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DIRICHLET AND NEUMANN EIGENVALUE PROBLEMS ON DOMAINS IN EUCLIDEAN SPACES

BY
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Abstract. We obtain here some inequalities for the eigenvalues of Dirichlet and Neumann value problems for general classes of operators (or system of operators) acting in $L^2(\Omega)$ (or $L^2(\Omega, \mathbb{C}^m)$), $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$.

1. Introduction

1. Let Ω be an open domain in $\mathbb{R}^d$, $d \geqslant 1$, and $0 < \lambda_1 < \lambda_2 \leqslant \dots$ be the eigenvalues of the Dirichlet boundary problem for the Laplace operator $-\Delta^{\mathcal{D}}$ in Ω. Denote by $|\Omega|$ the Lebesgue measure of the domain Ω and by $L_d^{cl} = v_d\,(2\pi)^{-d} = 2^{-d}\pi^{-d/2}/\Gamma(1 + d/2)$, where v_d is the volume of the unit ball in $\mathbb{R}^d$. Li and Yau [LY] proved that the eigenvalues λ_k satisfy the following inequality

$$\lambda_k \geqslant \frac{d}{d+2}(L_d^{cl}\,|\Omega|)^{-2/d}\,k^{\frac{2}{d}}, \qquad \forall k \in \mathbb{N}. \tag{1.1}$$

The constant L_d^{cl}, the so called "classical constant", appears in the Weyl asymptotic formula for the counting function of eigenvalues. The proof of (1.1) is based on a sharp inequality concerning the sum of the first eigenvalues

$$\sum_{j=1}^{k} \lambda_j \geqslant \frac{d}{d+2}(L_d^{cl}\,|\Omega|)^{-2/d}\,k^{\frac{d+2}{d}}, \qquad \forall k \in \mathbb{N}. \tag{1.2}$$

The constant in the right hand side of (1.2) cannot be improved because it coincides with the asymptotical constant for the sum in the left hand side of (1.2) as $k \to \infty$.

An opposite inequality can be obtained for the eigenvalues of the Neumann boundary problem. Let $0 = \mu_1 < \mu_2 \leqslant \dots$ be the eigenvalues of the Neumann Laplacian $-\Delta^{\mathcal{N}}$ in a bounded domain Ω with piecewise smooth boundary. By adapting the approach of Li and Yau to this problem, Kröger [K1] proved the upper estimate

$$\mu_{k+1} \leqslant \left(\frac{d+2}{d}\right)^{2/d}(L_d^{cl}\,|\Omega|)^{-2/d}\,k^{\frac{d+2}{d}}, \qquad \forall k \in \mathbb{N}. \tag{1.3}$$

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The key inequality here was the upper estimate for the sum of the first eigenvalues μ_j's

$$\sum_{k=1}^{k} \mu_k \leqslant \frac{d}{d+2} (L_d^{cl} \, |\Omega|)^{-2/d} \, k^{\frac{d+2}{d}}, \qquad \forall k \in \mathbb{N}.$$

In this paper we show that the inequalities (1.1) and (1.3) are corollaries of general (sharp) trace inequalities for convex functions of operators. In particular, (1.1) and (1.3) can be extended to the Dirichlet and Neumann boundary problems for various classes of (systems) differential and pseudodifferential operators with constant coefficients (for example $(-\Delta)^\alpha$, $\alpha > 0$, operator of classical elasticity, etc). This approach can be also easily extended to operators acting on functions with values in a Hilbert space. We shall not consider this case here only because it requires many additional notations and assumptions.

Notice that the inequality $\lambda_k \geqslant C_d \, |\Omega|^{-2/d} \, k^{-2/d}$ with a constant $C_d < d/(d+2) \, (L_d^{cl})^{-2/d}$ was proved for bounded domains in [BS] and [C] and later for arbitrary domains in [R1,2], [M] and [Lb1] (see also [L]).

G. Pólya conjectured in [P] that (1.1) should hold without the multiplier $d/(d+2)$. He proved this conjecture for "tiling" domains $\Omega \subset \mathbb{R}^2$, i.e. copies of Ω fill the plane without gaps. In Subsection 2.3 we notice that Theorem 2.1 allows us to justify this conjecture for domains $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $d_1 + d_2 = d$, $d_1 \geqslant 2$, $d_2 \geqslant 1$, as long as the Dirichlet Laplacian in $L^2(\Omega_1)$ satisfies the Pólya conjecture and Ω_2 is an arbitrary domain whose d_2 - Lebesgue measure is finite (see Theorem 2.8 and Corollary 2.9).

In [LP] the method of [LY] was applied to the Dirichlet boundary problem for (systems of) differential operators of a higher order. The method presented here, however, allows us to obtain the same constants for differential operators and better constants than in [LP] for systems of differential operators (see Corollary 2.9 and Remark 2.10).

