -\frac{1}{2}(\alpha - \tau\beta|\beta) \right).$$

Then

$$(3.24) \quad F_i^{\alpha,\beta} \left( \frac{a\tau + b}{c\tau + d} \right) = \sum_{j \in I} A_{ij} F_j^{d\alpha - b\beta, a\beta - c\alpha}(\tau).$$

*Proof.* If we set

$$\alpha - \beta \frac{a\tau + b}{c\tau + d} = \frac{\tilde{z}}{c\tau + d}, \text{ with } \tilde{z} = d\alpha - b\beta - (a\beta - c\alpha)\tau$$

then

$$F_i^{\alpha, \beta} \left( \frac{a\tau + b}{c\tau + d} \right) = F_i \left( \frac{a\tau + b}{c\tau + d}, \frac{\tilde{z}}{c\tau + d}, \tilde{u} - \frac{c(\tilde{z}|\tilde{z})}{2(c\tau + d)} \right)$$

where  $\tilde{u} = \frac{1}{2}(z|c\alpha - a\beta)$ . The law (3.24) then follows from (3.22).  $\square$

It is now straightforward to apply Lemma 3.1 to (3.18) to find the transformation formula of twisted  $\mathfrak{A}(G)$ -characters  $\chi_\Lambda^{\alpha, \beta}$  using the transformation formula for complete characters from the previous section. Introduce the following notation:

$$(3.25a) \quad S_{\Lambda, \Lambda'} = \prod_{j=0}^s S_{\Lambda_j, \Lambda'^j},$$

$$(3.25b) \quad m_\Lambda = \sum_{j=0}^s m_{\Lambda_j},$$

where the  $S_{\Lambda_j, \Lambda'^j}$  are given by (3.7) and (3.14) and the  $m_{\Lambda_j}$  are given by (3.9) and (3.15). Then we have

$$(3.26) \quad \chi_\Lambda^{\alpha, \beta} \left( -\frac{1}{\tau} \right) = e^{2\pi i(\alpha|\beta)_k} \sum_{\Lambda'} S_{\Lambda \Lambda'} \chi_{\Lambda'}^{\beta, -\alpha}(\tau),$$

$$(3.27) \quad \chi_\Lambda^{\alpha, \beta}(\tau + 1) = e^{2\pi i(m_\Lambda + \frac{1}{2}(\beta|\beta)_k)} \chi_\Lambda^{\alpha - \beta, \beta}(\tau).$$

### 3C. Small $\tau$ asymptotics of twisted characters of $\mathfrak{A}(G)$ .

The small  $\tau$  asymptotics will be used in the sequel for singling out non-trivial orbifold modules. Since the parameter  $\beta = \frac{2\pi}{i}\tau$  (which has a positive real part) can be interpreted as inverse temperature, the small  $\tau$  asymptotics can be interpreted as the high temperature behavior.

**Lemma 3.2.** (a) *The  $q$ -expansions of  $\Theta_{\lambda k}^{\alpha, \beta}(\tau)$ ,  $q^{c_k/24} c_\lambda^\Lambda(\tau)$  and  $q^{c_k/24} \chi_\Lambda^{\alpha, \beta}(\tau)$  involve only non-negative powers of  $q$ .*

- (b) *The  $q$ -expansion of  $\Theta_{\lambda k}^{\alpha, \beta}(\tau)$  has a non-zero constant term iff  $\lambda - k\beta \in kM$ . This constant term equals  $e^{2\pi i(\alpha|\beta)_k}$ .*
- (c) *The  $q$ -expansion of  $q^{c_k/24} c_\lambda^\Lambda(\tau)$  has a non-zero constant term iff  $\Lambda = k\Lambda_j$  with  $j \in J$  (see (1.33)) or  $\Lambda = 0$ , and  $\lambda - \Lambda \in kM$ . This constant term equals 1. (Recall that  $\Lambda_j$  are fundamental weights).*
- (d) *The  $q$ -expansion of  $q^{c_k/24} \chi_\Lambda^{\alpha, \beta}(\tau)$  has a non-zero constant term iff  $\Lambda = k\Lambda_j$  with  $j \in J$  or  $\Lambda = 0$ , and  $\Lambda - k\beta \in kM$ . This constant term equal  $e^{2\pi i(\alpha|\beta)_k}$ .*

*Proof.* (a) and (b) are clear. (c) is proved in [KW]. (d) follows from (b) and (c) by making use of (3.19).  $\square$

The modular inversion  $S$  relates low temperature to high temperature behavior and is a key to computing small  $\tau$  asymptotics.

By Lemma 3.2(a) and (d) each term in the expansion of  $e^{-\frac{\pi ic_k}{12\tau}} \chi_\Lambda^{\alpha, \beta} \left(-\frac{1}{\tau}\right)$  vanishes exponentially for  $\tau \downarrow 0$  unless  $\Lambda = k\Lambda_j$  with  $j \in J$  or  $\Lambda = 0$ , and  $\Lambda - k\beta \in kM$ , hence, by Lemma 3.2(d):

$$(3.28) \quad \lim_{\tau \downarrow 0} e^{-\frac{\pi ic_k}{12\tau}} \chi_\Lambda^{\alpha, \beta} \left(-\frac{1}{\tau}\right) = \begin{cases} e^{2\pi i(\alpha|\beta)_k} & \text{for } \Lambda = k\Lambda_j, j \in J, \text{ or } \Lambda = 0, \text{ and } \Lambda - k\beta \in kM, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have:

$$(3.29) \quad \lim_{\tau \downarrow 0} e^{-\frac{\pi ic_k}{12\tau}} \chi_\mu^{\alpha, \beta} \left(-\frac{1}{\tau}\right) = \begin{cases} e^{2\pi i(\alpha|\beta)} & \text{for } \beta - \mu \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting  $\tau$  by  $-\frac{1}{\tau}$  in (3.26):

$$\chi_\Lambda^{\alpha, \beta}(\tau) = e^{2\pi i(\alpha|\beta)_k} \sum_{\Lambda'} S_{\Lambda\Lambda'} \chi_{\Lambda'}^{\beta, -\alpha} \left(-\frac{1}{\tau}\right),$$

we find, using (3.28) and (3.29), that  $e^{-\frac{\pi ic_k}{12\tau}} \chi_\Lambda^{\alpha, \beta}(\tau) \sim |L^*/L|^{-1/2} e^{2\pi i(\Lambda^0|\alpha)} \prod_{j=1}^s S_{\Lambda^j, \gamma_j}$ , as  $\tau \downarrow 0$ , where  $\gamma_j = k_j \Lambda_i$  with  $i \in J^j$  or  $\gamma_j = 0$ , if  $\alpha + \sum_{j=1}^s \gamma_j \in L \oplus (\oplus_{j=1}^s M^j)$ , and tends to 0 otherwise. Recalling that [KW]

$$(3.30) \quad S_{\Lambda, k\Lambda_j} = S_{\Lambda, 0} e^{-2\pi i(\Lambda|\Lambda_j)} \text{ if } j \in J,$$

we arrive at the following result.

**Proposition 3.1.** *The high temperature asymptotics of twisted  $\mathfrak{A}(G)$  characters is given by*

$$\lim_{\tau \downarrow 0} e^{-\frac{\pi ic_k}{12\tau}} \chi_\Lambda^{\alpha, \beta}(\tau) = \begin{cases} S_{\Lambda, 0} e^{2\pi i(\Lambda|\beta)_k} & \text{if } \exp 2\pi i\alpha \in Z(G) \\ 0 & \text{otherwise.} \end{cases}$$

Here  $Z(G_c)$  is the finite central subgroup of  $G_c$  defined by (2.7) and we use (2.8).

#### 4A. Projection on a centralizer's irreducible representation. Asymptotic dimension.

Let as before  $\beta \in i\mathfrak{g}$  be such that  $b = \exp 2\pi i\beta \in G_c$  has finite order. Given a positive energy  $\mathfrak{A}(G)$ -module  $U$ , we have the  $b$ -twisted module  $U^{(\beta)}$  constructed in Sect. 2B. Consider the chiral subalgebra  $\mathfrak{A}(G)^b$  of fixed elements of  $\mathfrak{A}(G)$  with respect to  $Ad_b$ . When restricted to  $\mathfrak{A}(G)^b$ ,  $U^{(\beta)}$  becomes an untwisted  $\mathfrak{A}(G)^b$ -module. This simple, but important observation allows one to construct in many cases all untwisted modules of a chiral algebra (see e.g. [KR2]).

We shall use in the sequel the following orthogonality relations of irreducible characters of a finite group  $\Gamma$  :

$$(4.1) \quad \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \sigma^*(h) \sigma'(hg) = \frac{\sigma(g)}{\sigma(1)} \delta_{\sigma, \sigma'} , \quad \sigma, \sigma' \in \hat{\Gamma} ,$$

$$(4.2) \quad \frac{1}{|\Gamma_g|} \sum_{\sigma \in \hat{\Gamma}} \sigma^*(g) \sigma(h) = \delta_{\bar{g}, \bar{h}} , \quad g, h \in \Gamma .$$

Here and further  $\hat{\Gamma}$  denotes the set of all irreducible characters (= representations) of  $\Gamma$ ,  $\sigma^*$  stands for the complex conjugate character,  $\Gamma_g$  stands for the centralizer of  $g \in \Gamma$ . We shall also denote by  $\bar{g}$  the conjugacy class of  $g$  in  $\Gamma$ . Recall that  $|\Gamma| = |\Gamma_g| |\bar{g}|$ .

Let  $\Gamma$  be a finite subgroup of the compact group  $G_c$ . We shall consider  $\Gamma$  as the *gauge group* of our CFT and define the chiral subalgebra  $\mathfrak{A}^\Gamma$  of gauge invariant observables as the set of  $Ad_\Gamma$ -invariant elements of  $\mathfrak{A}(G)$ . This is called an *orbifold chiral algebra*. One can ensure that  $\mathfrak{A}^\Gamma$  only contains local Bose fields (even when  $\mathfrak{A}(L)$  involves fermionic vertex operators) replacing  $L$  by  $L_{\text{even}}$  (the maximal even sublattice of  $L$ ) and  $\Gamma$  by its extension by  $L/L_{\text{even}}$ . It will be the objective of this section to construct a set of positive energy representations of  $\mathfrak{A}^\Gamma$  which again give rise to an RCFT. That will be demonstrated in the next section by displaying the  $SL(2, \mathbb{Z})$  properties of their characters. (This is, in general, not the case if the subgroup  $\Gamma$  of  $G$  is infinite.) The  $\mathfrak{A}^\Gamma$ -modules in question are obtained by splitting the twisted  $\mathfrak{A}(G)$ -modules into  $\mathfrak{A}^\Gamma$ -invariant parts.

*Remark 4.1.* It is clear that  $\mathfrak{A}^\Gamma = \mathfrak{A}^{\Gamma Z(G_c)}$  where  $\Gamma Z(G_c)$  is the finite subgroup of  $G_c$  generated by  $\Gamma$  and  $Z(G_c)$ . Hence the orbifold model does not change if we enlarge  $\Gamma$  by the central group  $Z(G_c)$  and in principle we may assume that  $\Gamma$  contains  $Z(G_c)$  (but we shall not do that).

Pick  $b \in \Gamma$  and write it in the form  $b = \exp 2\pi i\beta$ , where  $i\beta \in \mathfrak{g}$ . Let  $\Gamma_\beta$  be the stabilizer of  $\beta$  in  $\Gamma$  with respect to the adjoint action of  $\Gamma$  on  $\mathfrak{g}$ . Then the twisted  $\mathfrak{A}(G)$ -module  $U^{(\beta)}$  becomes untwisted with respect to the chiral subalgebra  $\mathfrak{A}(G)^{\Gamma_\beta}$  of fixed elements with respect to  $\Gamma_\beta$ . It follows from the construction that the group  $\Gamma_\beta$  acts on  $U^{(\beta)}$ .

Let  $\sigma$  be an irreducible character of the group  $\Gamma_\beta$ . It follows from (4.1) that

the projector on the  $\sigma$ -isotypic component of a representation of  $\Gamma_\beta$  is given by

$$(4.3) \quad P_\sigma = \frac{\sigma(1)}{|\Gamma_\beta|} \sum_{h \in \Gamma_\beta} \sigma^*(h)h .$$

The subspace  $P_\sigma U^{(\beta)}$  is irreducible with respect to the pair  $(\Gamma_\beta, \mathfrak{A}(G)^{\Gamma_\beta})$ . This can be proved in the same way as Theorem 1.1 from [KR2]. It follows that the  $\mathfrak{A}(G)^{\Gamma_\beta}$ -module  $P_\sigma U^{(\beta)}$  is isomorphic to the sum of  $\sigma(1)$  copies of an irreducible module which we denote by  $U_\sigma^{(\beta)}$ .

Since the affine orbifold  $\mathfrak{A}(G)^\Gamma$  is contained in  $\mathfrak{A}(G)^{\Gamma_\beta}$ , we obtain a  $\mathfrak{A}(G)^\Gamma$ -module  $U_\sigma^{(\beta)}$  by restriction. Take now  $U = V_\Lambda$ . It follows from (3.18) and (4.3) that the character  $\chi_{\Lambda, \sigma}^\beta = \text{tr} q^{L_0 - \beta + \frac{1}{2}(\beta|\beta)_\kappa - \frac{c_k}{24}}$  of the  $\mathfrak{A}(G)^\Gamma$ -module  $V_{\Lambda, \sigma}^{(\beta)}$  is given by

$$(4.4) \quad \chi_{\Lambda, \sigma}^\beta(\tau) = \frac{1}{|\Gamma_\beta|} \sum_{\substack{h \in \Gamma_\beta \\ h = e^{2\pi i \alpha}, [\alpha, \beta] = 0}} \sigma^*(h) \chi_\Lambda^{\alpha, \beta}(\tau) .$$

Applying the orthogonality relation (4.2), we can invert (4.4):

$$(4.5) \quad \chi_\Lambda^{\alpha, \beta}(\tau) = \sum_{h \in \hat{\Gamma}_\beta} \sigma(h) \chi_{\Lambda, \sigma}^\beta(\tau) \text{ for } h = e^{2\pi i \alpha} .$$

Let  $Z = \Gamma \cap Z(G_c)$  denote the *small center* of the subgroup  $\Gamma$  of  $G_c$ .

**Theorem 4.1.** *The orbifold character  $\chi_{\Lambda, \sigma}^\beta(\tau)$  is nontrivial iff  $\Lambda$  and  $\sigma$  agree on  $Z$ :*

$$(4.6) \quad \Lambda|_Z = \sigma|_Z .$$

*Provided that (4.6) holds, one has:*

$$(4.7) \quad \lim_{\tau \downarrow 0} e^{-\frac{\pi ic}{12\tau}} \chi_{\Lambda, \sigma}^\beta(\tau) = S_{\Lambda, 0} \sigma(1) |Z| / |\Gamma_\beta| .$$

*Proof.* It is clear from the construction that  $V_{\Lambda, \sigma}^{(\beta)} = 0$  if (4.6) fails. Furthermore, by Proposition 3.1 and (4.4) we have:

$$\lim_{\tau \downarrow 0} e^{-\frac{\pi ic}{12\tau}} \chi_{\Lambda, \sigma}^\beta(\tau) = \frac{S_{\Lambda, 0}}{|\Gamma_\beta|} \sum_{\substack{h \in Z \\ h = e^{2\pi i \alpha}}} \sigma^*(h) e^{2\pi i(\Lambda|\alpha)_\kappa} .$$

It follows from the orthogonality (4.1) of characters of the group  $Z$  that this is zero unless (4.6) holds, in which case it is given by the right-hand side of (4.7). The latter is positive since  $S_{\Lambda, 0}$  is a positive real number (see discussion below).  $\square$

An important characteristic of a chiral algebra module  $V$  is its *asymptotic dimension* [KP2] [KW] and Sect. 13.13 of [K1]. It is defined as the coefficient  $a(V)$  of the leading term of the small  $\tau$  (or high temperature) expansion of the specialized character  $\chi_V$ :

$$(4.8) \quad \chi_V(\tau) = \text{tr}_V q^{(L_0 - \frac{c}{24})} \approx a(V) e^{\frac{\pi ic}{12\tau}} .$$

For example Theorem 4.1 states that the asymptotic dimension of the orbifold module  $V_{\Lambda, \sigma}^{(\beta)}$  is given by the right hand side of (4.7) provided that condition (4.6) holds. The positive reals  $a(V)$  have multifold interpretations. If  $\mathfrak{A}(V_1) \subset \mathfrak{A}(V_2)$  are two chiral algebras (with  $V_1 \subset V_2$ ) then  $a(V_2)/a(V_1)$  gives the *index of embedding* of the associated von Neumann algebras (see [R], [LR] and [RST] and references therein). If  $V_{\Lambda k}$  is an affine algebra module and  $V_{0k}$  the corresponding vacuum module of height  $h$  then  $a_h(\Lambda)/a_h(0)$  is the “quantum dimension” of  $V_{\Lambda}$  [V]. In the case at hand the knowledge of  $a(V)$  appears as an efficient tool for singling out non-trivial orbifold modules, and, as we shall see, for handling the splitting of reducible modules into irreducible components.

An  $\mathfrak{A}(G)$ -module  $V_{\Lambda}$  appears as an outer product of representation of the chiral algebras  $\mathfrak{A}(L)$  and  $\mathfrak{A}_{k_j}(\mathfrak{g}^j)$ . (We use the term *outer tensor product* to be distinguished from the tensor product of representations of a group  $G$  that is again regarded as a representation of  $G$  rather than as a representation of the direct product  $G \times G$ .)

The asymptotic dimension of an outer product of representations obviously equals the product of asymptotic dimensions of factors. Hence the asymptotic dimension  $a(V_{\Lambda})$  of a  $\mathfrak{A}(G)$ -module  $V_{\Lambda}$  is equal to the product of  $a(V_{\mu}(L))$  and  $a(V_{\Lambda^j}(\mathfrak{g}^j))$ ,  $j = 1, \dots, s$ . The asymptotic dimension of lattice modules is independent of  $\mu$ :

$$(4.9) \quad a(V_{\mu}(L)) = S_{\mu,0} = |L^*/L|^{-\frac{1}{2}} .$$

The asymptotic dimension of Kac-Moody modules is given by (see [KP],[KW] or [K1](13.8.10)):

$$(4.10) \quad a(V_{\Lambda}(\mathfrak{g})) = S_{\Lambda,0} = |M^*/hM|^{-\frac{1}{2}} \prod_{\alpha > 0} 2 \sin \frac{\pi(\Lambda + \rho|\alpha)}{h} .$$

This number is positive since  $(\Lambda|\alpha) \leq k$  and  $(\rho|\alpha) \leq (\rho|\theta) = g^{\vee} - 1$ , so that  $(\Lambda + \rho|\alpha) < h = k + g^{\vee}$ .

#### 4B. Affine orbifold models for non-exceptional $\Gamma$ . Action of $Z$ . Modular transformations.

In order to construct a modular invariant family of  $\Gamma$ -orbifold modules we need to impose some restrictions on the subgroup  $\Gamma$  of  $G_c$ . Let  $Z$  be the small center of  $\Gamma$ .

**Definition 4.1.** *An element  $b \in \Gamma$  is called non-exceptional if there exists  $\beta(b) \in i\mathfrak{g}$  such that  $b = \exp 2\pi i\beta(b)$  and  $\Gamma_b = \Gamma_{\beta}$ . The subgroup  $\Gamma$  of the compact group*

$G_c$  is called a non-exceptional subgroup if for any  $g \in \Gamma$  there exists  $\zeta \in Z$  such that  $\zeta g$  is a non-exceptional element of  $\Gamma$ .

An element  $g \in G_c$  is called *Ad-exceptional* element of  $G_c$  if it cannot be written in the form  $g = b\zeta$ , where  $b$  is a non-exceptional element of  $G_c$  and  $\zeta \in Z(G)$ . Obviously, a subgroup  $\Gamma$  of  $G_c$  containing  $Z(G_c)$  (recall that, due to Remark 4.1, we may assume that  $\Gamma \supset Z(G_c)$ ) which does not contain Ad-exceptional elements of  $G_c$  is a non-exceptional subgroup of  $G_c$ . We shall describe Ad-exceptional elements of a compact group  $G$  in Appendix B. Here we only note that  $U(n)$  contains no exceptional elements and  $SU(n)$  contains no Ad-exceptional elements. Any connected simple compact Lie group other than  $SU(n)$  does contain Ad-exceptional elements.

From now on let  $\Gamma$  be a non-exceptional finite subgroup of the compact Lie group  $G_c$ .

It follows from the definition that for each  $g \in \Gamma$  there exists a  $\zeta \in Z$  such that  $b = \zeta^{-1}g$  is non-exceptional. Moreover for each  $g$  of a conjugacy class  $\bar{g}$  we can choose the same  $\zeta \in Z$  and a map  $\beta : \bar{b} \rightarrow i\mathfrak{g}$  satisfying

$$(4.11) \quad b = e^{2\pi i\beta(b)}, \quad \beta(hbh^{-1}) = Ad_h\beta(b) \quad \text{for all } b \in \bar{b}, h \in \Gamma.$$

Note that a choice of  $\beta(b)$  such that  $\Gamma_b = \Gamma_{\beta(b)}$ , determines uniquely the map  $\beta$  satisfying (4.11).

