Semi-classical Limit for Random Walks

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Semi-classical Limit for Random Walks¹

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Abstract

Let (G, μ) be a discrete symmetric random walk on a compact Lie group G with step distribution μ and let T_{μ} be the associated transition operator on $L^{2}(G)$. The irreducibles V_{ρ} of the left regular representation of G on $L^{2}(G)$ are finite dimensional invariant subspaces for T_{μ} and the spectrum of T_{μ} is the union of the sub-spectra $\sigma(T_{\mu}'|_{V_{\rho}})$ on the irreducibles, which consist of real eigenvalues $\{\lambda_{\rho 1}, ..., \lambda_{\rho dim V_{\rho}}\}$. Our main result is an asymptotic expansion for the spectral measures

$$m^{\mu}_{
ho}(\lambda) := rac{1}{dim V_{
ho}} \sum_{j=1}^{dim V_{
ho}} \delta(\lambda - \lambda_{
ho j})$$

along rays of representations in a positive Weyl chamber \mathbf{t}_{+}^{*} , i.e. for sequences of representations $k\rho$, $k \in \mathbb{N}$ with $k \to \infty$. As a corollary we obtain some estimates on the spectral radius of the random walk. We also analyse the fine structure of the spectrum for certain random walks on U(n) (for which T_{μ} is essentially a direct sum of Harper operators).

0 Introduction

The spectral theory of a random walk (G, μ) on a compact Lie group G is concerned with the spectrum of the Markov (or transition) operator

$$T_{\mu} : L^{2}(G, d\mu_{0}) \to L^{2}(G, d\mu_{0})$$
$$T_{\mu}(f) = \mu \star f$$

where $d\mu_0$ is Haar measure on G and where μ is a positive measure of mass 1. If μ is symmetric, i.e., $d\mu(g) = d\mu(g^{-1})$ for all $g \in G$, the operator T_{μ} is self-adjoint on $L^2(G, d\mu_0)$. Moreover, if

$$L^2(G, d\mu_0) = \bigoplus_{
ho \in \hat{G}} V_{
ho} \otimes V_{
ho}^* \qquad (\hat{G} = ext{ unitary dual})$$

is the decomposition into irreducibles of the left regular representation of G on $L^2(G, d\mu_0)$, then

$$T_{\mu}: V_{\rho} \to V_{\rho} \quad \text{for all } \rho \in \hat{G},$$

that is, all irreducibles are invariant subspaces for T_{μ} . It follows that $\sigma(T_{\mu})$, the spectrum of T_{μ} , has the form

(0.1)
$$\sigma(T_{\mu}) = \bigcup_{\rho \in \hat{G}} \sigma(T_{\mu}'|_{V_{\rho}})$$

where $T_{\mu} |_{V_{\alpha}}$ is the restriction of T_{μ} to V_{ρ} . Equivalently, we have

(0.2)
$$\sigma(T_{\mu}) = \bigcup_{\rho \in \hat{G}} \sigma(\hat{\mu}(\rho))$$

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where

(0.3)
$$\hat{\mu}(\rho) = \int_{G} \rho(g) d\mu(g)$$

is the Fourier transform of μ at ρ . The decomposition (0.2) shows that $\sigma(T_{\mu}) \subset [-1, 1]$ consists entirely of eigenvalues. The main purpose of this paper is to exploit it further to determine some fine structure properties of the spectrum.

To state the problem more precisely, we first recall that a unitary irreducible representation ρ of G is uniquely determined by its highest weight, which for convenience we will also denote by ρ . Hence the unitary dual is parametrized

$$G = I^* \cap \mathbf{t}_{+}^*$$

by the integral points I^* in a positive Weyl chamber of the dual of a Cartan subalgebra t. For each $\rho \in I^* \cap \mathbf{t}^*_+$ we associate to $\sigma(\hat{\mu}(\rho)) = \{\lambda_{\rho i} : i = 1, 2, \dots, \dim V_{\rho}\}$ the probability measure

(0.4)
$$m_{\rho}^{\mu}(\lambda) := \frac{1}{\dim V_{\rho}} \sum_{j=1}^{\dim V_{\rho}} \delta(\lambda - \lambda_{\rho j})$$

in \mathbb{R} . Our interest is in determining the asymptotic properties of the measures m_{ρ}^{μ} as $|\rho| \to \infty$ along rays in \hat{G} , i.e., for sequences of representations $k\rho$, $k \in \mathbb{N}$ with $k \to \infty$. This generalizes the analysis of asymptotics of Fourier coefficients of measures on the circle group S^1 . Since the limit along rays has many (well-known) analogies to the semi-classical limit in quantum mechanics, we view the problem as that of determining the semi-classical limit of the random walk.

