On the Nodal Line Conjecture

M. Hoffmann-Ostenhof
T. Hoffmann-Ostenhof
N. Nadirashvili

Vienna, Preprint ESI 480 (1997) August 13, 1997

Supported by Federal Ministry of Science and Research, Austria
Available via http://www.esi.ac.at

ON THE NODAL LINE CONJECTURE

M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili

Abstract. We consider Dirichlet eigenfunctions of membrane problems. A counterexample to Payne's nodal line conjecture is given, i.e. a domain in $\mathbb{R}^2$ (not simply connected) whose second eigenfunction has a nodal set disjoint from the boundary. Also a domain in $\mathbb{R}^2$ is given whose second eigenvalue has multiplicity three.

Furthermore, some sufficient conditions are given which imply that an eigenfunction of a Dirichlet membrane problem in $\mathbb{R}^n$ has a zero set which hits the boundary.

1. Introduction

Let D be a bounded domain in $\mathbb{R}^n$ and consider the corresponding Dirichlet eigenvalue problem

$$-\Delta u_i = \lambda_i u_i, \quad i = 1, 2, \dots \tag{1.1}$$

with the eigenfunctions $u_i \in W_0^{1,2}(D)$ (the closure of $C_0^\infty(D)$ in the $W^{1,2}$-norm [**6**]), and with eigenvalues

$$\lambda_1 < \lambda_2 \le \lambda_3 \le \dots \tag{1.2}$$

Let us consider a solution u_i of (1.1). We denote by

$$\mathcal{N}(u_i) = \overline{\{x \in D : u_i = 0\}}$$

the nodal set of u_i. The nodal domains of u_i are the connected components of $D \backslash \mathcal{N}(u_i)$. Courant's nodal theorem shows that

$$\# \text{ nodal domains of } u_i \le i. \tag{1.3}$$

This holds also if we have degeneracy of eigenvalues in the following way: Suppose $\lambda_k = \lambda_{k+1} = \dots = \lambda_{k+l}$, then each u in the corresponding $l+1$-dimensional eigenspace has at most k nodal domains.

It is well known that u_1 can to be chosen to have one sign. u_2 must then have exactly 2 nodal domains.

There are many interesting problems concerning the eigenvalues and eigenfunctions of such membrane problems (see e.g., [**3, 17, 18**]). One which has been around for about 30 years is the nodal line conjecture first stated by Payne in 1967 [**14**].

1991 *Mathematics Subject Classification.* 35B05.

Supported by Federal Ministry of Science of Research, Austria.

Work partially supported by the European Union TMR grant FMRX-CT 96-0001.

Payne considered u_2 for the 2-dimensional one and conjectured that

$$\mathcal{N}(u_2) \cap \partial D = \{x_1, x_2\} \tag{1.4}$$

where $x_1, x_2 \in \partial D$ satisfy $x_1 \neq x_2$.

In 1982 Yau in his collection of problems [**17**] asked the same question for convex domains. Melas [**12**] has settled the convex case for C^∞ boundary and this was extended to general boundary by Alessandrini [**1**]. Earlier Jerison [**10**] had already shown that (1.4) holds for sufficiently long thin convex domains in $\mathbb{R}^n, n \geq 2$

$$\mathcal{N}(u_2) \cap \partial D \neq \emptyset. \tag{1.5}$$

Also, Yau [**18**] asked in his recent collection of problems whether there are suitable extensions of the nodal line conjecture in the sense of (1.5) for higher dimensions and eigenfunctions corresponding to higher eigenvalues. We should also mention the interesting results concerning the location of $\mathcal{N}(u_2)$ for long thin convex domains [**9**], [**7**].

In this paper we shall construct a counterexample to the nodal line conjecture in $\mathbb{R}^2$ for some non simply connected domains. As a consequence of this construction we shall give also an example of a membrane in $\mathbb{R}^2$ for which the second eigenvalue has multiplicity 3. This seems to be new; multiplicity 3 was only known for 2-dimensional Riemann surfaces such as S^2. These results will be given in section 2 below.[1]

In section 3 we give various sufficient conditions for

$$\mathcal{N}(u) \cap \partial D \neq \emptyset$$

including non convex domains and higher dimensional cases.

2. The Counterexample

We first describe the domain. We use polar coordinations $r = |x|$, $x_1 = r\cos\omega$, $x_2 = r\sin\omega$, $-\pi \leq \omega \leq \pi$.

Let $0 < R_1 < R_2$ and $B_{R_i} = \{x \in \mathbb{R}^2 : r < R_i\}, i = 1, 2$ and the annulus $M_{R_1,R_2} = B_{R_2} \setminus \overline{B_{R_1}}$. We pick R_1 and R_2 such that

$$\lambda_1(B_{R_1}) < \lambda_1(M_{R_1,R_2}) < \lambda_2(B_{R_1}) \tag{2.1}$$

where the $\lambda_i(\cdot)$ denote the corresponding Dirichlet eigenvalues. Let

$$D_{R_1,R_2} = B_{R_1} \cup M_{R_1,R_2}$$

then

$$\begin{aligned} \lambda_1(D_{R_1,R_2}) &= \lambda_2(B_{R_1}), \\ \lambda_2(D_{R_1,R_2}) &= \lambda_1(M_{R_1,R_2}) \\ \lambda_3(D_{R_1,R_2}) &= \min(\lambda_2(B_{R_1}), \lambda_2(M_{R_1,R_2})). \end{aligned} \tag{2.2}$$

Next we carve holes into ∂B_{R_1} so that D_{R_1,R_2}, which is not connected, becomes a domain. Let $N \in \mathbb{N}$ and $\varepsilon < \frac{\pi}{N}$. The domain $D_{N,\varepsilon}$ is defined by

$$D_{N,\varepsilon} = D_{R_1,R_2} \cup \cup_{J=0}^{N-1} \left\{ x \in \mathbb{R}^2 : r = R_1, \omega \in \left(\frac{2\pi j}{N} - \varepsilon, \frac{2\pi j}{N} + \varepsilon \right) \right\} \tag{2.3}$$

(see Figure 1 with $N = 4$).