A quadruple  $(\Lambda, b, \beta, \sigma)$  where  $\Lambda \in P_+^k$ ,  $b$  is a non-exceptional element of  $\Gamma$ ,  $\beta$  is a map satisfying (4.11) and  $\sigma \in \hat{\Gamma}_b$  is called an *admissible quadruple* if the compatibility condition (4.6) holds. Due to Theorem 4.1 the  $\mathfrak{A}^\Gamma$ -module  $V_{\Lambda, \sigma}^{(\beta(b))}$  is nontrivial for any admissible quadruple  $(\Lambda, b, \beta, \sigma)$ ; we shall denote it by  $V_{\Lambda, b, \sigma}^\beta$ . We have for any  $g \in \Gamma$  the identity

$$(4.12a) \quad V_{\Lambda, gbg^{-1}, \sigma^g}^{Ad_g\beta} = V_{\Lambda, b, \sigma}^\beta,$$

where  $\sigma^g \in \hat{\Gamma}_{gbg^{-1}}$  is defined by

$$(4.12b) \quad \sigma^g(h) = \sigma(g^{-1}hg).$$

We thus obtain the first equivalence of admissible quadruples:

$$(4.13) \quad (\Lambda, b, \beta, \sigma) \sim (\Lambda, gbg^{-1}, Ad_g\beta, \sigma^g).$$

Recalling that (4.11) defines a map  $\beta : \bar{b} \rightarrow i\mathfrak{g}$  and dropping the superscript  $g$  on  $\sigma$  we may denote the character of the module (4.12) by  $\chi_{\Lambda, \bar{b}, \sigma}^\beta$ . Furthermore, if  $\beta(b)$  is replaced by  $\beta(b) + m$  where

$$(4.14) \quad e^{2\pi im} = 1, \quad [\beta(b), m] = 0, \quad \Gamma_{\beta(b)+m} = \Gamma_b,$$

then

$$(4.15a) \quad V_{\Lambda, b, \sigma \otimes \sigma_m}^{\beta+m} = V_{\Lambda, b, \sigma}^\beta,$$

where  $\sigma_m$  is a 1-dimensional representation of  $\Gamma_b$  defined by

$$(4.15b) \quad \sigma_m(h) = e^{2\pi i(m|\alpha)_k} \quad \text{for } h = e^{2\pi i\alpha} \in \Gamma_b.$$

Here and further we are using the following simple fact.

**Lemma 4.1.** *Let  $G$  be a connected compact Lie group with Lie algebra  $\mathfrak{g}$  and let  $\lambda \in i\mathfrak{g}$  be a weight, i.e.*

$$(4.16a) \quad e^{2\pi i(\lambda|m)} = 1 \text{ if } e^{2\pi im} = 1 \text{ and } [\lambda, m] = 0 .$$

*Then  $\lambda$  defines a 1-dimensional representation  $\sigma_\lambda$  of its stabilizer  $G_\lambda$  by the formula*

$$(4.16b) \quad \sigma_\lambda(g) = e^{2\pi i(\lambda|\gamma)} \text{ for } g = e^{2\pi i\gamma} \in G_\lambda, \gamma \in i\mathfrak{g}_\lambda .$$

*Proof.* Since the group  $G_\lambda$  is connected, it is generated by elements  $g$  of the form (4.16b). The map  $\sigma_\lambda$  is independent of the choice of  $\gamma$  representing  $g$  due to (4.16a). If  $g_j = e^{2\pi i\gamma_j} \in G_\lambda$  where  $\gamma_j \in i\mathfrak{g}_\lambda$ ,  $j = 1, 2$ , then the Cambell-Hausdorff formula implies  $\sigma_\lambda(g_1 g_2) = \exp\{2\pi i[(\lambda|\gamma_1 + \gamma_2) + (\lambda|\gamma)]\}$  where  $\gamma$  is a linear combination of commutators  $[\gamma_1, \gamma_2], \dots, [[\gamma_{i_1}, \gamma_{i_2}], \dots]$ , for  $i_1, i_2, \dots \in \{1, 2\}$ . But  $(\lambda|[\gamma_1, \gamma_2]) = ([\lambda, \gamma_1]|\gamma_2) = 0$  and the same holds for multifold commutators of  $\gamma_j$ . Thus (4.16b) does indeed define a 1-dimensional representation of  $G_\lambda$ .  $\square$

The isomorphism (4.15) gives a second equivalence relation for admissible quadruples:

$$(4.17) \quad (\Lambda, b, \beta(b), \sigma) \sim (\Lambda, b, \beta(b) + m, \sigma \otimes \sigma_m)$$

provided that  $m \in i\mathfrak{g}$  satisfies (4.14). In deriving the equality of the corresponding characters we use the identity

$$(4.18) \quad e^{-2\pi i(m|\alpha)_k} \chi_\Lambda^{\alpha, \beta+m}(\tau) = \chi_\Lambda^{\alpha, \beta}(\tau) .$$

The least obvious equivalence relation appears when two non-exceptional elements of  $\Gamma$  are obtained from each other by multiplication with an element  $\zeta \in Z$ .

Every element of  $Z$  can be written in the form

$$\zeta = (\zeta_{j_0}^{(0)}, \dots, \zeta_{j_s}^{(s)}) \in Z_0 \times \dots \times Z_s, \quad \zeta_j^{(\nu)} = e^{2\pi i\Lambda_j^{(\nu)}} \text{ or } 1 .$$

Here  $\{\Lambda_j^{(0)}\}$  generate the finite abelian group  $L^*/L$ ; for each simple component  $\mathfrak{g}$  the fundamental weight  $\Lambda_j$  belongs to the set  $j$  of indices with  $a_j = 1$ , see (1.33). If both  $b$  and  $\zeta_j b$  are non-exceptional we can write

$$(4.19a) \quad k\beta(\zeta_j b) = k\beta(b) + k\Lambda_j + m$$

$$(4.19b) \quad Ad_{\Gamma_b}(k\Lambda_j + m) = k\Lambda_j + m, \quad e^{2\pi im} = 1 .$$

We proceed to define the action  $\zeta_j$  on  $\sigma$  and  $\Lambda$ . According to Lemma 4.1 the phase factor

$$(4.20) \quad \sigma_j(b') = e^{2\pi i(k\Lambda_j + m|\beta')} \text{ for } b' = e^{2\pi i\beta'}, \quad Ad_{\Gamma_b}\beta' = \beta'$$

gives rise to a 1-dimensional representation  $\sigma_j$  of  $\Gamma_b$ . The transformation  $\Lambda \rightarrow \zeta_j(\Lambda)$  of a lattice weight  $\Lambda \in L^*$  is given by  $\zeta_j(\Lambda) = (\Lambda + \Lambda_j) \pmod L$ . If  $\mathfrak{g}$  is a simple rank  $\ell$  Lie algebra and  $\Lambda \in P_+^k$ , then  $\zeta_j(\Lambda)$  is defined by

$$(4.21a) \quad \zeta_j(\Lambda) = k\Lambda_j + w_j\Lambda$$

where  $w_j$  is the unique element of the Weyl group  $W$  of  $\mathfrak{g}$  that permutes the set  $\{-\theta, \alpha_1, \dots, \alpha_\ell\}$  and satisfies

$$(4.21b) \quad -w_j\theta = \alpha_j.$$

**Theorem 4.2.** *The pair of non-exceptional quadruples*

(4.22)

$$x = (\Lambda, \bar{b}, \beta, \sigma), \quad \left( \Lambda = \sum_{\nu=0}^s \Lambda^\nu \right) \text{ and}$$

$$\zeta(x) = \left( \sum_{\nu} (w_{j_\nu} \Lambda^\nu + k_\nu \Lambda_{j_\nu}), \bar{\zeta} \bar{b}, \beta + \sum_{\nu} \left( \Lambda_{j_\nu} + \frac{m_\nu}{k_\nu} \right), \sigma \otimes (\otimes_{\nu} \sigma_{j_\nu}) \right)$$

*gives rise to the same orbifold module leaving the same corresponding character invariant.*

The action of the center on non-exceptional quadruples for which  $b$  and  $\zeta b$  belong to the same conjugacy class  $\bar{b}$  has no fixed points for level  $k = 1$  in the simply laced case, but may have a fixed point for higher levels. For  $G = \text{SU}(2)$  this happens for even  $k$  and  $\Lambda = \frac{1}{2}k$ . An example of this type is provided in Sect. 6 (see Example 6.4). The corresponding twisted orbifold module turns out to be reducible in this case. Understanding its splitting into irreducible components requires more work and will be postponed to a subsequent publication. Here we shall *restrict our attention to the case when  $Z$  acts on the admissible quadruples without fixed points* (thus including all level 1 orbifolds, all  $\text{SU}(p)$  orbifolds (with  $p$  prime) for levels not divisible by  $p$ , as well as all  $\Gamma \subset G$  orbifolds with a trivial small center).

We denote by  $\mathcal{X}$  the set of equivalence classes of all admissible quadruples with equivalence relations (4.13), (4.17) and (4.22).

One may use the following description of  $\mathcal{X}$ . Consider the action of  $Z \times \Gamma$  on  $\Gamma$  for which  $Z$  acts by multiplication and  $\Gamma$  by conjugation. Choose a subset  $B \subset \Gamma$  consisting of non-exceptional representatives of orbits of this action, and for each  $b \in B$  choose  $\beta(b) \in i\mathfrak{g}$  satisfying (4.11). We call such  $B$  an admissible subset of  $\Gamma$ . Then  $\mathcal{X}$  may be identified with the set of admissible quadruples  $(\Lambda, b, \beta(b), \sigma)$ , where  $\Lambda \in P_+^k, b \in B, \sigma \in \hat{\Gamma}_b$ , with the equivalence relation that occurs only if

$$(4.23a) \quad \zeta b = gbg^{-1} \text{ for some } \zeta \in Z \text{ and } g \in \Gamma.$$

Then we let (cf. (4.22)):

(4.23b)

$$(\Lambda, b, \beta(b), \sigma) \sim \left( \sum_{\nu} (w_{j_\nu} \Lambda^\nu + k \Lambda_{j_\nu}), b, \beta(b), \sigma \otimes \sigma_{\sum_{\nu} m_\nu}^* \otimes \sigma_{(1 - \text{Ad}_{g^{-1}})\beta(b)} \right).$$

We can state now our main result.

**Theorem 4.3.** (a) Under the modular inversion  $S$  the characters  $\chi_x (x \in \mathcal{X})$  transform among themselves:

$$(4.24a) \quad \chi_{\Lambda, \bar{b}, \sigma}^{\beta} \left( -\frac{1}{\tau} \right) = \sum_{\bar{g} = \zeta' b' \subset \Gamma} \sum_{\substack{b = e^{2\pi i \beta} \in \bar{b} \\ b' = e^{2\pi i \beta'} \in b' \\ [\beta, \beta'] = 0}} \sum_{\sigma' \in \bar{\Gamma}_b} \sum_{\Lambda'} S_{\Lambda \Lambda'} S_{\bar{b} \sigma, \bar{b}' \sigma'}^{\beta} \chi_{\Lambda', \bar{b}', \sigma'}(\tau)$$

where  $S_{\Lambda \Lambda'}$  is the affine Kac-Moody  $S$ -matrix (3.25a), and the “group theoretic” factor looks as follows:

$$(4.25) \quad S_{\bar{b} \sigma, \bar{b}' \sigma'}^{\beta} = \frac{1}{|\Gamma|} \sum_{\substack{b \in \bar{b}, b' \in \bar{b}' \\ bb' = b'b}} \sigma'(b) \sigma(b') e^{-2\pi i (\beta(b) | \beta(b'))_k} .$$

For levels and groups  $\Gamma \subset G$  for which the small center  $Z$  acts without fixed points each equivalence class of quadruples in  $\mathcal{X}$  is encountered  $|Z|$  times and we can write

$$(4.24b) \quad \chi_{\Lambda, \bar{b}, \sigma}^{\beta} \left( -\frac{1}{\tau} \right) = \sum_{(\Lambda', b', \beta', \sigma') \in \mathcal{X}} |Z| S_{\Lambda \Lambda'} S_{\bar{b} \sigma, \bar{b}' \sigma'}^{\beta} \chi_{\Lambda', \bar{b}', \sigma'}^{\beta'}(\tau) .$$

(b) If the lattice  $L$  is even then the characters  $\chi_x$  are eigenfunctions of the modular translation  $T$ :

$$(4.26) \quad \chi_{\Lambda, \bar{b}, \sigma}^{\beta}(\tau + 1) = \exp \left\{ 2\pi i \left( m_{\Lambda} + \frac{1}{2} (\beta(b) | \beta(b'))_k \right) \right\} \frac{\sigma^*(b)}{\sigma(1)} \chi_{\Lambda, \bar{b}, \sigma}^{\beta}(\tau) .$$

They are eigenfunctions of  $T^2$  also for odd lattices.

(c) The inverse matrix  $S^{-1}$  is complex conjugate to  $S$ . The matrix  $S$  in (4.24b) is manifestly symmetric and hence also unitary.

(d) The matrix elements of  $S$  and  $T$  remain unchanged under the equivalence relations (4.13), (4.17), (4.22), (4.23).

(e) The charge conjugation operator  $C = S^2$  gives rise to the following involutive permutation of the set  $\mathcal{X}$ :

$$(4.27a) \quad C : (\Lambda, b, \beta(b), \sigma) \longmapsto (\Lambda^c, b^{-1}, \beta(b^{-1}), \sigma^c)$$

where  $\Lambda^c = -\Lambda$  in the lattice case,  $\Lambda^c$  is the highest weight of the contragredient to  $\Lambda$  representation of  $\mathfrak{g}$  in the affine case, and

$$(4.27b) \quad \sigma^c(h) = \sigma^*(h) e^{2\pi i (\beta(b) + \beta(b^{-1}) | \alpha)_k} \text{ for } h = e^{2\pi i \alpha} \in \Gamma_b .$$

*Proof of Theorem 4.2.* We shall content ourselves with verifying the equality of characters for admissible quadruples (4.22). The crux of the argument is the proof of the relation

$$(4.28) \quad \chi_{k\Lambda_j + w_j\Lambda}^{\alpha, \beta + \Lambda_j + m}(\tau) = e^{2\pi i(\Lambda_j + m|\alpha)_k} \chi_{\Lambda}^{\alpha, \beta}(\tau)$$

(for an appropriate choice of  $m \in M$ ) in the case of a (rank  $\ell$ ) simple Lie algebra  $\mathfrak{g}$ . To prove it we use the Weyl-Kac formula for the affine characters ([K1] Chap. 10). We first extend the coroot and weight spaces of  $\mathfrak{g}$  by introducing the central element

$$(4.29) \quad K = \sum_{\nu=0}^{\ell} a_{\nu}^{\vee} \alpha_{\nu}^{\vee} \left( \leftrightarrow \alpha_0^{\vee} = K - \theta^{\vee} \right)$$

and the gradation operator  $d(\leftrightarrow -L_0)$  (see Chap. 7 of [K1]). The bilinear form  $(\cdot, \cdot)$  is extended to the resulting  $\ell + 2$  dimensional space by

$$(4.30) \quad (K|K) = (d|d) = 0 = (K|\alpha_i) = (d|\alpha_i), \quad i = 1, \dots, \ell; \quad (K|d) = 1.$$

The Weyl-Kac formula then gives:

$$(4.31) \quad \chi_{kd+\Lambda}^{\alpha, \beta}(\tau) = \frac{\sum_{\tilde{w}} \varepsilon(\tilde{w}) e^{2\pi i \left( \tau \left( \frac{|\beta|^2}{2} K - \beta - d \right) + \alpha |\tilde{w}(kd + \Lambda + \tilde{\rho}) \right)}}{\sum_{\tilde{w}} \varepsilon(\tilde{w}) e^{2\pi i \left( \tau \left( \frac{|\beta|^2}{2} K - \beta - d \right) + \alpha |\tilde{w}\tilde{\rho} \right)}}$$

where the sum is over the affine Weyl group  $W(\hat{\mathfrak{g}})$ ,  $\tilde{\rho}$  is defined by

$$(4.32) \quad \tilde{\rho} = g^{\vee} d + \rho, \quad \rho = \sum_{i=1}^{\ell} \Lambda_i,$$

and  $\varepsilon(\tilde{w}) = \pm 1$  according to the parity of  $\tilde{w}$ . We define the element  $\tilde{w}_j$  of the extended affine Weyl group  $\hat{W}$  as follows (cf. Sect. 1 of [FKW] and Appendix B below):

$$(4.33) \quad \begin{aligned} \tilde{w}_j &= t_j w_j, \quad t_j d = d + \Lambda_j - \frac{|\Lambda_j|^2}{2} K, \quad t_j v = v - (v|\Lambda_j)K \quad (v \in \mathfrak{h}), \\ w_j d &= d, \quad \tilde{w}_j K = K, \end{aligned}$$

(where  $w_j \in W(\mathfrak{g})$  is defined on  $\mathfrak{h}$  as above).

We shall use the following three properties of  $\tilde{w}_j$ :

- (i) it preserves the extended Killing form;
- (ii) it leaves  $\tilde{\rho}$  invariant;
- (iii) it normalizes  $W(\hat{\mathfrak{g}})$ .

They allow us to write down the exponent in the numerator of (4.31) as

$$\left( \tilde{w}_j \left\{ \tau \left( \frac{|\beta|^2}{2} K - \beta - d \right) + \alpha \right\} |w \{ \tilde{w}_j (kd + \Lambda) + \tilde{\rho} \} \right) = \tau \frac{k}{2} |w_j \beta + \Lambda_j|^2 - k (w_j \alpha | \Lambda_j + (w_j \alpha - \tau (d + \Lambda_j + w_j \beta)) |w \{ \tilde{\rho} + \tilde{w}_j (kd + \Lambda) \}) .$$

It follows that

$$(4.34) \quad \chi_{\tilde{w}_j (kd + \Lambda)}^{w_j \alpha, w_j \beta + \Lambda_j}(\tau) = e^{2\pi i k (w_j \alpha | \Lambda_j)} \chi_{kd + \Lambda}^{\alpha, \beta}(\tau) .$$

Observing on the other hand the invariance relation

$$\chi_{kd + \Lambda}^{w_j^{-1} \alpha, w_j^{-1} \beta}(\tau) = \chi_{kd + \Lambda}^{\alpha, \beta}(\tau)$$

and the fact that  $\tilde{w}_j (kd + \Lambda)$  can be substituted by  $\zeta_j (kd + \Lambda)$  in the expression (3.18) for the character, we complete the proof of (4.28). It remains to insert the result into (4.4) in order to conclude that

$$(4.35) \quad \chi_{w_j \Lambda + \Lambda_j, \tilde{\zeta}_j \bar{b}, \sigma \otimes \sigma_j}^{\beta + \Lambda_j + m}(\tau) = \chi_{\Lambda, \bar{b}, \sigma}^{\beta}(\tau) ,$$

thus proving Theorem 4.2.  $\square$

*Proof of Theorem 4.3.* We use the assumption that  $\Gamma$  is a non-exceptional subgroup of  $G$  in order to express  $h$  in the formula (4.4) for the orbifold character by a non-exceptional element  $b'^{-1}$ :

$$(4.36a) \quad h = \zeta b'^{-1} = e^{2\pi i (\alpha_\zeta + \beta'_-)}$$

where

$$(4.36b) \quad \zeta = e^{2\pi i \alpha_\zeta} \in Z , \quad [\alpha_\zeta, \beta(b)] = 0 , \quad b'^{-1} = e^{2\pi i \beta'_-} .$$

This allows to rewrite (4.4) in the form

$$(4.37) \quad \chi_{\Lambda, \bar{b}, \sigma}^{\beta}(\tau) = \frac{1}{|\Gamma|} \sum_{b \in \bar{b}} \sum_{\substack{h = \zeta b'^{-1} \in \Gamma_b \\ [\beta(b), \beta'_-] = 0}} \sigma(b') \chi_{\Lambda}^{\beta'_-, \beta}(\tau)$$

where we have used the relation

$$(4.38a) \quad \chi_{\Lambda}^{\alpha_\zeta + \beta'_-, \beta}(\tau) = e^{2\pi i (\Lambda | \alpha_\zeta)_k} X_{\Lambda}^{\beta'_-, \beta}(\tau)$$

for  $e^{2\pi i (m | \alpha_\zeta)} = 1$  whenever  $m \in M$ ,  $[\alpha_\zeta, m] = 0$ , implying

$$(4.38b) \quad \sigma^*(h) \chi_{\Lambda}^{\alpha_\zeta + \beta'_-, \beta}(\tau) = \sigma(b') \chi_{\Lambda}^{\beta'_-, \beta}(\tau)$$

for  $\sigma|_Z = \Lambda|_Z$  (we have also used  $\sigma^*(b'^{-1}) = \sigma(b')$ ). Inserting the modular inversion law (3.26) into (4.37) we find

$$(4.39a) \quad \chi_{\Lambda, \bar{b}, \sigma}^{\beta} \left( -\frac{1}{\tau} \right) = \frac{1}{|\Gamma|} \sum_{b \in \bar{b}} \sum_{\substack{h = \zeta b'^{-1} \\ b' \in \Gamma_{b\beta, -\beta'_-}}} \sigma(b') e^{2\pi i(\beta|\beta'_-)_k} \sum_{\Lambda'} S_{\Lambda\Lambda'} \chi_{\Lambda'}^{\beta, -\beta'_-}(\tau),$$

where, in view of (4.5), we can write

$$(4.39b) \quad \chi_{\Lambda}^{\beta, -\beta'_-}(\tau) = \sum_{\sigma^{-\beta'_-} \in \hat{\Gamma}_b} \sigma^{-\beta'_-}(b) \chi_{\Lambda', \bar{b}', \sigma^{-\beta'_-}}^{-\beta'_-}.$$

Finally, we would like to substitute the upper index of  $\chi$  by the phase  $\beta'$  of  $b'$  which differs from  $-\beta'_-$  by a coroot:

$$(4.40) \quad b' = e^{2\pi i\beta'} \Rightarrow e^{-2\pi i(\beta' + \beta'_-)} = 1 \quad ([\beta' + \beta'_-, \beta] = 0).$$

Applying (4.15) we obtain

$$(4.41) \quad \begin{aligned} & \chi_{\Lambda, \bar{b}, \sigma}^{\beta} \left( -\frac{1}{\tau} \right) \\ &= \frac{1}{|\Gamma|} \sum_{\substack{\bar{g} \subset \Gamma \\ \bar{g} = \zeta' \bar{b}'}} \sum_{\substack{b = e^{2\pi i\beta} \in \bar{b} \\ b' = e^{2\pi i\beta'} \in \bar{b}' \\ bb' = b'b}} \sum_{\sigma' \in \hat{\Gamma}_b} \sum_{\Lambda'} \sigma(b') \sigma'(b) e^{-2\pi i(\beta|\beta')_k} S_{\Lambda\Lambda'} \chi_{\Lambda', \bar{b}', \sigma'}^{\beta'}(\tau) \end{aligned}$$

where

$$(4.42) \quad \sigma'(b) = \sigma^{-\beta'_-}(b) e^{2\pi i(\beta' + \beta'_-|\beta)_k}.$$

If the small center  $Z$  acts on admissible quadruples for which  $\zeta b \in \bar{b}$  without fixed points, then each term in the sum is encountered exactly  $|Z|$  times and we end up with (4.24b), (4.25).