Asymptotic expansions for $m_{k\rho}^{\mu}$ cannot be expected, and indeed do not exist, for an arbitrary measure μ on G. For instance, in the case of S^1 there are just two rays in $\hat{S}^1 \simeq \mathbb{Z}$, \mathbb{Z}^+ and \mathbb{Z}^- , and $m_{\pm k}^{\mu}(\lambda) = \delta(\lambda - \hat{\mu}(\pm k))$. Under the symmetry condition, $\hat{\mu}(k) = \hat{\mu}(-k)$, so we have

$$\mu = 1 + 2\sum_{k \ge 1} \hat{\mu}(k) \cos k\theta$$

and any asymptotic expansion

$$\hat{\mu}(k) \sim a_0 k^m + a_{-1} k^{m-1} + \dots \qquad (k \to \infty)$$

of the Fourier coefficients $\hat{\mu}(k)$ would immediately entail an asymptotic expansion (in the sense of singularities)

$$\mu \sim 1 + a_0 \Re \chi_m + a_{-1} \Re \chi_{m-1} + \dots$$

of μ in terms of the homogeneous distributions

$$\chi_r(\theta) = \sum_{k=0}^{\infty} k^r e^{ik\theta}$$

on S^1 ($\Re \chi_r$ denotes the real part of χ_r). Such μ are clearly of a restricted class. However, this class is a very natural one which generalizes to any group – it is the class of "Lagrangean" measures of modern Fourier analysis (aka the theory of Fourier integral operators, see [Hö I–IV][D]). For such measures, asymptotic expansions similar to the one above *can* be expected.

Many random walks (G, μ) of interest in probability theory fit into this framework, for example, discrete random walks (i.e., with supp $\mu \subset G$ discrete) or random walks with μ supported on a conjugacy class (as studied in [R][P]). For such random walks it is possible to apply the methods of microlocal analysis to study the spectrum and smoothing properties of T_{μ} . In this paper we concentrate on discrete random walks, i.e. where μ is a discrete measure

$$\mu = \frac{1}{m_{\mu}} \sum_{i=1}^{n} c_i (\delta_{a_i} + \delta_{a_i^{-1}})$$

with $m_{\mu} = 2 \sum_{i=1}^{n} c_i$. In the sequel [P.Z] we extend the methods and results to certain continuous Lagrangean random walks, such as δ -functions on positively curved hypersurfaces or on sums of conjugacy classes. Discrete random walks are much simpler technically than continuous random walks and hopefully provide a clean introduction to the microlocalization of random walks to rays of representations.

Let us explain this notion in more detail before describing our methods and results. The key idea is to simplify the study of a random walk on a compact Lie group by localizing it to a ray in the cotangent bundle T^*G or in the unitary dual \hat{G} . The motivation for such a localization is that the Fourier transform $\hat{\mu}(\rho)$ can behave differently in different directions as $\rho \to \infty$ and one would like to isolate the behaviour in each direction. Each ray corresponds to a different co-adjoint orbit, and the localization to a ray restricts the dynamics of the random walk to this orbit. This kind of behaviour is well-known in the Fourier analysis of surface-carrying measures on \mathbb{R}^n (see e.g. [Stein, VIII: §5]) and arises for essentially the same reasons on any Lie group. It does not seem to have been studied before in the setting of non-abelian Lie groups (see however the paper of Taylor-Uribe [T.U] for a microlocal analysis of certain central measures).

Thus, our goal is to microlocalize a discrete random walk T_{μ} to a ray of representations $\mathbb{N}\rho$ and to determine the asymptotic behaviour of its Fourier transform as encoded by the spectral measures $m_{r\rho}^{\mu}$. Since this only involves the L^2 theory of T_{μ} we are free to conjugate to a realization of the ray $\mathbb{N}\rho$ best adapated to the problem of determining the semi-classical limit of the spectral measures $m_{r\rho}^{\mu}$. The relevant construction is the homogeneous generalization of the Kirillov-Kostant orbit theory due to Guillemin-Sternberg [G.S].

Recall that in the classical Borel-Weil-Bott- Kirillov-Kostant theory, an irreducible representation (ρ, V_{ρ}) corresponds to the classical "phase space" \mathcal{O}_{ρ} , the coadjoint orbit of ρ in g^* . In the homogeneous quantization theory of Guillemin-Sternberg [G.S-1], the entire ray $\mathbb{N}\rho$ corresponds to the Hardy subspace $H^2(B_{\rho}) \subset L^2(B_{\rho})$ where B_{ρ} is the so-called "prequantum" S^1 -bundle over \mathcal{O}_{ρ} . We defer the precise definitions of these objects to §2. For the time being, the main point is that B_{ρ} is a homogeneous space of G, and that $H^2(B_{\rho}) \equiv \bigoplus_{r=0}^{\infty} V_{r\rho}$. Thus, we can determine the semi-classical asymptotics of $m_{r\rho}^{\mu}$ along the ray by studying the concrete random walk $(G, \mu, H^2(B_{\rho}))$. The spectrum of T_{μ} on any homogeneous space, in particular on G, can then be synthesized from these special, but fundamental, cases. Concrete examples of such rays of representations will be presented in §4, along with rays such as $L^2(S^n)$ for G = SO(n+1) which do not arise from this construction.

Let us now state our results. In the following, we identify irreducibles with their highest weights ρ , which are lattice points in a positive Weyl chamber \mathbf{t}_{+}^{*} . The notation interior (\mathbf{t}_{+}^{*}) stands for the interior of the Weyl chamber. There is a slight distinction in the results between interior weights ρ and weights lying on the walls of the Weyl chamber (which we will refer to as 'wall weights.') For the sake of brevity, we concentrate on the interior case and do not describe the possible behaviours of wall weights. However, we should point out that the results for wall weights are often identical to the results for interior weights. For instance, in the case of G = U(n) only scalar weights behave differently from interior weights (see the Remark in §3).

