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TUBE DOMAINS IN STEIN SYMMETRIC SPACES

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Abstract. We formulate several problems about invariant tubes in Stein symmetric spaces whose edges are affine symmetric spaces.

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Slightly more then 20 years ago I.Gelfand and me discussed several problems on representations of real semisimple Lie groups. One of such questions was how to see on the group $G_{\mathbb{R}} = SL(2;\mathbb{R})$ itself that some part of the regular representation in $L^2(G_{\mathbb{R}})$ is decomposed on representations of the holomorphic discrete series? We found that

(1) functions out of corresponding subspace L^2_{hol} admit holomorphic extensions on the domain $G_+ \subset G_{\mathbb{C}} = SL(2;\mathbb{C})$ of contractions of upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C}; \Im z > 0\}$.

Of course G_+ is an open Stein submanifold of tube type in $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$ is its edge (Shilov's boundary) and in addition, for reasons which we did not understood then

(2) G_+ admits equivariant embedding as Zariski open part in the Siegel half-plane of rank 2 (complex symmetric matrixes of order 2 with positive imaginary parts). Correspondingly $SL(2;\mathbb{R}) \times SL(2;\mathbb{R})$ imbeds in $Sp(2;\mathbb{R})$.

(3) On holomorphic functions in G_+ a Hilbert norm of the Hardy type can be defined so that on the corresponding space $H^2(G_+)$, there are defined boundary values as an isometry on L^2_{hol}. It is also possible to explicitly construct a generalized Szegö operator of the projection of $L^2(G_{\mathbb{R}})$ on $H^2(G_+)$.

¿From my point of view the most surprising element of this construction is that in a natural problem of the theory of representations there appeares a manifold G_+ which is $G_{\mathbb{R}} \times G_{\mathbb{R}}$-invariant but inhomogeneous. We had many plans to develop this observation. Groups of Hermitian type were only a first natural step but our principal plans were of course connected with nonholomorphic discrete series. For certain reasons we postponed the work on this project and published only a short paper [GG]. It was not a standard publication: we had considered only the example of $SL(2;\mathbb{R})$ and did several general remarks. It was in a sense an invitation to the project. Thanks to the kindness of Ol'shanskii it is now sometimes called " Gelfand–Gindikin program". I myself never was seriously involved in its development but only from time to time thought about some things which can be connect with it. Nevertheless I want to use this an occasion and make a few remarks about how the situation looks for me 20 years later and which problems I believe are interesting. As I am not an expert in the area my references can be very noncomplete.

Our invitation was accepted. Several people constructed [O1], [S], [HOO] the complete theory of the Hardy spaces in the case of groups of Hermitian type and investigated the connection with holomorphic discrete series. They added several important points which we missed in our project. Ol'shanskii found that tubes which have groups of Hermitian type as edges are non unique. They have structure of semigroups. Using some Vinberg's results he gave their classification. We knew that G_+ is the set of contractions but we did not paid an attention to the fact that it is a semigroup. Hilgert, Olafsson and Ørsted generalized the construction of the tubes on the case of affine symmetric spaces of Hermitian type for which there exist holomorphic discrete series. The important component of the theory was the observation that on symmetric spaces of Hermitian type there is a causal structure (cf. below). The theory contained a construction of the Szegö projector but no explicit formula for the Szegö kernel. It is interesting and important point from which I want to start our discussion.

Szegö operators. Let us recall the construction of the Szegö operator in [GG]. We computed the character $S(g)$ of the (reducible) representation of $SL(2;\mathbb{R})$ in L^2_{hol} by the direct summation of characters of irreducible representations of the holomorphic discrete series with the corresponding Plancherel's coefficients. We remarked that the sum (which could be computed explicitly) can be (together with characters of irreducible representations) holomorphicaly extend in the tube G_+. This character $S(g)$ will give the kernel of the Szegö operator: the Szegö projector will be a convolution with it. The final expression for $S(g)$ looks very nice but unfortunately does not contain any hint how the Szegö kernels could be looked at the general case.

Last couple years this direct way of computations of Szegö kernels was realized by Molchanov for hyperboloids [Mo] and by Ol'shanskii for some classical groups [Ol3]. Nontrival computations give some nice explicit algebraic functions but their structure does not leave a lot of chances to guess a formula in the general case. Below we will discuss a more simple way to compute the kernel in special cases but for now I want to open my old plan of working with the Szegö kernels.

The Szego operator gives an integral formula which reconstructs holomorphic functions in G^+ through its boundary values on the edge $G_{\mathbb{R}}$. In multidimensional complex analysis there is a standard way to obtain integral formulas - the Cauchy–Fantappie formula. In the practice all known integral formulas can be realize as special cases of the Cauchy–Fantappie formula. I consider the following problem very important:

To give a construction of the Szegö kernels for semigroups and other tubes using the Cauchy–Fantappie formula.

It might be necessary to modify of the Cauchy–Fantappie formula since we have a unbounded domain in a manifold (cf. its projective version in [GKh]), but I do not see serious obstructions to realizing it this way. The result of such a construction must be an integral representation of the Szegö kernel through some geometrical characterics of the boundary. The explicit computation of this integral may not be simpler than the computation of the sum of characters, but I believe that this representation can be sufficient for computation of Plancherel's coefficients. Therefore the focus of the project is to find a geometrical way (without group invariance) of presentation of Szegö kernels and then to apply it to the computation of the "holomorphic" part of the Plancherel formula - the way which is just the opposite to one which uses for computations now.

There is an important analogy of this plan with the application of the integral geometry to the Plancherel formula for complex semisimple Lie groups (using horospherical transform) [GGr], [G3]. There we obtain the inversion of the horospherical transform as a special case of general inversion formula for problems of integral geometry without a direct usage of groups. For real groups the integral geometry does not worked in the general situation but it is natural for the holomorphic discrete series to consider the Cauchy–Fantappie formula as a surrogate of the integral geometry. Let us recall that there is a view that the Cauchy–Fantappie formula is a complex analog of the Radon inversion formula [G1]. It gives an extra support to this a project.