The  $T$ -transformation law (4.26) follows from Eq. (3.27):

$$(4.43) \quad \begin{aligned} \chi_{\Lambda, \bar{b}, \sigma}^{\beta}(\tau + 1) &= e^{2\pi i\{m_{\Lambda} + \frac{1}{2}(\beta|\beta)_k\}} \sum_{\substack{h \in \Gamma_b \\ h = e^{2\pi i\alpha} \\ [\alpha, \beta] = 0}} \sigma^*(h) \chi_{\Lambda}^{\alpha - \beta, \beta}(\tau) \\ &= e^{2\pi i\{m_{\Lambda} + \frac{1}{2}(\beta|\beta)_k - (\sigma|\beta)\}} \chi_{\Lambda, \bar{b}, \sigma}^{\beta}(\tau). \end{aligned}$$

Here we have used the fact that  $b$  is in the center of  $\Gamma_b$  and  $\sigma \in \hat{\Gamma}_b$  is irreducible, so that

$$(4.44a) \quad \sigma^*(h) = \sigma^*(hb^{-1}) \frac{\sigma^*(b)}{\sigma(1)}$$

where the last factor is a complex number of absolute value 1 which can be written as

$$(4.44b) \quad \frac{\sigma^*(b)}{\sigma(1)} =: e^{-2\pi i(\sigma|\beta)} .$$

(Equation (4.44b) thus defines a linear functional  $(\sigma|\beta)$  in  $\beta$  whose exponential agrees with the value of  $\Lambda$  on  $Z$ .)

Using once more Lemma 3.1 for the inverse transformation  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  to (3.4) we derive  $S^{-1} = S^*$ , where  $*$  stands for complex conjugate. The symmetry of  $S$  is manifest from the expressions for  $S_{\Lambda\Lambda'}$  and  $S_{\bar{b}\sigma, \bar{b}'\sigma'}^\beta$ .

To prove the invariance of  $S$ -matrix elements with non-exceptional entries under the equivalence relation (4.23) we use an extension of (3.30):

$$(4.45) \quad S_{\zeta_j(\Lambda), \Lambda'} = e^{-2\pi i(\Lambda_j|\Lambda')} S_{\Lambda\Lambda'} ,$$

(cf. [KW]) and the fact that  $\sigma'$  and  $\Lambda'$  coincide on the central element  $\Lambda_j$ . To verify that  $T$  is also invariant under  $\zeta_j$  one uses  $(\sigma|\Lambda_j + m) = (\Lambda|\Lambda_j + m)$  and  $(w_j\Lambda|\Lambda_j) = (\Lambda|w_j^{-1}\Lambda_j) = (\Lambda|\Lambda_j + m')$  ( $m' \in M$ ) to prove that the phase

$$\phi(kd + \Lambda, \beta, \sigma) = \frac{1}{2k}(kd + \Lambda + 2\tilde{\rho}|kd + \Lambda) - (\sigma|\beta) + \frac{k}{2}|\beta|^2$$

changes by an integer:

$$\begin{aligned} \Delta\phi &= \frac{k}{2}(|\Lambda_j|^2 - |\Lambda_j + m|^2) + (\Lambda|m' - m) + (\Lambda - \sigma|\Lambda_j + m) \\ &= (\Lambda|m' - m) - k\{(\Lambda_j|m) + \frac{1}{2}|m|^2\} \in \mathbb{Z} (\Rightarrow e^{2\pi i\Delta\phi} = 1) . \end{aligned}$$

We finally proceed to prove (4.27). To this end we compute  $C = S^2$  by applying Lemma 3.1 to the central element of  $SL(2, \mathbb{Z})$ .

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} .$$

This gives

$$\begin{aligned} (C\chi)_{\Lambda, \bar{b}, \sigma}^\beta(\tau) &= \sum_{\Lambda'} C_{\Lambda\Lambda'} \frac{1}{|\Gamma_b|} \sum_{b' \in \Gamma_b} \sigma^*(b') \chi_{\Lambda'}^{-\beta(b'), -\beta(b)}(\tau) \\ &= \sum_{\Lambda'} C_{\Lambda\Lambda'} \sum_{\bar{b}'\sigma'} C_{\bar{b}\sigma, \bar{b}'\sigma'}^{\beta, \beta'} \chi_{\Lambda', \bar{b}'\sigma'}^{\beta'}(\tau) , \end{aligned}$$

where  $C_{\Lambda\Lambda'} = \delta_{\Lambda^*\Lambda'}$  is known from the modular properties of affine Kac-Moody characters ([K1] Chap. 13), while the second factor is computed to be

$$(4.46) \quad C_{\bar{b}\sigma, \bar{b}'\sigma'}^{\beta, \beta'} = \delta_{\bar{b}^{-1}, \bar{b}'} \delta_{\sigma^c, \sigma'} \delta_{\beta, -\beta'} .$$

□

We note that the equivalence class  $v$  of the *vacuum admissible quadruple*, i.e. that corresponding to the vacuum  $\mathfrak{A}(G)^\Gamma$ -module, is selfconjugate:

$$v := \text{class of } (0, 1, 0, 1) = Cv.$$

Note also the following formula for any  $x = (\Lambda, b, \beta(b), \sigma) \in \mathcal{X}$  :

$$S_{x,v} = S_{\Lambda,0} \frac{|\bar{b}|}{|\Gamma|} \sigma(1) .$$

*Remark 4.2.* It follows from Lemma 3.2d that the eigenvalues of  $L_0$  are strictly positive in all  $\mathfrak{A}(G)$ -modules  $V_x$ ,  $x \in \mathcal{X}$ , except for the vacuum module  $V_v$ . The 0-th eigenspace of  $L_0$  in  $V_v$  is  $\mathbb{C}|0\rangle$ .

*Remark 4.3.* The  $\mathfrak{A}(G)^\Gamma$ -modules  $V_x$  and  $V_{C_x}$  ( $x \in \mathcal{X}$ ) are contragredient.

#### 4C. Fusion rules.

We can summarize the most important features of the outcome of the previous section as follows.

Starting with a compact Lie group  $G = (\mathbb{R}^r/L) \times G'$ , where  $G'$  is simply connected, and a negative definite integral invariant bilinear form on its Lie algebra which is even on the lattice  $L$ , we have constructed for every non-exceptional finite subgroup  $\Gamma$  of  $G$  a collection of  $\mathfrak{A}(G)^\Gamma$ -modules parametrized by a finite set  $\mathcal{X}$ . This set is equipped with an involutive permutation  $C$  (corresponding to taking a contragredient module) and a distinguished element  $v$  (corresponding to the vacuum module) such that  $Cv = v$ . We have also matrices  $S = (S_{xy})_{x,y \in \mathcal{X}}$  and  $T = (T_{xy})_{x,y \in \mathcal{X}}$  satisfying the following three properties, provided that the small center  $Z$  acts on  $\mathcal{X}$  without fixed points:

- (a)  $S$  is symmetric and  $T$  is diagonal,
- (b) the map  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow S, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow T, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow C$  give a unitary representation of the group  $SL_2(\mathbb{Z})$ .
- (c)  $S_{xv} > 0$  for all  $x \in \mathcal{X}$ .

Following Verlinde [V], introduce the *fusion algebra*  $A(\mathcal{X}) = \bigoplus_{x \in \mathcal{X}} \mathbb{C}x$  by the formula:

$$(4.47a) \quad xy = \sum_{z \in \mathcal{X}} N_{xyz} Cz ,$$

where the *fusion coefficients*  $N_{xyz}$  are defined by

$$(4.47b) \quad N_{xyz} = \sum_{a \in \mathcal{X}} S_{ax} S_{ay} S_{az} / S_{av} .$$

It follows from the above properties of  $S$  that the fusion algebra  $A(\mathcal{X})$  is a finite-dimensional commutative associative semisimple algebra with identity

element  $v$  and involutive automorphism  $C$ . All homomorphisms of the algebra  $A(\mathcal{X})$  to  $\mathbb{C}$  are labeled by elements  $y \in X$  and given by

$$(4.48) \quad ch_y(x) = S_{xy}/S_{vy} \quad (x \in \mathcal{X}).$$

The positive real number  $ch_v(x)$  is the relative (= quantum) dimension.

The basic observation of [V] is that the fusion algebras arising in a RCFT have the following fundamental property:

$$(d) \quad N_{xyz} \in \mathbb{Z}_+.$$

Denote by  $X^{af}$  the set  $P_+^k$  labeling all positive energy irreducible representations of the chiral algebra  $\mathfrak{A}(G)$  with vacuum element  $v = 0$ , conjugation  $C\Lambda = \Lambda^c$ ,  $S$ -matrix  $S^{af} = (S_{\Lambda\Lambda'})$  and  $T$ -matrix  $T^{af} = e^{2\pi i m_\Lambda} \delta_{\Lambda\Lambda'}$ . It follows from [KP2] that the properties (a)-(c) hold, and it is a very difficult theorem established by the efforts of many people that (d) holds as well. Denote by  $N_{\Lambda\Lambda'\Lambda''} (\in \mathbb{Z}_+)$  the fusion coefficients.

Similarly, let  $\mathcal{X}^{gr}$  denote the set of all pairs  $(\bar{g}, \sigma)$ , where  $\bar{g}$  is a conjugacy class of  $\Gamma$  and  $\sigma$  is an irreducible character of  $\Gamma_{\bar{g}}$ . Let  $v = (1, 1)$  be the vacuum element and let  $C(\bar{g}, \sigma) = (\overline{g^{-1}}, \sigma^c)$  where  $\sigma^c$  is defined by (4.27b). Let  $S_{\bar{g}\sigma, \bar{g}'\sigma'}^{gr}$  be the matrix defined by the right-hand side of (4.25) and let (cf. (4.26)):

$$(4.49) \quad T_{\bar{g}\sigma, \bar{g}'\sigma'}^{gr} = e^{2\pi i(\beta|\beta)_k} \frac{\sigma^*(b)}{\sigma(1)}.$$

It follows from the remarks of the previous section that the properties (a), (b) and (c) hold. It can be demonstrated by an appropriate, example of an  $SU(2)$  subgroup of level 1 (see Example 6.5), that property (d) does not hold in general.

Lusztig [Lus] studied the ‘‘limiting’’ case of our  $\mathcal{X}^{gr}$  when in (4.26), (4.27b) and (4.49) one sets all  $\beta(b)$  equal zero and  $b = g$ . In this case (d) holds due to his interpretation of the fusion algebra as the Grothendieck ring of the category of  $\Gamma$ -equivariant vector bundles.

Whenever the center of  $G$  is trivial like in the case of  $E_8$  the fusion rules factorize:  $N_{xx'x''} = N_{\Lambda\Lambda'\Lambda''} N_{\bar{g}''\sigma, \bar{g}'\sigma', \bar{g}\sigma''}$ . In particular, for a level 1 orbifold like  $\mathfrak{A}_1(E_8)^\Gamma$  they coincide with the group theoretic fusion rules which we proceed to compute.

The following cubic sum rule tells us that the fusion coefficient

$$(4.50) \quad N_{\bar{g}_1\sigma_1, \bar{g}_2\sigma_2, \bar{g}_3\sigma_3} = \sum_{\bar{h}\sigma} \frac{S_{\bar{g}_1\sigma_1, \bar{h}\sigma} S_{\bar{g}_2\sigma_2, \bar{h}\sigma} S_{\bar{g}_3\sigma_3, \bar{h}\sigma}}{S_{11, \bar{h}\sigma}}$$

vanishes unless there are triples  $g_j \in \bar{g}_j$ ,  $j = 1, 2, 3$  such that  $g_1 g_2 g_3 = 1$ .

**Proposition 4.4.** ([Gor] Theorem 2.12) *Let  $\bar{g}_i$ ,  $i = 1, 2, 3$ , be three conjugacy classes in a finite group  $\Gamma$ . The number  $n_{123}$  of triples  $g_i \in \bar{g}_i$  such that  $g_1 g_2 g_3 = 1$  is given by*

$$n_{123} = \frac{|\bar{g}_1| |\bar{g}_2| |\bar{g}_3|}{|\Gamma|} \sum_{\sigma \in \hat{\Gamma}} \frac{1}{\sigma(1)} \sigma(g_1) \sigma(g_2) \sigma(g_3).$$

In deriving the fusion rules we follow [DV<sup>3</sup>], but compute explicitly the phase factors.

**Theorem 4.5.** *The fusion rules (4.50) can be expressed in either of the two forms:*

$$(4.51a) \quad N_{\bar{b}_1\sigma_1, \bar{b}_2\sigma_2, \bar{b}_3\sigma_3} = \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \sum_{\substack{b_i \in \bar{b}_i \cap \Gamma_h \\ b_1 b_2 b_3 = 1}} \sigma_1(h) \sigma_2(h) \sigma_3(h) \mu(h | \Sigma \beta_i)$$

$$(4.51b) \quad N_{\bar{b}_1\sigma_1, \bar{b}_2\sigma_2, \bar{b}_3\sigma_3} = \sum_{O_{12}} \frac{1}{|\Gamma_{b_1, b_2}|} \sum_{h \in \Gamma_{b_1, b_2}} \sigma_1(h) \sigma_2(h) \sigma_3(h) \mu(h | \Sigma \beta_i) .$$

Here the multiplier  $\mu$  is given by

$$(4.52) \quad \mu(h | \Sigma \beta_i) = e^{2\pi i(\alpha | \Sigma \beta_i)_k} , \quad \beta_i = \beta(b_i) , \quad h = e^{2\pi i \alpha} .$$

The outer sum in (4.51b) is over different orbits  $O_{12}$  of pairs  $(b_1, b_2)$  under the adjoint action of  $\Gamma$ ; the number  $|O_{12}|$  of such orbits is determined from the relation

$$|O_{12}| |\Gamma_{12}| = |\Gamma| .$$

The proof uses the form

$$(4.53) \quad S_{\bar{g}_j \sigma_j, \bar{h} \sigma} = \frac{1}{|\Gamma_h|} \sum_{b_j \in \bar{b}_j \cap \Gamma_h} \sigma_j(h) \sigma(b_j) e^{-2\pi i(\alpha | \beta_j)_k}$$

of (4.25) for the three factors in the numerator of (4.47) and reduces to a straightforward application of Proposition 4.4 (noting the conjugation invariance of  $\mu$ ). (For  $x_3 = v$  (the vacuum module)  $\bar{b}_3 = \bar{1}$ ,  $\sigma_3 = 1$  ( $\beta_3 = 0$ ) we reproduce as a special case the charge conjugation matrix (4.46):  $C_{\bar{g}_1\sigma_1, \bar{g}_2\sigma_2} = N_{\bar{g}_1\sigma_1, \bar{g}_2\sigma_2, \bar{1}1}$ .)

The multiplier (4.52) does not depend on the choice of the phase  $\alpha$  of  $h$  provided it belongs to the stabilizer  $\mathfrak{g}_{b_1, b_2}$  of the pair  $(b_1, b_2)$  in  $\mathfrak{g}$ ;  $\mu$  thus defines a representation of  $\Gamma_{b_1, b_2}$  according to Lemma 4.1 applied to  $G = G_{b_1}$ ,  $\lambda = \beta_2$ .

## 5. $U(l)$ ORBIFOLDS AS RCFT EXTENSIONS OF $W_{1+\infty}$

What is now called  $W_{1+\infty}$  first appeared as the (unique nontrivial) central extension  $\hat{\mathcal{D}}$  of the Lie algebra  $\mathcal{D}$  of differential operators on the circle [KP1]. Its representation theory (including the classification of quasi-finite positive energy representations) was developed in [KR1,2] and [FKRW]. It has also attracted the attention of physicists, in particular, the most degenerate ‘minimal series’ of unitary representations of  $W_{1+\infty}$  of [FKRW] are being applied in the study of quantum Hall fluids [CTZ]. (More reference to both physical applications and related mathematical developments are cited in the above papers and in the bibliography to [AFMO].) The vacuum  $\hat{\mathcal{D}}$ -module (corresponding for unitary representation to a positive integer central charge  $c = l$ ) was shown [FKRW] to carry a canonical chiral (vertex) algebra structure. The resulting chiral algebra  $W_{1+\infty}^{(l)}$  was described in [BGT] in terms of a series of quasi primary fields of dimension  $\nu + 1$ ,  $\nu = 0, 1, \dots$ :

(5.1a)

$$V^\nu(z) = \sum V_n^\nu z^{-n-\nu-1} , \quad [L_m, V^\nu(z)] = z^m \left( z \frac{d}{dz} + \nu + 1 \right) V(z)^\nu , \quad m = 0, \pm 1,$$

satisfying local CR such that

(5.1b)

$$[V_m^\mu, V_n^\nu] = (\nu m - \mu n) V_{m+n}^{\mu+\nu-1} + \cdots + c \frac{(\nu!)^4}{(2\nu)!} \binom{m+\nu}{m-\nu-1} \delta_{m,-n} \delta_{\mu\nu}, \quad c = l.$$

The (quasi finite) irreducible positive energy modules  $V_{\vec{r}}$  of  $W_{1+\infty}^{(l)}$  are characterized by  $l$  exponents (see [KR1,2])  $\vec{r} = (r_1, \dots, r_l)$  that take real values for unitary representations. Each  $V_{\vec{r}}$  has a cyclic minimal energy vector  $|\vec{r}\rangle$  such that

$$(5.2) \quad V_n^\nu |\vec{r}\rangle = 0 \text{ for } n = 1, 2, \dots, \{V_0^\nu - v_\nu(\vec{r})\} |\vec{r}\rangle = 0$$

where

(5.3)

$$v_0(\vec{r}) = \sum_{i=1}^l r_i,$$

$$v_\nu(\vec{r}) = \frac{(\nu-1)! \nu!}{(2\nu)!} \sum_{j=0}^{\nu-1} \binom{\nu}{j} \binom{\nu}{j+1} \sum_{i=1}^l (r_i - j) \cdots (r_i + \nu - j - 1) r_i.$$

In particular  $V^1(z) = T(z)$  so that the ground state energy eigenvalue is  $v_1(\vec{r}) = \frac{1}{2} \vec{r}^2 = \sum_{i=1}^l r_i^2$ :

(5.4)

$$\left( L_0 - \frac{1}{2} \vec{r}^2 \right) |\vec{r}\rangle = 0 \quad (L_0 = V_0^1), \quad v_2(\vec{r}) = \frac{1}{3} \sum_{i=1}^l r_i^3, \quad v_3(\vec{r}) = \frac{1}{4} \sum_i (r_i^4 + \frac{1}{5} r_i^2).$$

The vacuum  $\hat{D}$ -module contains for  $c = l \in \mathbb{N}$  a unique singular vector of degree  $l+1$  such that the quotient by the submodule generated from this singular vector is irreducible [KR1]. This irreducible quotient (together with its chiral algebra structure) is isomorphic to a (level  $l$ )  $W(u_l)$  vacuum module — see [FKRW], Sect. 5. As a result, any irreducible representation of  $W_{1+\infty}^{(l)}$  has a canonical structure of an irreducible representation of  $W(u_l)$  of level  $l$ , and all irreducible representations of  $W(u_l)$  with central charge  $l$  arise in this way.