Main Theorem Let $\mu = \frac{1}{m_{\mu}} \sum_{i=1}^{n} c_i (\delta_{a_i} + \delta_{a_i^{-1}})$ be a discrete probability measure on a compact Lie group G. Then, for all $\rho \in I^* \cap \mathbf{t}^*_+$, the asymptotics of $m^{\mu}_{r\rho}$ is given as follows: (a) Interior case: If $\rho \in \text{interior}(\mathbf{t}^*_+)$, then

(0.5)
$$dm_{r\rho}^{\mu} \sim \sum_{z \in Z_G \cap \Gamma} \chi_{\rho}(z)^r dm^{\mu, z} + O(r^{-1})$$

where:

(i) Z_G denotes the center of G; (ii) $\Gamma = \langle a_i \rangle \subset G$ denotes the subgroup generated by $\{a_i\}$; (iii) χ_{ρ} denotes the highest weight character of ρ ; (iv)(e) $dm^{\mu,e}$ is the positive measure on [-1, 1] determined by the moments

$$(0.6.e) \qquad \qquad \int_{\mathbf{R}} x^p dm^{\mu,e} = \mu^{*p}(e) ;$$

 $(iv)(z) dm^{\mu,z}$ is a measure on [-1, 1] with the moments

(0.6.z)
$$\int_{\mathbb{R}} x^p dm^{\mu,z} = \mu^{*p}(\{z\});$$

(v) the remainder $O(r^{-1})$ holds pointwise for each $f \in C^{\infty}(\mathbb{R})$; for each $f \in C(\mathbb{R})$ the remainder is o(1).

(b) Wall case: Suppose ρ is a wall weight, and let (vi) $Z_{\rho} := \bigcap_{g \in G} g^{-1} G_{\rho} g$, where G_{ρ} is the stabilizer group of ρ . Then:

(0.7)
$$dm_{r\rho}^{\mu} \sim \sum_{z \in Z_{\rho} \cap \Gamma} \chi_{\rho}(z)^{r} dm^{\mu, z} + O(r^{-1})$$

with ii-v unchanged.

Above, the sums in (0.5), resp. (0.7) should be understood in the following sense: for each k, the kth moment of the measure on the left side equals the kth moment of the formal sum of measure on the right side. Note that for each k, only a finite number of measures $m^{\mu,z}$ have non-zero kth moment. The fact that $m^{\mu,z}$ is a measure for $z \neq e$ will be explained in Remark (3) below.

Let us now derive a number of consequences from the Main Theorem.

First, consider the asymptote $\sum_{z \in Z_G \cap \Gamma} \chi_{\rho}(z)^r dm^{\mu,z}$ in the interior case. It is natural to lump together the terms with a fixed value of $\chi_{\rho}(z)$. We therefore set

(0.8.1)
$$\Theta_{\rho} := \{ \theta \in [0, 2\pi) : \exists z \in Z_G \cap \Gamma, \chi_{\rho}(z) = e^{i\theta} \}$$

which is evidently a countable subset of angles $\{\theta_j\}$. For each $\theta \in \Theta_{\rho}$ we also put

$$(0.8.2) Z_{\rho,\theta} := \{ z \in Z_G \cap \Gamma : \chi_{\rho}(z) = e^{i\theta} \}.$$

Then the asymptote can be re-written in the form:

(0.9.1)
$$\sum_{z \in Z_G \cap \Gamma} \chi_{\rho}(z)^r dm^{\mu, z} = \sum_{\theta_j \in \Theta_{\rho}} e^{ir\theta_j} dm^{\mu}_{\theta_j}$$

with

$$(0.9.2) dm^{\mu}_{\theta_j} = \sum_{z \in Z_{\rho,\theta_j}} dm^{\mu,z}.$$

As is typical in the world of trace formulae, the terms in (0.9.2) can possibly cancel each other out. This certainly does not happen in the case of $\theta = 0$ since we have

(0.10.*o*)
$$dm_o^{\mu} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^N dm_{n\rho}^{\mu}$$

i.e the limit spectral measure with $\theta = 0$ is the Cesaro mean of the spectral measures along the ray. Since it is an average of positive measures it must be positive. However, the other limit spectral measures are given by

(0.10.j)
$$dm^{\mu}_{\theta_{j}} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} e^{-in\theta_{j}} dm^{\mu}_{n\rho}$$

and are consequently signed averages. There is no obvious reason why they cannot cancel out. To take this possibility into account, we henceforth write

(0.11)
$$\Theta_{\rho}^* := \{ \theta \in \Theta_{\rho} : dm_{\theta}^{\mu} \neq 0. \}$$