Basic classes of manifolds. As we will operate with several classes of manifolds and some of them have similar names, let us fixed basic objects:

(CH) *Compact Hermitian symmetric spaces.*

(NH) *Noncompact Hermitian symmetric spaces.* They are realized as domains in dual CH.

(FT) *NH of tube type.* They realized as tubes in $\mathbb{C}^n$. We will call them also *flat tubes* to differ from tubes which are the subject of this paper.

(AH) *Affine symmetric spaces of Hermitian type.* This class includes groups of Hermitian type which act in NH.

(R) *Symmetric R-spaces*- real forms of CH.

(CR) *Edges (Shilov's boundaries) of flat tubes.* It is a special case of symmetric R-spaces. They are compact Riemannian symmetric spaces (relative to maximal compact subgroup of the group of automorphisms). Let us emphasize that we take complete boundaries inside dual CH but not only inside $\mathbb{C}^n$.

(CT) *Curved tubes* -tubes in complexifications of AH which have these AH as the edges. In the case of groups they are semigroups in complex groups. In general case they are orbits of semigroups.

(SS) *Stein symmetric spaces.* These affine symmetric spaces of the form $Y = G/K$ where G is a complex semisimple Lie group and K is a involutive complex subgroup. As complex manifolds they are Stein manifolds. They are obtained as results of complexifications of affine symmetric spaces (of groups and isotropy subgroups). Complex semisimple groups are partial case of SS. Curve tubes (CT) are domains in SS - complexifications of their edges, at that time as flat tubes (FT) are domains in another class of complex symmetric spaces - CH.

Generalized conformal structures. Let us talk now about the observation in [GG] that the tube G_+ can be extended up to the Siegel half-plane. The key here lies in a generalized conformal structure [G2]. For a simplicity's sake we will give the definition in complex case. Let V be a conic (invariant relatively $(\mathbb{C}\backslash 0)^\times$) algebraic variety in $\mathbb{C}^n$. We call *a generalized conformal V-structure* on n-dimensional manifold M a field of varieties $V_z \subset T_z(M)$ (tangent spaces) which all are linear equivalent to V. In the case when V is the quadratic nongenerate cone we have usual conformal structure. Of course it is possible to consider the real version of this notion but it is important then to provide a possibility for a cone V to be imagine and it is convenient to start from complex structures and then to investigate their real forms (including imaginary ones). We can consider conformal morphisms of such structures and a generalized conformal V-structure on M is called *flat* if M with this structure is local isomorphic in neighborhoods of all points to domains in $\mathbb{C}^n$ where the structure is defined by translations of the cone V.

The classical result of Plücker is that the Grassmanian Gr$(2,4)$ of lines in $\mathbb{C}P^3$ has a canonical flat conformal structure and can be interpret as the conformal compactification of $\mathbb{C}^4$. Using the language of generalized conformal structures (GCS) it is possible to generalize this fact. Let us consider for an example the space $\mathbb{C}^{n^2} = M(n)$ of square matrixes of order n with the cone

$$V = \{z \in M(n); \det z = 0\}.$$

and the corresponding flat GCS. Then there is a conformal compactification of this manifold which is isomorphic to $Gr(n; 2n)$: conformal automorphisms of the compactification are exactly the automorphisms of the Grassmanian. Of course the corresponding GCS on the Grassmanian admits a geometrical description on the language of intersections of n-subspaces in $\mathbb{C}^{2n}$. The flatness of this structure can beeen established by a generalization of the stereographic projection (cf. below).

Around I985 I supposed that this a phenomena is a general one for compact Hermitian symmetric spaces of rank more than 1. Namely, on each such manifold there exists such a flat GCS that its automorphisms are exactly are automorphisms of the symmetric space. Moreover local conformal automorphisms can be extend up to global ones (generalized Liouville theorem). GCS connecting with Grassmanians (they are defined by Segre cones) were considered earlier by Akivis. Goncharov [Go] proved this conjecture for all compact Hermitian symmetric spaces of rank more than 1. As the cone V we take in this construction the closure of an orbit of the isotropy group on the tangent space. Goncharov worked with orbits of minimal dimension; another natural choice is orbits of codimension 1. Probably GCS for closure of any orbit of positive codimension defines the geometry of Hermitian spaces. Later Baston [B] also considered such GCS (he called them almost Hermitian symmetric). Last years were published several results about GCS connecting with Hermitian spaces. An interesting development was found by Neretin [N].

Between real forms of GCS the special interest have *causal* structures when real conic variety V is a boundary of a convex sharp (without lines inside) cone. Such structures were introduced by I.Segal. Ol'shanskii [Ol3] considered affine symmetric spaces with invariant causal structures. Symmetric spaces of Hermitian type (AH) are a subclass of causal ones [O]. Kaneyuki [K] considered flat causal structures on egdes (CR) of flat tubes (FT) and proved that their automorphisms can be extended as holomorphic automorphisms of tubes.

Let us illustrate the last fact. If in our example of $M(n)$ we take the Hermitian symmetric space - flat tubes (FT) - of matrixes z with positive skew Hermitian part:

$$1/2i(z - z^*) \gg 0 \tag{1}$$

then the induced GCS on the edge (Hermitian matrixes $z = z^*$) can be defined by the boundary of the cone of positive Hermitian matrixes. It will be causal and flat. Last years there were considered some theorems of Liouville type for GCS on symmetric R-spaces: Bertram [Be1] used the language of Jordan algebras, Kaneyuki and me - graded Lie algebras [GK].