Any  $V_{\vec{r}}$  splits into a tensor product of a  $W(\mathfrak{su}(l))$  module of central charge  $l-1$  and a  $W_{1+\infty}^{(1)}$ -module. To see this we rescale the  $u(1)$  current and split the stress energy tensor as in (1.39):

(5.5a)

$$J(z) = \frac{1}{\sqrt{l}} V^0(z), \quad T(z) = T_J(z) + T_{su}(z), \quad T_J = \frac{1}{2} : J^2 : \left( = \frac{1}{2l} : (V^0)^2 : \right),$$

(5.5b)

$$[T_{su}(z), J(w)] = 0.$$

The minimal eigenvalue of the energy of the second term,  $L_0^{su}$  is then given by the difference

$$(5.6) \quad \frac{1}{2} \vec{r}^2 - \frac{1}{2l} \left( \sum_{i=1}^l r_i \right)^2 = \frac{1}{2l} \sum_{i < j} (r_i - r_j)^2 =: \omega_l(\vec{r})$$

A  $W_{1+\infty}^{(l)}$ -module  $V_{\vec{r}}$  is *degenerate* if some of the differences  $r_i - r_j$  are integer. It is *maximally degenerate* if all  $r_i - r_j$  are integer (such representations are termed minimal [CTZ]); the representation of the second ( $su$ )-factor is indeed then a limit of the Zamolodchikov-Fateev-Lukyanov  $W_l(p)$ -models of central charge  $c = (l-1) \left\{ 1 - \frac{l(l+1)}{p(p+1)} \right\}$  as observed in [CTZ]. Since every  $V_{\vec{r}}$  can be viewed as a tensor product of maximally degenerate (including  $c = 1$ ) modules we shall turn our attention to the case of integer  $r_i - r_j$ . Assume that  $r_i - r_j \in \mathbb{Z}$ , we then arrange the  $r_i$ 's in a decreasing order and denote the set of such  $\vec{r}$ 's as  $P^+$ :

$$(5.7) \quad P^+ = \{ \vec{r} \in \mathbb{R}^l \mid r_1 \geq r_2 \geq \dots \geq r_l, r_i - r_j \in \mathbb{Z} \} .$$

If we interpret the ordered set  $\Lambda = (\lambda_1, \dots, \lambda_{l-1})$  of differences

$$(5.8) \quad \lambda_i = r_i - r_{i+1}, \quad i = 1, 2, \dots, l-1$$

as defining a highest weight of  $SU(l)$ , then for the fundamental weights  $\Lambda_1 = (1, 0, \dots, 0), \dots, \Lambda_{l-1} = (0, \dots, 0, 1)$  the ground state energy eigenvalues (5.6) coincide with the level 1 eigenvalues of the  $\hat{su}_l$  current algebra  $\mathfrak{A}_1(\mathfrak{su}(l))$ :

$$(5.9) \quad \omega_l(\vec{r}(i)) = \frac{(\Lambda_i + 2\rho|\Lambda_i)}{2(l+1)} \text{ for } r_j^{(i)} - r_{j+1}^{(i)} = \delta_{ij}$$

(which can be verified by a direct computation). It is natural to expect that the  $W(\mathfrak{su}_l)$  representations of such weights obey fusion rules given by the tensor product expansion formulae for  $SU(l)$  (see Conjecture 6.1 of [FKRW]).

It follows that a CFT with chiral algebra  $W(\mathfrak{su}(l))$  and a highest weight module  $V_{\vec{r}}$  with  $r_i - r_j$  non-zero integers has an infinite number of sectors and hence is not a rational CFT. (We are using here the basic property of any quantum field theory to be closed under fusion.) This ‘irrationality’ can also be seen from an analysis of the characters of these representations (computed in [FKRW]). The orbifold construction of the previous sections allows to define a large class of RCFT extensions of  $W_{1+\infty}$  with the same stress energy tensor.

In fact the embedding of the vacuum module of  $W_{1+\infty}^{(l)}$  into the Fock space  $\mathcal{F}_l$  of  $l$  free complex fermion fields, used from the outset in [FKRW] and [KR2], does provide one such (chiral superalgebra) extension. So does its even (bosonic) part which coincides with the level 1 current algebra of the rank  $l$  (even) orthogonal group  $\mathfrak{A}_1(\mathfrak{so}(2l))$ . (Indeed, if we separate the real and imaginary part of the free fermions writing them as

$$(5.10) \quad \psi_j = \frac{1}{\sqrt{2}}(\varphi_{2j-1} - i\varphi_{2j}), \quad j = 1, \dots, l, \quad \text{then } J_{jk}(z) = i\varphi_j(z)\varphi_k(z) \quad (j < k)$$

satisfy the commutation relations of level 1  $\mathfrak{so}(2l)$  currents. The complex structure selects a Cartan subalgebra that includes  $V^0$ :

$$(5.11) \quad H^j(z) =: \psi_j^*(z)\psi_j(z) := J_{2j-1,2j}(z), \quad V^0(z) = \sum_{j=1}^l H^j(z) .$$

Then we can define  $W_{1+\infty}^{(l)}$  as the  $U(l)$ -invariant subalgebra of  $\mathfrak{A}_1(\mathfrak{so}(2l))$  ( $u(l)$  and  $\mathfrak{so}(2l)$  sharing the same Cartan subalgebra). A more general RCFT extension of  $W_{1+\infty}^{(l)}$  is provided by the chiral algebra associated with the compact group  $U(l)$ , equipped with a lattice structure  $Q$  (see Sect. 1). Here  $Q$  is an  $l$ -dimensional even integral lattice whose sublattice of vectors of length square 2 includes the (rank  $l-1$ )  $\mathfrak{su}(l)$  lattice. (The root lattices of rank  $l$  semi-simple Lie algebras-  $\mathfrak{so}(2l)$ ,  $\mathfrak{su}(l+1)$ ,  $\mathfrak{su}(l) \oplus \mathfrak{su}(2)$ ) — appear then as special cases. Note that the  $\mathfrak{su}(l)$  Cartan currents are orthogonal to  $V^0$  (5.11) (or  $J$  (5.5)); they are

$$(5.12) \quad H^{\alpha_i}(z) = H^i(z) - H^{i+1}(z), \quad i = 1, \dots, l-1,$$

$\alpha_1, \dots, \alpha_{l-1}$  being the simple roots of  $\mathfrak{su}(l)$ . Any of the extensions  $\mathfrak{A}(Q)$  of  $W_{1+\infty}^{(l)}$  where  $Q$  is a (rank  $l$ ) lattice with the above properties admits a finite set of positive energy CFT representations whose characters span a (finite dimensional) representation of  $SL(2, \mathbb{Z})$ . All these extensions involve, in particular,  $l$  commuting  $u(1)$  currents and can be thus related to the approach of Fröhlich, Thiran et al. to the quantum Hall effect (see [FT] and references therein). A large family of intermediate observable algebras is provided by  $\Gamma$  orbifolds of  $\mathfrak{A}(Q)$  where  $\Gamma$  is any finite subgroup of  $U(l)$ . If  $\Gamma$  is not contained in any proper Lie subgroup of  $SU(l)$  then  $\mathfrak{A}^\Gamma$  only involves a single  $u(1)$  current — the one belonging to  $W_{1+\infty}^{(l)}$ . Such  $\mathfrak{A}^\Gamma$  could be viewed as RCFT extensions of minimal  $W_{1+\infty}^{(l)}$  models (exploited in [CTZ]).

We proceed to state the precise results for the Fock space  $\mathcal{F}_l$  of  $l$  free (complex) fermions and its orbifolds.

**Theorem 5.1** [FRKW]. *The fermion Fock space  $\mathcal{F}_l$  viewed as a representation of the pair  $(U(l), W_{1+\infty}^l)$  splits into an infinite direct sum of tensor products*

$$(5.13) \quad \mathcal{F}_l = \bigoplus_{\vec{r} \in P_+} F(\vec{r}) \otimes L(\vec{r}),$$

where  $P_+ = \{\vec{r} = (r_1, \dots, r_l) \in \mathbb{Z}^l | r_1 \geq \dots \geq r_l\}$ ,  $F(\vec{r})$  is the finite dimensional irreducible  $U(l)$ -module of highest weight  $\vec{r}$ ,  $L(\vec{r})$  is a unitary  $W_{1+\infty}^{(l)}$  positive energy module with exponents  $\vec{r}$  and specialized character

$$(5.14) \quad \chi_{\vec{r}}(\tau) = \text{tr}_{L(\vec{r})} q^{L_0 - \frac{1}{24}} = q^{\frac{1}{2}\vec{r}^2} \eta^{-l}(\tau) \prod_{1 \leq i < j \leq l} (1 - q^{r_i - r_j + j - i}).$$

The following result is a specialization of Theorem 3.2 applied to the chiral algebra  $\mathfrak{A}(\mathbb{Z}^l)^\Gamma$  where  $\mathbb{Z}^l$  is the integral lattice with the standard bilinear form, and  $\Gamma$  is a finite subgroup of  $U(l)$ . Recall that  $\mathfrak{A}(\mathbb{Z}^l)$  has a unique irreducible representation, hence we may skip the index  $\Lambda$ .

**Theorem 5.2.** *Let  $\Gamma$  be a finite subgroup of  $U(l)$ . Write each  $b \in \Gamma$  in the form  $b = \exp 2\pi i \beta$  where  $i\beta \in u(l)$  is fixed by  $\text{Ad}_{\Gamma_b}$ . Let  $\{\beta_i(\vec{r})\}$  denote the set of eigenvalues of  $\beta$  in  $F(\vec{r})$ . Given an irreducible character  $\sigma$  of  $\Gamma_b$ , let*

$$m_{\vec{r}, \sigma, \beta}(q) = q^{\frac{1}{2}(\beta|\beta)} \sum_i m_{\vec{r}, \sigma, \beta_i(\vec{r})} q^{-\beta_i(\vec{r})},$$

where  $m_{\vec{r},\sigma,\beta_i(\vec{r})}$  is the multiplicity of  $\sigma$  in the  $\beta_i(\vec{r})$ -eigenspace of  $\beta$  in  $F(\vec{r})$ . Then the  $\mathfrak{A}(\mathbb{Z}^l)^\Gamma$ -characters can be written in the following form:

$$(5.15) \quad \chi_{\vec{b},\sigma}^\beta(\tau) = \sum_{\vec{r} \in P^+} m_{\vec{r},\sigma,\beta}(q) \chi_{\vec{r}}(\tau).$$

All these characters are modular functions and their  $\mathbb{C}$ -span is invariant under the transformation  $\tau \mapsto -\frac{1}{\tau}$ .

In particular, for  $\vec{b} = \vec{1}$ , we have  $\beta = 0$  and all  $m_{\vec{r},\sigma,\beta}(q) \in \mathbb{Z}_+$  and we find the characters of untwisted orbifold modules, which, unlike  $\chi_{\vec{r}}$  are modular functions of  $\tau$ . This special case of Theorem 5.2 provides a family of solutions to the following problem: find non negative integers  $n(\vec{r})$  such that

$$\sum_{\vec{r} \in P^+} n(\vec{r}) \chi_{\vec{r}}(\tau)$$

is a modular function of  $\tau$ . Each pair  $\Gamma \subset U(n)$  ( $\Gamma$  finite subgroup),  $\sigma \in \hat{\Gamma}$  gives a solution to this problem with  $n(\vec{r}) = n_\Gamma^\sigma(\vec{r})$  being the multiplicity of  $\sigma$  in  $F(\vec{r})$  viewed as a  $\Gamma$ -module.

*Proof of Theorem 5.2.* In view of (4.4) and (3.18) we can write

$$(5.16) \quad \chi_{\vec{b},\sigma}^\beta(\tau) = \frac{1}{|\Gamma_b|} \sum_{\substack{a \in \Gamma_b \\ a = \exp 2\pi i \alpha \\ [\alpha, \beta] = 0}} \sigma^*(a) \chi(\tau, \alpha - \beta\tau, \frac{1}{2}(\beta|\beta\tau)),$$

where, due to (5.13)

$$(5.17) \quad \chi(\tau, z, u) = e^{2\pi i u} \sum_{\vec{r} \in P^+} \chi_{\vec{r}}(\tau) \text{tr}_{F(\vec{r})} e^{2\pi i z}.$$

Hence we have:

$$(5.18) \quad \chi_{\vec{b},\sigma}^\beta(\tau) = \sum_{\vec{r} \in P^+} \chi_{\vec{r}}(\tau) q^{\frac{1}{2}(\beta|\beta)} \sum_{a \in \Gamma_b} \sigma^*(a) \text{tr}_{F(\vec{r})}(aq^{-\beta}).$$

Since  $\Gamma_b$  fixes  $\beta$ , each eigenspace of  $\beta$  in  $F(\vec{r})$  is  $\Gamma_b$ -invariant. The contribution of the  $\beta_i(\vec{r})$ -eigenspace to the inner sum of (5.18) is clearly equal  $m_{\vec{r},\sigma,\beta_i(\vec{r})} q^{-\beta_i(\vec{r})}$ . This proves (5.15).  $\square$

*Remark 5.1.* Theorem 5.2 can be generalized to any simply laced simple Lie algebra  $\mathfrak{g}$  of rank  $l$  and  $\Lambda \in P_+^1$ . Namely, formula (5.15) holds for any non-exceptional element  $b$ , where the sum is taken over  $\lambda \in (\Lambda + Q) \cap P_+^1$ , and (see [K1], Exercise 12.17):

$$\chi_\lambda(\tau) = q^{\frac{1}{2}(\lambda|\lambda)} \eta^{-l}(\tau) \prod_{\alpha > 0} (1 - q^{(\lambda + \rho|\alpha)}).$$

We have:

$$(5.19) \quad \chi_{\Lambda, \bar{b}, \sigma}^{\beta}(\tau) = \sum_{\lambda \in (\Lambda + Q) \cap P_+} m_{\lambda, \sigma, \beta}(q) \chi_{\lambda}(\tau).$$

The character  $\chi_{\Lambda, \sigma, b}^{\beta}(\tau)$  is a modular function and their  $\mathbb{C}$ -span is  $SL_2(\mathbb{Z})$ -invariant provided that  $\Gamma$  is a non-exceptional finite subgroup of our simple Lie group.

*Remark 5.2.* Taking  $\Gamma = \{1\}$  in Remark 5.1 we arrive at the following curious identity by comparing two expressions for  $\Gamma$ -orbifold characters for each weight  $\Lambda$  and real number  $m$ :

$$|\{\lambda \in \Lambda + Q | (\lambda|\lambda) = m\}| = \sum_{\substack{\lambda \in \Lambda + Q \\ (\lambda|\lambda) = m}} \prod_{\alpha > 0} \frac{(\lambda + \rho|\alpha)}{(\rho|\alpha)}.$$

## 6. EXAMPLES.

### 6A Lattice current algebras for $c = 1$ .

The simplest ( $c = 1$ ) case of a lattice current algebra is worth singling out for at least two reasons: (1) the basic  $\theta$ -functions encountered here also appear in the  $SU(2)$  affine orbifold model; (2) the lattice part of a  $U(l)$  orbifold encountered in a  $W_{1+\infty}$  theory is of this ( $U(1)$ -)type.

A 1-dimensional lattice  $L = \mathbb{Z}\omega$  is characterized by a single natural number  $m = |\omega|^2$ ; we shall denote  $\mathfrak{A}(L; |\omega|^2 = m)$  by  $\mathfrak{A}(m)$ . Note that  $m$  is twice the dimension of the basic charged fields  $Y(e^{\pm\omega}, z)$ , while  $v(z) = m^{-\frac{1}{2}}\omega(z)$  is the corresponding  $u(1)$  current (see Sect. 1B). The dual lattice is  $L^* = \mathbb{Z}\omega^*$  where  $(\omega^*|\omega) = 1 \Rightarrow |\omega^*|^2 = \frac{1}{m}$ . The factor group  $L^*/L$  is the cyclic group of order  $m$ ; there are, correspondingly,  $m$  untwisted modules whose weights will be labeled by minimal length representatives

$$(6.1) \quad \mu\omega^* \in L^*/L, \quad \frac{m-1}{2} \leq \mu \leq \frac{m}{2}, \quad \mu \in \mathbb{Z}.$$

The specialized character of the positive energy  $\mathfrak{A}(m)$ -module  $V_{\mu}$  (of ground state  $|\mu\omega^*\rangle$ ) is given by (see [DFSZ] [PT])

$$(6.2) \quad K_{\mu}(\tau, m) = \frac{1}{\eta(\tau)} \Theta_{\mu 1}^L(\tau, 0, 0) = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{\frac{m}{2}(n + \frac{\mu}{m})^2}.$$

This set spans a representation of  $SL(2, \mathbb{Z})$  in the case of a bosonic algebra ( $m$  even) and requires supplementing it with *Ramond sector* ( $\mathbb{Z}_2$  twisted) modules corresponding to half-odd integer  $\mu$ 's in the interval (6.1) for  $m$  odd and splitting each integer  $\mu$  character into two (corresponding to summing over even and odd  $n$ 's in (6.2)).

For  $m = 2s$  even the modular transformation law for  $K_{\mu}$  is given, according to (3.15), by

$$(6.3) \quad K_{\mu}(\tau + 1, 2s) = e^{i\pi(\frac{\mu^2}{2s} - \frac{1}{12})} K_{\mu}(\tau, 2s)$$

$$(6.4) \quad K_\mu \left( -\frac{1}{\tau}, 2s \right) = \frac{1}{\sqrt{2s}} \sum_{\nu=1-s}^s e^{-i\pi \frac{\mu\nu}{s}} K_\nu(\tau, 2s).$$

*Example 6.1.* A  $\mathbb{Z}_N$ -orbifold of  $\mathfrak{A}(m)$  is given by the chiral algebra  $\mathfrak{A}(N^2m)$  (and its positive energy modules). If indeed we introduce the inner automorphism

$$(6.5a) \quad \mathfrak{A}(m) \ni A \rightarrow UAU^{-1}, \quad U = e^{2\pi i \omega_0^*/N}$$

$$(6.5b) \quad Ue^\omega U^{-1} = e^{2\pi i/N} e^\omega \quad (UJ(z)U^{-1} = J(z))$$

( $J$  being the  $u(1)$  current  $J(z) = Y(t^{-1} \frac{\omega}{\sqrt{m}}, z)$ ,  $\omega_0^* = \frac{1}{\sqrt{m}} J_0$ , cf. Sect. 1B), then the vertex operators  $Y(e^{\pm N\omega}, z)$  generate the gauge invariant subalgebra

$$(6.6) \quad \mathfrak{A}(m)^{\mathbb{Z}_N} = \mathfrak{A}(N^2m).$$

The  $\mathbb{Z}_2$ -orbifold of  $\mathfrak{A}(m)$  with  $m$  odd has an even gauge invariant subalgebra  $\mathfrak{A}(4m)$ . The representation theory of  $\mathfrak{A}(m)$ ,  $m = 2\rho + 1$ ,  $\rho \in \mathbb{Z}_+$ , can be deduced from this remark.

*Example 6.2.* Modular properties of characters of  $\mathfrak{A}(m = 2\rho + 1)$  derived from those for  $\mathfrak{A}(4m)$ . The characters  $K_\mu(\tau, m)$ ,  $m$  odd,  $\mu = \frac{1}{2}\mathbb{Z} \bmod m$  are expressed in terms of  $K_\nu(\tau, 4m)$  as follows:

$$(6.7) \quad K_\mu(\tau, m) = K_{2\mu}(\tau, 4m) + K_{2\mu+2m}(\tau, 4m).$$

The periodicity relation

$$(6.8) \quad K_{\nu+m}(\tau, m) = K_\nu(\tau, m)$$

allows to replace (if necessary) the indices in the right hand side of (6.7) by equivalent ones in the canonical interval (6.1). The resulting  $SL_2(\mathbb{Z})$  transformation properties of  $K_\mu(\tau, m)$  then read

$$(6.9a) \quad K_\mu(\tau + 1, m) = e^{i\pi \left( \frac{\mu^2}{m} - \frac{1}{12} \right)} \{ K_{2\mu}(\tau, 4m) + (-1)^{2\mu+m} K_{2\mu+2m}(\tau, 4m) \}$$

$$(6.9b) \quad \begin{aligned} K_\mu \left( -\frac{1}{\tau}, m \right) &= \frac{1}{\sqrt{m}} \sum_{\substack{\nu=1-m \\ (\nu \in \mathbb{Z})}}^m e^{-2\pi i \mu \nu / m} K_{2\nu}(\tau, 4m) \\ &= \frac{1}{\sqrt{m}} \sum_{\nu \bmod m} e^{-\frac{2\pi i \mu \nu}{m}} K_\nu(\tau, m). \end{aligned}$$

Thus, for  $m$  odd, only the entire set of  $4m$  characters  $K_\nu(\tau, 4m)$  is closed under  $SL_2(\mathbb{Z})$ . The original set  $\{K_\mu(\tau, m), \mu \in \mathbb{Z}/m\mathbb{Z}\}$ , corresponding to the Neveu-Schwarz sector of the supersymmetric theory, is however invariant under the

subgroup of the modular group generated by  $T^2(\tau \rightarrow \tau + 2)$  and  $S$ . It is remarkable that the diagonal partition function (in which we restore the dependence on the  $u(1)$  variable  $z$ ),

$$(6.10) \quad Z(\tau, z) = \sum_{\mu \bmod m} \chi_{\mu m}(\tau, z) \bar{\chi}_{\mu m}(\tau, z)$$

where

$$(6.11) \quad \chi_{\mu m}(\tau, z) = \sum_n q^{\frac{1}{2m}(mn+\mu)^2} e^{2\pi i z \frac{mn+\mu}{m}}$$

is related to the Laughlin plateaus of the quantum Hall effect (corresponding to filling factor  $\nu = \frac{1}{m}$ , charge  $\frac{n}{m}$  and fractional spin  $J = \frac{n^2}{2m}$ ,  $n \in \mathbb{Z}$  — see [CZ]). (The characters used in [CZ] differ from (6.11) by a non-analytic factor,  $\exp\{-\frac{\pi}{m} \frac{(Imz)^2}{Im\tau}\}$  corresponding to a modified Hamiltonian and ensuring invariance under  $z \rightarrow z + \tau$ .)

