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Paweł Nurowski
Maciej Przanowski

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A FOUR DIMENSIONAL EXAMPLE OF RICCI FLAT METRIC ADMITTING ALMOST-KÄHLER NON-KÄHLER STRUCTURE *

Paweł Nurowski

Dipartimento di Scienze Matematiche
Universita degli Studi di Trieste, Trieste, Italy †

Maciej Przanowski

Instytut Fizyki, Politechnika Lódzka
Wólczańska 219, 93-005 Łódź, Poland

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Abstract

We construct an example of Ricci-flat almost-Kähler non-Kähler structure in four dimensions.

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†Permanent address: *Instytut Fizyki Teoretycznej, Wydział Fizyki, Uniwersytet Warszawski, ul. Hoża 69, Warszawa, Poland, e-mail: nurowski@fuw.edu.pl*

1. Let $\mathcal{M}$ be a 4-manifold equipped with a metric g of signature $(++++)$. The pair $(\mathcal{M},\ g)$ is called a Riemmanian 4-manifold.

An almost hermitian structre on $(\mathcal{M},\ g)$ is a tensor field $J : T\mathcal{M} \to T\mathcal{M}$ such that $J^2 = -id$ and $g(JX, JY) = g(X, Y)$. An almost hermitian structure $(\mathcal{M},\ g,\ J)$ is called hermitian if J is integrable. Due to the Newlander-Nirenberg theorem this is equivalent to the vanishing of the Nijenhuis tensor $N_J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$ for J.

Given an almost hermitian structure $(\mathcal{M},\ g,\ J)$ one defines the fundamental 2-form ω by $\omega(X, Y) = g(X, JY)$. An almost hermitian structure $(\mathcal{M},\ g,\ J)$ is called almost-Kähler if its fundamental 2-form is closed. If, in addition, J is integrable then such structure is called Kähler.

This paper is motivated by the following conjecture [5].

Goldberg's Conjecture
The almost Kähler structure of a compact Einstein manifold is necessarilly Kähler.

The conjecture was proven in the case of non-negative scalar curvature of the Einstein manifold by K. Sekigawa in [10].

In this paper we show that the assumption about compactness of the Einstein manifold is essential for the Goldberg conjecture. In particular, we give an explicit example of a Ricci-flat almost-Kähler non-Kähler structure on a noncompact 4-manifold. This result is given by Theorem 1 of paragraph 4.

2. Let $\mathcal{U}$ be an open subset of $\mathbf{R}^4$. Let $\theta^i = (M, \bar{M}, N, \bar{N})$ be four complex-valued 1-forms on $\mathcal{U}$ such that $M \wedge \bar{M} \wedge N \wedge \bar{N} \neq 0$. Using θ^i we define a metric g on $\mathcal{U}$ by

$$g = 2(M\bar{M} + N\bar{N}) := M \otimes \bar{M} + \bar{M} \otimes M + N \otimes \bar{N} + \bar{N} \otimes N.$$

Clearly $(\mathcal{U},\ g)$ is a Riemannian 4-manifold.
The Weyl tensor W of the metric g splits onto self-dual (W^+) and anti-self-dual (W^-) parts. $(\mathcal{U}, g)$ is said to be (anti-)self-dual iff $(W^+ \equiv 0)$ $W^- \equiv 0$. If $(W^+ \neq 0)$ $W^- \neq 0$ then in every point of $\mathcal{U}$ it defines at most two spinor directions $([\alpha^+, \beta^+])$ $[\alpha^-, \beta^-]$; see e.g. [6, 9]. (W^+) W^- is said to be of type D if (α^+) α^- coincides with (β^+) β^-.

Let $e_i = (m, \bar{m}, n, \bar{n})$ be a basis dual to $\theta^i = (M, \bar{M}, N, \bar{N})$. For any $\xi \in \mathbf{C}$ it is convenient to consider 1-forms

$$M_\xi = \frac{M - \bar{\xi}\bar{N}}{\sqrt{1+\xi\bar{\xi}}} \qquad N_\xi = \frac{N + \bar{\xi}\bar{M}}{\sqrt{1+\xi\bar{\xi}}}$$

and vector fields

$$m_\xi = \frac{m - \xi\bar{n}}{\sqrt{1+\xi\bar{\xi}}} \qquad n_\xi = \frac{n + \xi\bar{m}}{\sqrt{1+\xi\bar{\xi}}}.$$

The following Lemma is well known (see for example [6, 9]).

