A Remark About Static Space Times

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1 Introduction

We study the overdetermined equation

\[ Ddf + (\Delta f)g - fRic = 0 \quad (S), \]

whose unknowns are the metric \( g \) and the (non identically vanishing) function \( f \), on a compact manifold \( M \). This equation is well-known in general relativity. In that situation, \((M, g)\) is a space-like section of a static space time. Namely, let \((X, G)\) be a four dimensional Ricci-flat Lorentz manifold admitting a timelike Killing vector field, whose orthogonal distribution is integrable. Then any leaf of this foliation gives a solution of \((S)\) : \( g \) is the (riemannian) metric induced by \( G \), and \( f \) is the length of the Killing vector. For more information, we refer to [8] and [2].

It is also important in riemannian geometry (in any dimension), since the metrics for which \((S)\) has a non trivial solution are just the critical points of the scalar curvature map \( g \mapsto \text{scal}_g \). See [6] for a beautiful use of this fact, and the introduction of [7] for further comments.

Not much is known about this equation, although it is highly overdetermined. The present knowledge is as follows.

- The scalar curvature \( s \) is constant, and \( f \) is an eigenfunction of the Laplace operator, whose associate eigenvalue is \( s/(n - 1) \) (Bourguignon, cf. [3] or [1], ch. 4 ; with our sign convention, the Laplace operator of the real line is \( -f'' \)).

- The only compact Einstein manifold which satisfies \((S)\) is the standard sphere (Obata, cf. [9]).

- Conformally flat solutions have been classified independently by the author in [7] and O. Kobayashi in [4].

- If \((f, g)\) is a non trivial solution on a compact three dimensional manifold, then \( f^{-1}(0) \) has positive Euler characteristic (Shen, cf. [10]). Shen proves this result by using a tricky integration argument, together with Gauss–Bonnet theorem, but omits to check that \( f^{-1}(0) \) is actually a manifold.

In this paper, we prove the following partial result.
**Theorem 1.1** Let $(M, g)$ a compact 3-dimensional Riemannian manifold such that the vector space of solutions of (S) has dimension at least 2. Then $(M, g)$ is isometric to a standard product $S^1 \times S^2$, $S^1 \times P^2 \mathbb{R}$ or to the standard 3-sphere.

As a by-product, we give examples of non conformally flat solutions of (S) on open manifolds. Some of them are complete, but $s < 0$ in that case. Although the existence of such examples seems to be a common belief among relativists, explicit ones do not seem to be known.

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**2 Some formulas**

The following property will be used repeatedly. Given a Riemannian manifold $(L, g^L)$, take on $L \times I$ the metric $g = g^L + u^2 dt^2$, where $u \in C^\infty(L)$ never vanishes. This is just a local model for a metric which admits a Killing vector field whose orthogonal distribution is integrable.

Denote with the superscript $L$ the Riemannian invariants of $g^L$, and set $T = \partial / \partial t$. Then $D_X U = D_X^L Y$ when the vector fields $X$ and $Y$ are tangent to the leaves, while $D_X T = \frac{du}{u}(X) T$ and $D_T T = -u \nabla^L u$. A straightforward computation shows that the Ricci and the scalar curvature of $g$ are just

\[
Ric = Ric^L - \frac{D^L du}{u} + u \Delta^L u dt^2
\]

\[
s = s^L + 2 \frac{\Delta^L u}{u}
\]

Moreover, if $F \in C^\infty(L \times I)$ is given by $F(y, t) = a(t) f(y)$, with $a \in C^\infty(I)$ and $f \in C^\infty(L)$, we have

\[
DdF = \begin{cases} 
  a D^L df & \text{along } L \times \{t\} \\
  (a^2 f + a u g^L(\nabla^L u, \nabla^L f)) dt^2 & \text{along } \{x\} \times I \\
  a'(t) [df - f \frac{du}{u} \otimes dt] & \text{transversally.}
\end{cases}
\]

We infer that

\[
\Delta F = a \Delta^L f - \frac{a' f}{u^2} - a g^L(\nabla^L u, \nabla^L f).
\]

**3 Proof of the theorem**

We need the following property (which by the way fills the gap in [10]).