*Example 6.3. Charge conjugation orbifolds.* The involutive lattice conjugation

$$(6.12) \quad C_L : e^\omega \rightarrow e^{-\omega}, \quad J \rightarrow -J$$

provides, for  $m \neq 2$ , an example of an outer automorphism of the chiral algebra  $\mathfrak{A}(m)$ . Our construction of orbifold modules does not apply, strictly speaking, to this case. Nevertheless, it is easy to construct a modular invariant set of  $C_L$ -orbifold characters. We shall write them down for the bosonic ( $m = 2s$ ,  $s \in \mathbb{N}$ ) case.

The  $C_L$ -orbifold chiral algebra  $\mathfrak{A}(2s)^{C_L}$  is generated by a single primary field  $\phi = \phi(z, \omega)$  with respect to its  $\mathfrak{A}(S \otimes 1)^{C_L}$  subalgebra, the real part of the vertex operator  $Y(e^\omega, z)$ :

$$\phi(z, \omega) = \frac{1}{\sqrt{2}} \{Y(e^\omega, z) + Y(e^{-\omega}, z)\} .$$

Here  $\mathfrak{A}(S \otimes 1)$  is the  $u(1)$  chiral current subalgebra corresponding to the subspace  $S \otimes 1$  (1.16). The operator product expansion of two  $\phi$ 's involves the stress energy tensor  $T$  and the Virasoro primary field  $:J^4(z):$  that generates  $\mathfrak{A}(S \otimes 1)^{C_L}$ . The chiral algebra splits into a  $C_L$ -even and a  $C_L$ -odd parts. The vacuum module character splits, accordingly, into two pieces:

$$(6.13a) \quad K_0(\tau, 2s) = K_0^+(\tau, 2s) + K_0^-(\tau, 2s)$$

where

$$(6.13b) \quad K_0^\pm(\tau, 2s) = \frac{1}{2} \{K_0(\tau, 2s) \pm (K_0(\tau, 8) - K_4(\tau, 8))\} .$$

The difference of  $\mathbb{Z}_2$  twisted level 1  $A_1^{(1)}$  characters (that appears in parentheses) can be written in the form

$$(6.13c) \quad K_0(\tau, 8) - K_4(\tau, 8) = \frac{1}{\eta(\tau)} \sum_n (-q)^{n^2} .$$

Each pair of representations of weights  $\pm\mu\omega^*$  of  $\mathfrak{A}(2s)(|\omega^*|^2 = \frac{1}{2s})$  for  $1 \leq \mu \leq s-1$  gives rise to a single representation of the gauge invariant subalgebra  $\mathfrak{A}(2s)^{C_L}$ . The characters  $K_0^\pm$  (6.13), being expressed in terms of  $K_\mu$ , have known modular transformation properties; in particular,

$$(6.14) \quad K_0^\pm(-\frac{1}{\tau}, 2s) = \frac{1}{2\sqrt{2s}} \left\{ K_0^+(\tau, 2s) + K_0^-(\tau, 2s) + K_s(\tau, 2s) + 2 \sum_{\mu=1}^s K_\mu(\tau, 2s) \right\} \pm \frac{1}{\sqrt{2}} (K_1(\tau, 8) + K_3(\tau, 8)).$$

Analyzing this relation together with the unitarity requirement for the  $S$ -matrix one concludes that there are altogether  $s+7$  inequivalent representations of  $\mathfrak{A}(2s)^{C_L}$  (see [DV<sup>3</sup>]) corresponding to  $s+3$  untwisted and 4 twisted orbifold modules. The  $\mu = s$   $\mathfrak{A}(2s)$ -module splits, in particular, into two  $\mathfrak{A}(2s)^{C_L}$ -modules with the same specialized character

$$(6.15) \quad \frac{1}{2} K_s(\tau, 2s) = \frac{1}{\eta} \sum_{n=0}^{\infty} q^{s(n+\frac{1}{2})^2}$$

Similarly, there are two pairs of twisted representations with characters  $K_i(\tau, 8)$ ,  $i = 1, 3$ , each  $K_i$  appearing twice (with a coefficient  $\pm \frac{1}{2\sqrt{2}}$ ) in (6.14).

For  $s = 1$  the model reduces to a  $\mathbb{Z}_2$  affine orbifold. For  $s = 2, 3, 4$  and 6 it has been identified with known models in [DV<sup>3</sup>]. We conjecture that these  $C_L$ -orbifolds can be shown to exist for all values of  $s$  using the vertex operator construction of Sect. 1A.

### 6B. SU(2) orbifolds.

The finite subgroups of SU(2) being thoroughly studied,<sup>1</sup> the  $\mathfrak{A}_k(\mathfrak{su}(2))$  orbifold characters and their modular properties can be worked out quite explicitly. Noting that the Cartan subalgebra of  $\mathfrak{su}(2)$  is 1-dimensional we can express its elements  $\alpha, \beta, \gamma, \lambda$  by (rational) numbers identifying each of them with the coefficient to  $\Lambda_1^\vee = \frac{1}{2}\sigma_3$  ( $\sigma_j$  are the Pauli matrices — see (6.23)); then

$$(6.16) \quad |\gamma - \beta|^2 = \frac{1}{2} \left( 2n + \frac{\lambda}{k} - \beta \right)^2, \quad n \in \mathbb{Z}, \quad \lambda = 1 - k, \dots, 0, 1, \dots, k,$$

$$(\gamma|\alpha) = \left( n + \frac{\lambda}{2k} \right) \alpha, \quad \alpha, \beta \in \mathbb{Q}.$$

The character (4.36), (3.18), (3.3) can be written in the form

$$(6.17) \quad \chi_{\Lambda, \bar{b}, \sigma}^\beta(\tau) = \sum_{\substack{\lambda=1-k \\ \Lambda-\lambda \in 2\mathbb{Z}}}^k c_\lambda^\Lambda(\tau) \Theta_{\lambda, k, \sigma}^\beta(\tau),$$

---

<sup>1</sup>For a modern treatment based on the McKay correspondence — see [Kos].

where

$$(6.18) \quad \Theta_{\lambda,k,\sigma}^{\beta}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{k}{4}(2n + \frac{\lambda}{k} - \beta)^2} \sigma_{2kn+\lambda} ,$$

$$(6.19) \quad \sigma_{2kn+\lambda} = \frac{1}{|\Gamma_b|} \sum_{\substack{h \in \Gamma_b \\ \text{tr } h = 2 \cos \pi \alpha}} \sigma^*(h) e^{i\pi(2kn+\lambda)\alpha} .$$

For  $b \neq 1$  and non-exceptional,  $\Gamma_b$  is abelian and  $h$  can be assumed diagonal.

We have treated in Sects. 2, 3 and 6A the case of a  $\mathbb{Z}_N$  orbifold (as an automorphism group of  $\mathfrak{A}(\text{SU}(2))$ ,  $\mathbb{Z}_N$  appears as a subgroup of  $\text{SO}(3)$ ;  $\Gamma$  in this case should be identified with its double cover  $\mathbb{Z}_{2N} \subset \text{SU}(2)$ ). Each  $\mathbb{Z}_N$  automorphism group leaves a  $u(1)$  (Cartan) current invariant. The remaining *non-abelian subgroups of  $\text{SO}(3)$*  can be described as groups on two generators,  $s$  and  $t$ , obeying three relations:

$$(6.20) \quad s^{n_1} = t^{n_2} = (st)^{n_3} = 1 , \quad \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{|\text{Ad } \Gamma|} > 1$$

( $n_1, n_2, n_3$  are natural numbers and we denote the group unit by 1). The double cover  $\Gamma (\subset \text{SU}(2))$  of  $\text{Ad } \Gamma$  is again generated by two elements  $s$  and  $t$  but the group unit in the first relation (6.20) is replaced by the non-trivial central element  $\varepsilon$  of  $\text{SU}(2)$ :

$$(6.21) \quad s, t \in \Gamma \Rightarrow s^{n_1} = t^{n_2} = (st)^{n_3} = \varepsilon , \quad \varepsilon^2 = 1 (|\Gamma| = 2|\text{Ad } \Gamma|) .$$

*Example 6.4. The  $\mathbb{H}_8 \subset \text{SU}(2)$  orbifold.* The abstract group of quaternion units has 8 elements,  $\{1, \varepsilon, q_i, \varepsilon q_i, i = 1, 2, 3\}$ ; they obey multiplication rules  $q_i^2 = \varepsilon$ ,  $q_1 q_2 = q_3$  which fit (6.21) with  $n_1 = n_2 = n_3 = 2$ . It corresponds (according to McKay) to the affine Dynkin diagram  $D_4^{(1)}$  (see [K1] Chap. 4, Table Aff 1). The dimensions of its non-trivial representations coincide with the coefficients  $a_j$  in the expansion of the highest root  $\theta$  of  $D_4$  in terms of simple roots:

$$(6.22) \quad \theta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 .$$

We shall denote the (equivalence classes of) irreducible representations (IR) of  $\mathbb{H}_8$  by the simple roots  $\alpha_\nu$  of  $D_4^{(1)}$  ( $\alpha_0$  corresponding to the trivial representation). Then  $\alpha_2$  maps  $\mathbb{H}_8$  into a subgroup of  $\text{SU}(2)$ :

$$(6.23) \quad \alpha_2(q_j) = \frac{1}{i} \sigma_j , \quad j = 1, 2, 3 \quad \left( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \sigma_2 = i\sigma_1\sigma_3 \right) .$$

We reproduce in Table 1, for reader's convenience, the character table for  $\Gamma = \mathbb{H}_8$  also indicating the centralizer  $\Gamma_g$  of an element in each conjugacy class (CC).

TABLE 1.  $\Gamma = \mathbb{H}_8$ : characters and centralizers.

$IR \backslash cc$	1	$\varepsilon$	$\{q_1, \varepsilon q_1\}$	$\{q_2, \varepsilon q_2\}$	$\{q_3, \varepsilon q_3\}$
$\alpha_0$	1	1	1	1	1
$\alpha_1$	1	1	1	-1	-1
$\alpha_2$	2	-2	0	0	0
$\alpha_3$	1	1	-1	1	-1
$\alpha_4$	1	1	-1	-1	1
$\Gamma_g$	$\Gamma$	$\Gamma$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$

Using Table 1 and symmetrizing with respect to  $2kn + \lambda$  we compute the sum (6.19) for the *untwisted characters* (i.e., for  $\Gamma_g = \Gamma$ ,  $\beta = 0$ ):

$$\begin{aligned}
(\alpha_0)_{2kn+\lambda} &= \frac{1}{8} [1 + (-1)^\lambda] [1 + 3(-1)^{kn} i^\lambda], \\
(\alpha_j)_{2kn+\lambda} &= \frac{1}{8} [1 + (-1)^\lambda] [1 - (-1)^{kn} i^\lambda], \quad j = 1, 3, 4, \\
(\alpha_2)_{2kn+\lambda} &= \frac{1}{4} [1 - (-1)^\lambda].
\end{aligned}
\tag{6.24}$$

Inserting these expressions in (6.17), (6.18) we recover for  $k = 1$  the characters (6.13) of the  $C_L$ -orbifold for  $s = 4$ :

(6.25a)

$$\begin{aligned}
k = 1 : \chi_{0,1,\alpha_0}(\tau) &= \frac{1}{4\eta(\tau)} \sum_n [1 + 3(-1)^n] q^{n^2} = K_0^+(\tau, 8), \\
\chi_{0,1,\alpha_j}(\tau) &= \frac{1}{2\eta(\tau)} \sum_n q^{(2n+1)^2} = \frac{1}{2} K_4(\tau, 8), \quad j = 1, 3, 4, \quad m = 0, 1, \\
\chi_{1,1,\alpha_2}(\tau) &= \frac{(-1)^m}{2\eta(\tau)} \sum_n q^{\frac{1}{4}(2n+1)^2} = K_2(\tau, 8) \left( = \frac{1}{2} K_1(\tau, 2) \right),
\end{aligned}$$

where

$$(6.25b) \quad K_0^+(\tau, 8) = K_0(\tau, 8) - \frac{1}{2} K_4(\tau, 8).$$

The characters of the  $\mathbb{Z}_2$ -twisted orbifolds are also computed from (6.17), (6.18) for  $\beta = \frac{1}{2}$  and  $\sigma(q_j^\mu) = i^{\sigma\mu}(q_j^\mu)$ ,  $\mu \in \mathbb{Z}/4\mathbb{Z}$  is the general form of an element of the centralizer  $\mathbb{Z}_4$  of  $q_j$ ). Equation (6.19) then gives

$$(6.26) \quad \sigma_{2kn+\lambda} = \frac{1}{4} \sum_{\mu \bmod 4} i^{2kn+\lambda-\sigma}\mu = \frac{1 + (-1)^{\lambda-\sigma}}{4} [1 + (-1)^{kn} i^{\lambda-\sigma}] ,$$

reproducing, for  $k = 1$  the  $C_L$ -twisted characters of  $\mathfrak{A}(8)$ :

$$(6.27) \quad \begin{aligned} \chi_{0,\bar{q}_j,0}(\tau) &= \sum_n q^{\frac{1}{4}(4n-\frac{1}{2})^2} = K_1(\tau, 8) = \chi_{1,\bar{q}_j,1}(\tau) , \quad j = 1, 2, 3 , \\ \chi_{0,\bar{q}_j,2}(\tau) &= K_3(\tau, 8) = \chi_{1,\bar{q}_j,-1} . \end{aligned}$$

(We label throughout the irreducible representations of  $\mathbb{Z}_4$  — and their characters — by the exponents  $\sigma = 0, \pm 1, 2$ .)

The number of inequivalent orbifold modules of a level 1 current algebra (for a simple  $\mathfrak{g}$ ) is

$$(6.28) \quad N(\Gamma \subset G; k = 1) = \frac{1}{|Z|} \sum_{\bar{g} \subset \Gamma} |\hat{\Gamma}_{\bar{g}}| .$$

In the case at hand it is  $\frac{1}{2}(5 + 5 + 3 \times 4) = 11$  thus coinciding with the number  $s + 7$  of  $C_L$ -orbifold modules for  $s = 4$ .

Equations (6.24) and (6.26) also allow to compute orbifold characters for higher levels; in particular, for  $k = 2, g = 1$ , we obtain (expressing the string functions  $c_\lambda^\Lambda$  in terms of the *branching coefficients*  $b_\lambda^\Lambda = \eta^l c_\lambda^\Lambda$ , for a rank  $l$   $\mathfrak{g}$  — see [K1] Sect. 12.12):

$$(6.29a) \quad \begin{aligned} \chi_{\Lambda,1,\alpha_0}(\tau) &= \frac{1}{\eta(\tau)} \left\{ b_0^\Lambda(\tau) \sum_n q^{2n^2} - \frac{1}{2} b_2^\Lambda(\tau) \sum_n q^{\frac{1}{2}(2n+1)^2} \right\} \\ &= b_0^\Lambda(\tau) K_0(\tau, 4) - \frac{1}{2} b_2^\Lambda(\tau) K_2(\tau, 4) , \quad \Lambda = 0, 2 , \end{aligned}$$

$$(6.29b) \quad \chi_{\Lambda,1,\alpha_j}(\tau) = \frac{1}{2} b_2^\Lambda(\tau) K_2(\tau, 4) , \quad j = 1, 3, 4 , \quad \Lambda = 0, 2 ,$$

$$(6.29c) \quad \chi_{1,1,\alpha_2}(\tau) = b_1^1(\tau) K_1(\tau, 4) \text{ (since } b_\lambda^\Lambda = b_{-\lambda}^\Lambda \text{)} .$$

Similarly, using (6.26), we can evaluate the twisted characters. For those permuted by the action of the centre we find

$$(6.30a) \quad \begin{aligned} \chi_{0,\bar{q}_j,0}(\tau) &= b_0^0(\tau) K_1(\tau, 4) = \chi_{2,\bar{q}_j,2}(\tau) , \\ \chi_{2,\bar{q}_j,0}(\tau) &= b_0^2(\tau) K_1(\tau, 4) = \chi_{0,\bar{q}_j,2}(\tau) , \quad j = 1, 2, 3 . \end{aligned}$$

The remaining twisted characters are split by the action of the centre, and we only obtain their sums:

$$(6.30b) \quad \begin{aligned} \chi_{1, \bar{q}_j, 1}^+(\tau) + \chi_{1, \bar{q}_j, 1}^-(\tau) &= b_1^1(\tau) K_0(\tau, 4), \\ \chi_{1, \bar{q}_j, -1}^+(\tau) + \chi_{1, \bar{q}_j, -1}^-(\tau) &= b_1^1(\tau) K_2(\tau, 4). \end{aligned}$$

Here the branching coefficients can be expressed in terms of the Virasoro characters  $\chi_\Delta(\tau, c)$  of the Ising model (corresponding to  $c = \frac{1}{2}, \Delta = 0, \frac{1}{16}, \frac{1}{2}$ ):

$$(6.30c) \quad \begin{aligned} b_0^0(\tau) &= b_2^2(\tau) = \chi_0\left(\tau, \frac{1}{2}\right), \\ b_0^2(\tau) &= b_2^0(\tau) = \chi_{\frac{1}{2}}\left(\tau, \frac{1}{2}\right), \\ b_1^1(\tau) &= b_{-1}^1(\tau) = \chi_{\frac{1}{16}}\left(\tau, \frac{1}{2}\right). \end{aligned}$$

It follows from (6.29) and (6.30) that there are  $2 \times 4 + 1 = 9$  untwisted and  $3 \times 6 = 18$  twisted level 2 orbifold modules or altogether  $27 \mathfrak{A}_2(\mathfrak{su}(2))^{\mathbb{H}_8}$ -representations.

*Example 6.5.* Group theoretic  $S$ -matrix and fusion rules for  $\mathbb{H}_8 \subset \text{SU}(2)$  and for  $\mathbb{H}_8 \subset \text{SU}(2) \subset E_8$ . The simply connected compact group  $E_8$  is singled out (among the Lie groups with simple simply laced Lie algebras) for having a trivial centre. The corresponding current algebra has a single level 1 representation, the vacuum  $\mathfrak{A}_1(E_8)$  module; the modular  $S$ -matrix is then the identity operator (multiplication by 1). Hence, if  $\Gamma$  is a (non-exceptional) finite subgroup of  $E_8$  then the  $\Gamma \subset E_8$  group theoretic  $S$ -matrix coincides with the  $\mathfrak{A}_1(E_8)^\Gamma$  orbifold  $S$ -matrix. The possibility to embed the pair  $\mathbb{H}_8 \subset \text{SU}(2)$  in  $E_8$  thus provides an additional justification for the study of the group theoretic  $S$ -matrix per-se.

We observe that the  $S$ -matrix elements depend on both the Lie group  $G$  containing the pair  $\mathbb{H}_8 \subset \text{SU}(2)$  and on the *level of embedding* of  $\text{SU}(2)$  in  $G$  which is defined as follows. Let the bases in  $\mathfrak{su}(2)$  and  $\mathfrak{g}$  be chosen in such a way that the Cartan generator  $H$  of  $\mathfrak{su}(2)$  is expressed as a linear combination of the Cartan generators  $H^i$  with non-negative integer coefficients  $m_i$ :  $H = \sum_{i=1}^l m_i H^i$ . Then the integers  $m_i$  satisfy the quadratic relation

$$\frac{1}{2} \sum_{i,j=1}^l m_i a_{ij} m_j = \sum_{j=1}^l m_j =: N,$$

where, for a simply laced  $\mathfrak{g}$ ,  $(a_{ij})$  is its Cartan matrix. The positive integer  $N$  is the level of embedding of  $\mathfrak{su}(2)$  in  $\mathfrak{g}$ .

For a level 1 embedding the  $S$ -matrix elements involving at least one non-exceptional entry are independent of  $G$ . In the case of  $\mathbb{H}_8$  the phase factor in (4.25) for a non-exceptional  $b$  and an arbitrary  $g$  is only non-trivial if both  $b$  and  $g$  belong to the same conjugacy class  $q_j$ . We shall then set

$$(6.31) \quad \beta(\varepsilon^m q_3) = \frac{(-1)^{m-1}}{4} \sigma_3 \Rightarrow \exp\{-2\pi i k(\beta(\varepsilon^m q_j) | \beta(\varepsilon^n q_j))\} = \exp\left\{(-1)^{m+n} \frac{k\pi}{4i}\right\}$$

Omitting the upper index  $\beta$  on  $S$  (for this fixed choice) we obtain

$$(6.32a) \quad 4S_{\varepsilon^m \alpha_\mu, \bar{q}_j \sigma} = (-1)^{m\sigma} \alpha_\mu(q_j)$$

$$(6.32b)$$

$$2S_{\bar{q}_j \sigma, \bar{q}_j \sigma'} = \frac{i^{\sigma+\sigma'}}{2} \left\{ e^{-i\frac{k\pi}{4}} + (-1)^{\sigma+\sigma'} e^{i\frac{k\pi}{4}} \right\} = \cos \left\{ \left( \sigma + \sigma' - \frac{k}{2} \right) \frac{\pi}{2} \right\} .$$

(In computing the sum in the 2 elements  $b = \pm q_j$  of the conjugacy class  $\bar{q}_j$  in the expression (4.25) for  $S$  it is important to change at the same time  $\sigma$  according to (4.12). This yields (6.32b).)