In Section 4 we obtain some more inequalities on the eigenvalues of $-\Delta^{\mathcal{D}}$ and, in particular, we give an upper bound for the eigenvalues λ_k, assuming only that the spectrum of $-\Delta^{\mathcal{D}}$ in $L_2(\Omega)$ is discrete.

In what follows we shall be dealing with different classes of vector functions on $\mathbb{R}^d$ with values in $\mathbb{C}^m$, $\mathbb{R}_+ = (0, +\infty)$, $D = -i\partial/\partial x$. By φ_λ we denote the following convex function

$$\varphi_\lambda(t) = (\lambda - t)_+ = \begin{cases} \lambda - t, & t < \lambda, \\ 0, & t \geqslant \lambda. \end{cases}$$

Assuming that a selfadjoint operator $B \geqslant 0$ has a discrete spectrum accumulating at infinity, we denote by $N(\lambda, B)$ its counting function of the spectrum

$$N(\lambda, B) = \#\{k : \, \lambda_k < \lambda\}.$$

If A is an $m \times m$ complex matrix, then A^* is its adjoint matrix.

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2. Dirichlet boundary value problem

1. Let Ω be an open measurable subset in $\mathbb{R}^d$. We shall deal with various classes of functions with values in $\mathbb{C}^m$, $m \in \mathbb{N}$. The norm and the scalar product in $\mathbb{C}^m$ is denoted by $\|\cdot\|$ and $(\cdot,\cdot)$ respectively. Let

$$L^2(\Omega, \mathbb{C}^m) = \Big\{ u : \int_\Omega \|u(x)\|^2\, dx < \infty \Big\}.$$

The class of smooth vector valued function with compact support $C_0^\infty(\Omega, \mathbb{C}^m) \subset L^2(\Omega, \mathbb{C}^m)$ is dense in $L^2(\Omega, \mathbb{C}^m)$.

Let $A(\xi)$ be a complex $m\times m$ matrix function, $\xi \in \mathbb{R}^d$. We assume, for simplicity, that there is $\varkappa \in \mathbb{R}_+$ and a constant C such that

$$0 \leqslant \|A(\xi)\| \leqslant C|\xi|^\varkappa, \qquad \xi \in \mathbb{R}^d. \tag{2.1}$$

Let $\hat u$ be the Fourier transform of the vector function $u \in L^2(\mathbb{R}^d, \mathbb{C}^m)$. We introduce a sesqui-linear form $\mathfrak{B}_\Omega$ defined on the vector functions from the class $C_0^\infty(\Omega, \mathbb{C}^m)$

$$\begin{aligned} \mathfrak{B}_\Omega[u,v] &= (2\pi)^{-d} \int_{\mathbb{R}^d} (A(\xi)\hat u(\xi), A(\xi)\hat v(\xi))\, d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} (B(\xi)\hat u(\xi), \hat v(\xi))\, d\xi, \qquad u,v \in C_0^\infty(\Omega, \mathbb{C}^m), \end{aligned} \tag{2.2}$$

where $B(\xi) = A^*(\xi)A(\xi)$. The completion of the class $C_0^\infty(\Omega, \mathbb{C}^n)$ with respect to the quadratic form $\mathfrak{B}_\Omega[u,u] + \gamma\|u\|^2$, $\gamma > 0$, defines a Hilbert space $\mathcal{H}_\gamma[\mathfrak{B}_\Omega] \subset L^2(\Omega, \mathbb{C}^m)$. ¿From (2.1) it follows that the Sobolev space $H_0^\varkappa(\Omega, \mathbb{C}^m)$ is a subspace of $\mathcal{H}_\gamma[\mathfrak{B}_\Omega]$. The closed quadratic form $\mathfrak{B}_\Omega$ defined on $\mathcal{H}_\gamma[\mathfrak{B}_\Omega]$, gives a selfadjoint pseudodifferntial operator which we denote by $B_{\mathcal{D}}$.

If $\Omega = \mathbb{R}^d$, then the above construction leads to a closed quadratic form $\mathfrak{B}_{\mathbb{R}^d}$. The selfadjoint operator defined by $\mathfrak{B}_{\mathbb{R}^d}$ is denoted by B. Both operators $B_{\mathcal{D}}$ and B can be considered as the Friedrichs extension of the pseudodifferntial operator

$$B_0(D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} B(\xi) u(y)\, dy d\xi$$

defined on the intersection $C_0^\infty(\mathbb{R}^d, \mathbb{C}^m) \cap L^2(\Omega, \mathbb{C}^m)$ and $C_0^\infty(\mathbb{R}^d, \mathbb{C}^m)$ respectively. We naturally identify the extention $B_{\mathcal{D}}$ with Dirichlet boundary value problem for B in Ω.