[1] The counterexample is also going to appear in a forthcoming paper [**8**]

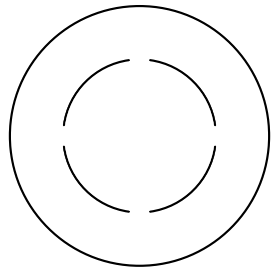

FIGURE 1

THEOREM 2.1. *Pick R_1 and R_2 so that* (2.1) *holds. There is an $N_0 \in \mathbb{N}$ such that for $N \geq N_0$ and sufficiently small $0 < \varepsilon = \varepsilon(N)$ the following holds:*
(i) $\lambda_2(D_{N,\varepsilon})$ is simple
(ii) The second eigenfunction $u_2(D_{N,\varepsilon})$ has a closed nodal line in $D_{N,\varepsilon}$, i.e.,

$$\mathcal{N}(u_2) \cap \partial D_{N,\varepsilon} = \emptyset.$$

REMARK 2.1. As can be seen from the proof below, one can replace $D_{N,\varepsilon}$ by a domain which is obtained by first picking $0 < R_1' < R_2' < R_3'$ such that $\lambda_1(B_{R_1'}) < \lambda_1(M_{R_2',R_3'}) < \lambda_2(B_{R_1'})$ and then by opening passages between $B_{R_1'}$ and $M_{R_2',R_3'}$ as in the construction of $D_{N,\varepsilon}$.

REMARK 2.2. $D_{N,\varepsilon}$ has $N+1$-boundary components. We have not tried to get an explicit bound on the constant N_0 which occurs in our theorem. This would require controling various quantities simultaneously in our proof, and would probably lead to an astronomical number.

Clearly the interesting question is whether there exists a simply connected domain for which u_2 has a closed nodal line. We conjecture that this cannot happen. The more general question is: what is the smallest possible N_0 for which a domain with $N_0 + 1$ boundary components exists such that the corresponding u_2 has a closed nodal line. Before starting the proof of Theorem 2.1 we want to give some heuristic argument. Consider first $u_2(D)$. Since D is not connected

$$u_2 = \begin{cases} u_1(M_{R_1,R_2}) & \text{in } M_{R_1,R_2} \\ 0 & \text{in } B_{R_1}. \end{cases} \tag{2.4}$$

If we carve a small hole into ∂B_{R_1}, u_2 of the resulting domain D_δ will have both signs in D_δ and will live for small δ almost entirely in M_{R_1,R_2}. (Here δ denotes as in the construction of $D_{N,\varepsilon}$ the width of the hole). If we assume that $\mathcal{N}(u_2) \cap \partial D_\delta \neq \emptyset$ we expect $\mathcal{N}(u_2)$ to look as indicated in either of the diagrams in Figure 2

Now if we believe that nodal lines are not too curved without reason (this is admittedly a very vague statement), $\mathcal{N}(u_2)$ should rather look like the $\mathcal{N}(u_2)$ depicted in the left-hand figure than in the right-hand figure. If we now carve two holes into ∂B_{R_1}, both small and close to each other, we would expect on the same grounds that $\mathcal{N}(u_2)$ should touch ∂B_{R_1} only twice. If we finally carve many little holes into ∂B_{R_1} in a regular fashion, then $\mathcal{N}(u_2)$ should not hit ∂B_{R_1} at all. Of course it would be nice to make this heuristic argument rigorous.

PROOF OF THEOREM 2.1. We first note some well known properties of the zero set of u_2:

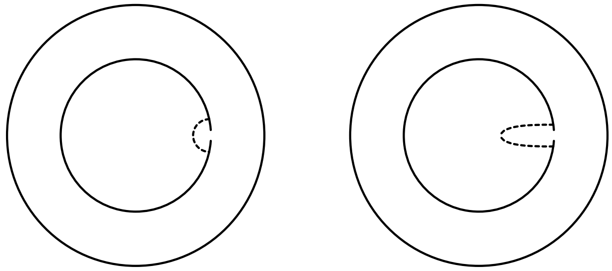

FIGURE 2

(i) If $u_2(x_0) = 0$ for $x_0 \in D_{N,\varepsilon}$, then it must have both signs in a neighbourhood of x_0.

(ii) u_2 cannot have a zero of order ≥ 2, [**4**], [**13**], since this would lead to more than 2 nodal domains of u_2 contradicting Courant's nodal theorem. □

The boundary of $\partial D_{N,\varepsilon}$ is not at all smooth, but we have

PROPOSITION 2.1. *For fixed N the eigenvalue $\lambda_i(D_N, \varepsilon), i = 1, 2$ are monotonically decreasing in ε and converge to $\lambda_i(D)$ as $\varepsilon \downarrow 0$. The corresponding $u_i(D_{N,\varepsilon})$ converge pointwise to $u_i(D)$.*

PROOF OF PROPOSITION 2.1. This follows from a recent result of Stollman [**16**]. □

By construction $D_{N,\varepsilon}$ has following symmetry properties: It is invariant with respect to reflections through the N lines labelled by the angles $\omega_i = (i-1)\frac{\pi}{N}$, $i = 1, 2 \ldots N$, which pass through the origin. This implies also by composition of such reflections that $D_{N,\varepsilon}$ is invariant with respect to rotations with angle $\frac{2\pi j}{N}$, $j = 1, 2 \ldots N-1$.

From Proposition 1 it follows immediately that $\lambda_2(D_{N,\varepsilon})$ for ε sufficiently small. This is simple because $\lambda_2(D_{N,\varepsilon}) \to \lambda_2(D) = \lambda_1(M_{R_1,R_2})$ as $\varepsilon \downarrow 0$ and $\lambda_2(D)$ is simple according to (2.1). This implies also that $u_2(D_{N,\varepsilon})$ enjoys all the symmetry properties of $D_{N,\varepsilon}$ for ε sufficiently small.

From now on we assume $N > 1$ and that ε is sufficiently small so that $\lambda_2(D_{N,\varepsilon})$ is simple.