The only  $G$  dependence appears if the central element  $\varepsilon$  of  $\Gamma$  is present in both entries:

$$(6.33a) \quad 8S_{\varepsilon^m \alpha_\mu, \varepsilon^n \alpha_\nu} = p_\varepsilon^{mnk} (-1)^{n\delta\mu_2 + m\delta\nu_2} 2^{\delta\mu_2 + \delta\nu_2}, \quad p_\varepsilon := e^{-2\pi i |\beta(\varepsilon)|^2},$$

where  $\beta(\varepsilon) = 0$  if  $G = \text{SU}(2)$ , or, more generally, if it is an exceptional element of  $\Gamma \subset G$ , while

$$(6.33b) \quad p_\varepsilon = -1 \text{ if } \Gamma_{\beta(\varepsilon)} = \Gamma_\varepsilon$$

(in a level 1 embedding). It turns out that the fusion rules involving a pair of  $q_j$  and an  $\varepsilon$  are integer iff  $\varepsilon$  is a regular element of  $\Gamma \subset G$  (i.e., if (6.33b) takes place). Indeed we have

$$N_{\bar{q}_j \sigma_1, \bar{q}_j \sigma_2, 1\alpha_\mu} = \frac{1 + (-1)^{\sigma_1 + \sigma_2 + \delta\mu_2}}{4} \alpha_\mu(1) + \frac{\alpha_\mu(q_j)}{2} \cos \pi \frac{\sigma_1 - \sigma_2}{2},$$

which is a  $k$  independent non-negative integer, but

$$N_{\bar{q}_j \sigma_1, \bar{q}_j \sigma_2, \varepsilon \alpha_\mu} = \frac{1 + p_\varepsilon^k (-1)^{\sigma_1 + \sigma_2 + \delta\mu_2}}{4} \alpha_\mu(1) + \frac{\alpha_\mu(q_j)}{2} \cos \left( \frac{k - \sigma_1 - \sigma_2}{2} \pi \right)$$

which is only integer for odd  $k$  if  $p_\varepsilon = -1$ .

*Remark 6.1.* Equation (6.33b) always takes place for a level 1 embedding  $\text{SU}(2) \subset E_8$ . In spite of the fact that  $\varepsilon$  is an involution ( $\varepsilon^2 = 1$ ) and every involution in  $E_8$  is exceptional (as a consequence of the description of finite order automorphisms of a simple Lie algebra presented in Appendix B)  $\varepsilon$  is not exceptional in  $\Gamma \subset \text{SU}(2) \subset E_8$  whenever  $\text{SU}(2)$  is generated by a pair of opposite roots of  $E_8$  — which is always the case (up to conjugation) for a level 1 embedding. In other words  $(E_8)_{\beta(\varepsilon)}$  is strictly smaller than  $(E_8)_\varepsilon$  but  $\text{SU}(2) \cap (E_8)_{\beta(\varepsilon)} = \text{SU}(2) \cap (E_8)_\varepsilon$ . By contrast, for the maximal embedding  $\text{SU}(2) \subset E_8$  given by

$$E = \sum_{i=1}^8 E^{\alpha_i}, \quad H = 2\rho,$$

$\varepsilon$  is exceptional in  $\Gamma \subset E_8$ . However, the level of this embedding,

$$N = 2(\rho|\rho) = g^\vee \dim E_8/6 = 1240 ,$$

is divisible by 4, hence the group theoretic fusion rules (with  $\beta(\varepsilon) = 0 = 1 - p_\varepsilon$ ) coincide with those of the Grothendieck ring proven to be non-negative integers in [Lus1]. We have an exceptional subgroup  $\Gamma \subset \text{SO}(3) \subset E_8$  in this case. The image of any 4-th order element of  $\tilde{\Gamma} \subset \text{SU}(2)$  is an involution whose centralizer in  $\text{SO}(3)$  is disconnected (see the discussion at the end of Appendix B). It is likely that at least in the case when orders of all elements of  $\Gamma$  divide  $N$  the corresponding twisted orbifold modules do exist and the resulting modular  $S$ -matrix consider with the one for the Grothendieck ring. To compute the fusion rules for the  $\mathfrak{A}_k(\text{SU}(2))^\Gamma$  orbifold we shall use the (non-factorizable)  $|\mathcal{X}| \times |\mathcal{X}|$   $S$ -matrix of the full theory.

For the *level 1 orbifold* ordering the states as  $(\Lambda, \bar{b}, \sigma) = (0, 1, \alpha_\nu), \nu = 0, 1, 3, 4, (1, \bar{1}, \alpha_2), (0, \bar{q}_j, 0) (\simeq (1, \bar{q}_j, 1)), (1, \bar{q}_j, -1) (\simeq (0, \bar{q}_j, 2)), j = 1, 2, 3$ , we can write the  $11 \times 11$   $S$ -matrix as

$$2\sqrt{2}S = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & \sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & -\sqrt{2} \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} \end{bmatrix} .$$

The resulting fusion rules differ, in general, from the group theoretic ones even for admissible entries. We have, for instance,

$$N_{0\bar{q}_j 0, \Lambda \bar{q}_j - \Lambda, 11\alpha_2} = 1 \text{ for } \Lambda = 0, 1 , \text{ while } N_{\bar{q}_j 0, \bar{q}_j - \Lambda, \bar{1}\alpha_2} = \frac{1 + (-1)^{1-\Lambda}}{2} ,$$

$$N_{0\bar{q}_j 0, 1\bar{q}_j - 1, 0\bar{1}\alpha_\mu} = \frac{1 - \alpha_\mu(q_j)}{2} \text{ for } \mu \neq 2 .$$

$$\text{while } N_{\bar{q}_j 0, \bar{q}_j - 1, 1\alpha_\mu} = 0 \text{ for } \mu \neq 2 .$$

*Example 6.6. The  $\mathfrak{A}_2(\text{su}(2))^{\mathbb{H}_3}$  orbifold and its Clifford algebra extension.* The study of level 2  $\text{SU}(2)$ -orbifolds is simplified by the observation that  $\mathfrak{A}_2 \equiv \mathfrak{A}_2(\text{su}(2))$  is the even part of the Clifford algebra  $Cl_3$  of 3 anticommuting Majorana-Weyl spinor fields  $\psi_j(z)$ ,  $j = 1, 2, 3$ . Indeed, the  $\Lambda = 2$   $\mathfrak{A}_2$ -module is generated by an ‘‘isotopic triplet’’ of primary fields of dimension  $\Delta_\Lambda = \frac{1}{4h}\Lambda(\Lambda + 2)$  (for  $\Lambda = k = 2, h = k + 2 = 4$ ), the Virasoro central charge being  $c = 3\frac{k}{h} = \frac{3}{2}$ . The fields  $\psi_j(z)$  are single-valued in the vacuum (Neveu-Schwarz) sector and satisfy the canonical anticommutation relations (and hermiticity)

$$[\psi_i(z), \psi_j(w)]_+ = \delta_{ij}\delta(z - w) , \quad \psi_j^* = \psi_j , \quad i, j = 1, 2, 3 .$$

The  $\mathbb{Z}_2$  graded algebra  $Cl_3$  (with odd generators  $\psi_j(z)$ ) provides a superconformal extension of  $\mathfrak{A}_2$  whose  $SU(2)$  invariant subalgebra is generated by the  $\Delta = \frac{3}{2}$  partner

$$(6.34a) \quad G(z) = i\psi_1(z)\psi_2(z)\psi_3(z) (= G^*(z))$$

of the stress energy tensor

$$(6.34b) \quad T(z) = T_1(z) + T_2(z) + T_3(z), \quad T_j(z) = \frac{1}{4} : [\partial\psi_j(z), \psi_j(z)]$$

which can be viewed as a composite of two  $G$ -fields. The generator  $G(z)$  of the super-Virasoro algebra is a primary field with respect to  $T$  but not with respect to  $\mathfrak{A}_2$ ; its commutator with a Cartan current is

$$[J(z), G(w)] = \delta'(z-w)\psi_3(w) \text{ for } J(z) = -i\psi_1(z)\psi_2(z).$$

It intertwines the  $\Lambda = 0$  and  $\Lambda = 2$  Neveu-Schwarz modules mapping the  $\Lambda = 1$  Ramond sector into itself.

Each subgroup  $\Gamma$  of  $SU(2)$  acts on  $Cl_3$  by automorphisms which form the adjoint group

$$(6.35) \quad Ad_\Gamma = \Gamma/\mathbb{Z}_2 \subset SO(3); \text{ for } \Gamma = \mathbb{H}_8, \quad Ad_\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

In the (orthonormal  $SO(3)$ ) basis  $\{\psi_j\}$  the non-trivial elements  $E_j = \alpha_2(q_j)$  of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  act as diagonal matrices:

$$(6.36) \quad E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_3 = E_1 E_2.$$

The  $Ad_\Gamma (= \mathbb{Z}_2 \times \mathbb{Z}_2)$  invariant subalgebra  $Cl_3^\Gamma$  ( $\Gamma = \mathbb{H}_8$ ) of the  $Cl_3$  superalgebra is generated by  $G$  and by the individual stress-tensors  $T_j$  of the 3 ‘‘Ising models’’ (associated with each  $\psi_j$ ) — see (6.34b). The 3 commuting ( $\Delta = 2$ ) field operators  $T_j(z)$  give rise to the even part  $\mathfrak{A}_2^\Gamma$  of this superalgebra. Its positive energy representations are tensor products of irreducible representations of the 3 (minimal) Ising models. There are, as expected,  $3^3 = 27$  such  $\mathfrak{A}_2^\Gamma$  orbifold modules. In particular, the characters of the fixed point modules split into a sum of two irreducible characters:

$$(6.37) \quad \begin{aligned} \chi_{1, \bar{q}_j, 1}(\tau) &= b_1^1(\tau) K_0(\tau, 4) = b_1^1(\tau) \{ [b_0^0(\tau)]^2 + [b_0^2(\tau)]^2 \}, \\ \chi_{1, \bar{q}_j, -1}(\tau) &= b_1^1(\tau) K_2(\tau, 4) = 2b_1^1(\tau) b_0^2(\tau) b_0^0(\tau). \end{aligned}$$

The asymptotic dimensions of  $b_1^1(b_0^\Lambda)^2$  for  $\Lambda = 0, 2$  indeed coincide, (the quantum dimension of the  $(c = \frac{1}{2}, \Delta = \frac{1}{2})$  module being 1. Here we have used the

expression (6.30) of the Ising model characters in terms of the branching coefficients. The remaining orbifold modules are identified in the tensor product  $(\Delta_1, \Delta_2, \Delta_3)(\Delta_i = 0, \frac{1}{16}, \frac{1}{2})$  of three Ising modules as follows:

$$\begin{aligned}
(0, \bar{1}, \alpha_0) &= (0, 0, 0) & (2, 1, \alpha_0) &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \\
(0, \bar{1}, \alpha_1) &= \left(0, \frac{1}{2}, \frac{1}{2}\right) & (2, 1, \alpha_1) &= \left(\frac{1}{2}, 0, 0\right) \\
(0, \bar{1}, \alpha_3) &= \left(\frac{1}{2}, 0, \frac{1}{2}\right) & (2, 1, \alpha_3) &= \left(0, \frac{1}{2}, 0\right) \\
(0, \bar{1}, \alpha_4) &= \left(\frac{1}{2}, \frac{1}{2}, 0\right) & (2, 1, \alpha_4) &= \left(0, 0, \frac{1}{2}\right) \\
(1, \bar{1}, \alpha_2) &= \left(\frac{1}{16}, \frac{1}{16}, \frac{1}{16}\right) \\
(0, \bar{q}_1, 0) &= \left(0, \frac{1}{16}, \frac{1}{16}\right) & (2, \bar{q}_1, 0) &= \left(\frac{1}{2}, \frac{1}{16}, \frac{1}{16}\right) \\
(0, \bar{q}_2, 0) &= \left(\frac{1}{16}, 0, \frac{1}{16}\right) & (2, \bar{q}_2, 0) &= \left(\frac{1}{16}, \frac{1}{2}, \frac{1}{16}\right) \\
(6.38a) \quad (0, \bar{q}_3, 0) &= \left(\frac{1}{16}, \frac{1}{16}, 0\right) & (2, \bar{q}_3, 0) &= \left(\frac{1}{16}, \frac{1}{16}, \frac{1}{2}\right) ;
\end{aligned}$$

the reducible (fixed point) modules with characters (6.37) split according to the law

$$\begin{aligned}
(1, \bar{q}_1, 1) &= \left(\frac{1}{16}, 0, 0\right) + \left(\frac{1}{16}, \frac{1}{2}, \frac{1}{2}\right) \\
(6.38b) \quad (1, \bar{q}_1, -1) &= \left(\frac{1}{16}, \frac{1}{2}, 0\right) + \left(\frac{1}{16}, 0, \frac{1}{2}\right) , \text{ etc.}
\end{aligned}$$

The  $\mathfrak{A}_2^{\mathbb{H}_8}$   $S$ -matrix is the tensor product of 3 Ising model  $S$ -matrices of the form

$$(6.39) \quad S_{\text{Ising}} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} .$$

We note that while  $S_{1\bar{q}_1\sigma, 1\bar{q}_2\sigma'} = 0$  according to (4.25) (since the conjugacy classes  $\bar{q}_1$  and  $\bar{q}_2$  do not contain commuting elements) the corresponding split  $S$ -matrix elements do not vanish:

$$\begin{aligned}
(6.40) \quad \left(S_{\frac{1}{16}\Delta_2\Delta_3, \Delta'_1\frac{1}{16}\Delta'_3}\right) &= \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} , \\
(\Delta_i, \Delta_j) &= (0, 0), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right) .
\end{aligned}$$

Note that the sum of  $\mathfrak{A}_2^\Gamma$ -modules in each line of equation (6.38) is irreducible with respect to the conformal superalgebra  $Cl_3^\Gamma$ . The characters of the subset of Neveu-Schwarz modules spanned by the direct sum of  $\Lambda = 0$  and  $\Lambda = 2$  representations give rise to a 7-dimensional representation of the subgroup  $\Gamma^0(2)$  of  $SL_2(\mathbb{Z})$  generated by  $T^2$  and  $S$ . In particular, the Neveu-Schwarz  $S$ -matrix is

$$(6.41) \quad S_{NS} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 \\ 2 & 2 & -2 & -2 & 0 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & 0 & 0 \\ 2 & -2 & -2 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

The importance of this example stems from the fact that it has a bearing on other  $SU(2)$  orbifold models. The three conjugacy classes of imaginary quaternion units  $\{\pm q_j, j = 1, 2, 3\}$  of  $\mathbb{H}_8$  combine in a single 6-element conjugacy class in the binary tetrahedral group  $\tilde{A}_4$  which in turn is a part of a 12-element conjugacy class of the binary octahedral group  $\tilde{S}_4$  and of a 30 element class of the binary icosahedral group  $\tilde{A}_5$ . Here  $S_n$  is the permutation group of  $n$  letters,  $A_n$  is its alternating invariant subgroup,  $\tilde{G} \subset SU(2)$  denotes, in general, the double cover of a subgroup  $G$  of  $SO(3)$ . In all three cases the centralizer  $\Gamma_{q_j}$  of an element  $q_j$  of this conjugacy class is  $\mathbb{Z}_4$ . Hence, the reducible character  $\chi_{1\bar{q}_j\sigma}$  is the same for all three orbifold modules and splits in the same way — according to (6.37) — for all three binary polyhedral groups. There are no other conjugacy classes  $\bar{b}$  in either  $\tilde{A}_4$  or  $\tilde{A}_5$  such that both  $b$  and  $\varepsilon b$  belong to  $\bar{b}$ . Furthermore, for all finite  $SU(2)$  subgroups  $\Gamma$  the Neveu-Schwarz module of  $Cl_3^\Gamma$  contain no fixed points and give rise to a  $\Gamma^0(2)$ -invariant subset of characters. Furthermore, a similar argument extends to a level  $n$  representation of  $SU(n)$  which also involves fixed points of the action of the centre. Indeed, there is a conformal embedding

$$\mathfrak{A}_n \equiv \mathfrak{A}_n(\mathfrak{su}(n)) \subset \mathfrak{A}_1(\mathfrak{spin}(n^2 - 1)) \left( c = \frac{1}{2}(n^2 - 1) \right)$$

allowing to extend an  $\mathfrak{A}_n^\Gamma$  orbifold to a  $Cl_{n^2-1}^\Gamma$ -orbifold.

The  $k = 1$  tetrahedral ( $\tilde{A}_4 \subset SU(2)$ ) orbifold and its fusion rules are displayed in [DV<sup>3</sup>]. The octahedral ( $\tilde{S}_4 \subset SU(2)$ ) and the icosahedral ( $\tilde{A}_5 \subset SU(2)$ ) orbifolds can be studied with equal ease. We shall reproduce in Table 2 for a later reference the character table for the 120 element binary icosahedral group  $\tilde{A}_5$  (associated with  $E_8^{(1)}$  under the McKay correspondence).

Equation (6.28) implies:  $N(\tilde{A}_5 \subset SU(2); k = 1) = \frac{1}{2}(9 \times 2 + 10 \times 4 + 6 \times 2 + 4) = 37$ . It is a straightforward exercise to write down, using Table 2, the characters of  $\mathfrak{A}_1(\mathfrak{su}(2))^{\tilde{A}_5}$ .

TABLE 2. Characters of  $A_5 = \tilde{A}_5/\mathbb{Z}_2$  and of its double cover  $\Gamma = \tilde{A}_5$ .

$cc \backslash IR$	1	$\overline{\{p, p^4\}}$	$\overline{\{p^2, p^3\}}$	$\overline{\{t, t^2\}}$	$\overline{E} = \overline{\alpha_2(q)}$				
$\alpha_0$	1	1	1	1	1				
$\alpha_2$	3	$x_+$	$x_-$	0	-1				
$\alpha_4$	5	0	0	-1	1				
$\alpha_6$	4	-1	-1	1	0				
$\alpha_8$	3	$x_-$	$x_+$	0	-1				
$(A_5)_g$	$A_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2$				
$cc \backslash IR$	1	$\varepsilon$	$\bar{p}$	$\bar{p}^4$	$\bar{p}^2$	$\bar{p}^3$	$\bar{t}$	$\bar{t}^2$	$\bar{q}$
$\alpha_1$	2	-2	$x_+$	$-x_+$	$-x_-$	$x_-$	1	-1	0
$\alpha_3$	4	-4	1	-1	-1	1	-1	1	0
$\alpha_5$	6	-6	-1	1	1	-1	0	0	0
$\alpha_7$	2	-2	$x_-$	$-x_-$	$-x_+$	$x_+$	1	-1	0
$\Gamma_g$	$\Gamma = \tilde{A}_5$		$\mathbb{Z}_{10}$		$\mathbb{Z}_{10}$		$\mathbb{Z}_6$		$\mathbb{Z}_4$

$$A_5 = \alpha_2(\tilde{A}_5), \quad x_{\pm} = \frac{1 \pm \sqrt{5}}{2}, \quad \Gamma = \alpha_1(\tilde{A}_5) \simeq \tilde{A}_5, \quad p^5 = t^3 = q^2 = \varepsilon, \\ \theta^{E_8} = 2(\alpha_1 + \alpha_7) + 3(\alpha_2 + \alpha_8) + 4(\alpha_3 + \alpha_6) + 5\alpha_4 + 6\alpha_5.$$

6C. A LEVEL 1  $SU(3)$  ORBIFOLD. CHARGE CONJUGATION  
ASSOCIATED WITH A NON-ABELIAN CENTRALIZER.

We shall consider the subgroup  $\Gamma$  of  $SU(3)$  of order  $|\Gamma| = 1080$  which is a non-trivial central extension of the simple alternating group  $A_6 : 1 \rightarrow \mathbb{Z}_3 \rightarrow \Gamma \rightarrow A_6 \rightarrow 1$ . It is generated by the (60 element) isoahedral group  $A_5 \subset SO(3)$  and by one more element of order 2. In a basis in which a selected  $\mathbb{Z}_2 \times \mathbb{Z}_2$  subgroup of  $A_5$  (see Table 2) is generated by any two of the matrices  $E_i (= \alpha_2(q_i), i = 1, 2, 3)$  given by (6.36) while the generators of its  $\mathbb{Z}_3$  and  $\mathbb{Z}_5$  subgroups are chosen as

$$(6.42) \quad t = \frac{1}{2} \begin{pmatrix} x_- & 1 & -x_+ \\ 1 & x_+ & -x_- \\ x_+ & x_- & -1 \end{pmatrix}, \quad p = \frac{1}{2} \begin{pmatrix} -x_- & 1 & -x_+ \\ -1 & x_+ & -x_- \\ x_+ & -x_- & 1 \end{pmatrix},$$

where  $t^3 = p^5 = (tp)^2 = 1$ ,  $tp = E_2$ , and the additional involutive generator  $E_4$  of  $\Gamma$  is given by ( $\omega^2 + \omega + 1 = 0$ ):

$$(6.43) \quad E_4 = - \begin{pmatrix} 0 & \omega & 0 \\ \bar{\omega} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4^2 = 1 = (E_3 E_4)^2.$$

It is the 360 element factor group  $A_6 = \Gamma/\mathbb{Z}_3$  that acts by non-trivial automorphisms on the  $\mathfrak{su}(3)$  current algebra. There are 17 conjugacy classes of  $\Gamma$  versus 7 of  $A_6$ . Both are listed in the combined character table below (see Table 3).