The next statement deals with a trace type inequality which is a partial case of the Berezin-Lieb inequality (see [Bz1,2], [Lb2], [S] and for its generalizations [LS]). We include the proof of this statement for the sake of completeness.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure, $|\Omega| < \infty$ and let the spectrum of the operator $B_{\mathcal{D}}$ consist of eigenvalues $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \dots$ such that $\lambda_k \to \infty$, as $k \to \infty$. Assume that for some $\lambda > 0$*

$$\int_{\mathbb{R}^d} \operatorname{Tr} \varphi_\lambda(B(\xi))\, d\xi < \infty. \tag{2.3}$$

Then the following inequality holds

$$\text{(2.4)} \quad \operatorname{Tr}\varphi_\lambda(B_{\mathcal D}) = \sum_k (\lambda-\lambda_k)_+ \leqslant (2\pi)^{-d}|\Omega| \int_{\mathbb{R}^d} \operatorname{Tr}(\lambda - (B(\xi))_+\, d\xi$$
$$= (2\pi)^{-d}|\Omega| \int_{\mathbb{R}^d} \operatorname{Tr}\varphi_\lambda(B(\xi))\, d\xi.$$

Proof. Let $\omega_1, \omega_2, \dots$ be the orthonormal basis in $L^2(\Omega, \mathbb{C}^m)$ consisting of the eigenfunctions of the operator $B_{\mathcal D}$ whose corresponding eigenvalues are $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \dots$ and let I be the unit matrix in $\mathbb{C}^m$. Then

$$\sum_k (\lambda-\lambda_k)_+ = \sum_k \Big(\lambda I - \big(\mathfrak{B}_\Omega[\omega_k,\omega_k]\big)\Big)_+$$
$$= \sum_k \Big((2\pi)^{-d} \int_{\mathbb{R}^d} \big((\lambda I - B(\xi))\,\hat\omega_k(\xi), \hat\omega_k(\xi)\big)\, d\xi\Big)_+$$
$$\leqslant (2\pi)^{-d} \sum_k \int_{\mathbb{R}^d} \big((\lambda I - B(\xi))\,\hat\omega_k(\xi), \hat\omega_k(\xi)\big)_+\, d\xi.$$

Denote by $\{\nu_j(\xi)\}_{j=1}^m$ and $\{\tau_j(\xi)\}_{j=1}^m$ the eigenvalues and the eigenvectors of the matrix $B(\xi)$. The right hand side of the last inequality can be rewritten as

$$(2\pi)^{-d} \sum_{j=1}^m \sum_k \int_{\mathbb{R}^d} (\lambda-\nu_j(\xi))_+ |(\hat\omega_k(\xi), \tau_j(\xi))|^2\, d\xi$$
$$= (2\pi)^{-d} \sum_{j=1}^m \sum_k \int_{\mathbb{R}^d} (\lambda-\nu_j(\xi))_+ \Big| \int_\Omega \Big(e^{i(x,\xi)}\tau_j(\xi), \omega_k(x)\Big)\, dx\Big|^2 d\xi$$
$$= (2\pi)^{-d} \sum_{j=1}^m \int_{\mathbb{R}^d} (\lambda-\nu_j(\xi))_+ \|e^{i(\cdot,\xi)}\tau_j(\xi)\|^2_{L^2(\Omega,\mathbb{C}^m)}\, d\xi$$
$$= (2\pi)^{-d}\,|\Omega| \sum_{j=1}^m \int_{\mathbb{R}^d} (\lambda-\nu_j(\xi))_+\, d\xi = (2\pi)^{-d}\,|\Omega| \int_{\mathbb{R}^d} \operatorname{Tr}(\lambda I - B(\xi))_+\, d\xi.$$

The proof is complete. □

Remark. The proof of Theorem 2.1 remains almost the same if instead of $\mathbb{C}^m$ we consider an infinite dimensional Hilbert H.

Definition 2.2. We say that $B(\xi)$ is a *positively homogeneous symbol of degree* α, $\alpha > 0$, if there exists a family of unitary in $\mathbb{C}^m$ matrix-function $U(\lambda,\xi)$, such that

$$B(\lambda\xi) = \lambda^\alpha U^*(\lambda,\xi) B(\xi) U(\lambda,\xi), \qquad \lambda > 0.$$

If $B(\xi)$ is now a homogeneous symbol, then

$$\operatorname{Tr}\varphi_\lambda(B(\xi)) = \lambda \operatorname{Tr}\varphi_1\big(\lambda^{-1}B(\xi)\big)$$
$$= \lambda \operatorname{Tr}\varphi_1\Big(U(\lambda^{-1/\alpha},\xi)B(\lambda^{-1/\alpha}\xi)U^*(\lambda^{-1/\alpha},\xi)\Big) = \lambda \operatorname{Tr}\varphi_1\big(B(\lambda^{-1/\alpha}\xi)\big).$$