It is evident from the Main Theorem and from (0.10.j) that the angles $\theta_j \in \Theta_{\rho}^*$ and the measures $dm_{\theta_j}^{\mu}$ are spectral invariants of the random walk T_{μ} along the ladder $\mathbb{N}\rho$. They are analogous to the so-called principal *band invariants* of a self-adjoint Toeplitz operator on contact manifold with periodic characteristic flow, or of the Laplacian on a Zoll manifold (cf. [BdM.G, §13] for background and references). However, the Markov operator T_{μ} is a kind of Fourier Integral Toeplitz operator associated to a Markov chain (the random walk), whereas the previously studied Toeplitz or pseudodifferential operators have no dynamical aspect (they are associated to the identity map). Hence the analogy only goes so far. In particular, we have not one band invariant, but a whole family $dm_{\theta_j}^{\mu}$ of band invariants. To give them a name, we will refer to them as the *limit spectral measures* of the random walk on the ray. The name is justified by the following:

Corollary 1 In the notation of the Main Theorem, suppose that ρ is an interior weight. Then:

$$\bigcup_{\theta_j \in \Theta_{\rho}} supp(dm_{\theta_j}^{\mu}) \subset \overline{\sigma(T_{\mu}'|_{\mathbb{N}^{\rho}})}$$

 $(\overline{S} \text{ denotes the closure of the set } S).$

The proof is just to observe that if $\lambda \notin \overline{\sigma(T_{\mu}|_{\mathbb{N}\rho})}$, then there is an open set $(\lambda - \epsilon, \lambda + \epsilon)$ disjoint from $\overline{\sigma(T_{\mu}|_{\mathbb{N}\rho})}$ and hence all the band spectral measures $dm_{r\rho}^{\mu}$ annihilate the functions in $C_{c}(\lambda - \epsilon, \lambda + \epsilon)$. Hence so must each limit measure $dm_{\theta_{j}}^{\mu}$. The result in the wall case is analogous and is left to the reader to formulate.

We note that the converse \subset relation is a very difficult problem, since a priori the limit spectral measures only capture points of density in the spectrum of the random walk. Whether or not all points of $\sigma(T_{\mu}'|_{\mathbb{N}\rho})$ are points of density is a question that requires a detailed study in each special case.

Two extreme cases occur when $\chi_{\rho}(z) \equiv 1$ and when $\chi_{\rho}(z) \neq 1$ for $z \neq e$. In the first case we obviously have:

Corollary 2 If $\chi_{\rho}(z) = 1$ for all $z \in Z_G \cap supp \mu_p$ (for example, if Z_G is trivial), then, for interior $\rho \neq$, there exists a weak limit

$$m^{\mu}_{ro} \xrightarrow{w^*} m^{\mu}$$

with moments

(0.12)
$$\int_{\mathbb{R}} x^p dm^{\mu} = \sum_{z \in Z_G} \mu^{*p}(\{z\})$$

In the second case, we can deduce results about the spectral radius

$$\beta_{\rho}^* := \sup\{|\lambda| : \lambda \in \sigma(T_{\mu}'|_{\mathbb{IN}\rho})\}$$

of the random walk (G, μ) along the ray $\mathbb{N}\rho$. The key point here is that the limit measures $dm^{z,\mu}$ depend only on the discrete group $\Gamma := \langle a_i^{\pm 1}, i = 1, ..., n \rangle$, on the generating set $A = \{a_i\}$ and on the set Z_G . In particular, the limit measure $dm^{\mu,e}$ corresponding to the identity element e is the Kesten spectral measure of the random walk (Γ, μ) defined by μ on $l^2(\Gamma)$. As is well-known, this spectral measure encodes many properties of the pair (Γ, A) . See [K.V] and [H.R.V] for background. From the known results on the spectrum of (Γ, μ) , we conclude:

Corollary 3 Suppose $\chi_{\rho}(z) \neq 1$ for $z \in Z_G \cap \Gamma \setminus \{id\}$. Then, with the above notation and assumptions: (i) Suppose Γ is an amenable group. Then $\beta_{\rho}^* = 1$.

(ii) Suppose Γ is a free group and $\{a_i, i = 1, ..., n\}$ is a free set of generators. Then $\left[-\frac{\sqrt{2n-1}}{n}, \frac{\sqrt{2n-1}}{n}\right] \subseteq \overline{\sigma(T_{\mu'}|_{\mathbb{N}_{\rho}})}$.

Remarks:

(1) The Main Theorem may be viewed as connecting the spectral theory of the random walk determined by μ along rays of representations of G with the spectral theory of the random walk (Γ, μ) defined by μ on $l^2(\Gamma)$. Essentially, we are looking at different representations of the element $T_{\mu} = \frac{1}{m_{\mu}} \sum c_i a_i \in \mathbb{C}[\Gamma]$. For the random walk (Γ, μ) , we are considering the left regular representation λ of Γ on $l^2(\Gamma)$. Along the ray, we are considering the representation $H^2(B_{\rho})$ of G, restricted to Γ . For simplicity of notation, let us denote this restricted representation of Γ by π_{ρ} . Then in favorable cases, our results allow us to compare the spectra $Sp \lambda(T_{\mu})$ and $Sp \pi_{\rho}(T_{\mu})$.