Now we will give an example of causal structures on symmetric spaces of Hermitian type (AH) and its complexifications. On $SL(2)$ we take in the unit point e the cone of singular elements

$$v = \{g; \det(g - e) = 0\},$$

its image V in Lie algebra and the field of translations V_g of the last cone in points of the group. It is invariant GCS on $SL(2;\mathbb{C})$ and its restriction on $SL(2;\mathbb{R})$ is causal.

The next important fact is that between invariant GCS on affine symmetric spaces there can exist non trivial local isomorphisms. Let us include $SL(2;\mathbb{R}) = Sp(1;\mathbb{R})$ in the family of symplectic groups ($Sp(n;\mathbb{R})$ acts on the Siegel half-plane and has the Hermitian type) and *establish a local conformal isomorphism of the group* $Sp(n;\mathbb{C})$ *and the Lagrangian Grassmanian* $\mathcal{L}_n = LGr(n)$ on which the group $Sp(2n;\mathbb{C})$ acts by biholomorphic automorphisms. Let us describe this an example in more details. We start with the Grassmanian $Gr(n;2n)$ which we realize as the manifold of equivalency classes of $n \times 2n$ -matrixes Z relative to the relation

$$Z \sim uZ, u \in Gl(n;\mathbb{C}).$$

Then $\mathcal{L}_n$ is defined by the equation

$$ZJZ^T = 0, \tag{2}$$

where J is a fixed nondegenerate real symmetric $2n \times 2n$-matrix; for example,

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where $I = I_n$ is the unit matrix of the order n. The subgroup in the group $SL(2n;\mathbb{C})$ of automorphisms of the Grassmanian which conserve (1) is $Sp(n;\mathbb{C})$. Different J give equivalent realizations of $\mathcal{L}_n$ in the Grassmanian. In the intersection with the coordinate chart $Z = (I, z)$, where z is an $n \times n$-matrix, $\mathcal{L}_n$ coincides with the space of symmetric matrices $\mathrm{Sym}(n)$. The dual symmetric domain D_+ in $\mathcal{L}_n$ (Siegel half-plane) can be define by the condition

$$1/2i\, ZJZ^* \gg 0 \tag{3}$$

and in the coordinate chart of symmetric matrices by the condition

$$\Im z \gg 0.$$

Its group of automorphisms will be $Sp(n;\mathbb{R})$.

Let us for $Z \in \mathcal{L}_n$ define

$$v_Z = \{Z_1 \in \mathcal{L}; ZJZ_1^T = 0\}$$

be a field of conic subvarieties in $\mathcal{L}_n$ and V_Z be their tangent cones. The field V_Z defines on $\mathcal{L}_n$ a generalized conformal structure. In the chart of symmetric matrixes the field has the form

$$V_z = \{w; \det(z - w) = 0\}$$

Therefore *we have on the space of symmetric matrixes the flat GCS and $\mathcal{L}_n$ is its conformal compactification.*

On the group $G = Sp(n; \mathbb{C})$ we consider the field of conic submanifolds

$$v_g = \{g_1 \in G; \det(g_1 g^{-1} - I) = 0\}$$

and their tangent cones V_g. We obtain GCS and $Sp(n; \mathbb{C})$ admits conformal imbedding in $\mathcal{L}_n$. We need only take the definition of $Sp(n; \mathbb{C})$:

$$g\tilde{J}g^T = \tilde{J}, \qquad g \in Gl(n; \mathbb{C}), \tag{4}$$

where $\tilde{J}$ is a real symmetric matrix of the order n. If we take $\mathcal{L}_n$ corresponding

$$J = \begin{pmatrix} \tilde{J} & 0 \\ 0 & -\tilde{J} \end{pmatrix}$$

then $Sp(n; \mathbb{C})$ will coincide with the intersection with the coordinate chart $Z = (g, I)$. Correspondingly we have the embedding of the automorphism groups: $Sp(n; \mathbb{C}) \times Sp(n; \mathbb{C})$ in $Sp(2n; \mathbb{C})$. This embedding commutes with taking of the real forms $Sp(n; \mathbb{R})$. As a consequence we obtain that GCS on Sp is flat (which was not evident apriori) and our embedding is a matrix analog of the stereographic projection. It turns out that *the preimage G_+ of the tube D_+ (3) for this imbedding is exactly the semigroup with the edge $Sp(n; \mathbb{R})$.*

This construction put our observation from [GG] about $SL(2; \mathbb{R})$ in more broad context. Such an extension exists for several classical groups and symmetric manifolds of Hermitian type. The GCS on $GL(n; \mathbb{C})$ is flat and equivalent to the GCS on the Grassmanian $Gr(n; 2n)$ (cf. above), any $U(p; q)$ is locally isomorphic to the manifold of Hermitian matrixes of the order $n = p + q$ and corresponding semigroup G_+ is compactified up to the symmetric space (1) [G4].

Embeddings of one symmetric space in another one with corresponding embeddings of groups in some classical examples were known for a long time. Let us mention results of Makarevich on embeddings of symmetric spaces in symmetric R-spaces [M]. Rallis and Piatetski–Shapiro [PR] applied the "doubling" of the symplectic and other classical groups to constructions of L-functions. May be a new element was only the interpretation on the language of GCS and the remark on the extentions of (inhomogeneous) curve tubes up to flat ones. Bertram [Be2] considered the general problem about the equivalency of causal structures to flat structures on R-spaces and connected the classification with the results of Makarevich. Let us emphasize that not all symmetric spaces of Hermitian type (including groups) admit conformal extension up to R-spaces and correspondingly *not all curved tubes can be extend up to flat tubes*. May be it is true, that

Causal structures on affine symmetric spaces of Hermitian type are flat if and only if they are isomorphic to causal structures on symmetric R-spaces (their complexifications are then isomorphic to structures on compact Hermitian symmetric spaces) and then curved tubes extend up to flat ones and the corresponding Stein symmetric spaces compactify up to compact Hermitian ones

It is possible to formulate such a conjecture for arbitrary causal symmetric spaces. Classification from [Be2] can be the essential step for this fact.