If we integrate both sides of the last equality with respect to ξ and change the variables $\lambda^{-1/\alpha}\xi \to \xi$, then we derive the following statement:

Corollary 2.3. *Let $B(\xi)$ be a positively homogeneous symbol of degree α. Then under the conditions of Theorem 2.1 we obtain*

$$\sum_k (\lambda - \lambda_k)_+ \leqslant \lambda^{1+\frac{d}{\alpha}} (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_1(B(\xi)) \, d\xi, \qquad \lambda > 0. \tag{2.5}$$

Remark 2.4. The constant in the right hand side of (2.5) is the best possible since it appears in the corresponding asymptotic formula for $\sum_k (\lambda - \lambda_k)_+$, as $\lambda \to \infty$.

2. We use the results of Subsection 2.1 in order to deduce an upper estimate for the counting function of the spectrum of the operator $B_{\mathcal{D}}$.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure, $|\Omega| < \infty$ and $B(\xi)$ be a positively homogeneous symbol of degree α. Then*

$$N(\lambda, B_{\mathcal{D}}) \leqslant \lambda^{\frac{d}{\alpha}} (2\pi)^{-d} \, |\Omega| \, \frac{d}{\alpha} \left(1 + \frac{\alpha}{d}\right)^{1+\frac{d}{\alpha}} \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_1(B(\xi)) \, d\xi. \tag{2.6}$$

Proof. Obviously

$$N(\eta - \rho, B_{\mathcal{D}}) \leqslant \frac{1}{\rho} \int_0^\infty (\eta - \nu)_+ \, dN(\nu, B_{\mathcal{D}}) = \frac{1}{\rho} \sum_k (\eta - \lambda_k)_+, \qquad \eta \geqslant \rho > 0.$$

Therefore Corollary 2.3 implies

$$N(\eta - \rho, B_{\mathcal{D}}) \leqslant \frac{\eta^{1+d/\alpha}}{\rho} \, (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_1(B(\xi)) \, d\xi.$$

Choose $\eta = (1 + \tau)\lambda$ and $\rho = \tau\lambda$. Then

$$N(\lambda, B_{\mathcal{D}}) \leqslant \lambda^{d/\alpha} \frac{(1+\tau)^{1+d/\alpha}}{\tau} \, (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_1(B(\xi)) \, d\xi. \tag{2.7}$$

The minimum value of $(1+\tau)^{1+d/\alpha}\tau^{-1}$ is reached at $\tau = \alpha/d$. By substituting this value in (2.7) we obtain (2.6). $\square$

Let $m = 1$ and $B(\xi) = |\xi|^\alpha$. Then the operator $B_{\mathcal{D}}$ coincides with the operator of Dirichlet boundary problem for $(-\Delta^{\alpha/2})$. In this case

$$(2\pi)^{-d} \, \frac{d}{\alpha} \left(1 + \frac{\alpha}{d}\right)^{1+\frac{d}{\alpha}} \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_1(B(\xi)) \, d\xi \; = L_d^{cl} \left(1 + \frac{\alpha}{d}\right)^{\frac{d}{\alpha}}.$$

and we obtain

Corollary 2.6. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure, $|\Omega| < \infty$. Then*

$$N(\lambda, (-\Delta^{\alpha/2})_{\mathcal{D}}) \leqslant \lambda^{d/\alpha} \, L_d^{cl} \left(1 + \frac{\alpha}{d}\right)^{\frac{d}{\alpha}} |\Omega|. \tag{2.8}$$

Remark 2.7. If $\alpha = 2$, then (2.8) is equivalent to the inequality (1.1) proved by Li and Yau in [LY].

3. We show here that in some special cases Theorem 2.1 implies the Pólya conjecture.

Theorem 2.8. *Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, where $d_1 + d_2 = d$, $d_1 \geqslant 2$, $d_2 \geqslant 1$. Suppose that the operator of the Dirichlet boundary problem in $L^2(\Omega_1)$ satisfies the Pólya conjecture and Ω_2 is an arbitrary domain whose d_2-Lebesgue measure is finite. Then*

$$N(\lambda, -\Delta^{\mathcal{D}}) \leqslant \lambda^{d/2}\, L_d^{cl}\, |\Omega|, \qquad \lambda > 0,$$

or equivalently,

$$\lambda_k \geqslant (L_d^{cl}\, |\Omega|)^{-2/d}\, k^{\frac{2}{d}}, \qquad k \in \mathbb{N}.$$