LEMMA 2.1. *If $\mathcal{N}(u_2) \cap D_{N,\varepsilon} \neq \emptyset$ then for $x \in \cup_{j=1}^N \mathcal{J}_j$*

$$u_2(x) \neq 0$$

where the line segments $\mathcal{J}_j$ are given by

$$\mathcal{J}_j = \left\{ x \in \mathbb{R}^2 : r \in [0, R_1), \omega = (2j-1)\frac{\pi}{N} \right\}.$$

This means that one of the two nodal domains of u_2 is contained in

$$\mathcal{D}_{N,\varepsilon} = D_{N,\varepsilon} \setminus \cup_{j=1}^N \mathcal{J}_j$$

(see Figure 3 *for $N = 4$).*

PROOF OF LEMMA 2.1. The simplicity of $\lambda_2(D_{N,\varepsilon})$ follows from Proposition 2.1. This then implies the symmetry of u_2. □

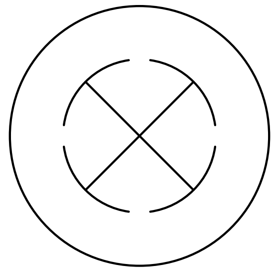

FIGURE 3

If the nodal line of u_2 hits the boundary of $\partial D_{N,\varepsilon}$ Lemma 2.1 implies that

$$\lambda_1(\mathcal{D}_{N,\varepsilon}) < \lambda_2(D_{N,\varepsilon}). \tag{2.5}$$

We shall eventually obtain a contradiction to (2.5) for sufficiently large N and small $\varepsilon > 0$. This can be interpreted as making the heuristics above for this special case rigorous.

Obviously we have

$$\lambda_1(M_{R_1,R_2}) = \lambda_2(D) > \lambda_2(D_{N,\varepsilon}).$$

For given $\delta > 0$ we can close ε so that $\lambda_1(M_{R_1}, R_2) - \lambda_2(D_{N,\varepsilon}) < \delta$ by using Proposition 2.1. This observation also implies that there is an $R = R_{N,\varepsilon}$ such that

$$\lambda_1(B_R) = \lambda_2(D_{N,\varepsilon}), \quad R < R_1. \tag{2.6}$$

$R_{N,\varepsilon}$ tends for $\varepsilon \downarrow 0$ to $R_0, R_0 < R_1$ with $\lambda_1(B_{R_0}) = \lambda_1(M_{R_1,R_2})$, again this follows from Proposition 2.1. We can also require that $|R_0 - R| < \delta$ for given $\delta > 0$ by picking ε sufficiently small. Let

$$\Omega_{N,\varepsilon} = D_{N,\varepsilon} \backslash \overline{B_R}.$$

It is easy to see that the variational principle implies

$$\lambda_1(\Omega_{N,\varepsilon}) \geq \lambda_2(D_{N,\varepsilon}). \tag{2.7}$$

From (2.5) and (2.7) it follows now that

$$\lambda_1(\mathcal{D}_{N,\varepsilon}) \leq \lambda_1(\Omega_{N,\varepsilon}). \tag{2.8}$$

To keep notation simple we will frequently suppress the dependence of various quantities on N and ε, assuming always that $\varepsilon > 0$ is chosen sufficiently small.

Let f and g be the positive Dirichlet ground states of Ω and $\mathcal{D}$ respectively, so that

$$\begin{aligned} -\Delta f &= \lambda_1(\Omega) f \\ -\Delta g &= \lambda_1(\mathcal{D}) g \end{aligned} \tag{2.9}$$

By symmetry f is also the ground state of

$$T = T_{N,\varepsilon} = \mathcal{D}_{N,\varepsilon} \cap \Omega_N \cap \left\{\omega : |\omega| < \frac{\pi}{N}\right\} \tag{2.10}$$

with suitable boundary conditions. Let ∂T be given by

$$\partial T = A^{\pm} \cup B^{\pm} \cup C \cup D \cup E$$

where (we use polar coordinates)

$$A^{\pm} = \{x \in \mathbb{R}^2 : r \in (R, R_1), \omega = \pm\frac{\pi}{N}\}$$
$$B^{\pm} = \{x \in \mathbb{R}^2 : r \in (R_1, R_2), \omega = \pm\frac{\pi}{N}\}$$
$$C = \{x \in \mathbb{R}^2 : r = R_2, |\omega| \leq \frac{\pi}{N}\}$$
$$D = \{x \in \mathbb{R}^2 : r = R_1, \omega \in [\frac{-\pi}{N}, -\varepsilon] \cup [\varepsilon, \frac{\pi}{N}]\}$$
$$E = \{x \in \mathbb{R}^2 : r = R, \omega \leq \frac{\pi}{N}\}.$$

Obviously we have $f = 0$ in $E \cup D \cup C$ and $\frac{\partial f}{\partial n} = 0$ in $A^{\pm} \cup B^{\pm}$ where $\frac{\partial g}{\partial n}$ denotes the outward directed normal derivative. So f is the ground state of T with these mixed Neumann and Dirichlet boundary conditions.

Moreover, g is the ground state of $T^{(1)}_{N,\varepsilon} = \mathcal{D}_{N,\varepsilon} \cup \{|\omega| < \frac{\pi}{N}\}$ where we have everywhere Dirichlet boundary conditions at $\partial T^{(1)}_{N,\varepsilon}$ except for $B^{\pm}$ where we have Neumann conditions.

We will arrive at a contradiction if with the above boundary conditions on $T_{N,\varepsilon}$ respectively $T^1_{N,\varepsilon}$

$$\lambda_1(T_{N,\varepsilon}) > \lambda_1(T^1_{N,\varepsilon}) \tag{2.11}$$

for sufficiently large N and small $\varepsilon > 0$.

Using (2.9) and noting that $T \subset T^1$ we have that

$$(\lambda_1(\Omega) - \lambda_1(\mathcal{D})) \int_T fg dx = -\int_E g\frac{\partial f}{\partial n} d\sigma + \int_{A^+ \cup A^-} f\frac{\partial g}{\partial n} d\sigma. \tag{2.12}$$

We assume f and g to be positive.

In the following we will show that the right hand side becomes negative for large N and small $\varepsilon(N)$. This proves (2.11) which contradicts (2.8) and hence finally the assumptions in Lemma 2.1 that $\mathcal{N}(u_2) \cap \partial D_{N,\varepsilon} \neq \emptyset$.

We now investigate f and $\frac{\partial f}{\partial n}$ respectively g and $\frac{\partial g}{\partial n}$.