Lemma 1

i) *For any value of the complex parameter $\xi \in \mathbf{C} \cup \{\infty\}$ the expressions*

$$J^+_\xi = i(\overline{M_\xi} \otimes \overline{m_\xi} - M_\xi \otimes m_\xi + \overline{N_\xi} \otimes \overline{n_\xi} - N_\xi \otimes n_\xi)$$

$$J^-_\xi = i(M_\xi \otimes m_\xi - \overline{M_\xi} \otimes \overline{m_\xi} + \overline{N_\xi} \otimes \overline{n_\xi} - N_\xi \otimes n_\xi)$$

define almost hermitian structures on $(\mathcal{U}, g)$.

ii) *The fundamental 2-forms corresponding to J^+_ξ and J^-_ξ are respectively given by*

$$\omega^+_\xi = i(M_\xi \wedge \overline{M_\xi} + N_\xi \wedge \overline{N_\xi})$$

$$\omega^-_\xi = i(\overline{M_\xi} \wedge M_\xi + N_\xi \wedge \overline{N_\xi}).$$

iii) *Any almost hermitian structure on $(\mathcal{U}, g)$ is given either by one of J^+_ξ or by one of J^-_ξ. Structures J^+_ξ are different from J^-_ξ; also, different ξs correspond to different structures.*

iv) *If the metric g is not self-dual then among J^+_ξs only at most four structures, corresponding to specific four values of the parameter ξ, may be integrable. Analogously, if the metric g is not anti-self-dual then only at most four J^-_ξs may be integrable.*

3. Let (x^1, x^2, x^3, x^4) be Euclidean coordinates on $\mathcal{U}$. Define

$$z_1 = x^1 + ix^2 \qquad z_2 = x^3 + ix^4. \tag{1}$$

Let $\partial_k = \frac{\partial}{\partial z_k}$ and $\partial_{\bar{k}} = \frac{\partial}{\partial \overline{z_k}}$, $k = 1, 2$.

Consider two 1-forms M and N on $\mathcal{U}$ defined by

$$M = f(\mathrm{d}z_1 + h\mathrm{d}z_2) \qquad N = \frac{1}{f}\mathrm{d}z_2, \tag{2}$$

where $f \neq 0$ (real) and h (complex) are functions on $\mathcal{U}$.

Since $M \wedge \bar{M} \wedge N \wedge \bar{N} = \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2} \neq 0$ then the metric $g = 2(M\bar{M} + N\bar{N})$ equipes $\mathcal{U}$ with the Riemannian structure. Consider almost hermitian structures J^+_ξ for such $(\mathcal{U}, g)$. It is interesting to note that if $\xi = \mathrm{e}^{i\phi}$ =const then the corresponding fundamental 2-form $\omega^+_{\mathrm{e}^{i\phi}}$ reads

$$\omega^+_{\mathrm{e}^{i\phi}} = i(\mathrm{e}^{i\phi}\mathrm{d}z_2 \wedge \mathrm{d}z_1 - \mathrm{e}^{-i\phi}\mathrm{d}\overline{z_2} \wedge \mathrm{d}\overline{z_1})$$

and is closed. Thus, for any $\mathrm{e}^{i\phi} \in \mathbf{S}^1$ we constructed an almost-Kähler structure $(\mathcal{U}$, g, $J^+_{\mathrm{e}^{i\phi}})$. If the functions f and h are general enough, then the metric g has no chance to be self-dual. Moreover, since in such case there is a finite number of hermitian structures among J^+_ξ, then most of our structures must be non-Kähler. Summing up we have the folowing Lemma.

Lemma 2 *Let $(z_1, \overline{z_1}, z_2, \overline{z_2})$ be coordinates on $\mathcal{U}$ as in (1). Then for each value of the real constant $\phi \in [0, 2\pi[$ the metric*

$$g = 2f^2(\mathrm{d}z_1 + h\mathrm{d}z_2)(\mathrm{d}\overline{z_1} + \bar{h}\mathrm{d}\overline{z_2}) + 2\frac{1}{f^2}\mathrm{d}z_2\mathrm{d}\overline{z_2} \tag{3}$$

and the almost complex structure

$$J^+_{\mathrm{e}^{i\phi}} = 2\mathrm{Re}\{i\mathrm{e}^{i\phi}[f^2(\mathrm{d}z_1 + h\mathrm{d}z_2) \otimes (\partial_{\bar{2}} - \bar{h}\partial_{\bar{1}}) - \frac{1}{f^2}\mathrm{d}z_2 \otimes \partial_{\bar{1}}]\} \tag{4}$$

define an almost-Kähler structure on $\mathcal{U}$.
If the functions f and h are general enough to prevent the metric of beeing self-dual then these structures are non-Kähler for almost all values of ϕ.