Proof. Let $-\Delta_j^{\mathcal{D}}$ be the Dirichlet Laplacian in Ω_j, $j = 1, 2$. Since $\Omega = \Omega_1 \times \Omega_2$, the eigenvalues of $-\Delta^{\mathcal{D}}$ in Ω are equal to

$$\lambda_{lk} = \rho_l + \eta_k, \qquad l, k \in \mathbb{N},$$

where ρ_l and η_k are the eigenvalues of $-\Delta_1^{\mathcal{D}}$ and $-\Delta_2^{\mathcal{D}}$ respectively. Our assumptions on $-\Delta_1^{\mathcal{D}}$ imply

$$N(\rho, -\Delta_1^{\mathcal{D}}) \leqslant \rho^{d_1/2}\, L_{d_1}^{cl}\, |\Omega_1|.$$

Therefore

$$\begin{aligned} (2.9) \quad N(\lambda, -\Delta^{\mathcal{D}}) &= \#\{(l,k) \in \mathbb{N} \times \mathbb{N} \colon\ \rho_l + \eta_k < \lambda\} \\ &= \#\{(l,k) \in \mathbb{N} \times \mathbb{N} \colon\ \rho_l < (\lambda - \eta_k)_+\} \leqslant L_{d_1}^{cl}\, |\Omega_1| \sum_k (\lambda - \eta_k)^{d_1/2}. \end{aligned}$$

Let us first assume that $d_1 = 2$. Then by applying (2.5) to $-\Delta_2^{\mathcal{D}}$ we find

$$\begin{aligned} N(\lambda, -\Delta^{\mathcal{D}}) &\leqslant \lambda^{1+d_2/2}\, L_2^{cl}\, |\Omega_1|\, (2\pi)^{-d_2}\, |\Omega_2| \int_{\mathbb{R}^{d_2}} (1 - |\xi|^2)_+ d\xi \\ &= \lambda^{1+d_2/2}\, L_2^{cl}\, L_{d_2}^{cl}\, \frac{2}{(d_2+2)}\, |\Omega_1|\, |\Omega_2| = \lambda^{d/2}\, L_d^{cl}\, |\Omega|. \end{aligned}$$

Let

$$\mathcal{B}(p,q) = \int_0^1 \nu^{p-1}(1-\nu)^{q-1}\, d\nu = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

be the Beta function. If $d_1 > 2$, then using the same arguments we find

$$\begin{aligned} (2.10) \quad \sum_k (\lambda - \eta_k)_+^{d_1/2} &= \mathcal{B}(d_1/2 - 1, 2)^{-1} \sum_{k:\lambda > \eta_k} \int_0^\infty \nu^{d_1/2-2}(\lambda - \eta_k - \nu)_+\, d\nu \\ &\leqslant \mathcal{B}(d_1/2 - 1, 2)^{-1} \sum_k \int_0^\infty \nu^{d_1/2-2}(\lambda - \nu - \eta_k)_+\, d\nu \\ &\leqslant \mathcal{B}(d_1/2 - 1, 2)^{-1} |\Omega_2|\, \frac{2}{d_2+2}\, L_{d_2}^{cl} \int_0^\infty \nu^{d_1/2-2}(\lambda - \nu)_+^{d_2/2+1}\, d\nu \\ &= \lambda^d\, |\Omega_2|\, \frac{2}{d_2+2}\, L_{d_2}^{cl}\, \mathcal{B}(d_1/2-1, 2)^{-1}\, \mathcal{B}(d_1/2-1, d_2/2+2). \end{aligned}$$

Collecting together all the constants in (2.9) and (2.10) we complete the proof. $\square$

Corollary 2.9. *Under the conditions of Theorem 2.8 if $\Omega_1 \subset \mathbb{R}^2$ is a tiling domain, then the Pólya conjecture holds true.*

4. Let us consider the eigenvalue problem for the equations of classical elasticity

$$-a\Delta u_j - (a+b)\frac{\partial}{\partial x_j}(\nabla \cdot u) = \lambda u_j, \tag{2.11}$$

$$u_j\Big|_{\partial\Omega} = 0, \qquad j = 1,2,3, \quad x \in \Omega \subset \mathbb{R}^3, \tag{2.12}$$

where a and b denote the Lame constants, $a, b > 0$ and $u = (u_1, u_2, u_3)$ is the elastic displacement vector. In this case $B(\xi)$ is equal to the matrix

$$B(\xi) = (a+b) \cdot \begin{pmatrix} \frac{a}{a+b}|\xi|^2 + \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_2\xi_1 & \frac{a}{a+b}|\xi|^2 + \xi_2^2 & \xi_2\xi_3 \\ \xi_3\xi_1 & \xi_3\xi_2 & \frac{a}{a+b}|\xi|^2 + \xi_3^2 \end{pmatrix}, \qquad \xi \in \mathbb{R}^3.$$

Its eigenvalues are

$$\nu_1 = a|\xi|^2, \quad \nu_2 = a|\xi|^2, \quad and \quad \nu_3 = (2a+b)|\xi|^2.$$