Lemma 2.2. *Suppose that*

$$\sup_{x \in M_{r_1,r_2}} f = 1 \tag{2.13}$$

where

$$r_1 = R + \frac{1}{3}(R_1 - R), \quad r_2 = R + \frac{2}{3}(R_1 - R).$$

Then

$$\inf_{x \in M_{r_1,r_2}} f(x) > C_1(\lambda, R, R_1) \tag{2.14}$$

where C_1 is bounded away from zero uniformly for large N. Furthermore there is a constant $C_2(\lambda_1, R, R_1) < \infty$ such that

$$\left|\frac{\partial f}{\partial n}\right| < C_2 \text{ for } x \in E_{N,\varepsilon(N)} \tag{2.15}$$

Here $\lambda = \lambda_1(\Omega_{N,\varepsilon})$.

Proof of Lemma 2.2. Inequality (2.14) is an immediate consequence of Harnack's inequality [**6**]. That $C_1(\lambda, R, R_1)$ is bounded away from zero uniformly in N follows from the fact that, as mentioned above, $|R - R_0|$ is small for ε small.

Next we consider $|\frac{\partial f}{\partial n}(R\omega)|$. Suppose φ is a positive radial C^2-function satisfying $\varphi(R) = 0$ and $\varphi(r_2) = 1$. If in addition

$$-\Delta\varphi \geq \lambda_1(\Omega)\varphi \tag{2.16}$$

in M_{R,r_2}, then the following standard argument implies $\varphi \geq f$ in M_{R,r_2}: Assume indirectly that $G = \{x \in M_{R,r_2} : f > \varphi\} \neq \emptyset$, then $-\Delta(f-\varphi) \leq \lambda_1(f-\varphi)$ in G. So $\int_G |\nabla(f-\varphi)|^2 dx \leq \lambda_1(\Omega) \int_G |f-\varphi|^2 dx$ contradicting then $\lambda_1(\Omega) < \lambda_1(G)$, so G must be empty. Now let φ be the Dirichlet ground state in M_{R,R_1} (which is radially symmetric) and normalize φ such that $\varphi(r_2) = 1$. Obviously $\lambda_1(M_{R,R_1}) > \lambda_1(\Omega)$ and φ satisfies (2.16), hence $\varphi \geq f$ in M_{R,r_2} and in particular $|\frac{\partial\varphi}{\partial r}| > |\frac{\partial f}{\partial r}|$ for $x \in E$ proving (16) since $\frac{\partial\varphi}{\partial r}(R)$ remains uniformly bounded for large N. □

Next we investigate g. We start with a simple result we shall need to bound $|\frac{\partial g}{\partial n}|$ from below for $x \in A$.

Lemma 2.3. *Suppose $\Omega \subset \mathbb{R}^2$ and $B_\rho(y) = \{x \in \mathbb{R}^2 : |x-y| < \rho\} \subset \Omega$ with $\partial B_\rho \cap \partial\Omega \neq \emptyset$. Suppose $u > 0$ in Ω, $u \in C^0(\overline{\Omega})$ and $-\Delta u = \lambda u$ in Ω, $\lambda \geq 0$. If $x_0 \in \partial B_\rho \cap \partial\Omega$ and $u(x_0) = 0$ then*

$$\left|\frac{\partial u}{\partial n}(x_0)\right| \geq \frac{1}{3\rho} u(y) \tag{2.17}$$

where $\frac{\partial}{\partial n}$ denotes the outward directed normal derivative with respect to $\partial\Omega$.

Remark. This Lemma can be viewed as an explicit variant of Hopf's boundary point lemma [**6**].

Proof of Lemma 2.3. Without loss we may assume $y = 0$. Since $\lambda \geq 0$, $v = u(x_1, x_2)e^{\sqrt{\lambda}x_3}$ is harmonic in $\Omega \times \mathbb{R}^1$. We use Harnack's inequality for harmonic functions in $\overline{B}_\rho = \{x \in \mathbb{R}^3 : |x| < \rho\} \subset \Omega \times \mathbb{R}^1$. We have [**2**]

$$v(x) \geq \left(1 - \frac{|x|}{\rho}\right)\left(1 + \frac{|x|}{\rho}\right)^{-2} v(0) \quad \forall x \in \tilde{B}_\rho.$$

In particular for $x_3 = 0$ this becomes

$$u(x) \geq (1 - |x|/\rho)(1 + |x|/\rho)^{-2} u(0) \quad \forall x \in B_\rho. \tag{2.18}$$

Now fix $0 < \rho_0 < \rho$ and let $\alpha = \rho_0/\rho$. Define

$$h(x) = h(r) := \frac{1-\alpha}{(1+\alpha)^2} \frac{1}{|\ln \alpha|} u(0)(\ln \rho - \ln r), \quad (r = |x|).$$

Then $\Delta h = 0$ and (19) implies since $h(x) = 0 \leq u(x)$ for $|x| = \rho$, by the maximum principle that $h(x) \leq u(x)$ in $\overline{B\rho \backslash B_{\rho_0}}$. So

$$\left|\frac{dh}{dr}(\rho)\right| = \left|\frac{\partial h}{\partial n}(x)\right| \leq \left|\frac{\partial u}{\partial n}(x)\right|$$

for $x \in \partial B\rho \cap \partial\Omega$ if $u(x) = 0$. Now pick $\alpha = 1/4$, then (2.17) follows. □

We normalize g so that

$$\sup_{|\omega|\leq \frac{\pi}{N}} g(r_0,\omega) = 1 \tag{2.19}$$

with

$$r_0 := \frac{R+R_1}{2}.$$

Using Lemma 4 we now show that

$$\int_{A^+\cup A} |\frac{\partial g}{\partial n}| f d\sigma \geq C_3 > 0 \tag{2.20}$$

for sufficiently small ε and where C_3 does not depend on N. Let

$$Q_{N,\bar{r}} = \left\{ x\varepsilon\mathbb{R}^2 : |\omega| < \frac{\pi}{N}, r < \bar{r} \right\}. \tag{2.21}$$