(2) To enlarge on the comparison, we consider it from the C*-algebraic point of view (cf. [H.R.V]). Given a unitary representation π of Γ on a Hilbert space \mathcal{H} we can take the norm closure of $\mathbb{C}[\Gamma]$ in the algebra $\mathcal{L}(\mathcal{H})$ of bounded operators on \mathcal{H} , to obtain a C* algebra $C_{\pi}^*(\Gamma)$. In the case of the left regular representation, we get the reduced C*-algebra $C_{\lambda}^*(\Gamma)$. In the case of π_{ρ} we get a representation $C_{\pi_{\rho}}^*$.

Consider for instance the spectral measure $dm^{\mu,e}$ from the C^{*} point of view. Its moments may be expressed in the form

$$\int_{\mathbf{R}} x^p dm^{\mu,e} = \langle \lambda(T_\mu)^p \delta_e, \delta_e \rangle$$

The linear functional $\tau(A) = \langle \lambda(A)\delta_e, \delta_e \rangle$ on $\mathbb{C}[\Gamma]$ is continuous with respect to the operator norm of $\lambda(A)$ on $l^2(\Gamma)$ and hence extends to a continuous trace on $C^*_{\lambda}(\Gamma)$. It further extends to a continuous trace on the weak closure $W^*_{\lambda}(\Gamma)$. In terms of the spectral resolution dE_x of $\lambda(T_{\mu})$ we may write

$$m^{\mu,e}(B) = \tau(E_B)$$

for any Borel $B \subset \mathbb{R}$. It is known that the support of this measure $m^{\mu,e}$ equals the spectrum $Sp_{\lambda}(T_{\mu})$ of $\lambda(T_{\mu})$ as an element of $C^{*}_{\lambda}(\Gamma)$ (cf. [H.R.V, Proposition 5].) This fact allows one to relate $Sp_{\lambda}(T_{\mu})$ to the properties of (Γ, A) (loc.cit.)

The fact that $m^{\mu_i e}$ arises in the asymptotics of $m^{\mu}_{r\rho}$ also allows us to relate $Sp_{\pi_{\rho}}(T_{\mu})$ to the properties of (Γ, A, π_{ρ}) . The Corollary above gives one example. The other spectral measures arising in the asymptotics can be used in a similar way, at least when the oscillatory factors $\chi_{\rho}(z)^r$ can be separated out from each other.

(3) We note that the other spectral measures $m^{\mu,z}$ with $z \neq e$ also arise from traces on $C^*_{\lambda}(\Gamma)$. They are defined by $\tau^z(a) := \langle a\delta_e, \delta_z \rangle$. In other words $\tau^z(\lambda(\gamma)) = 1$ if $\gamma z = e$ and is otherwise zero.

The trace condition $\tau^{z}(ab) = \tau^{z}(ba)$ follows easily from the condition $z \in Z_{G}$, and continuity on $C_{\lambda}^{*}(\Gamma)$ is obvious. However, $\tau^{z}(I) = 0$ if $z \neq e$ so the traces are not states on the algebra.

The traces τ^z induce measures on $Sp \lambda(T_\mu)$ as in the case of z = e: namely, we define $m^{\mu,z}$ on $C(Sp \lambda(T_\mu))$ by

$$\int f dm^{\mu,z} := \tau^z (f(\lambda(T_\mu))).$$

Since $||f(\lambda(T_{\mu})|| = ||f||_{\infty}$ (the sup-norm on the set $Spec(\lambda(T_{\mu}))$), it is evident that $m^{\mu,z}$ does actually define a measure on $Sp\lambda(T_{\mu})$.

(4) The traces τ^z originally arise as residue (Dixmier) traces on the algebra of Toeplitz operators on $H^2(B_\rho)$. They are in general not continuous traces on the C^{*} algebra (norm closure) of the Toeplitz algebra. When restricted to $C^*_{\pi_\rho}$ they may be viewed as defining unbounded traces on this algebra.

Examples:

Examples of rays of representations of compact semi-simple Lie groups and of their discrete random walks will be given in §4. In particular, we will analyze in detail the spectrum of a specific discrete random walk on the unitary group (the pseudo-Harper operator) by means of the spectral measures obtained in the Main Theorem.

For the time being, let us consider some well-known examples from the literature on the spectral theory of discrete random walks on Lie groups.

(Ex.1) A special case of a discrete random walk on a ray of representations is given by the Hecke operators on $L^2(S^2)$ studied by Lubotzky, Phillips, and Sarnak [L.P.S] (see also [CV]) In this example, the group is G = SO(3) and $L^2(S^2)$ consists of just one ray. The random walk has transition operator $T = \frac{1}{6} \sum T_{a_i} + T_{a_i^{-1}}$ where the a_i are rotations around three perpendicular axes and with angle α given by $\cos \alpha = -\frac{3}{5}$. The group Γ is a free group, and the rate of equidistribution of orbits of the random walk is optimal. Since G has trivial center, only the measure corresponding to z = e shows up in limit spectral measure, i.e. the limit spectral measure of this random walk is the Kesten spectral measure for the left regular representation.

Our methods could also be extended to non-compact Lie groups such as $G = SL(2, \mathbb{R})$, and would then apply to the (true) Hecke operators T_p , acting in the discrete or continuous ray (series) of representations of the quotient $G/SL(2, \mathbb{Z})$. The results are just as in the previous example.