Thus we obtain

$$\int_{\mathbb{R}^3} \operatorname{Tr} \varphi_1(B(\xi))\, d\xi = \frac{8\pi}{15}\Big(a^{3/2} + (2a+b)^{3/2}\Big).$$

Applying Theorem 2.5 with $\alpha = 2$ we derive

Corollary 2.10. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure, $|\Omega| < \infty$. If $B_{\mathcal{D}}$ is the operator of classical elasticity (2.11), (2.12), then*

$$N(\lambda, B_{\mathcal{D}}) \leqslant \lambda^{3/2}(2\pi^2)^{-1}5^{3/2}3^{-5/2}\Big(a^{3/2} + (2a+b)^{3/2}\Big)\,|\Omega|,$$

or equivalently

$$\lambda_k \geqslant \frac{3a}{5}\Big(\frac{3}{2+(2+b/a)^{-3/2}}\Big)^{2/3} \cdot \Big(\frac{2\pi^2 k}{|\Omega|}\Big)^{2/3}. \tag{2.13}$$

Remark 2.11. Formula (2.13) is an improvement of the inequality (1.19) obtained in [LP]. This became possible because the right hand side in (2.6) involves the trace $\operatorname{Tr}\varphi_1(B(\xi))$ rather than $m \cdot \max_{j=1,\dots,m} \nu_j(B(\xi))$, where $\nu_j(B(\xi))$ are the eigenvalues of the matrix $B(\xi)$.

3. Neumann boundary value problem

1. Let us consider a differential operator

$$A(D)u(x) = \sum_{\beta \leqslant l} A_\beta D^\beta u(x), \qquad u \in C^\infty(\bar{\Omega}, \mathbb{C}^m),$$

where $\Omega \subset \mathbb{R}^d$ is an open set and the coeffitients A_β are $m \times m$-matrices independent of $x \in \Omega$. Let us introduce a quadratic form

$$\mathfrak{B}_{\bar{\Omega}}[u,u] = \int_\Omega \|A(D)u\|^2 \, dx, \qquad u \in C^\infty(\bar{\Omega}, \mathbb{C}^m),$$

where $\bar{\Omega}$ is the closure of the set Ω. This form is semibounded from below. We assume that the form $\mathfrak{B}_{\bar{\Omega}}$ is closable. Then $\mathfrak{B}_{\bar{\Omega}}$ defines a semibounded selfadjoint operator in $L^2(\Omega, \mathbb{C}^m)$ which we denote by $B_{\mathcal{N}}$. The symbol of this operator is $B(\xi) := A^*(\xi)A(\xi)$. The operator $B_{\mathcal{N}}$ can be naturally considered as an operator of the Neumann boundary problem in the domain Ω for the differential operator whose symbol is equal to $B(\xi)$.

Let us assume that the spectrum of this operator is discrete, consists of $0 = \mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \dots$, and $\mu_k \to \infty$, as $k \to \infty$.

We put aside the problem of the existence of the closure of the form $\mathfrak{B}_{\bar{\Omega}}$ and the discreteness of the spectrum of $B_{\mathcal{N}}$. For example, in the scalar case when $B(\xi) = |\xi|^{2l}$ the discreteness of the spectrum of this operator is equivalent to the compactness of the embedding $H^l(\Omega) \to L^2(\Omega)$. The latter requires some restrictive assumption on Ω. The precise conditions of the compactness of this embedding are given in [Mz].

Therem 3.1. *Let $|\Omega| < \infty$,*

$$\int_{\mathbb{R}^d} \operatorname{Tr} \varphi_\mu(B(\xi))\, d\xi < \infty \tag{3.1}$$

for some $\mu > 0$ and assume that the spectrum of the operator $B_{\mathcal{N}}$ is discrete, $\mu_k \to \infty$ as $k \to \infty$. Then

$$\sum_k (\mu - \mu_k)_+ \geqslant (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_\mu(B(\xi))\, d\xi. \tag{3.2}$$

Proof. Let ω_k, be the orthonormal basis of eigenfunctions of the operator $B_{\mathcal{N}}$ whose respective eigenvalues are μ_k, $k = 1, 2, \dots$. Denote

$$e_\xi(x) = \begin{cases} \exp(ix\xi), & \text{as } x \in \Omega, \\ 0, & \text{as } x \notin \Omega, \end{cases}$$

and introduce the orthonormal basis $\{\tau_j(\xi)\}_{j=1}^m$ consisting of the eigenvectors of the matrix $B(\xi)$. Then

$$\sum_k (\mu-\mu_k)_+ = \operatorname{Tr}\varphi_\mu(B_{\mathcal N}) = \sum_k \varphi_\mu(\mu_k)\int_\Omega \|\omega_k(x)\|^2\,dx \tag{3.3}$$
$$= (2\pi)^{-d}\sum_k \varphi_\mu(\mu_k)\int_{\mathbb R^d} \|\hat\omega_k(\xi)\|^2\,d\xi$$
$$= (2\pi)^{-d}\sum_k\sum_{j=1}^m \varphi_\mu(\mu_k)\int_{\mathbb R^d} |(\hat\omega_k(\xi),\tau_j(\xi))|^2\,d\xi.$$