We note that $g(r,\omega) = g(r,-\omega)$ in Q_{N,R_1}. Let ω_0 be chosen such that $g(r_0,\omega_0) = 1$ and suppose without loss that $\omega_0 > 0$. We have with $\rho = r_0 \sin(\frac{\pi}{N} - \omega_0)$ and $y_0 = (r_0\cos\omega_0, r_0\sin\omega_0)$ that $B_\rho(y_0) \subset Q_{N,R_1}$ provided N is sufficiently large. Let $y_+ = (r_0\cos(\frac{\pi}{N}-\omega_0)\cos\frac{\pi}{N},\ r_0\cos(\frac{\pi}{N}-\omega_0)\sin\frac{\pi}{N})$ then $y_+ \in \partial B_\rho(y_0)\cap A^+$. Lemma 2.3 applies and gives

$$\left|\frac{\partial g}{\partial n}(y_+)\right| \geq \frac{1}{3r_0\sin(\frac{\pi}{N}-\omega_0)} \geq C_4 N \tag{2.22}$$

for some positive constant not depending on N. (2.18) implies now that

$$g \geq \frac{2}{9} \text{ for } x \in B_{\rho/2}(y_0) \tag{2.23}$$

and we can use Lemma 2.3 again to obtain

$$\left|\frac{\partial g}{\partial n}\right| \geq C_5 N \text{ for } x \in A^+ \cap B_{C_6/N}(y_+)$$

for some N-independent positive constants C_5, C_6. This gives

$$\int_{A^+} \left|\frac{\partial g}{\partial n}\right| d\sigma \geq C_5 \int_{A^+\cap B_{C_6/N}(y_+)} N\, d\sigma \geq C_7 > 0 \tag{2.24}$$

with C_7 again N-independent. Inequality (2.20) now follows from (2.24) and (2.14).

The proof that the right hand side of (2.12) is negative and hence the proof of our result will be complete once we show that

$$\int_E g\frac{\partial f}{\partial n} d\sigma \to 0 \tag{2.25}$$

for large N. This will follow using (2.15) by showing that g satisfies for $x \in E$

$$g(R\omega) \leq C_8 \left(\frac{R}{r_0}\right)^{C_9 N} \tag{2.26}$$

with some N-independent constants C_8, C_9, where r_0 was defined in (2.19). To see that (2.26) holds let $v = \sqrt{2}\left(\frac{r}{r_0}\right)^\gamma \cos\frac{N}{4}\omega$ with

$$\gamma = \left\{\frac{1}{4} - \lambda_1(\mathcal{D}_{N,\varepsilon})r_0^2 + \left(\frac{N}{4}\right)^2\right\}^{1/2} - \frac{1}{2}$$

(we assume N so large that γ is real). A simple calculation shows that

$$-\Delta v \geq \lambda_1(\mathcal{D}_{N,\varepsilon})v$$

for $x \in Q_{N,r_0}$ and that $v \geq g$ in $\partial Q_{N,r_0}$. Since $\lambda_1(\mathcal{D}_{N,\varepsilon})r_0^2$ stays bounded for large N we again can use a comparison argument as in the proof of Lemma 2.2 to conclude that $v \geq g$ in Q_{N,r_0} showing (27).

This finally proves Theorem 2.1. □

The construction of our counterexample also lends itself to an example where λ_2 has multiplicity 3.

THEOREM 2.2. *Let $N \geq 3$ then $\lambda_2(D_{N,\varepsilon})$ has multiplicity 3 for suitable $\varepsilon > 0$.*

REMARK 2.3. To our knowledge this is the first example of a domain in $\mathbb{R}^2$ where the second eigenvalue has multiplicity 3. In [**11**] it is claimed that the multiplicity of λ_2 is always at most 2, but probably the author had the simply connected case in mind.

PROOF OF THEOREM 2.2. We first note that the eigenfunction of a disk B_{R_2} satisfy

$$\lambda_1(B_{R_2}) < \lambda_2(B_{R_2}) = \lambda_3(B_{R_2}) < \lambda_4(B_{R_2}) = \lambda_5(B_{R_2}) < \lambda_6(B_{R_2}). \tag{2.27}$$

u_1 and u_6 are radially symmetric, hence invariant with respect to inversion at the origin, whereas u_2, u_3, u_4, u_5 are antisymmetric with respect to inversion.

Now consider $D_{N,\varepsilon}$. We have

$$D_{N,\frac{\pi}{N}} = B_{R_2}.$$

We also can distinguish between eigenfunctions which are symmetric, respectively antisymmetric with respect to inversion of the origin. For $\varepsilon = 0$, $u_1(D_{N,0})$, $u_2(D_{N,0})$ are symmetric and $u_3(D_{N,0})$ is antisymmetric with respect to inversion at the origin. Hence $\lambda_2(D_{N,\varepsilon})$ will approach $\lambda_6(B_{R_2})$ for $\varepsilon \to \frac{\pi}{N}$, whereas $\lambda_3(D_{N,\varepsilon})$ will tend to $\lambda_2(B_{R_2})$, hence these two curves must cross leading to a $\lambda_2(D_{N,\varepsilon})$ with multiplicity 3 for a suitable ε. □

3. Sufficient conditions

In the following we give some sufficient conditions such that the nodal set of an eigenfunction of a Dirichlet problem hits the boundary. Thereby we shall not strive for generality but rather present the main ideas together with some examples.

Let $D \subset \mathbb{R}^n$ be a bounded domain and suppose that ∂D is C^2. Suppose that ∂D has N components such that

$$\partial D = \cup_{i=1} \partial D^{(i)}. \tag{3.1}$$

Our sufficient conditions for the nodal line conjecture to hold will be based on the following simple observation.

THEOREM 3.1. *Suppose D satisfies the assumptions above. Let $u_k(D)$ be an eigenfunction of the corresponding Dirichlet problem*

$$\Delta u_k(D) + \lambda_k(D)u_k(D) = 0 \tag{3.2}$$

for some k.

Suppose there exists a function $f : \overline{D} \to \mathbb{R}$ such that

$$f \in C^2(D) \cap C^1(\overline{D}) \tag{3.3}$$

with

$$(\Delta + \lambda_k)f = 0. \tag{3.4}$$

Suppose further that there is one component of ∂D, say ∂D^*, such that

$$f \geq 0 \ \textit{in} \ \partial D^*,$$

$$f \mid \partial D^* \not\equiv 0 \tag{3.5}$$

and

$$f \equiv 0 \ \textit{in} \ \partial D \backslash \partial D^*. \tag{3.6}$$

Then

$$\mathcal{N}(u_k(D)) \cap \partial D^* \neq \emptyset.$$

Remarks 3.1. (i) Our regularity conditions for ∂D and $f \mid \partial D$ are certainly much too strong but we keep them in order to avoid certain technicalities.