Let us illustrate these constructions on the simplest example of classical conformal structures (cf. details in [G6]). Let Q be the hyperboloid in $\mathbb{R}^n$:

$$m(x,x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2 = 1, \quad n = p+q. \tag{5}$$

Let $Q_{\mathbb{C}}$ be its complexification and

$$Q_{\pm} = \{z \in Q_{\mathbb{C}}; m(z,\bar{z}) \gtrless 0\}. \tag{6}$$

The group $SO(p,q)$ acts transitively on Q and Q will be an affine symmetric space relative to this action. On $Q_{\mathbb{C}}$ the conformal structure is defined by the field of intersections of $Q_{\mathbb{C}}$ with tangent hyperplanes and on Q by intersections with real tangent hyperplanes. The last cones for $q = 0$ will be imaginary. The conformal compactification of Q will be the hyperboloid $\tilde{Q}$ in the projective space $\mathbb{R}P^n$:

$$x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_n^2 - x_0^2 = 0. \tag{7}$$

Corresponding domains $\tilde{Q}_{\pm}$ are pseudo-Hermitian symmetric manifolds and for $p = 2$ $\tilde{Q}_+$ is a Hermitian symmetric. For $p = q = 2$ the hyperboloid Q coincides with $SL(2;\mathbb{R})$. Between affine forms of $\tilde{Q}$ (their conformal compactifications are $\tilde{Q}$) there is a paraboloid P:

$$x_1 = x_2^2 + \dots + x_p^2 - x_{p+1}^2 - \dots x_n^2.$$

In this a situation the symmetric domains P_+ and $\tilde{Q}_+$ are isomorphic and they both are conformal extensions of Q_+. The group $SO(p,q)$ is not transitive on Q_+. For $p = 2$ the hyperboloid Q will be a symmetric manifold of Hermitian type and Q_+ will be the corresponding tube.

Some problems on GCS.

We will start from much more strong version of the problem from the last section.

1. On Hermitian symmetric compact spaces (CH) of rank more than 1 there are flat GCS. I am sure that the condition of the existence of a conformal compactification is extremely strong and I do not see obstructions to think that

CH of rank more than 1 are only compact complex manifolds with flat GCS.

More realistic is to consider this conjecture under conditions that this manifold is homogeneous relative to automorphisms of the structure or to put some conditions of homogeneity on the conic variety, defining the structure. Nevertheless it would be extremely interesting to find a geometrical characterization of Hermitian symmetric spaces without the group language.

2. One more problem in a similar direction. As explained by old geometers, starting with Sofus Lie, the isotropic submanifolds (whose tangent spaces lie inside of isotropy cones V, defining the conformal structure) are most important objects in the conformal geometry. It would be interesting to investigate systematically maximal isotropic submanifolds for GCS. The 4-dimensional (complex) flat conformal manifold C^4 it is possible to realize either as a nondegenerate quadric in $\mathbb{C}P^5$ or as the Grassmanian $Gr(2;4)$ of lines in $\mathbb{C}P^3$. There are 2 one-parametric families of isotropic submanifolds of dimension 2 in it (corresponding to points and planes) and they are in the focus of Penrose's theory of twistors. It is not difficult to prove that if on a 4-manifold with GCS, $\operatorname{codim} V = 1$, there are 2 one-parametric families of isotropic 2-submanifolds such that submanifolds in one family do not intersecting and from different families have one-dimensional intersections then this structure is the classical conformal flat structure (with qudratic cone). Hint: if on a surface in the projective 3-space there are 2 one-parametrical families of lines such that lines of one family do not intersect, but of different families do then this surface is a hyperboloid. I believe that it is not difficult *to generalize this result for Grassmanians*. It would interesting

to characterize Hermitian symmetric compact manifolds on the language of maximal isotropic submanifolds.

3. Almost everything what we know about GCS is restricted by flat ones and often only causal. I believe now is the time to think more about nonflat GCS and noncausal GCS. One of first natural problems is to investigate invariant GCS on arbitrary pseudo-Riemannian symmetric spaces $X = G_0/K_0$. In non-Hermitian case they are apparently must be non flat. Again it is natural to start with the complexification - the Stein symmetric space $Y = G/K$ where G, K are complexifications of G_0, K_0. Real forms of GCS on Y can be on X imaginary. Natural candidates for the conic varieties V are orbits of K in the adjont representations. If there is a geometrical realization of points of Y then usually it is possible to describe the conic variety V_x with the vertex in a point x as the tangent cone to the variety of points in a nongeneric position to x. The first problem is

to find for arbitrary Stein symmetric spaces (SS) GCS satisfying the Liouville condition: the group of their automorphisms is the complete group G of automorphisms of the symmetric manifold Y.

Such a GCS can exist in symmetric spaces of rank 1. In [G9] we explain that in the case of $X = \mathbb{C}P^n$ the (conformal) Fubini–Study metric satisfies this condition and we generalize this construction on the case of Grassmanians. Restrictions of such structures on Riemannian symmetric spaces probably are always imaginary.