(Ex.2) A well-known example of a discrete random walk on the three- dimensional reduced Heisenberg group $N_{\mathbb{R}}$ is given by the measure $\frac{1}{4}[\delta_U + \delta_U^* + \delta_V + \delta_V^*]$, where U, V are elements satisfying $UV = e^{i\theta}VU$. The corresponding transition operator is the Harper operator $H_{\theta} = \frac{1}{4}[U + U^* + V + V^*]$. It is much-studied as a model for a periodic Schrodinger operator in a magnetic field; see [B][B.V.Z] [C.E.Y][Su] for a variety of results and points of view.

As is well-known, the spectrum has a very interesting dependence on θ : for rational values $\theta = \frac{p}{q}$, the spectrum is a union of q + 1 intervals in [-1, 1] with q gaps between them; when θ is irrational, the spectrum is often (and conjecturally always) a Cantor set. These facts can be proved by studying the C^* -algebra A_{θ} generated by these unitaries (the so-called rotation algebra, or non-commutative torus). See the articles cited above.

The Harper operator could be studied from the point of view of this paper as a discrete random walk on a Lie group (namely, $N_{\rm I\!R}$). However, our concern will be with a compact analogue on the unitary group U(q). Namely, for $\theta = \frac{p}{q}$ with (p,q) = 1, the algebra A_{θ} has an irreducible representation π taking U to the diagonal matrix $u := \text{diag}(1, e^{i\theta}, \ldots, e^{(q-1)\theta})$ and V into the cyclic permuation operator v on the standard basis $e_2 \to e_1, \ldots, e_n \to e_{n-1}, e_1 \to e_n$. These matrices are unitary, hence $\pi(H_{\theta})$ is the transition operator of a random walk

$$\mu_{\theta} := \frac{1}{4} (\delta_u + \delta_{u^*} + \delta_v + \delta_{v^*})$$

on U(q). Given a ray of representations $\mathbb{N}\rho$ of U(q) we then get an example of a random walk $(U(q), \mu_{\theta}, H^2(B_{\rho}))$ to which the Main Theorem applies.

The relation between the spectrum of the random walk μ_{θ} and the spectrum of the Harper operator is not so immediate, however. The problem is that along the ray, u, resp. v, act by the translation operators T_u , resp. T_v , on $H^2(B_{\rho})$. The commutation relation between T_u and T_v along the ray is then given by

$$T_u T_v = T_z T_v T_u$$

where T_z is translation by $z = e^{i\theta}$. Evidently T_z is not multiplication by a scalar so the algebra $\langle T_u, T_v \rangle$ generated by T_u, T_v is not the rotation algebra.

Nevertheless, as will be explained in §4, the spectrum of T_{μ} along a ray can be related to the spectra of a finite number of Harper operators. Hence the analogy to the rational Harper operator is not too far fetched. From this example, we see that the spectrum of discrete random walks along rays of representations can have interesting multiple band-gap structures.

This article is intended for readers with an interest in random walks but with little or no background in microlocal analysis. For this reason, we have included rather lengthy expository sections summarizing the relevant material from homogeneous quantization theory and Fourier integral operators (Sections 1–2). Much of the proof of the theorem above will be given, as examples and illustrations, in the course of the exposition, so it should be regarded as an integral portion of the text. The microlocalization of a random walk to a ray of representations actually requires the theory of Toeplitz operators of [BdM.G]. We will be building on this material as well in the sequel.

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1 Review of Microlocal analysis

The purpose of this section is to present the microlocal machinery that we will need for the proof of the main theorem. IIMoreover, much of the proof will actually be carried out in the course of introducing the basic notions. Let us begin with some heuristic background.

Microlocal analysis is, among other things, a rigorous version of quantization theory and provides a table of correspondences between notions of classical and quantum mechanics. For the purposes of this paper, classical mechanics means the study of the dynamics of canonical transformations f and Hamiltonian group actions T_g on symplectic manifolds (M, ω) . Quantum mechanics means the study of the spectral theory of their quantizations as unitary Fourier Integral operators U_f , resp. U_g , on an associated Hilbert space.

Specifically, the classical mechanics will in part consist of the (Hamiltonian) action of G by conjugation on the co-adjoint orbits \mathcal{O}_{ρ} and on certain symplectic cones Y encompassing all the orbits in the ray through ρ . But more interestingly it will consist in the dynamics of the random walk itself – i.e. the discrete dynamical system generated by the Markov operator T_{μ} – on these symplectic manifolds. The quantum mechanics will consist in part of the representation theory of G on the Hilbert space corresponding to \mathcal{O}_{ρ} , namely V_{ρ} and on the Hilbert space $H^2(B)$ corresponding to Y, the ray of representations $\bigoplus_{r=1}^{\infty} V_{r\rho}$ thru ρ . But the focus is on the spectrum of the Markov operator T_{μ} on the Hilbert spaces V_{ρ} and $H^2(B)$.

1.1 Symplectic manifolds.

Throughout the paper, we assume familiarity with basic notions from differential geometry. For background see, for example, [N]. Let M be a compact connected manifold of dimension n. M is called *symplectic* if M possesses a symplectic form ω , i.e., a nondegenerate closed ($d\omega = 0$) differential 2-form. Note that the

non-degeneracy of ω implies that n be even. For each $x \in M$, the tangent space $T_x M$ at x is isomorphic to the cotangent space T_x^*M via the map

$$v_x \to \iota(v_x)\omega_x$$

where ι denotes the usual insertion map (here, $\iota(v_x)\omega_x = \omega_x(v_x, \cdot)$).