Let E_ν, $\nu\in\mathbb R$, be the spectral projection of the selfadjoint operator $B_{\mathcal N}$. We can now rewrite (3.3) as

$$\operatorname{Tr}\varphi_\mu(B_{\mathcal N})$$
$$= \int_{\mathbb R^d}\sum_k \varphi_\mu(\mu_k)\sum_{j=1}^m\int_\Omega\int_\Omega \big(\omega_k(x),\tau_j(\xi)e_\xi(x)\big)\big(\tau_j(\xi)e_\xi(y),\omega_k(y)\big)\,dy\,dx\,d\xi$$
$$= \sum_{j=1}^m\int_{\mathbb R^d}\int_0^\infty \varphi_\mu(\nu)\,(dE_\nu e_\xi\tau_j(\xi), e_\xi\tau_j(\xi))\,d\xi.$$

Since

$$|\Omega|^{-1}\int_0^\infty (dE_\nu e_\xi\tau_j, e_\xi\tau_j) = 1,\qquad \forall\xi\in\mathbb R^d,\quad j=1,2,\dots,m,$$

then by using the Jensen inequality we obtain

$$\operatorname{Tr}\varphi_\mu(B_{\mathcal N}) \geqslant |\Omega|\int_{\mathbb R^d}\sum_{j=1}^m \varphi_\mu\Big(\frac{1}{|\Omega|}\int_0^\infty \nu\,(dE_\nu e_\xi\tau_j(\xi), e_\xi\tau_j(\xi))\Big)\,d\xi.$$

Notice that

$$\int_0^\infty \nu\,(dE_\nu e_\xi\tau_j, e_\xi\tau_j) = \mathfrak B_{\bar\Omega}[e_\xi\tau_j(\xi), e_\xi\tau_j(\xi)]$$
$$= \int_\Omega \|A(D)e_\xi\tau_j(\xi)\|^2\,dx = |\Omega|\,\|A(\xi)\tau_j(\xi)\|^2 = |\Omega|\,(B(\xi)\,\tau_j(\xi),\tau_j(\xi)).$$

Since $\tau_j(\xi)$ are the eigenvectors of the matrix-function $B(\xi)$, we have

$$\sum_{j=1}^m \varphi_\mu\Big(\frac{1}{|\Omega|}\int_0^\infty \nu\,(dE_\nu e_\xi\tau_j(\xi), e_\xi\tau_j(\xi))\Big) = \operatorname{Tr}\varphi_\mu(B(\xi)).$$

This leads to (3.2) and completes the proof. $\square$

2. Apply now the inequality (3.2) to the counting function of the spectrum of the operator $B_{\mathcal N}$.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure, $|\Omega| < \infty$ and $B(\xi)$ be a positively homogeneous symbol of degree $2l$. Then under the assumptions of Theorem 3.1 we have*

$$N(\mu, B_{\mathcal{N}}) \geqslant \mu^{\frac{d}{2l}} (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_1(B(\xi)) \, d\xi, \qquad \mu > 0. \tag{3.4}$$

Proof. Since the first eigenvalue of the operator $B_{\mathcal{N}}$ is equal to zero we obtain that

$$N(\mu, B_{\mathcal{N}}) \geqslant \frac{1}{\mu} \sum_k (\mu - \mu_k)_+, \qquad \mu > 0.$$

Theorem 3.1 and the homogeneity of the matrix $B(\xi)$ lead us to

$$N(\mu, B_{\mathcal{N}}) \geqslant \frac{1}{\mu} (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_\mu(B(\xi)) \, d\xi$$
$$= \mu^{d/2l} (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_1(B(\xi)).$$

$\square$

Let $m = 1$ and $B(\xi) = |\xi|^{2l}$, $l \in \mathbb{N}$. Then the operator $B_{\mathcal{N}}$ coincides with the operator of the Neumann boundary problem for $(-\Delta^l)$. In this case

$$(2\pi)^{-d} \int_{\mathbb{R}^d} \varphi_1(B(\xi)) \, d\xi \; = L_d^{cl} \frac{2l}{(d + 2l)}.$$

and (3.3) implies

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure, $|\Omega| < \infty$. Then*

$$N(\mu, (-\Delta^l)_{\mathcal{N}}) \geqslant \mu^{d/2l} \, L_d^{cl} \, \frac{2l}{d + 2l} \, |\Omega|. \tag{3.5}$$

Remark 3.4. If $l = 1$, then (3.5) is equivalent to the inequality (1.3) proved in [K1].