It is important to be flexible in the work with GCS. Invariant GCS on symmetric spaces might not be equivalent: GCS corresponding to different orbits on tangent spaces can be nonequivalent. On one hand it is interesting to consider "non-Liouville" GCS whose (local) automorphisms can not be automorphisms of symmetric spaces and as the result we can consider conformal extensions. On the other hand sometimes it is important to find such GCS for which automorphisms are only authomorphisms of a symmetric manifold and defines in such a way its geometry. For example, if X is Hermitian compact space (G_0 is compact) then its complexification Y will be a Zariski open part in $X \times \bar{X}$. On Y there is of course the flat GCS (the direct product of structures on X and $\bar{X}$) relative to which $X \times \bar{X}$ will be the conformal compactification of Y (and this extension is useful for some constructions of representations. On other hand, as examples demonstrate, on Y can exist GCS with Liouville property (its automorphisms are automorphisms of Y) . Above mentioned Fubini-Study structures [G9] have this property. They are more informative for representations: Hua and Schmid's elliptic differential equations on X are confomal invariant relative to the 2nd structure and as the result there is an important connection with discrete series of G_0.

4. Compact Hermitian symmetric manifolds are partial case of flag manifolds $F = G/P$, G is a complex group, P is a parabolic subgroup. It is natural to investigate invariant GCS on flag manifolds which again must be nonflat if F is not symmetric. Let us consider the simplest example of $F(\mathbb{C}P^2)$: flags on the projective plane: pairs $(f = (a, l); a \in l)$, where a is a point, l is a line. The isotropic cone in the tangent space in a flag f is induced by a conic variety of flags which are not in generic position with f (there is a incidence between corresponding points and lines). Apparently conformal automorphisms of this structure will correspond to maps of $\mathbb{C}P^2$ conserving the set of lines and the corresponding Liouville theorem will be the Darbox theorem (which call sometimes the fundamental theorem of the projective geometry): all such maps are projective transformations. There is an essential connection between GCS on flag manifolds and the Borel-Bott-Weil theorem [GW] .

Analytic applications of GCS on compact Hermitian symmetric spaces. The idea of [G4] was to use the embedding of some curved tubes in flat ones for investigation of Hardy spaces and Szegö kernels. Namely flat tubes are symmetric and Hardy spaces for them were studied in details including explicit formulas for the Szegö kernels. The crucial step is to extend functions on curve tubes and their edges on their conformal extensions.

There is a pretty standard way to do it. If we have function on a manifold with GCS, say on hyperboloid Q with usual conformal structure, we can extend them on the conformal compactification ($\tilde{Q}$) as sections of an appropriate line bundle (in our example as homogeneous functions of some degree in homogeneous coordinates in the projective space). These sections can be then restricted on another affinization where all considerations can be simpler (e.g. it can be homogeneous). In our example of the paraboloid P there is a big abelian subgroup of automorphisms which gives us the ability to apply the Fourier integral. For $p = 2$ the domain P_+ is a (flat) tube domain (with the light cone) and we have the embedding of the curve tube Q_+ in the flat tube P_+. The idea was to use homogeneous functions to transfer functions from P, P_+ on Q, Q_+, including Hardy spaces and Szegö kernels.

This plan has a dangerous point which I missed in [G4] and which was observed by Koufany and Ørsted [KO] . Namely it is necessary to be careful when we transform holomorphic functions in sections on conformal extensions. In the case of the paraboloid P we need to extend functions out of L^2 for the invariant measure as homogeneous functions of the degree of $-(n+1)/2$ (it is a consequence of the "homogeneous" extension of the invariant measure on Q). So if n is even for holomorphic functions on P_+ we obtain 2-valued holomorphic homogeneous functions and this property will conserve when we restrict them on Q_+. So only for odd n this trick works and gives the correct Hardy space and the Szegó kernel on Q_+. For even n we obtain formulas on Q_+ for 2-valued (odd) holomorphic functions. This space as well as spaces of holomorphic sections of other invariant line bundles are interesting from a point of view of the theory of representations but we need to seek other ways to work with the Hardy space in Q_+. We have the similar situation for some other curve tubes admitting generalized conformal extentions. Therefore only under extra restrictions can we transfer Szegö kernels from flat tubes on embedded curve tubes but in any case it demonstrates several examples where the Szegö kernel can be explicitely compute.

Stein neighborhoods of Riemannian symmetric manifolds. I already mentioned that from my point of view the most unusual element of these constructions is the appearance of inhomogeneous complex manifolds in the theory of representations. There is one more example of such a phenomena. In the theory of discrete series the crucial role play *flag domains* - open G_0-orbits on flag manifolds $F = G/P$. Noncompact Herimitian and pseudo-Hermitian symmetric spaces are special cases of flag domains when F is symmetric and it is important that we go outside of the class of symmetric spaces. In $\bar{\partial}$-cohomology in flag domains there are realized discrete series and some other important representations. An efficient method of investigation of $\bar{\partial}$-cohomology is the integration of them on compact complex submanifolds (the Penrose transform). It is the reason why it is important to investigate the manifold Z of such cycles. Wells and Wolf [WW] started to consider it. If G_0 is the group of Hermitian type then Z sometimes coincides with the corresponding Hermitian symmetric space; this situation is degenerate (but important nonetheless). If G_0 is non Hermitian type then the corresponding Riemannian symmetric space $X = G_0/K_0$ (K_0 is compact) is non Hermitian and Z is G_0-invariant but inhomogeneous. What we can say then about Z? Wolf [W] proved that Z is the Stein manifold. If we exclude the cases when G_0 is Hermitian type and $Z = X$, then *the parametrical space Z is an open submanifold in the Stein symmetric space $Y = G/K$ and Z contains X*. Wolf and Zierau announced that if the group G_0 is of Hermitian type then in the generic situation $Z = X \times \bar{X}$.

Relative to the non-Hermitian case the first question is to find a way to describe hypotetic inhomogeneous manifold Z. The first conjecture is that *Z is independent of a choice of a flag domain for G_0*. So if we were to believe in this conjecture then it must be an universal Stein neighborhood Z of X - *the Stein crown* -with some remarkable properties. Akhiezer and me [AG] tried to start from another end and to find G_0-invariant Stein neighborhoods of X in Y with some extra conditions. We described some extensions on the language of G_0-orbits and between them one is especially nice and, I believe, it coincides with Z . We could not prove in general situation that this remarkable extension is a Stein manifold (it can be very difficult to work with a union of orbits of a noncompact real group and prove in particular that it is a Stein manifold!).