We now assume there exists a group action $G \times M \to M$, $(g, x) \to g \cdot x$ of the Lie group G on the symplectic manifold (M, ω) . For each $(g, x) \in G \times N$, the differential of the group action induces a linear map $T_g G \times T_x N \to T_{g \cdot x} M$. Take in particular g = e, the identity element of G. Then for each $X \in \mathbf{g} = T_e G$, the Lie algebra of G, we get a vector field $X^{\#}$ on M (the image of (X, 0) under this map). We call the group action symplectic if it preserves the symplectic structure, i.e., if

$$D_{X^{\#}}\omega = 0 \qquad (\Leftrightarrow d(\iota(X^{\#})\omega = 0)).$$

If for all $X \in \mathbf{g}$ the form $(\iota(X^{\#})\omega)$ is *exact*, i.e., there exists a smooth function ϕ_X on M such that $\iota(X^{\#})\omega = d\phi_X$ (the $X^{\#}$'s are symplectic gradient vector fields), we call the group action Hamiltonian. The corresponding moment map $\Phi: M \to \mathbf{g}^*$ (where \mathbf{g}^* is the dual of the Lie algebra \mathbf{g}) is defined by

$$\langle X, \Phi(x) \rangle = \phi_X(x)$$
.

 $(\langle X, \xi \rangle$ with $\xi \in \mathbf{g}^*$ and $X \in \mathbf{g}$ denotes the natural pairing $\xi(X)$.) The following two examples will be crucial.

Example 1. The cotangent bundle of a manifold. Let M be a smooth manifold, T^*M its cotangent bundle, and π the natural projection $\pi: T^*M \to M$. T^*M carries a natural 1-form, α , given by

$$\alpha_{(x,\xi)} = \xi \circ d\pi_{(x,\xi)} \qquad (\Leftrightarrow \alpha_{(x,\xi)} = \pi^*(x,\xi)) .$$

Here (x,ξ) are the local coordinates of a point in T_x^*M (short for $(x_1,\ldots,x_n,\xi_1,\ldots,\xi_n)$). In these local coordinates, α can be written as $\alpha = \sum_{i=1}^n \xi_1 dx_1$. The canonical 2-form on T_x^*M is $\omega = d\alpha$ (= $\sum_{i=1}^n d\xi_i \wedge dx_i$).

Example 2. Coadjoint orbits. Let G be a Lie group, g its Lie algebra (which can be identified with the tangent space T_eG at the identity $e \in G$), and g^* the dual of g. Recall that G acts on g via the adjoint representation Ad:

$$g \cdot X = \operatorname{Ad}(g)X := d(c_g)_e X$$
 for all $X \in \mathbf{g}, g \in G$

where $c_g: G \to G$ is the conjugation map $c_g(h) = ghg^{-1}$ and $d(c_g)$ denotes its differential. Hence by duality we get the coadjoint action Ad^{*} of G on g^{*}:

$$\langle X, \mathrm{Ad}^*(g)\xi \rangle = \langle \mathrm{Ad}(g^{-1})X, \xi \rangle$$

for all $g \in G$, $X \in \mathbf{g}$, and $\xi \in \mathbf{g}^*$. For any $\xi \in \mathbf{g}^*$, the corresponding coadjoint orbit $\mathcal{O}_{\xi} = \{ \operatorname{Ad}^*(g)\xi : g \in G \}$ is a symplectic manifold and the (coadjoint) action of G on \mathcal{O}_{ξ} is Hamiltonian. The symplectic structure is given by ω with

$$\omega_{\eta}(X_{\eta}^{\#}, Y_{\eta}^{\#}) = \langle [X, Y], \eta \rangle \quad \text{for all } \eta \in \mathcal{O}_{\xi}$$

and the moment map $\Phi: \mathcal{O}_{\xi} \to \mathbf{g}^*$ is the natural inclusion map *i*. See [Ki] for details and proofs.

Let (M, ω) and (M', ω') be two symplectic manifolds and $f : M \to M'$ a diffeomorphism. f is called a symplectomorphism or a canonical transformation if $f^*\omega' = \omega$.

Example 3. The lift of a diffeomorphism. Let X, X' be two manifolds and $f: X \to X'$ a diffeomorphism. We can lift f to a diffeomorphism $F: T^*X \to T^*X'$ of the cotangent bundles by defining $F(x,\xi) = (f(x), (f^{-1})^*\xi)$. This map F is a canonical transformation with respect to the canonical 2-forms ω and ω' on T^*X and T^*X' , respectively.