4. We look for not-self-dual Ricci-flat metrics among the metrics of Lemma 2. For this pourpose it is convenient to restrict to the metrics (3) whose anti-self-dual part of the Weyl tensor is strictly of type D. Such a restriction guarantees that all structures (4) are non-Kähler [6, 9].

We recall a useful Lemma [7].

Lemma 3 *Let g be a Ricci-flat Riemannian metric in four dimensions. Assume that the anti-self-dual part of the Weyl tensor for g is strictly of type D. Then, locally there always exist complex coordinates (z_1, z_2) and a real function $K = K(v, z_2, \overline{z_2})$, $v = z_1 + \overline{z_1}$ such that the metric can can be written as*

$$g = \frac{\varepsilon K_{vv}}{(K_v)^{3/2}}(\mathrm{d}z_1 + \frac{K_{v2}}{K_{vv}}\mathrm{d}z_2)(\mathrm{d}\overline{z_1} + \frac{K_{v\bar{2}}}{K_{vv}}\mathrm{d}\overline{z_2}) + 4\mathrm{e}^{-K}\frac{(K_v)^{1/2}}{\varepsilon K_{vv}}\mathrm{d}z_2\mathrm{d}\overline{z_2}, \tag{5}$$

where $K_{v\bar{2}} = \frac{\partial^2 K}{\partial v \partial \overline{z_2}}$, etc. The function K satisfies

$$K_{vv}K_{2\bar{2}} - K_{v\bar{2}}K_{v2} - 2\mathrm{e}^{-K}(K_{vv} + 2(K_v)^2) = 0, \tag{6}$$

$$K_v > 0, \qquad \varepsilon K_{vv} > 0 \tag{7}$$

where ε is either plus or minus one.
Also, every function $K = K(v, z_2, \overline{z_2})$ satisfying (6)-(7) defines, via (5), a Ricci-flat metric. This metric has the anti-self-dual part of the Weyl tensor of strictly type D.

We ask when the metric (3) can be written in the form (5). Identifying coordinates (z_1, z_2) in both metrics we see that it is possible if

$$2f^2 = \frac{\varepsilon K_{vv}}{(K_v)^{3/2}} \qquad \text{and} \qquad \frac{2}{f^2} = 4\mathrm{e}^{-K}\frac{(K_v)^{1/2}}{\varepsilon K_{vv}}.$$

These two equations are compatible only if $K_v \mathrm{e}^K = 1$. It is a matter of straightforward integration that, modulo the coordinate transformations, the general solution of this equation which simultaneously satisfies the equation (6) is[1] $K = \log(v - 2z_2\overline{z_2})$. Using such K we easily find that in the region

$$\mathcal{U}' = \{\mathcal{U} \ni (z_1, z_2) \quad \text{s.t.} \quad v - 2z_2\overline{z_2} > 0\}$$

[1]This solution was already known to Sławomir Białecki in 1984 [1].

the metric (3) with

$$f = \frac{1}{\sqrt{2}(v - 2z_2\overline{z_2})^{1/4}}, \qquad h = -2\overline{z_2},$$

is Ricci-flat and strictly of type D on the anti-self-dual side of its Weyl tensor. The explicit expression for such g reads

$$g = \frac{1}{(v - 2z_2\overline{z_2})^{1/2}}(\mathrm{d}z_1 - 2\overline{z_2}\mathrm{d}z_2)(\mathrm{d}\overline{z_1} - 2z_2\mathrm{d}\overline{z_2}) + 4(v - 2z_2\overline{z_2})^{1/2}\mathrm{d}z_2\mathrm{d}\overline{z_2}, \tag{8}$$

To have a better insight into this metric we choose new coordinates

$$x = (v - 2z_2\overline{z_2})^{1/2}, \qquad y = z_2 + \overline{z_2}, \qquad z = i(\overline{z_2} - z_2), \qquad q = \frac{z_1 - \overline{z_1}}{2i}$$

on $\mathcal{U}'$. These coordinates are real. The metric (8) in these coordinates reads

$$g = x(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \frac{1}{x}(\frac{1}{2}z\mathrm{d}y - \frac{1}{2}y\mathrm{d}z + \mathrm{d}q)^2.$$

This shows that it belongs to the Gibbons-Hawking class [4] and that its self-dual part of the Weyl tensor vanishes.