Now I want to explain another way to construct Stein neighborhoods of X (joint work with H.-W.Wang). Let P_i be a maximal parabolic subgroup of G_0. They correspond to simple restricted roots. A function $D_i(x)$ on X is called an *determinant* *function* if

$$D_i(xp) = \alpha(p)D_i(X),$$

where $\alpha(p)$ is a character of P_i. Then D_i are algebraic functions and D_i have no zeroes on X. We can extend them on Y and choose D_i such a way that it will holomorphic on Y. Let $\tilde{X}$ be the connected component of the set

$$\{z \in Y; D_i(zg) \neq 0, g \in G_0\},$$

containing X. For precaution let us take all determinant functions in this definition, but examples show however that in non-Hermitian case it is enough to take any one of them (in the Hermitian case we can need two). It is evident that $\tilde{X}$ is the Stein manifold and G_0-invariant. *The conjecture is that the Stein crown*

$$Z = \tilde{X}$$

and it coincides with the domain constructed in [AG]. This conjecture is proved in several cases. I believe that $\tilde{X}$ universally arises in all analytic extensions from Riemannian symmetric space X (e.g. it is the joint domain of holomorphy of all solutions of Schmid elliptic equations).

Let us give a few examples. For $G_0 = SL(n;\mathbb{R})$ the manifold X will be the manifold of positive symmetric forms Q on $\mathbb{R}^n$ with the determinant 1, Y will be the manifold of complex symmetric forms also with determinant 1 and $\tilde{X}$ will be the component of the set of such forms $Q \in Y$ that corresponding quadric $Q(u,u) = 0$ has no real points different from 0.

For $G_0 = SO(p,q)$ if twe were to fix a real form Q of signature (p,q) then points of X are p-subspaces in $\mathbb{R}^n$ on which Q is positive; points of Y are p-subspaces of $\mathbb{C}^n$ on which Q is nondegenarate, and $\tilde{X}$ is the component of the set of such subspaces that intersections with Q-quadric have no real points different from 0.

If we were to compare this with the situation from which we started, we can see that we have two pictures: in one we construct for a pseudo-Riemannian symmetric space invariant but inhomogeneous Stein tube which has this manifold as the edge. In other case we include a Riemannian symmetric space inside an invariant Stein manifold. Both Stein manifolds lie in the Stein symmetric space Y. I thought many years about these two very special constructions and could not understand why they lie in "different baskets". Only not long time ago, when I started to understand more about Stein neighborhoods, I realized suddenly they are from the same basket. Namely examples show that domains $\tilde{X}$ are indeed (at least in many cases) are tubes whose edges are some pseudo-Riemannian symmetric spaces with the same group G_0. Thus for $SO(p,q)$ the edge is $SO(p,q)/SO(p-1,1) \times SO(1,q-1)$, for $SL(n;\mathbb{R})$ the edge is disconnected and its components are $SL(n;\mathbb{R})/SO(p,q), p \neq 0, q \neq 0$. We can see that these edges are causal symmetric manifolds but of non Hermitian type. Thus such causal spaces also are associated with some curved tubes but they are not orbits of semigroups and do not connected with holomorphic discrete series (which do not exist for these groups). They are connected with continuous series of representations. It is surprising that they did not appeared earlier. The reason was probably that in the difference with causal structures it was not an intrinsic definition of tubes: the definition was constructive and as far as all holomorphic discrete series were included it gave an impression that all curved tubes were described. It would be interesting

to give an intrinsic definition of tubes in Stein symmetric spaces and connect them with causal symmetric spaces.

I am sure that domains $\tilde{X} = Z$ are very important for Harmonic Analysis and deserve very serious attention. It is enough to mention that all representations of discrete series of corresponding groups are realized in holomorphic subrepresentations in these tubes. One from first tasks here to investigate their geometry, especially their boundary components.

Non-Stein tubes in symmetric Stein spaces. We will discuss in the end the problem which I believe is in the focus of this area. We will consider an affine symmetric manifold $X = G_0/K_0$ (important special case: $X = G_1$ is a real semisimple Lie group and $G_0 = G_1 \times G_1$ and $K_0 = G_0$ in the diagonal embedding). We consider the complexification - the Stein symmetric space $Y = G/K$ (in the case of groups they will be the complex groups). In the case when the manifold X is of Hermitian type (in particular when it is the group of Hermitian type) in Y there are the G_0-invariant curve Stein tubes T with the edge X (orbits of semigroups) and the part of the regular representation on X, corresponding to the holomorphic discrete series, can be holomorfically extend in the smallest of them . This situation was the subject of detailed investigation for the last 20 years.

There are also other G_0-invariant tubes with the edge X in Y. We saw examples of such tubes when consider the example of hyperboloids. They will not be already Stein manifolds so following to the standard philosophy of multidimensional complex analysis we need to consider in them instead of holomorphic functions $\bar{\partial}$-cohomology of appropriate dimension. I tried for 20 years to promote the problem on functional spaces of $\bar{\partial}$-cohomology in such invariant “nonconvex” tubes.

My basic conjecture is that

there is a number of invariant tubes $T_1, \dots, T_l$ such that in appropriate Hardy spaces of $\bar{\partial}$-cohomology, such that in them can be realized all parts of the regular representation on X corresponding to different series of representations.