**Proposition 3.1** Let $f$ be a non-trivial solution of the equation (S). Then $f^{-1}(0)$ is a smooth submanifold, which is totally geodesic. The same properties occur for $\cap_{i=1}^p f_i^{-1}(0)$, if the $f_i$ are $p$ linearly independent solutions.
For any geodesic $\gamma$, the function $\phi = f \circ \gamma$ satisfies a second order differential equation of the type

$$\ddot{\phi} + F \phi = 0,$$

where $F$ is a smooth function. Now, let $a \in X$ such that $f(a) = 0$ and $df_a = 0$. For any $v \in T_a X$, the function $\phi(t) = f(\exp_a tv)$ satisfies such an equation, with $\phi(0) = \phi'(0) = 0$, and therefore vanishes identically, so that $f$ should be zero. For $p$ independent solutions, the same argument works. Of course, in that case the submanifold may be empty.

Now, $f^{-1}(0)$ is clearly totally geodesic, since $Ddf_a$ vanishes whenever $f(a) = 0$. Moreover, the intersection of totally geodesic submanifolds is totally geodesic, as soon as it is a submanifold. □

**Proposition 3.2** Let $f_1$ and $f_2$ be two independent solutions of (S). Then the vector field $\xi = f_1 \nabla f_2 - f_2 \nabla f_1$ is a Killing field. Moreover, its orthogonal distribution is integrable, and gives a totally geodesic foliation of $M \setminus f_1^{-1}(0) \cap f_2^{-1}(0)$.

The dual one form of $\xi$ is $f_1 df_2 - f_2 df_1$, whose symmetric covariant derivative is just $f_1 Ddf_2 - f_2 Ddf_1$, which clearly vanishes, which proves the first part.

Now, the zero set of $f_1 df_2 - f_2 df_1$ is $f_1^{-1}(0) \cap f_2^{-1}(0)$. Indeed, take a point $a$ where $f_1(a)(df_2)_a = f_2(a)(df_1)_a = 0$ but $(f_1(a), f_2(a)) \neq (0, 0)$. Then, by looking at the restrictions of $f_1$ and $f_2$ to geodesics from $a$, the same differential equation argument as in 3.1 shows that $f_1$ and $f_2$ must be proportional, a contradiction. □

**Remark.** It is easy to check directly (and standard) that the orthogonal distribution of a Killing field is totally geodesic as soon as it is integrable.

As a free gift, we see that the closure of each leaf is a manifold of type $(\lambda_1 f_1 + \lambda_2 f_2)^{-1}(0)$, with $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{0\}$. Set $L = f_1^{-1}(0)$, and $L' = L \setminus f_1^{-1}(0) \cap f_2^{-1}(0)$.

**Lemma 3.3** Under the assumptions of proposition 3.2, let $\Phi(x,t)$ be the flow of $\xi$. Then

1. The pull-back of the metric to $L' \times I$ is $g^L + u^2 dt^2$ (where $u = |\xi|$, and $g^L$ is the induced metric on $L$.)

2. The pull back of any solution of (S) is of type $au$, with $a \in C^\infty(I)$.

The first part is elementary: indeed $\Phi_t^{-1} \xi = \frac{\partial}{\partial t}$, so that the leaves in $L' \times I$ are just $L' \times \{t\}$.

Now, let $f$ be a solution of (S) for $(L' \times I, g^L + u^2 dt^2)$. Using the formulas of section 1, we know that $D^2 f(v, \frac{\partial}{\partial t}) = 0$ whenever $v$ is tangent to $L'$. On the other hand

$$D^2 f(v, \frac{\partial}{\partial t}) = v, f'_t - df(Dv \frac{\partial}{\partial t}) = v, f'_t - \frac{du}{u} (v) f'_t,$$

so that $f'_t$ and $u$ are proportional along each leaf. On each connected component of $L' \times I$, we have $f = au + b$, with $a \in C^\infty(I)$, $b \in C^\infty(L)$.
\[ f_i = a_i u + b_i, \ i = 1, 2 \] are two independent solutions of (S), a long but straightforward computation, that we prefer to omit, shows that \( b_1, b_2 \) and \( a \) generate a 1-dimensional vector-space.\[ \Box \]