(ii) Actually we have a stronger result: $\frac{\partial u}{\partial n}$ must change sign in ∂D^*. (Here $\frac{\partial u}{\partial n}$ denotes the outward directed normal derivative.)

(iii) Theorem 3.1 has obvious generalizations to Dirichlet problems on manifolds with boundary or Schrödinger operators on bounded domains with zero-Dirichlet boundary conditions.

(iv) Theorem 3.1 is a sufficient condition but not a necessary one. It is easy to construct examples where a domain has an eigenvalue of multiplicity > 1 and where in the corresponding eigenspace there is one function u with $\mathcal{N}(u) \cap \partial D = \emptyset$. But by linear combination of the eigenfunction in this eigenspace one can always force a function to have a zero hitting the boundary. Such an example is given for the square membrane in the book of Courant and Hilbert [**5**].

Proof of Theorem 3.1. Equations (3.2) and (3.4) imply that

$$u\Delta f - f\Delta u = 0$$

(We suppress the indices.) By Green's second identity we have using (3.6)

$$\int_{\partial D} u\frac{\partial f}{\partial n}d\sigma - \int_{\partial D} f\frac{\partial u}{\partial n}d\sigma = -\int_{\partial D^*} f\frac{\partial u}{\partial n}d\sigma = 0.$$

This implies that $\frac{\partial u}{\partial n}$ changes sign in ∂D^* or $\frac{\partial u}{\partial n} \equiv 0$ in ∂D^*. But this is impossible since by assumption $\partial D \in C^2$, hence D satisfies an interior sphere condition, which in turn means that Hopf's boundary point lemma [**6**] applies implying that $\frac{\partial u}{\partial n} \neq 0$ for all $y \in \partial D^* \backslash \mathcal{N}(u)$. Hence if $\mathcal{N}(u) \cap \partial D^* = \emptyset$ we have the desired contradiction. □

Theorem 3.1 looks nice, but given a domain D it is not at all clear how to check for a specific eigenfunction and eigenvalue whether we can find a function f satisfying (3.3-3.6).

In the following we describe two families of domains for which we can apply our results.

The first family is related to balls in $\mathbb{R}^n$

$$B_R^{(n)} = \{x \in \mathbb{R}^n : |x| < R\}.$$

We also need Bessel functions and denote by $j_{\nu,l}$ the l-th zero of the Bessel function $J_\nu(r)$. $(r = |x|)$. The $J_\nu(r)$ are in a well known way related to the Dirichlet eigenfunctions of balls [**3**].

Let

$$q_n = \min\left(\frac{j_{\frac{n}{2},1}}{j_{\frac{n}{2}-1,1}}, \frac{j_{\frac{n}{2}-1,2}}{j_{\frac{n}{2},1}}\right). \tag{3.8}$$

E.g.

$$q_2 \sim 1.43, \quad q_3 \sim 1.39, \ldots, q_8 \sim 1.11. \tag{3.9}$$

THEOREM 3.2. *Let $D \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ with only one boundary component and $\partial D \in C^2$.*

Let

$$B_{R_1} \subset D \subset B_{R_2} \tag{3.10}$$

and

$$\int_{B_{R_2} \setminus D} dx > 0. \tag{3.11}$$

If

$$1 \leq \frac{R_2}{R_1} \leq q_n \tag{3.12}$$

then

$$\mathcal{N}(u_k(D)) \cap \partial D \neq \emptyset$$

for $k \leq n+1$.

PROOF OF THEOREM 3.2. We prove just the 2-dimensional case, the n-dimensional is almost identical.

Let $D \subset \mathbb{R}^2$ be simply connected and consider $u_2(D)$. (The proof for $u_3(D)$ is identical.) First we note that (3.10) implies

$$\lambda_2(D) \in (\lambda_2(B_{R_2}), \lambda_2(B_{R_1})]. \tag{3.13}$$

This follows from the well known fact that if $D_1 \subset D_2$ the Dirichlet eigenvalues satisfy $\lambda_k(D_1) \geq \lambda_k(D_2)$. Now consider the Dirichlet eigenvalues of the disk. We have (see also (2.27))

$$\lambda_1(B_R) < \lambda_2(B_R) = \lambda_3(B_R) < \lambda_4(B_R) = \lambda_5(B_R) < \lambda_6(B_R).$$

With the above defined $j_{\nu,l}$ we have [**3**]

$$\sqrt{\lambda_1(B_R)} = \frac{j_{0,1}}{R}, \sqrt{\lambda_2(B_R)} = \frac{j_{1,1}}{R}, \sqrt{\lambda_4(B_R)} = \frac{j_{2,1}}{R}, \sqrt{\lambda_6(B_R)} = \frac{j_{0,2}}{R}, \tag{3.14}$$

$u_1(B_R)$ and $u_6(B_R)$ have spherical symmetry and $\mathcal{N}(u_6)$ is a circle. Equation (3.13) implies that there is an $R \in (R_1, R_2]$ such that $\lambda_2(B_R) = \lambda_2(D)$. We can also find $\underline{R} < \overline{R}$ such that

$$\lambda_2(B_R) = \lambda_1(B_{\underline{R}}) = \lambda_6(B_{\overline{R}}). \tag{3.15}$$

This means that

$$\mathcal{N}(u_6(B_{\overline{R}})) = \partial B_{\underline{R}}.$$

So if

$$B_{\underline{R}} \subset D \subset B_{\overline{R}}$$

we are done for $f := u_6(B_{\bar{R}})$ will do the job since with the appropriate choice of sign $f \geq 0$ in ∂D and $f \not\equiv 0$ in ∂D because of (3.11).

Let us define $\overline{R_1}$ and $\overline{R_2}$ by

$$\lambda_2(B_{R_1}) = \lambda_6(B_{\overline{R_1}})$$
$$\lambda_2(B_{R_2}) = \lambda_1(B_{\overline{R_2}})$$

It suffices to show that

$$\underline{R_2} \leq R_1 < R_2 \leq \overline{R_1} \tag{3.16}$$

is satisfied. Using (3.14) this means that

$$\underline{R_2} = \frac{j_{0,1}}{j_{1,1}} R_2 \leq R_1$$

$$\overline{R_1} = \frac{j_{0,2}}{j_{1,1}} R_1 \geq R_2$$

must hold. But this is exactly the requirement.

$$\frac{R_2}{R_1} \leq q_2$$

proving our result for $u_2(D)$. For $u_3(D)$ the proof is identical.