Let us emphasize that we talk here about all series not only discrete. The examples of such tubes are domains $Q_\pm$ on the complex quadric $Q_\mathbb{C}$ with the real quadric Q as the edge. For $p = 2$ Q_+ has 2 components which are Stein manifolds corresponding to holomorphic and nonholomorphic series. The case $p = q = 2$ corresponds to $SL(2;\mathbb{R})$. Then we have 3 tubes: 2 components of Q_+ and Q_-; continuous principal series are realized in 1-dimensional $\bar{\partial}$–cohomology of Q_-.

It is not difficult to describe these tubes T_j. Usually they are unions of regular orbits of G_0 on Y and are parameterized by Weyl chambers in some Cartanian subgroups. We can use the detailed results of Matsuki [Ma] about the parametrization of orbits; Bremigan [Br] gave a convenient description in the case of groups. The next step is to find for these tubes the invariant q for which they are q-pseudoconcave and to understand better their complex geometry.

Analytic problems have needed much more attention. We need to define a structure of Hardy space on cohomology and to define an operator of boundary values from the Hardy space of cohomology to $L^2(X)$. Already the tubes from the last section which correspond to continuous series show that the situation can be more complicate. When many years ago I started to think about this problem I realized that standard constructions of $\bar{\partial}$-cohomology do not help too much in this situation. It was very unusual from point of view of multidimensional complex analysis that $\bar{\partial}$-cohomology in tubes with totally real edges can have functions as boundary values on edges. Of course there was one remarkable example of such a phenomena - Sato's definition of hyperfunctions as $\bar{\partial}$-cohomology, but I did not found other examples.

The strategy was to start from the "flat case". It turns out that in this general (non-Stein) situation there are not only curved tubes in Stein symmetric spaces but also flat tubes in compact Hermitian symmetric spaces. We can omit the condition homogeneity in the beginning and consider tubes in $\mathbb{C}^n$: $T = \mathbb{R}^n + iV$, where V is a cone in $\mathbb{R}^n$ not necessarilly convex. Under very strong geometrical conditions on cones V (in a sense they must by a very regular way to unify convexity and concavity) I developed the theory of Hardy spaces of $\bar{\partial}$–cohomology in T including boundary values in functions in $L^2(\mathbb{R}^n)$. In these restrictions it is possible to represent $L^2(\mathbb{R}^n)$[G5], [G6] as a direct sum of boundary values of $\bar{\partial}$–cohomology in some tubes. The crucial component of this construction was a new language for the description of $\bar{\partial}$-cohomology different from Čech and Dolbeault languages. It uses essentially that we have infinite coverings by Stein manifolds (convex tubes) and we work with de Rham complex of differential forms on the manifold parameterizing the covering which also holomorphically depend on parameters in corresponding manifolds of covering (continuos Čech cohomology). This part of results was generalized in the general situation and one of the final products is a purely holomorphic language for $\bar{\partial}$-cohomology [G7], [EGW1], [EGW2] Another development of these constructions is a theory of boundary values of cohomology in local curved tubes [CGT].

Very interesting examples of nonconvex cones are cones V which are affine symmetric spaces. Examples of such cones: nondegenerate symmetric matrixes of fixed signature, cones bounded by quadrics (corresponding tubes are equivalent to domains $P_{\pm}$ on complex quadrics), the cone of real matrixes with the positive determinants etc. D'Atri and me were considered some classical cones and corresponding tubes [DG]; Faraut and me [FG] - the general case. The remarkable property of these tubes is that they are Zariski open parts of some pseudo-Hermitian symmetric spaces. Faraut and me work now on the theory of $\bar{\partial}$–cohomology in these tubes. One of the central problems here is the possibility of extension of cohomology in (flat) tubes up to cohomology in pseudo-Hermitian symmetric spaces with coefficients in appropriate line bundles. Troubles here are similar to those which we discussed in the connection of extension of holomorphic functions from curve (Stein) tubes to flat ones. This problem is connected with interesting phenomena in the theory of representation which was discovered by Kashiwara and Vergne.

These constructions for flat tubes give many hints on how to work on the conjecture for tubes in Stein symmetric spaces, but it is necessary to do many things for its proof. There is a situation when the conjecture is proved: when curved tubes are extensions of some flat ones and the corresponding extension of cohomology is possible. The simplest of such examples are hyperboloids in odd-dimensional space.

I am sure that the problems which we discussed are very important for more clear understanding of representations, first of all in the direction of their connection with the complex analysis. Of course the connection of representations and complex analysis is well known (e.g. realizations of discrete series) but I believe it is much broader. From other side these problems must feed back multidimensional complex analysis since they supplied by new phenomenas of complex geometry.

References

[AG] Akhiezer D.N., Gindikin S.G., *On Stein extensions of real symmetric spaces*, Math.Ann. **286** (1990), 1-12.

[B] Baston R.J., Almost Hermitian symmetric manifolds, I:Local twistor theory; II: Differential invariants, Duke Math.J. **63** (1991), 81-11, 113-138.

[Be1] Bertram W., *Un théorème de Liouville pour les algébres de Jordan*, Bulletin Soc.Math.Française **124** (1996), 299-327.

[Be2] Bertram W., *On some causal and conformal groups*, J.Lie Theory **6** (1996), no. 2, 215-247.

[Br] Bremigan, *Invariant analytic domains in complex semisimple groups*, Transformation Groups **1** (1996), no. 4, 279-305.

[CGT] Cordaro P., Gindikin S., Treves F., *Hyperfunctions as boundary values of cohomology classes*, J.Funct.Anal. **131**, no. 1, 183-227.

[DA] D'Atri J.E., Gindikin S., *Siegel domains realization of pseudo-Hermitian symmetric manifolds*, Geometriae Dedicata **46** (1993), 91-126.

[EGW1] Eastwood M.G., Gindikin S.G., Wong H.-W., *Holomorphic realization of $\overline{\partial}$-cohomology and constructions of representations*, J.Geom.Phys. **17** (1995), 231-244.