4. More eigenvalue estimates for the Dirichlet Laplacian

Let $B_{\mathcal{D}} = -\Delta^{\mathcal{D}}$ in $L^2(\Omega)$, $\Omega \subset \mathbb{R}^d$ and let us assume that the spectrum of this operator is discrete. Let $\omega_1, \omega_2, \dots$ be the orthonormal basis of eigenfunctions of the operator $B_{\mathcal{D}}$ whose respective eigenvalues are $0 < \lambda_1 < \lambda_2 \leqslant \dots$. Denote

$$\widetilde{\omega} = \sup_{x \in \Omega} |\omega_1(x)|. \tag{4.1}$$

Using the argument similar to those we used in Section 3 we can prove the following statement.

Theorem 4.1. *Let the spectrum of the operator $-\Delta^{\mathcal{D}}$ in $L^2(\Omega)$ be discrete. Then for any $\lambda > 0$ we have*

$$\operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}}) = \sum_k (\lambda - \lambda_k)_+ \geqslant (\lambda - \lambda_1)_+^{1+d/2} \, L_d^{cl} \frac{2}{d+2} \widetilde{\omega}^{-2}. \tag{4.2}$$

Proof. The functions

$$\theta_\xi(x) := \omega_1 e^{-i(x,\xi)}, \qquad \xi \in \mathbb{R}^d,$$

belong to the domain of the operator $-\Delta^{\mathcal{D}}$. Obviously

$$\begin{aligned}\operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}}) = \sum_k \varphi_\lambda(\lambda_k) \int |\omega_k|^2 \, dx \geqslant \widetilde{\omega}^{-2} \sum_k \varphi_\lambda(\lambda_k) \int |\omega_1 \, \omega_k|^2 \, dx \\ = (2\pi)^{-d} \widetilde{\omega}^{-2} \sum_k \varphi_\lambda(\lambda_k) \int \Big| \int \omega_k \theta_\xi(x) \, dx \Big|^2 \, d\xi.\end{aligned}$$

If the spectral projection of the operator $-\Delta^{\mathcal{D}}$ is denoted by E_ν, then the last expression can be rewritten as

$$\operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}}) \geqslant (2\pi)^{-d} \widetilde{\omega}^{-2} \int_{\mathbb{R}^d} \int_0^\infty \varphi_\lambda(\nu) (dE_\nu \, \theta_\xi, \theta_\xi) \, d\xi.$$

Clearly

$$\int_0^\infty (dE_\nu \, \theta_\xi, \theta_\xi) = \|\theta_\xi\|^2_{L^2(\Omega)} = \|\omega_1\|^2_{L^2(\Omega)} = 1,$$

and by using the Jensen inequality we obtain

$$\operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}}) \geqslant (2\pi)^{-d} \widetilde{\omega}^{-2} \int \varphi_\lambda \Big(\int_0^\infty \nu \, (dE_n \, \theta_\xi, \theta_\xi) \Big) \, d\xi. \tag{4.3}$$

A simple calculation gives

$$\int_0^\infty \nu \, (dE_n \, \theta_\xi, \theta_\xi) = \int_{\mathbb{R}^d} |\nabla \theta_\xi|^2 \, dx = (|\xi|^2 + \lambda_1). \tag{4.4}$$

Combining (4.4) in (4.3) we arrive at

$$\operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}}) \geqslant (2\pi)^{-d} \, \widetilde{\omega}^{-2} \int_{\mathbb{R}^d} \varphi_\lambda(|\xi|^2 + \lambda_1) \, d\xi = (\lambda - \lambda_1)_+^{1+d/2} \, L_d^{cl} \frac{2}{d+2} \widetilde{\omega}^{-2}.$$

The theorem is proved. $\square$

In particular, if $\lambda = \lambda_2$ in (4.2), then we obtain the following upper estimate for the difference of the two first eigenvalues for the Dirichlet Laplacian.

Corollary 4.2. *Under the conditions of Theorem 4.1 we have*

$$\lambda_2 - \lambda_1 \leqslant \Big(L_d^{cl} \frac{2}{d+2} \Big)^{-2/d} \widetilde{\omega}^{4/d}.$$

Remark 4.3. Some other upper estimates on $\lambda_2 - \lambda_1$ were studied in [PPW] and [SWYY] (see also [SY]).

If $\lambda > \lambda_1$, then

$$N(\lambda, -\Delta^{\mathcal{D}}) \geqslant \frac{1}{\lambda - \lambda_1} \operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}})$$

and by using (4.2) we derive

Corollary 4.4. *If the conditions of Theorem 4.1 are satisfied, then for any $\lambda > \lambda_1$ we obtain*

$$N(\lambda, -\Delta^{\mathcal{D}}) \geqslant (\lambda - \lambda_1)^{d/2} L_d^{cl} \frac{2}{d+2} \widetilde{\omega}^{-2}.$$

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