We observe that to each of the first 5 conjugacy classes in  $A_6$  correspond 3 such classes (of the same size) in  $\Gamma$  while the last two are mapped into classes of triple size:  $|\bar{t}_\Gamma| = 3|\bar{t}_{A_6}| = 3 \times 40 (= |\bar{t}'_\Gamma|)$ . The essential difference between  $A_6 = \Gamma/\mathbb{Z}_3$  and the subgroups  $\Gamma/\mathbb{Z}_2$  of  $SO(3)$  is the presence of elements  $E (\in \bar{E})$  with a non-abelian centralizer  $\Gamma_8$ . Table 4 is its character table ( $E_5 = E_3 E_4$ ,  $q = E_4 E_2$ ,  $q^3 = E_2 E_4$ ).

We note that the centralizer  $\mathbb{Z}_4$  of  $q$  in  $A_6$  is a normal subgroup of  $\Gamma_8$ .

There are (according to (6.28)) altogether  $\frac{1}{3} \sum_{\hat{g} \in \Gamma} |\hat{\Gamma}_g| = 17 + 3 \cdot 15 + 12 + 3 + 3 = 80$  level 1  $\Gamma \subset SU(3)$  orbifold modules. Although it is not practical to write down the  $80 \times 80$   $S$ -matrix, one can extract the relevant information about  $E$ -twisted orbifolds.

The multipliers  $\mu(h | \sum \beta_i)$  give rise to a new notion of conjugation whenever the class  $\bar{E}$  of involutions labels a sector. To display this fact we first observe that the set of  $(45)^2$  pairs  $(E, E')$  splits into 9 different orbits displayed in Table 5.

The stabilizer  $\Gamma_{E, E' E}$  of the pair  $E, E' E$  in  $\Gamma$  is the direct product of the central subgroup  $\mathbb{Z}_3$  with the above  $\Gamma_{E, E' E}^{(0)} \subset A_6$ . To verify the data of Table 5 one needs to construct a representative pair in each orbit. The number of elements of such an orbit is  $|A_6| = 360$  divided by  $|\Gamma_{E, E' E}^{(0)}|$ . For instance, the orbit  $O_{\bar{p}}$  is obtained by conjugation of the pair  $(E_p, E_1)$  where

$$E_p = p^{-1} E_3 p = \frac{1}{2} \begin{pmatrix} -x_- & 1 & x_+ \\ 1 & -x_+ & -x_- \\ x_+ & -x_- & -1 \end{pmatrix},$$

$$E_1 E_p = E_2 p^{-1} E_2 \in \bar{p} \quad \left( x_{\pm} = \frac{1 \pm \sqrt{5}}{2} \right).$$

We shall now prove that the oppositely ordered pairs  $(E_2, E_4)$  and  $(E_4, E_2)$  belong to different orbits  $O_{\bar{q}}$  although they belong to the same  $SU(3)$  orbit. To this end we construct the most general  $u \in SU(3)$  such that

$$(6.44a) \quad uE_2u^* = E_4, \quad uE_4u^* = E_2;$$

it is given by a 2(real) parameter family.

$$(6.44b) \quad u = \begin{pmatrix} u_1 & u_2 & 0 \\ -\zeta\bar{u}_2 & \zeta\bar{u}_1 & 0 \\ 0 & 0 & \bar{\zeta} \end{pmatrix} \text{ with } |\zeta|^2 = 2|u_1|^2 = 2|u_2|^2 = 1, \quad 2\bar{\zeta}u_1u_2 = -\omega.$$

TABLE 3.  $\hat{A}_6 \subset \hat{\Gamma}$ : Zero versus non-zero triality representations

TABLE 3A.  $\hat{A}_6$ .

$IR \setminus cc$	1	$\bar{E}(E^2 = 1)$	$\bar{q}(q^2 \in \bar{E})$	$\bar{p}(p^5 = 1)$	$\bar{p}^2$	$\bar{t}(t^3 = 1 = t'^3)$	$\bar{t}'$
1	1	1	1	1	1	1	1
5	5	1	-1	0	0	2	-1
5'	5'	1	-1	0	0	-1	2
8	8	0	0	$x_+$	$x_-$	-1	-1
8'	8'	0	0	$x_-$	$x_+$	-1	-1
9	9	1	1	-1	-1	0	0
10	10	-2	0	0	0	1	1
$(A_6)_g$	$A_6$	$\Gamma_8$	$\mathbb{Z}_4$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$

TABLE 3B.  $\hat{\Gamma}$ .

$IR \setminus cc$	1	$\omega$	$\omega^2$	$\overline{E}$	$\overline{\omega E}$	$\overline{\omega^2 E}$	$\overline{q}$	$\overline{\omega q}$	$\overline{\omega^2 q}$	$\overline{p}$	$\overline{\omega p}$	$\overline{\omega^2 p}$	$\overline{p^2}$	$\overline{\omega p^2}$	$\overline{\omega^2 p^2}$	$\overline{t}$	$\overline{t'}$
$3_\omega$	3	$3\omega$	$3\omega^2$	-1	$-\omega$	$-\omega^2$	1	$\omega$	$\omega^2$	$x_+$	$\omega x_+$	$\omega^2 x_+$	$x_-$	$\omega x_-$	$\omega^2 x_-$	0	0
$3_\omega^*$	3	$3\omega^2$	$3\omega$	-1	$-\omega^2$	$-\omega$	1	$\omega^2$	$\omega$	$x_+$	$\omega^2 x_+$	$\omega x_+$	$x_-$	$\omega^2 x_-$	$\omega x_-$	0	0
$3'_\omega$	3	$3\omega$	$3\omega^2$	-1	$-\omega$	$-\omega^2$	1	$\omega$	$\omega^2$	$x_-$	$\omega x_-$	$\omega^2 x_-$	$x_+$	$\omega x_+$	$\omega^2 x_+$	0	0
$3'^*_\omega$	3	$3\omega^2$	$3\omega$	-1	$-\omega^2$	$-\omega$	1	$\omega^2$	$\omega$	$x_-$	$\omega^2 x_-$	$\omega x_-$	$x_+$	$\omega^2 x_+$	$\omega x_+$	0	0
$6_\omega$	6	$6\omega$	$6\omega^2$	2	$2\omega$	$2\omega^2$	0	0	0	1	$\omega$	$\omega^2$	1	$\omega$	$\omega^2$	0	0
$6_\omega^*$	6	$6\omega^2$	$6\omega$	2	$2\omega^2$	$2\omega$	0	0	0	1	$\omega^2$	$\omega$	1	$\omega^2$	$\omega$	0	0
$9_\omega$	9	$9\omega$	$9\omega^2$	1	$\omega$	$\omega^2$	1	$\omega$	$\omega^2$	-1	$-\omega$	$-\omega^2$	-1	$-\omega$	$-\omega^2$	0	0
$9_\omega^*$	9	$9\omega^2$	$9\omega$	1	$\omega^2$	$\omega$	1	$\omega^2$	$\omega$	-1	$-\omega^2$	$-\omega$	-1	$-\omega$	$-\omega^2$	0	0
$15_\omega$	15	$15\omega$	$15\omega^2$	-1	$-\omega$	$-\omega^2$	-1	$-\omega$	$-\omega^2$	0	0	0	0	0	0	0	0
$15_\omega^*$	15	$15\omega^2$	$15\omega$	-1	$-\omega^2$	$-\omega$	-1	$-\omega^2$	$-\omega$	0	0	0	0	0	0	0	0
$\Gamma_g$	$\Gamma$			$\mathbb{Z}_3 \times \Gamma_8$			$\mathbb{Z}_{12}$			$\mathbb{Z}_{15}$			$\mathbb{Z}_{15}$			$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$
$ \overline{g} $	1			45			90			72			72			120	120

It remains to prove that this family of  $3 \times 3$  matrices does not intersect our group  $\Gamma$ . To this end we note that  $|\text{tr}(u + uE_1)| = |2u_1| = \sqrt{2}$ ; a glance at Table 3 tells us that this cannot be the case for  $u \in \Gamma$ . It turns out that the same 2-parameter family of  $u$ 's is the most general subset of  $\text{SU}(3)$  elements that transforms the two  $O_{\overline{E}}$  orbits among themselves:

$$(6.45) \quad uE_3u^* = E_3 \Rightarrow uE_1u^* = uE_2E_3u^* = E_4E_3 = E_5 .$$

This completes the proof that each of the two pairs of representatives in the last column of Table 5 belongs to a different  $\Gamma$ -orbit. We finally note that the sum of all  $|O_{\overline{g}}|(4.360 + 2.180 + 2.90 + 45)$  adds up, as it should, to  $(45)^2 = 2025$ .

TABLE 4. Characters of  $\Gamma_8 = \Gamma_{E_3} \subset A_6$ .

$IR \backslash cc$	1	$E_3$	$E_1, E_2$	$E_4, E_5$	$q, q^3$
$1_0$	1	1	1	1	1
$1_1$	1	1	1	-1	-1
$1_2$	1	1	-1	1	-1
$1_3$	1	1	-1	-1	1
2	2	-2	0	0	0
$\Gamma_{E_3, g}$	$\Gamma_8$	$\Gamma_8$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_4$

**Proposition 6.1.** *The charge conjugation matrix (4.27) for the  $\mathfrak{A}_1(\mathfrak{su}(3))^\Gamma$  orbifold involves a non-trivial involution  $\sigma \rightarrow \sigma^c$  for  $b \in E$ ,  $\sigma \in \hat{\Gamma}_E$ :*

$$(6.46) \quad C_{\Lambda_0 E_3 \sigma, \Lambda_0 E_3 \sigma'} = \delta_{\sigma' \sigma^c}, \quad \sigma_c = \sigma^* \otimes \sigma^E, \quad \sigma^E = e^{2\pi i(2\beta_3 | \beta(h))}$$

where  $\Gamma_E = \mathbb{Z}_3 \times \Gamma_8$ ,  $\sigma^* = \sigma$  (i.e.,  $\sigma(\omega h) = \sigma(h)$  for  $h \in \Gamma_E$ ),

$$(6.47) \quad \beta_i = \beta(E_i) : \beta_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \Rightarrow$$

$$\beta_1 = \frac{1}{2} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\beta_4 = \frac{1}{4} \begin{pmatrix} -1 & \pm 3\omega & 0 \\ \pm 3\bar{\omega} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

*Proof.* The statement is a straightforward consequence of (4.27) (Theorem 4.3e) and of the observation that  $\beta_3 = \beta(E_3) = \beta(E_3^{-1})$ . The representation  $\sigma$  is trivial on  $\mathbb{Z}_3$  (and hence, selfconjugate; see Table 4), since it has to agree with the representation  $\Lambda_0 = 0$  of  $SU(3)$  on the small center.  $\square$

*Remark 6.2.* The appearance of a non-trivial conjugation depends on the choice of a representative in a class of equivalent quadruples. Had we chosen instead of

TABLE 5. Orbits  $O_{E'E}^{(i)}$  of pairs  $(E, E') \subset \bar{E}$   
and their stabilizers ( $i = 1, 2$ ):

$CC$ of $E'E$	$\Gamma_{E, E'E}^{(0)} \subset A_6$	$ O_{E'E}^{(i)} $	Representative pairs
$E'E = 1 (E = E')$	$\Gamma_8$	45	
$E'E = EE' \in \bar{E}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$ O_{\bar{E}}^{(i)}  = 90$ , $i = 1, 2$	$O_{\bar{E}}^{(1)} = O(E_1, E_2)$ , $O_{\bar{E}}^{(2)} = O(E_5, E_4)$
$E'E \in \bar{q}$	$\mathbb{Z}_2$	$ O_{\bar{q}}^{(i)}  = 180$ , $i = 1, 2$	$O_{\bar{q}}^{(1)} = O(E_2, E_4)$ , $O_{\bar{q}}^{(2)} = O(E_4, E_2)$
$E'E \in \bar{p}^n$ , $n = 1, 2$	$\{1\}$	360	
$E'E \in \bar{t}$ or $\bar{t}'$	$\{1\}$	360	

the involution element  $E_3 \in \bar{E}$  a representative of a minimal phase like  $\bar{\omega}E_3 \in \omega^2\bar{E}$  for which

$$(6.48) \quad \tilde{\beta}_3 := \beta(\bar{\omega}E_3) = \frac{1}{6} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ so that } |\tilde{\beta}_3|^2 = \frac{1}{6} \left( = \frac{1}{9} |\beta_3|^2 \right),$$

then we would have dealt with complex representations since

$$(6.49a) \quad \chi_{\Lambda_0, \bar{E}, \sigma}^{\beta_3}(\tau) = \chi_{\Lambda_2, \omega^2\bar{E}, \sigma_2}^{\tilde{\beta}_3}(\tau) \text{ with } \sigma_2(h) = \sigma(h)e^{2\pi i(\Lambda_2|\alpha)}$$

where  $\Lambda_2$  is the fundamental weight of the ‘‘antiquark’’ representation  $3^*$ ,

$$(6.49b) \quad \Lambda_2 = \frac{2}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \tilde{\beta}_3 - \beta_3, \quad h = e^{2\pi i\alpha}, \quad [\alpha, \Lambda_2] = 0 (= [\alpha, \beta_3]).$$

The charge conjugation matrix in these new labels would assume its usual form with non-zero entry

$$(6.50) \quad C_{\Lambda_2 b \sigma_2, \Lambda_2^* b^{-1} \sigma_2^*} = 1, \quad b \in \omega^2\bar{E}, \quad b^{-1} \in \omega\bar{E}.$$

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## APPENDIX A. ACTION OF THE CENTER OF A SIMPLY CONNECTED SIMPLE LIE GROUP ON THE COROOTS AND FUNDAMENTAL WEIGHTS

We shall display the action of  $w_j$  for the classical Lie algebras as well as for  $E_6$  and  $E_7$  (the simply connected groups with Lie algebras  $G_2, F_4$  and  $E_8$  have a trivial center). We let  $\tilde{J} = J \cup \{0\}$ ,  $a_0 = a_0^\vee = 1$ .

### A1. Simply laced algebras ( $\alpha_i^\vee = \alpha_i, a_i^\vee = a_i$ ).

The center  $\mathbb{Z}_{l+1}$  of  $SU(l+1)$  acts on both the (co) roots and weights of  $A_l^{(1)}$  via cyclic permutations:

$$(A.1) \quad \begin{aligned} w_1(\alpha_0, \alpha_1, \dots, \alpha_l) &= (\alpha_1, \alpha_2, \dots, \alpha_l, \alpha_0), & w_j &= w_1^j \\ \tilde{w}_1(\tilde{\Lambda}_0, \tilde{\Lambda}_1, \dots, \tilde{\Lambda}_l) &= (\tilde{\Lambda}_1, \tilde{\Lambda}_2, \dots, \tilde{\Lambda}_l, \tilde{\Lambda}_0), & w_1^{l+1} &= 1. \end{aligned}$$

Here  $\tilde{\Lambda}_\nu$  are the extended fundamental weights

$$(A.2) \quad \tilde{\Lambda}_\nu = d + \Lambda_\nu + \kappa_\nu K$$

chosen to have equal norm squares:

$$(A.3) \quad |\tilde{\Lambda}_\nu|^2 = 2\kappa_\nu + \frac{\nu(l+1-\nu)}{l+1} = 2\kappa_0.$$

The set  $\tilde{J}$  consists of all indices  $0, 1, \dots, l$ . The element  $w_1$  is a Coxeter element of the finite Weyl group  $W(A_l) = S_{l+1}$ . In terms of the elementary Weyl reflections  $s_i$  it is written as:

$$(A.4) \quad w_1 = s_1 \dots s_l \Rightarrow w_1 \Lambda_j = \Lambda_j - \alpha_1 - \dots - \alpha_j.$$

The center of the simply connected group  $\text{Spin}(2l)$  with Lie algebra  $D_l$  is  $\mathbb{Z}_2 \times \mathbb{Z}_2$  for  $l$  even and  $\mathbb{Z}_4$  for  $l$  odd. To exhibit its action on roots and weights of  $D_l^{(1)}$  it is convenient to use an orthonormal basis  $\{e_i\}$  in the  $l$  dimensional root space of  $D_l$  setting

$$(A.5) \quad \alpha_i = e_i - e_{i+1}, \quad i = 1, \dots, l-1, \quad \alpha_l = e_{l-1} + e_l, \quad \alpha_0 = K - e_1 - e_2,$$

$$(A.6) \quad \Lambda_i = \sum_{s=1}^i e_s, \quad \Lambda_{l-1} = \Lambda_l - e_l, \quad \Lambda_l = \frac{1}{2} \sum_{i=1}^l e_i.$$

The set  $\tilde{J}$  of indices  $\mu$  for which  $a_\mu = 1$  consists of 4 elements:  $0, 1, l-1, l$ . Writing again

$$(A.7a) \quad \tilde{\Lambda}_\nu = a_\nu d + \Lambda_\nu + \kappa_\nu K$$

we restrict  $\kappa_\nu$  demanding that the norm squares of  $\tilde{\Lambda}_\mu$  ( $\mu \in \tilde{J}$ ) coincide:

$$(A.7b) \quad |\tilde{\Lambda}_0|^2 = 2\kappa_0 = |\tilde{\Lambda}_j|^2 = 1 + 2\kappa_1 = \frac{l}{4} + 2\kappa_{l-1} = \frac{l}{4} + 2\kappa_l .$$

We shall first determine the finite part  $w_l$  of  $\tilde{w}_l$  defined by  $w_l \alpha_0 = w_l(-\theta) = \alpha_l$  and hence (being a permutation of  $\alpha_\mu, \mu \in \tilde{J}$ ),  $w_l \alpha_1 = \alpha_{l-1}$ . As a consequence of invariance of inner products we further deduce  $w_l \alpha_i = \alpha_{l-i}, i = 1, \dots, l-2$ ; hence, in view of (A.5),

$$(A.8a) \quad w_l e_i = -e_{l+1-i}, \quad i = 1, \dots, l-1 ;$$

$w_l e_l$  is then determined from the condition that an element of  $W(D_l)$  should involve an even number of reflections:

$$(A.8b) \quad w_l e_l = -(-1)^l e_1$$

As a result, we have  $w_l^2 = w_1$  for  $l$  odd,  $w_l^2 = 1$  for  $l$  even; in both cases  $\bar{w}_1^2 = 1$ ;

$$(A.9) \quad w_l(e_1, e_2, \dots, e_{l-1}, e_l) = (-e_1, e_2, \dots, e_{l-1}, -e_l)$$

The corresponding permutations of fundamental weights are

$$(A.10) \quad \begin{aligned} \tilde{w}_l \tilde{\Lambda}_0 &= \tilde{\Lambda}_l, & \tilde{w}_{l-1} \tilde{\Lambda}_1 &= \tilde{\Lambda}_{l-1}, & \tilde{w}_l \tilde{\Lambda}_{l-\nu} &= \begin{cases} \tilde{\Lambda}_\nu & \text{for } l \text{ even} \\ \tilde{\Lambda}_{1-\nu} & \text{for } l \text{ odd} \end{cases} \quad \nu = 0, 1 ; \\ \tilde{w}_1 \tilde{\Lambda}_0 &= \tilde{\Lambda}_1, & \tilde{w}_1 \tilde{\Lambda}_l &= \tilde{\Lambda}_{l-1}, & \tilde{w}_1^2 &= 1, & \tilde{w}_{l-1} &= \tilde{w}_1 \tilde{w}_l . \end{aligned}$$

The center of the group  $E_6$  is  $Z_3$ . Choosing a basis of simple roots of  $E_6$  in such a way that the highest root is  $\theta = \alpha_2 + \alpha_4 + 2(\alpha_1 + \alpha_3 + \alpha_5) + 3\alpha_6$  we have  $\tilde{J} = \{0, 2, 4\}$ . The center acts on an arbitrary weight  $\Lambda$  according to the law  $\tilde{w}_j \Lambda = k\Lambda_j + w_j \Lambda$ ,  $j = 2, 4$ , where

$$(A.11a) \quad w_2(-\theta, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (\alpha_2, \alpha_3, \alpha_4, \alpha_5, -\theta, \alpha_1, \alpha_6), \quad w_2^2 = w_4$$

$$(A.11b) \quad w_2 \Lambda_2 = \Lambda_4 - \Lambda_2 = \frac{1}{3}(\alpha_5 - \alpha_3 + 2\alpha_4 - 2\alpha_2) \Rightarrow \tilde{w}_2 \tilde{\Lambda}_2 = \tilde{\Lambda}_4, \quad w_2^3 = 1 .$$

Here we have used the expressions for the fundamental weights in terms of simple roots:

$$\begin{aligned} \Lambda_2 &= \alpha_1 + \frac{1}{3}(4\alpha_2 + 5\alpha_3 + 2\alpha_4 + 4\alpha_5 + 6\alpha_6), \\ \Lambda_4 &= \alpha_1 + \frac{1}{3}(2\alpha_2 + 4\alpha_3 + 4\alpha_4 + 5\alpha_5 + 6\alpha_6), \quad \left( |\Lambda_2|^2 = |\Lambda_4|^2 = \frac{4}{3} \right) \end{aligned}$$

as well as the relations  $\tilde{\Lambda}_\nu = d + \Lambda_\nu + \kappa_\nu K$  with

$$|\tilde{\Lambda}_0|^2 = |\tilde{\Lambda}_2|^2 = |\tilde{\Lambda}_4|^2 \Rightarrow 2\kappa_0 = \frac{4}{3} + 2\kappa_2 = \frac{4}{3} + 2\kappa_4 \text{ or } \kappa_2 = \kappa_4 = \kappa_0 - \frac{2}{3} .$$

The center of  $E_7$  is  $\mathbb{Z}_2$ . Choosing a basis of simple roots of  $E_7$  such that the highest root is  $\theta = \alpha_6 + 2(\alpha_1 + \alpha_5 + \alpha_7) + 3(\alpha_2 + \alpha_4) + 4\alpha_3$ , we have  $\tilde{J} = \{0, 6\}$ . The non-trivial element of the center is  $\tilde{w}_6 = t_6 w_6$  where

$$(A.12a) \quad w_6(-\theta, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (\alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, -\theta, \alpha_7)$$

$$(A.12b)$$

$$w_6 \Lambda_6 = -\Lambda_6 \Rightarrow \tilde{w}_6 \tilde{\Lambda}_6 = \tilde{\Lambda}_0 \text{ for } \Lambda_6 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_5 + \frac{1}{2}(5\alpha_4 + 3\alpha_6 + 3\alpha_7) .$$

Here again  $\tilde{\Lambda}_\nu = d + \Lambda_\nu + \kappa_\nu K$  where  $|\tilde{\Lambda}_6|^2 = 2\kappa_6 + \frac{3}{2} = 2\kappa_0$ .