[EGW2] Eastwood M.G., Gindikin S.G., Wong H.-W., *A holomorphic realization of analytic cohomology*, C.R.Acad.Sci **322** (1996), no. Serie 1, 529-534.

[FG] Faraut J., Gindikin S, Pseudo-Hermitian symmetric spaces of tube type, Topics in Geometry. In memory of Joseph D'Atri (Gindikin S., ed.), Birkhäuser Boston, 1996, pp. 123-154.

[GG] Gelfand I.M., Gindikin S.G., *Complex manifolds whose skeletons are real semisimple groups and holomorphic discrete series*, Funct.Anal.Appl. **11** (1977), 19-27.

[GGr] Gelfand I.M., Graev M.I., *Complexes of k-dimensional planes in the space $\mathbb{C}^n$ and Plancherel's formula for the group $GL(n,\mathbb{C}^n)$*, Sov.Math.Dokl. **9** (1968), 394-398.

[G1] Gindikin S., *Integral formulas and integral geometry for $\overline{\partial}$-cohomology in $\mathbb{C}P^n$*, Funct. Anal. Prilozh. **18** (1984), no. 2, 26-33 (Russian); Engl.transl.: Funct.Anal.Appl. **18** (1984).

[G2] Gindikin S.G., *Generalized conformal structures*, Twistors in Mathematics and Physics (T.N.Bailey and R.J.Baston,, eds.), London Math.Soc., London; Lect.Notes Ser. **156** (1990), 36-52.

[G3] Gindikin S., *Integral geometry on symmetric manifolds*, Amer.Math.Soc.Transl.(2) **148** (1991), 29-37.

[G4] Gindikin S., *Generalized conformal structures on classical real Lie groups and related problems on the theory of representations*, C.R.Acad.Sci.Paris **315 série I** (1992), 675-679.

[G5] Gindikin S., *Fourier transform and Hardy spaces of $\overline{\partial}$-cohomology in tube domains*, C.R. Acad.Sci.Paris **415 série I** (1992), 1139-1143.

[G6] Gindikin S., *Conformal harmonic analysis on hyperboloids*, J.Geometry and Physics **10** (1993), no. 2, 175-184.

[G7] Gindikin S.G., *Holomorphic language for $\overline{\partial}$-cohomology and representations of real semisimple Lie groups*, Contemporary Math. **154** (1993), 103-115.

[G8] Gindikin S., *The Radon transform from cohomological point of view*, 75 years of Radon transform (S.Gindikin, P.Michor, eds.), International Press, 1994, pp. 123-128.

[G9] Gindikin S., *Fubini-Study structures on Grassmanians*, Preprint IHES (1995).

[GK] Gindikin S., Kaneyuki S., *On the automorphism group of the generalized conformal structure of a symmetric R-space . Preprint,* (1995).

[GKh] Gindikin S, Khenkin G., *The Cauchy–Fantappie formula on projective space*, Amer.Math. Soc.Transl.(2) **146** (1990), 23-32.

[GW] Gindikin S., Wong H.-W., *A Holomorphic version of Borel-Weil-Bott theorem*, Advances in Mathematics **130** (1997).

[Go] Goncharov A.B., *Generalized conformal structures on manifolds*, Selecta Math.Soviet. **6** (1987), 308-340.

[HOO] Hilgert J., Ólafsson G., Ørsted B.,, *Hardy spaces on affine symmetric spaces*, J.reine and angew.Math. **415** (1991), 189-218.

[K] Kaneyuki S., *On the causal structures of the Shilov boundaries of symmetric bounded domains*, Lect.Notes in Math. **1468** (1991), Springer–Verlag, 127-159.

[KO] Koufany K., Ørsted B., *Espace de Hardy sur le semi-groupe métaplectique*, C.R.Acad.Sci.Paris **322** (1996), 113-116.

[M] Makarevich B.O., *Open symmetric orbits of reductive groups in symmetric R-space*, Math.Sbornik **20** (1973), 406-418.

[Ma] Matsuki, *Double coset decompositions of algebraic groups arising from two involutions I,*, J.Algebra **175** (1995,), 865-925; II, preprint.

[Mo] Molchanov S..

[N] Neretin Yu.A., *Integral operators with Gaussian kernels and symmetries of canonical commutation relations*, Amer.Math.Soc.Transl.(2) **175** (1996), 97-135.

[O] Ólafsson G., *Symmetric spaces of Hermitian type*, Differential Geometry Appl. **1** (1991), 195-233.

[Ol1] Ol'shanskii G.I., *Invariant cones in Lie algebras, Lie semigroups, and the holomorphic discrete series*, Funct.Anal.Appl. **15** (1981), 275-285.

[Ol2] Ol'shanskii G.I., *Convex cones in symmetric Lie algebras, Lie semigroups and invariant causal (order) structures on pseudo-Riemannian symmetric spaces*, Soviet Math.Dokl. **26** (1982), 97-101.

[Ol3] Ol'shanskii G.I., *Cauchy-Szegö kernels for Hardy spaces on simple Lie groups*, J. Lie Theory **5** (1996), 241-273.

[PR] Piatetski–Shapiro, Rallis S., *L-functions for classical groups*, Springer Lecture Notes in Math. **1254** (1987), 1-52.

[S] Stanton R.J., *Analytic extension of the holomorphic discrete series*, Amer.J.Math **10** (1986), 1411-1424.

[W] Wolf J, *The Stein condition for cycle spaces of open orbits on complex flag manifolds*, Ann.Math. **136** (1992), 541-555.

[WW] Wells R.O.(Jr.), Wolf J., *Poincaré series and automorphic cohomology on flag domains*, Ann.Math. **105** (1977), 397-448.

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