For $D \subset \mathbb{R}^n$ we just note that $\lambda_2(B_R^{(n)})$ is n-fold degenerate and that in (3.14) the corresponding zero's of the Bessel functions have to be used so that (3.8) turns up. □

Remarks 3.2. (i) Again we stress that the assumption $\partial D \in C^2$ can be weakened considerably.

(ii) Due to the n-fold degeneracy of $\lambda_2(B_R^{(n)})$ we get the result for $u_k(D)$, $k \in [3, n+1]$ for free. But at the expense that $q_n \to 1$ for $n \to \infty$.

(iii) Similar arguments allow us to treat also higher eigenvalues. For instance if for $D \subset \mathbb{R}^2$, $B_{R_1} \subset D \subset B_{R_2}$ and $\frac{R_2}{R_1} - 1$ is sufficiently small, $\mathcal{N}(u_4)$ and $\mathcal{N}(u_5)$ must hit ∂D (provided D is simply connected).

One way to interpret the underlying construction which led to Theorem 3.2 (e.g., for $n = 2$) is the following: we took a disk B_{ρ_1} and considered its ground state $u_1(B_{\rho_1})$ with eigenvalues $\lambda_1(B_{\rho_1})$. Then we noticed that $u_1(B_{\rho_1})$ can be extended so that it does not change sign in the annulus $B_{\rho_2} \setminus B_{\rho_1}$. If we have now a simply connected domain D such that $\lambda_2(D) = \lambda_1(B_{\rho_1})$ we can apply Theorem 2.1 with f being the extension of $u_1(B_{\rho_1})$ to B_{ρ_2}.

But instead of B_{ρ_1} we can consider other domains and try to extend their ground states to larger domains.

We shall use reflections to extend domains whose boundaries have flat pieces. Thereby we shall illustrate the main ideas by examples rather than stating some general theorems (which would have to be quite complicated and lengthy). Figure 4 demonstrates the underlying principle.

Let D be again a bounded domain and λ some Dirichlet eigenvalue and u one of the corresponding eigenfunctions. Suppose we can cut D into two pieces (or as will be seen later chop off several pieces) as shown in Figure 4. We again assume $\partial D \in C^2$ and for simplicity that ∂D consists of only one component, such that $D = D_1 \cup D_2 \cup \Gamma$, where D_2 is the domain to the right of the dashed line Γ and D_1

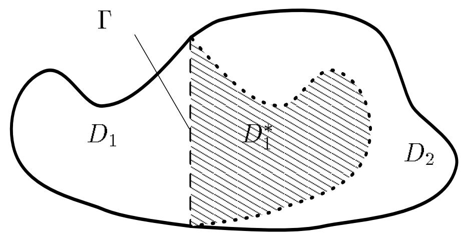

FIGURE 4

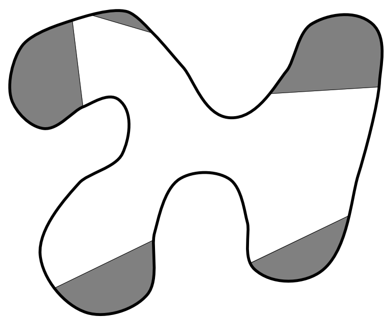

FIGURE 5

the domain to the left. Suppose that $\lambda_1(D_2) = \lambda(D)$ and that the shaded reflection D_1^* of D_1, satisfies $D_1^* \subset D_2$ and $\int_{D_2 \setminus D_1^*} dx > 0$ then

$$\mathcal{N}(u) \cap \partial D_1 \setminus \Gamma \neq \emptyset. \tag{3.17}$$

To see why (3.17) holds in this case we consider φ the Dirichlet ground state of D_2 and assume that $\Gamma = \{x \in D : x_1 = 0\}$ and that $D_2 \subset \{x \in \mathbb{R}^n : x_1 > 0\}$, then

$$f = \begin{cases} \varphi(x_1, x_2, \ldots, x_n) & \text{for } x \in D_2 \\ -\varphi(-x_1, x_2, \ldots, x_n) & \text{for } x \in D_1 \end{cases} \tag{3.18}$$

can be used to apply Theorem 3.1 since f does not change sign in $\partial D_1 \setminus \Gamma$ since $\partial D_1 \setminus \Gamma_1 \subset D_2^*$ and $\Delta f + \lambda f = 0$ in D.

Naturally by the same reasoning we could chop off many pieces, as illustrated in Figure 5. If the domain without the shaded pieces has a Dirichlet first eigenvalue λ which coincides with some Dirichlet eigenvalue λ of the whole domain D we can argue as above to show that the corresponding u satisfies $\mathcal{N}(u) \cap \partial D \neq \emptyset$. But in general it is not clear how to check for a given domain whether we can for some eigenvalue λ chop off pieces in the way described above. However, for domains which consist of separated identical pieces which are connected by thin channels (hence a semi-classical situation) we can often use the approach above. To be more precise let us give an explicit example.

Let

$$D_0 = \cup_{i=1}^N B_R(x_i) \tag{3.19}$$

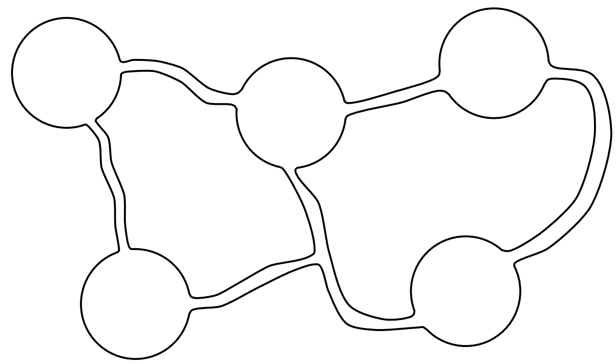

FIGURE 6

where the $B_R(x_i) = \{x \in \mathbb{R}^n : |x - x_i| < R\}$ and

$$|x_i - x_i| > 2CR \text{ for some } C > 1. \tag{3.20}$$

We make D_0 into a domain by connecting the $B_R(x_i)$ with thin channels. So let M be these channels and

$$D = M \cup D_0 \cup (\overline{M} \cap \overline{D_0}) \tag{3.21}$$

where $M = \cup_{k=1}^{K} M^{(k)}$, K finite. We illustrate this in Figure 6.