## A2 $\mathbb{Z}_2$ action on $B_l$ and $C_l$ .

The simple roots, the highest root and the fundamental weights of  $B_l$  can be written in an orthonormal basis  $\{e_i\}$  as

$$(A.13) \quad \begin{aligned} \alpha_i &= e_i - e_{i+1}, \quad i = 1, \dots, l-1, \\ \alpha_l &= e_l, \quad \theta = \alpha_1 + 2(\alpha_2 + \dots + \alpha_l) = e_1 + e_2 \\ \Lambda_i &= \sum_{s=1}^i e_s, \quad i = 1, \dots, l. \end{aligned}$$

The center  $\mathbb{Z}_2$  of the simply connected group  $\text{Spin}(2l+1)$  acts on  $(\alpha_0 = K - \theta, \alpha_i)$  and on  $(\Lambda_0, \Lambda_i)$  as  $\tilde{w}_1 = t_1 w_1$  where

$$(A.14a) \quad w_1(e_1, e_2, \dots, e_l) = (-e_1, e_2, \dots, e_l)$$

$$(A.14b)$$

$$t_1 \alpha^\vee = \alpha^\vee - (\alpha^\vee | \Lambda_1) K \text{ for } \alpha^\vee \in M, \quad t_1 \tilde{\Lambda}_\nu = \tilde{\Lambda}_\nu + (\tilde{\Lambda}_\nu | K) \tilde{\Lambda}_1 \text{ for } \tilde{\Lambda}_\nu \in M^* ;$$

thus

$$\begin{aligned} t_1 w_1 \alpha_0 &= t_1 (K + \alpha_1) = \alpha_1, \quad w_1 \alpha_i = \alpha_i = t_1 w_1 \alpha_i \text{ for } i = 2, \dots, l, \\ t_1 w_1 \alpha_0 &= t_1 (-\theta) = -\theta + K = \alpha_0 ; \\ t_1 w_1 \tilde{\Lambda}_0 &= t_1 \tilde{\Lambda}_0 = \tilde{\Lambda}_0 + \tilde{\Lambda}_1, \\ t_1 w_1 (\tilde{\Lambda}_0 + \tilde{\Lambda}_1) &= t_1 (\tilde{\Lambda}_0 - \tilde{\Lambda}_1) = \Lambda_0 - \Lambda_1 + \Lambda_1 = \Lambda_0 \end{aligned}$$

The simple roots, the highest root and the fundamental weights for  $C_l$  are expressed as

$$(A.14) \quad \begin{aligned} \alpha_i &= \frac{1}{\sqrt{2}}(e_i - e_{i+1}), \quad i = 1, \dots, l-1, \\ \alpha_l &= \sqrt{2}e_l, \quad \theta = 2 \sum_{i=1}^{l-1} \alpha_i + \alpha_l = \sqrt{2}e_1 \\ \Lambda_i &= \sqrt{2} \sum_{s=1}^i e_s, \quad i = 1, \dots, l-1, \quad \Lambda_l = \frac{1}{\sqrt{2}} \sum_{s=1}^l e_s. \end{aligned}$$

The non-trivial element  $\tilde{w}_l = t_l w_l$  of the center  $\mathbb{Z}_2$  of  $Sp(2l)$  acts on these orthonormal basis  $e_i$  as

$$(A.16) \quad w_l(e_1, e_2, \dots, e_{l-1}, e_l) = (-e_l, -e_{l-1}, \dots, -e_2, -e_1) ;$$

hence

$$(A.17a) \quad w_l(-\theta, \alpha_1, \dots, \alpha_l) = (\alpha_l, \alpha_{l-1}, \dots, \alpha_1, -\theta) ,$$

$$(A.17b)$$

$$\begin{aligned} \tilde{w}_l \tilde{\Lambda}_0 &= \tilde{\Lambda}_l (= d + \Lambda_l + \kappa_l K) , & w_l \Lambda_l &= -\Lambda_l \Rightarrow \tilde{w}_l \tilde{\Lambda}_l = \tilde{\Lambda}_0 \\ \left( |\tilde{\Lambda}_0|^2 = 2\kappa_0 = |\tilde{\Lambda}_l|^2 = 2\kappa_l + \frac{l}{2} \right) . \end{aligned}$$

## APPENDIX B. EXCEPTIONAL ELEMENTS OF A COMPACT LIE GROUP.

Let  $G$  be a connected compact Lie group with a simple Lie algebra  $\mathfrak{g}$  of rank  $l$ , and let  $Ad_G$  denote the adjoint group. An element  $g \in G$  is called *ad-exceptional* if it cannot be written in the form  $g = \exp 2\pi i \beta$ , where  $\beta \in i\mathfrak{g}$  is such that  $Ad_g x = x$  iff  $[\beta, x] = 0$  for all  $x \in \mathfrak{g}$ . Note that an element  $g \in G$  is *Ad-exceptional* iff it is *ad-exceptional* or its centralizer in  $G$  is not connected. (Recall that in a simply connected  $G$  the centralizer of any element is connected.) In this Appendix we classify *ad-exceptional* elements of finite order of the group  $Ad_G$ .

The finite order inner automorphisms of the simple Lie algebra  $\mathfrak{g}$  belong to  $Ad_G$  and can be described as follows (see Theorem 8.6 and Proposition 8.6b of [K1]).

**Proposition B.1.** *Each order  $N$  inner automorphism of  $\mathfrak{g}$  is conjugate to*

$$(B.1) \quad Ad_{b(s)} , \quad b(s) = \exp 2\pi i \beta(s) ,$$

where

$$(B.2) \quad \beta(s) = \frac{1}{N} \sum_{j=1}^l s_j \Lambda_j^\vee$$

and  $s_0, s_j, j = 1, \dots, l$  are relatively prime non-negative integers such that:

$$(B.3) \quad s_0 + \sum_{j=1}^l a_j s_j = N .$$

Here  $\Lambda_j^\vee$  are the fundamental co-weights:

$$(B.4) \quad (\alpha_i | \Lambda_j^\vee) = (\alpha_i^\vee | \Lambda_j) = \delta_{ij} , \quad i, j = 1, \dots, l .$$

**Proposition B.2.** *The centralizer of  $Ad_{b(s)}$  in  $\mathfrak{g}$  is generated by the  $E^{\pm\alpha_\nu}$ ,  $\nu = 0, 1, \dots, l$ , for which  $s_\nu = 0$  and by the Cartan subalgebra.*

According to Definition 4.1 an element  $b \in \Gamma$  is exceptional if there is no  $\beta \in \mathfrak{g}$  such that

$$(B.5) \quad b = e^{2\pi i\beta} \text{ and } \Gamma_b = \Gamma_\beta .$$

As noted,  $G = U(l)$  has no exceptional elements. By contrast, for each partition of the positive integer  $n \geq 2$  of the type

$$(B.6) \quad n = k_1 + \dots + k_\rho, \quad k_{\min} = \min(k_1, \dots, k_\rho) = 2$$

there are exceptional elements of  $SU(n)$  conjugate to diagonal matrices with  $k_j$  eigenvalues  $\exp(2\pi i \frac{\nu_j}{N})$ ,  $j = 1, \dots, \rho$ , where the  $\nu_j$  are subject to the conditions: (i)  $(\nu_1, \dots, \nu_\rho, N) = 1$  (i.e. these  $\rho + 1$  integers have no common factor) and  $\sum_j k_j \nu_j = kN$  with  $1 \leq k < k_{\min}$ . For  $n = 2, 3$  all such elements belong to the center  $Z_n$  of  $SU(n)$ . More generally, for any  $n$ , one can find an element  $\zeta \in Z_n$  such that  $g = b\zeta$  is non-exceptional. (In the above example it suffices to choose  $\zeta = \exp(-2\pi i \frac{k}{n})$ .) This agrees with the remark (of Sect. 4B) that  $SU(n)$  contains no exceptional subgroups.

Recall that an element  $b \in G$  is *Ad-exceptional* if  $b\zeta$  is exceptional for any choice of  $\zeta \in Z(G)$ . The following theorem describes all finite order *ad-exceptional* elements of  $Ad_G$  (for a simple  $\mathfrak{g}$ ), and hence all finite order *Ad-exceptional* elements of a simply connected  $G$ .

**Proposition B.3.** *The finite order automorphism  $Ad_{b(s)}$  is ad-exceptional iff the marks  $a_\nu$  with  $s_\nu > 0$  have a non-trivial common factor.*

*Proof.* It follows from Proposition B.2 that it suffices to study the commutator of  $\beta(s)$  with  $E^{\alpha_\nu}$  for those  $\nu (= 0, \dots, l)$  for which  $s_\nu = 0$ .

This commutator is trivial for  $j = 1, \dots, l$  and  $s_j = 0$  since Eqs. (B. 1-4) imply

$$(B.7) \quad [\beta(s), E^{\alpha_j}] = (\alpha_j | \beta(s)) E^{\alpha_j}, \quad (\alpha_j | \beta(s)) = s_j = 0 .$$

Thus  $Ad_b$  can only be *ad-exceptional* if  $s_0 = 0$ ; in this case

$$(B.8) \quad [\beta(s), E^{\alpha_0}] = [\beta(s), E^{-\theta}] = \left( \frac{s_0}{N} - 1 \right) E^{\alpha_0} = -E^{\alpha_0} .$$

This is still not sufficient to assert that  $Ad_b$  is *ad-exceptional* since  $\beta(s)$  is not unique: we can add to it  $\sum_{i=1}^l m_i \Lambda_i^\vee$  for  $m_i \in \mathbb{Z}$  without changing the automorphism. That would give

$$[\beta(s) + \sum_i m_i \Lambda_i^\vee, E^{-\theta}] = \left( -1 - \sum_{\substack{i \\ s_i \neq 0}} a_i m_i \right) E^{-\theta}$$

which can be made zero iff the  $a_i$  in the sum have no common factor.  $\square$

Proposition B.3 shows that  $SU(l)$  has no  $Ad$ -exceptional elements, whereas all other simple simply connected compact groups do. Examples of  $Ad$ -exceptional  $b$  are provided by the *special elements* with  $\beta(s) = \frac{1}{a_j} \Lambda_j^\vee$  for  $a_j > 1$ , corresponding to  $s_\nu = \delta_{\nu j}$ . Such is, for instance, the diagonal symplectic matrix

(B.9)

$$b_1 = e^{2\pi i \Lambda_1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in Sp(4) = \{g \in SU(4) \mid {}^t g C g = C\},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_1 = \frac{1}{2} \Lambda_1^\vee = \frac{1}{2} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$$

$$(\quad = \alpha_1 + \frac{1}{2} \alpha_2).$$

( $\Lambda_1$  is only stabilized by  $U(2)$  while the centralizer of  $b_1$  in  $Sp(4)$  is  $SU(2) \times SU(2)$ ).

If  $\Gamma_1 \subset SU_2$  is the binary icosahedral group, then  $\Gamma = \langle b_1, -1 \rangle \times \Gamma_1 \times \Gamma_1 \subset Sp(4)$  is clearly an exceptional subgroup containing the center of  $Sp(4)$ .

The simplest example of a non-special  $Ad$ -exceptional element is provided by the simply laced Lie algebra  $D_5$  (corresponding to the simply connected group  $Spin(10)$ ). If we label the nodes of the affine diagram  $D_5^{(1)}$  so that  $a_2 = a_3 = 2$  (while  $a_0 = a_1 = a_4 = a_5 = 1$ ) then the non-special  $Ad$ -exceptional element of  $Spin(10)$  correspond to  $\beta = \frac{1}{4}(\Lambda_2^\vee + \Lambda_3^\vee)$ .

An example of an element of  $SO(3) = Ad_{SU(2)}$  with a disconnected centralizer is provided by either of the diagonal matrices  $E_i$ ,  $i = 1, 2, 3$  of Eq. (6.36). Indeed, there is no Cartan subalgebra of  $SO(3)$  containing the infinitesimal generators of both  $E_1$  and  $E_2$ . Note that the preimages of  $E_i$  in the simply connected double cover  $SU(2)$  of  $SO(3)$  do not commute (in fact, they anticommute). This example extends to the  $n^3$  element Heisenberg subgroup  $H_n$  of  $SU(n)$  generated by the  $n \times n$  matrices  $a$  and  $b$  satisfying

$$(B.10) \quad a^n = b^n = 1, \quad ab = e^{\pi i/n} ba.$$

Clearly,  $Ad_a$  and  $Ad_b$  commute but their infinitesimal generators do not. This happens since  $Ad_{SU(n)}$  (unlike  $SU(n)$ ) is not simply connected and the centralizer of either  $Ad_a$  or  $Ad_b$  is disconnected.

## REFERENCES

- [AFMO] J.H. Awata, M. Fukuma, Y. Matsuo, S. Odake, *Representation theory of the  $W_{1+\infty}$* , vol. 118, Prog. Theor. Phys., Proc. Suppl., 1995, pp. 343–373.
- [BGT] B.N. Bakalov, L.S. Georgiev, I.T. Todorov, *A QFT approach to  $W_{1+\infty}$* , New Trends in Quantum Field Theory, Proc. of the 1995 Razlog (Bulgaria) Workshop, A. Ganchev et al. (eds.), Heron Press, Sofia, 1996, pp. 147–158.

- [BPZ] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B241** (1984), 333–381.
- [Bor] R. Borcherds, *Vertex algebras, Kac-Moody algebras and the Monster*, Proc. Natl. Acad. Sci. USA **83** (1986), 3068–3071; *Monstrous moonshine and monstrous Lie Superalgebras*, Invent. Math. **109** (1992), 405–444.
- [BMP] P. Bouwknegt, J. McCarthy, K. Pilch, *Semi-infinite cohomology and  $w$ -gravity*, J. Geom. Phys. **11** (1993), 225–249.
- [BMT] J.D. Buchholz, G. Mack, I.T. Todorov, *The current algebra on the circle as a germ of local field theory*, Nucl. Phys. B (Proc. Suppl.) **5B** (1988), 20–56.
- [CTZ] A. Cappelli, C.A. Trugenberger, G.R. Zemba, *Stable hierarchical quantum Hall fluids on  $W_{1+\infty}$  minimal models*, Nucl. Phys. **B448** [FS] (1995), 470–504;  *$W_{1+\infty}$  dynamics of edge excitations*, cond-mat **9407095**.
- [CZ] A. Cappelli, G. Zemba, *Modular invariant partition functions in the quantum Hall effect*, hep-th/9605127.
- [DFSZ] P. Di Francesco, H. Saleur, J.-B. Zuber, *Modular invariance in non-minimal two-dimensional conformal theories*, Nucl. Phys. **B285** [FS19] (1987), 454–480.
- [DV<sup>3</sup>] J.K. Dijkgraaf, C. Vafa, E. Verlinde, H. Verlinde, *The operator algebra of orbifold models*, Commun. Math. Phys. **123** (1989), 485–526.
- [DGM] L. Dolan, P. Goddard, P. Montague, *Conformal field theory of twisted vertex operators*, Nucl. Phys. **B338** (1990), 529–601.
- [FFK] W.M. Fairbairn, T. Fulton, W.H. Klink, *Finite and disconnected subgroups of  $SU_3$  and their application to the elementary particle spectrum*, J. Math. Phys. **5** (1964), 1038–1051.
- [F] M. Flohr,  *$W$ -algebras, new rational models and completeness of the  $c = 1$  classification*, Commun. Math. Phys. **157** (1993), 179–212.
- [FK] I.B. Frenkel, V.G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980), 23–66.
- [FKRW] E. Frenkel, V. Kac, A. Radul, W. Wang,  *$W_{1+\infty}$  and  $W(gl_N)$  with central charge  $N$* , Commun. math. Phys. **170** (1995), 337–357.
- [FKW] E. Frenkel, V. Kac, M. Wakimoto, *Characters and fusion rules for  $W$ -algebra via quantized Drinfeld-Sokolov reduction*, Commun. Math. Phys. **147** (1992), 295–328.
- [FLM] I.B. Frenkel, J. Lepowsky, A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, New York, 1988.
- [FZ] I.B. Frenkel, Y. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebra*, Duke Math. J. **66** (1992), 123–168.
- [FT] J. Fröhlich, E. Thiran, *Integral quadratic forms, Kac-Moody algebras, and fractional quantum Hall effect; An ADE-O classification*, J. Stat. Phys. **76** (1994), 209–283.
- [FST] P. Furlan, G.M. Sotkov, I.T. Todorov, *Two-dimensional conformal quantum field theory*, Riv. Nuovo Gim **12:6** (1989), 1–202.
- [Gep] D. Gepner, *New conformal field theories associated with Lie algebras and their partition functions*, Nucl. Phys. **B285** [FS20] (1987), 10–24.
- [G] P. Ginsparg, *Curiosities at  $c = 1$* , Nucl. Phys. **B295** [FS21] (1988), 153–170.
- [Go] P. Goddard, *Meromorphic conformal fields theory*, Infinite-dimensional Lie algebras and groups (V. Kac, ed.), Adv. Ser. Math. Phys. **7**, World Sci., Singapore, 1989, pp. 556–587.
- [Gor] D. Gorenstein, *Finite Groups*, Harper & Row, New York, 1968.
- [H] G. Harris,  *$SU(2)$  current algebra orbifolds of the Gaussian model*, Nucl. Phys. **B300** [FS22] (1988), 588–610.
- [K1] V.G. Kac, *Infinite Dimensional Lie Algebras*, Third edition, Cambridge Univ. Press, 1990.
- [K2] V.G. Kac, *Vertex algebras*, in: New Trends in Quantum Field Theory, Proceedings of the 1995 Razlog (Bulgaria) Workshop, A. Ganchev et al. editors, Heron Press, Sofia, 1996, pp. 261–358.
- [KP0] V.G. Kac, D.H. Peterson, *Affine Lie algebras and Hecke modular forms*, Bull. Amer. Math. Soc. **3** (1980), 1057–1061.

- [KP1] V.G. Kac, D.H. Peterson, *Spin and wedge representations of infinite dimensional Lie algebras and groups*, Proc. Nat. Acad. Sci. USA **78** (1981), 3308–3312.
- [KP2] V.G. Kac, D.H. Peterson, *Infinite dimensional Lie algebras, theta-functions and modular forms*, Adv. in Math. **53** (1984), 125–264.
- [KR1] V.G. Kac, A. Radul, *Quasi-finite highest weight modules over the Lie algebra of differential operators on the circle*, Commun. Math. Phys. **157** (1993), 429–457.
- [KR2] V.G. Kac, A. Radul, *Representation theory of the vertex algebra  $W_{1+\infty}$* , Transformation groups 1, 1996, pp. 41–70.
- [KW] V.G. Kac, M. Wakimoto, *Modular and conformal invariant constraints in representation theory of affine algebras*, Advances in Math. **70** (1988), 156–236.
- [Kos] B. Kostant, *The McKay correspondence, the Coxeter element and representation theory*, in: *The Mathematical Heritage of Eli Cartan*, Soc. Math. de France, Asterisque, hors series (1985), 209–255.
- [LZ] B.H. Lian, G. Zuckerman, *Commutative quantum operator algebras*, J. Pure Appl. Alg. **100** (1995), 117–139.
- [LR] R. Longo, K.-H. Rehren, *Nets of subfactors*, Rev. Math. Phys. **7** (1995), 567–597.
- [Lus1] G. Lusztig, *Unipotent representations of finite Chevalley groups of type  $E_8$* , Quart. J. Math. Oxford (2) **30** (1979), 315–338.
- [Lus2] G. Lusztig, *Leading coefficients of character values of Hecke algebras*, in: Arcata Conference of Representations of Finite Groups, Proceedings of Symposium in Pure Math (P. Fong, ed.), vol. 47, 1987.
- [MST] L. Michel, Ya.S. Stanev, I.T. Todorov,  *$D - E$ -classification of the local extensions of the  $su_2$  current algebras*, Teor Mat. Fiz. **92** (1992), 507–521; (American edition: Theor. Math. Phys. **92** (1993), 1063).
- [R] K.-H. Rehren, *A new view of the Virasoro algebra*, Lett. Math. Phys. **30** (1994), 125–130.
- [RST] K.-H. Rehren, Ya.S. Stanev, I.T. Todorov, *Characterizing invariants for local extensions of current algebras*, hep-th/9409165, Commun. Math. Phys. **174** (1996), 605–633.
- [V] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. **B300** [FS22] (1988), 360–376.