We also recall [8] that a suitable Lie-Backlund transformation brings equation (6) to the Boyer-Finley-Plebański [2, 3] equation [2]

$$F_{yy} + F_{zz} + (\mathrm{e}^F)_{xx} = 0$$

for one real function $F = F(x, y, z)$ of three real variables. It is interesting to note that the metric (8) corresponds to the simplest solution $F = 0$ of this equation.

Summing up we have the following theorem.

Theorem 1 *Let $(z_1, \overline{z_1}, z_2, \overline{z_2})$ be coordinates on $\mathcal{U} \subset \mathbf{R}^4 \cong \mathbf{C}^2$. The Riemannian manifold ($\mathcal{U}'$, g), where*

$$\mathcal{U}' = \{\mathcal{U} \ni (z_1, z_2) \quad \text{s.t.} \quad v - 2z_2\overline{z_2} > 0, \quad v = z_1 + \overline{z_1}\}$$

[2] also known to describe the $\mathbf{SU}(\infty)$ Toda lattice

and

$$g=\frac{1}{(v-2z_2\overline{z_2})^{1/2}}(\mathrm{d}z_1-2\overline{z_2}\mathrm{d}z_2)(\mathrm{d}\overline{z_1}-2z_2\mathrm{d}\overline{z_2})+4(v-2z_2\overline{z_2})^{1/2}\mathrm{d}z_2\mathrm{d}\overline{z_2},$$

is Ricci-flat, anti-self-dual and has the anti-self-dual part of the Weyl tensor of type D. Moreover, ($\mathcal{U}'$, g) admits a circle of almost-Kähler non-Kähler structures

$$J^+_{\mathrm{e}^{i\phi}}=2\mathrm{Re}\{i\mathrm{e}^{i\phi}[\frac{1}{2(v-2z_2\overline{z_2})^{1/2}}(\mathrm{d}z_1-2\overline{z_2}\mathrm{d}z_2)\otimes(\partial_{\bar{2}}+2z_2\partial_{\bar{1}})-2(v-2z_2\overline{z_2})^{1/2}\mathrm{d}z_2\otimes\partial_{\bar{1}}]\}.$$

These structures are parametrized by the real constant $\phi\in[0,2\pi[$. Their fundamental 2-forms are given by

$$\omega^+_{\mathrm{e}^{i\phi}}=i(\mathrm{e}^{i\phi}\mathrm{d}z_2\wedge\mathrm{d}z_1-\mathrm{e}^{-i\phi}\mathrm{d}\overline{z_2}\wedge\mathrm{d}\overline{z_1}).$$

5. Interestingly, our examples can be globalized.
Indeed, the transformation

$$t=\frac{1}{2}\log(v-2z_2\overline{z_2}),\qquad y=z_2+\overline{z_2},\qquad z=i(\overline{z_2}-z_2),\qquad q=\frac{z_1-\overline{z_1}}{2i}$$

brings the structures $(g,J^+_{\mathrm{e}^{i\phi}},\omega^+_{\mathrm{e}^{i\phi}})$ of Theorem 1 to a form which is regular for all the values of the real parameters $(t,y,z,q)\in\mathbf{R}^4$.

6. Finally, we observe that the metric (8), as beeing anti-self-dual, possesses a strictly Kähler structure. This is given by

$$J=i[(\mathrm{d}z_1-2\overline{z_2}\mathrm{d}z_2)\otimes\partial_1-(\mathrm{d}\overline{z_1}-2z_2\mathrm{d}\overline{z_2})\otimes\partial_{\bar{1}}+\mathrm{d}\overline{z_2}\otimes(\partial_{\bar{2}}+2z_2\partial_{\bar{1}})-\mathrm{d}z_2\otimes(\partial_2+2\overline{z_2}\partial_1)]$$

and belongs to the structures of opposite orientation that $J^+_{\mathrm{e}^{i\phi}}$. It is interesting whether there exist Ricci-flat metrics that admit almost-Kähler non-Kähler structures but do not admit any strictly Kähler structure.

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