**End of the proof.**

Putting \( a(t)u(m) \) into the equation, we get

\[ 2D^L du + \frac{s}{n-1} u g^L - u Ric^L = 0 \]

If \( u \) is constant, since \( \dim L = 2 \), we see that \( L \) has constant positive curvature, and we are in the case of product metrics on \( S^1 \times S^2 \) or \( S^1 \times P^2 \mathbb{R} \).

If \( u \) is not constant, taking the computations of section 2 into account, and knowing that \( Ric^L = \frac{L}{2} g^L \), we get

\[ D^L du + \frac{\Delta^L u}{2} g^L = 0, \]

in \( L' \) first, then on \( L \).

This equation, which is of course strongly related to (S), has been extensively studied, cf. [5].

If \( u \) is not constant solution and if \( L \) is compact, then \( (L, g^L) \) is the standard sphere, and \( a \) a first order spherical harmonic (cf. [5], p. 133), so that \( M \) is the standard three-sphere.\[ \Box \]

### 4 A local non conformally flat solution

To get local solutions, we use the classification of [5], pp. 115, of local solutions of the differential equation \( D^2 u + (\Delta u/2)g \). In a domain where \( du \) does not vanish, the metric \( g^L \) is just the metric of some (abstract) surface of revolution, and can be written as

\[ g^L = d\sigma^2 + (a'(\sigma))^2 dx^2. \]

For the solution of (S) we are looking for, \( (M, g) = (L \times I, g^L + u^2 dt^2) \), while \( f \) is of type \( f(t, \sigma) = a(t)u(\sigma) \). We shall use the following elementary fact.

**Lemma 4.1** The couple \( (f, g) \) is a solution of (S) if and only if \( g \) is a constant scalar curvature metric such that

\[ D df + \frac{s}{n-1} g - f Ric = 0. \]

So we get the system

\[
\begin{align*}
D^L du + \frac{\Delta^L u}{2} g^L &= 0 \\
a''(t) &= ca(t) \\
\Delta^L u^2 - su^2 &= 2c \\
s^L + 2 \frac{\Delta^L u}{u} &= s,
\end{align*}
\]

\( 4 \)
where \( c \) and \( s \) are constant. Here, \( s^L = -2u^{(3)}/u' \), and \( \Delta^L u = -2u'' \). The first equation above has already been checked, and the second just gives \( a \). The third and the fourth boil down to

\[
4uu'' + 2u'^2 + sa^2 = -2c \quad \text{and} \quad -2\frac{u^{(3)}}{u'} - 4\frac{u''}{u} = s \]

The constant \( s \) being given arbitrarily, any fonction \( u \) satisfying the first equation also satisfies the second. Of course, we recover in particular products (for \( u \) constant) and the standard sphere (for \( u(\sigma) = \cos \sigma \)). The general solution is given by elliptic functions : if \( u \) is not constant, a standard device (taking \( u' \) as unknown function of the variable \( u \)) leads to the differential equation

\[
u'^2 = -c - s\frac{u^2}{6} + \frac{\lambda}{u} \quad \text{where} \ \lambda \ \text{is a further constant.}
\]

It cannot give a conformally flat metric : the metric is conformal to

\[
dt^2 + \frac{d\sigma^2}{u(\sigma)^2} + \left(\frac{u'(\sigma)}{u(\sigma)}\right)^2 dx^2.
\]

Such a metric is conformally flat if and only if the 2–dimensional factor has constant curvature. An alternative way to prove that these examples are not conformally flat is to check that \( \nabla f \) is not an eigenvector of the Ricci tensor, since it can be proved that this last property forces conformal flatness.

\textbf{Final remark.} If we allow the scalar curvature \( s \) to be negative, suitable choices of \( \lambda \) and \( c \) yield non conformally flat complete solutions of (S).

\textbf{References}


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