We also assume that $\partial D \in C^2$ (also not essential). Now if M has sufficiently small measure ε then for $i \leq N$

$$\lambda_i(D) < \lambda_2(B_R) \tag{3.22}$$

since

$$\lambda_i(D_0) = \lambda_i(B_R) \text{ for } i = 1, \ldots, N$$

and

$$\lim_{\varepsilon \to 0} \lambda_i(D) = \lambda_1(B_R).$$

This can be easily seen by Dirichlet Neumann bracketing [**15**], but here $C > 1$ in (3.20) is important.

Now we assume that there is one component ∂D^* of ∂D such that for each i, $\partial D^* \cap \partial B_R(x_i)$ contains a set η_i which after a suitable rigid motion to shift $B_R(x_i)$ into $B_R(0)$ is given by

$$\eta = \{x \in \partial B_R : x_1 > \gamma R\} \text{ for some } 0 < \gamma < 1.$$

By picking ε sufficiently small (depending on γ)

$$\mathcal{N}(u_k) \cap \partial D^* \neq \emptyset \tag{3.23}$$

where $k \leq N$. To prove (3.23) we chop off from each $B_R(x_i)$ the set (after rigid motion to $B_R(0)$) $x_1 > \gamma R$ so that we obtain a domain without the shaded regions Ω. See Figure 7. The Dirichlet ground state energy will be larger than $\lambda_N(D)$ for sufficiently small ε. But we can individually make the shaded regions smaller so that we eventually arrive at domains D_i such that

$$\lambda_1(D_i) = \lambda_i(D) \quad i = 2, \ldots, N$$

and so for each D_i we can use the reflections described above. Hence the ground state of D_i continued by reflections in each $B_R(x_i)$ in the way described by (3.18) will serve as a function f allowing to apply Theorem 3.1.

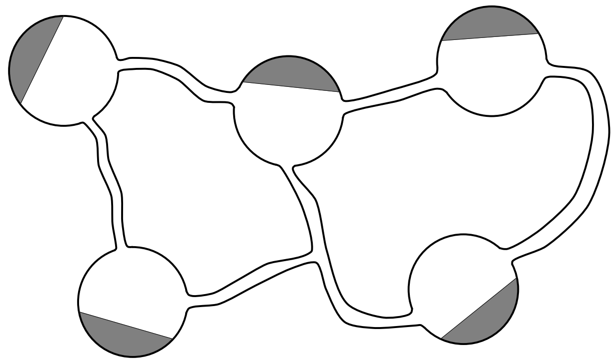

FIGURE 7

The above example is rather special. We could have replaced the $B_R(x_i)$ by individually different domains satisfying some spectral and geometrical conditions etc., and again $\partial D \in C^2$ can be relaxed considerably.

It seems to be difficult to characterize the domains for which

$$\mathcal{N}(u_2)(D) \cap \partial D \neq \emptyset$$

can be shown using Theorem 3.1 together with the reflection procedure. In particular we do not know whether convex domains in $\mathbb{R}^2$ or $\mathbb{R}^n$ can be treated in this way.

Acknowledgement

It is a pleasure to thank M. Ashbaugh, F. Gesztesy, M. P. Owen and J. P. Solovej for discussions.

References

[1] G. Alessandrini. Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains. *Comment. Math. Helvetici*, 69:142–154, 1994.

[2] S. Axler, P. Bourdon, and W. Ramey. *Harmonic function theory*. Springer, 1992.

[3] C. Bandle. *Isoperimetric inequalities and applications*. Pittman, 1980.

[4] S-Y Cheng. Eigenfunctions and nodal sets. *Commentaric Math. Helv.*, 51:43–55, 1976.

[5] R. Courant and D. Hilbert. *Methoden der Mathematischen Physik I*. Springer, 1968.

[6] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*. Springer, 1983.

[7] D. Grieser and D. Jerison. Asymptotics of the first nodal line of a convex domain. *Invent. Math.*, 125:197–219, 1996.

[8] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili. The nodal line of the second eigenfunction of the Laplacian in $\mathbb{R}^2$ can be closed. To appear.

[9] D. Jerison. The diameter of the first nodal line of a convex domain. *Annal of Math.*, 141:1–33, 1995.

[10] D. Jerison. The first nodal set of a convex domain. *Princeton University Press*, 42:225–249, 1995. In: Ch. Fefferman ed., Essays in honour of Elias Stein.

[11] Zhang Liqun. On the multiplicity of Laplacian in $\mathbb{R}^2$. *Comm. in Analysis and Geometry*, 3:273–296, 1995.

[12] A.D. Melas. On the nodal line of the second eigenfunction of the Laplacian in $\mathbb{R}^2$. *J. Diff. Geom.*, 35:255–263, 1992.

[13] N.S. Nadirashvili. On the multiplicity of the eigenvalues of the Neumann problem. *Soviet. Math. Dokl.*, 33(1):281–282, 1986.

[14] L.E. Payne. Isoperimetric inequalities and their applications. *Siam Review*, 9:453–488, 1967.

[15] M. Reed and B. Simon. *Methods of modern mathematical physics*, volume IV: Analysis of operators. Academic Press, 1978.

[16] P. Stollmann. A convergence theorem for Dirichlet forms with applications to boundary value problems with varying domains. *Math. Zeitschrift*, 219:275–287, 1995.

[17] S.-T. Yau. Problem section in 'Seminar on Differential Geometry'. *Ann. of Math. Studies*, 102:669–706, 1982.

[18] S.T. Yau. Open problems in geometry. *Proc. Symp. Pure. Math.*, 54.1:1–28, 1993.

Institut für Mathematik, Universität Wien

Institut für Theoretische Chemie, Universität Wien and International Erwin Schrödinger Institute for Mathematical Physics

International Erwin Schrödinger Institute for Mathematical Physics