1.2 Lagrangean submanifolds

Let $\Lambda \subset M$ be a smooth submanifold of the symplectic manifold (M, ω) . For each $x \in \Lambda$, the tangent space $T_x \Lambda$ is a subspace of $T_x M$. We denote the orthogonal complement of $T_x \Lambda$ in $T_x M$ with respect to ω_x by $T_x^-\Lambda$. Λ is called isotropic if for all $x \in \Lambda$, $T_x \Lambda \subseteq T_x^-\Lambda$, i.e., $\omega'|_{\Lambda} = 0$ $(\omega'|_{\Lambda} = i^*\omega$ where $i : \Lambda \to M$ is the natural inclusion map). If Λ is maximal isotropic, i.e., if $T_x \Lambda = T_x^-\Lambda$ for all $x \in \Lambda$, we call Λ Lagrangean. An isotropic subspace Λ is Lagrangean if and only if dim $\Lambda = \frac{1}{2} \dim M$.

Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. Clearly, $M'_2 = (M_2, -\omega_2)$, $M = (M_1 \times M_2, \omega_1 + \omega_2)$, and $M' = (M_1 \times M_2, \omega_1 - \omega_2)$ are also a symplectic manifolds. We call a Lagrangean submanifold $\Lambda' \subset M'$ a canonical relation.

Example 4. Graphs of canonical transformations. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds and $f : M_1 \to M_2$ a canonical transformation. Since $f^*\omega_2 = \omega_1$, $\omega_1 - \omega_2$ vanishes on Graph $(f) = \{(m_1, f(m_1)) | m_1 \in M_1\} \cong M_1$. Furthermore, dim Graph $(f) = \dim M_1 = \frac{1}{2}\dim (M_1 \times M_2)$. Thus Graph (f) is a Lagrangean submanifold of M'.

Of particular relevance in this article are the homogeneous canonical transformations on cotangent bundles: that is, canonical transformations $\chi : T^*X \to T^*Y$ satisfying $\chi(x, r\xi) = r\chi(x, \xi)$ for r > 0. The simplest examples are lifts F to the cotangent bundle T^*X of diffeomorphisms $f : X \to X$. The graph Graph χ of any homogeneous canonical transformation is naturally diffeomorphic to T^*X under either of the projections

(1.2.1)
$$\begin{array}{ccc} \Lambda \subset T^*(X \times X) \\ \pi_1 \swarrow & \searrow & \pi_2 \\ T^*X & & & T^*X \end{array}$$

Example 5. The conormal bundle of a submanifold. Let X be a manifold and $Y \subset X$ a smooth submanifold. The conormal bundle $N^*Y \subset T^*X$ is defined as the set

$$N^*Y = \{(y,\xi) | y \in Y, \xi(v) = 0 \text{ for all } v \in T_yY\}$$

We can define local coordinates x_1, x_2, \ldots, x_n in X such that Y is given by $x_1 = x_2 = \ldots = x_k = 0$. Thus, with respect to the coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ in T^*X , N^*Y is defined by $x_1 = \cdots x_k = \xi_{k+1}, \cdots, \xi_n = 0$ and of dimension n. It is clear that the natural 1-form α (see Example 1) is 0 on $T(N^*Y)$, therefore also the canonical 2-form $\omega = d\alpha$, and N^*Y is a Lagrangean submanifold of T^*X .

A manifold M is called *conic*, if there exists a free \mathbb{R}^+ action on M. If M_1, M_2 and Λ' are all conic, we call Λ' a *homogeneous canonical relation*. Here, conic and homogeneous are synonyms. In this context, homogeneous has nothing to do with "homogeneous space".

Two canonical relations $\Lambda'_1 \subset M_1 \times M_2$ and $\Lambda'_2 \subset M_2 \times M_3$ can be composed to

$$\Lambda' = \Lambda'_1 \circ \Lambda'_2 = \{(\gamma_1, \gamma_3) | (\gamma_1, \gamma_2) \in \Lambda_1 \text{ and } (\gamma_2, \gamma_3) \in \Lambda_2 \text{ for some } \gamma_2 \}$$

Equivalently, Λ' is the image of $\Lambda'_1 \times \Lambda'_2 \cap M_1 \times \Delta(M_2) \times M_3$ under the projection

$$\pi: M_1 \times M_2 \times M_2 \times M_3 \to M_1 \times M_3$$
.

(Here $\Delta(M_2)$ denotes the diagonal $\{(\gamma_2, \gamma_2) | \gamma_2 \in M_2\}$.) In general, $\Lambda' = \Lambda'_1 \circ \Lambda'_2$ is not a canonical relation. However, a *clean intersection* hypothesis gives a sufficient condition. Before we state this result, let us record the following definition. **Definition 1.2.1** Let X, Y, Z be smooth manifolds and $f : X \to Z$ and $g : Y \to Z$ two maps. The fiber product

(1.2.2)
$$F = \{(x, y) \in X \times Y | f(x) = g(y)\}$$

can be represented by the diagram

We say the two maps f and g intersect *cleanly* if

1) F is a submanifold of $X \times Y$ and

2) for each $p \in F$, p = (x, y), the tangent space $T_p F$ is the fiber product of the tangent spaces $T_x X$ and $T_y Y$, i.e., the diagram

$$(1.2.4) \begin{array}{cccc} T_p F & \longrightarrow & T_x X \\ \downarrow & & \downarrow & df_x \\ T_y Y & \longrightarrow & T_z Z \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ \end{array}$$

is a fiber diagram. The number $e = \dim F + \dim Z - (\dim X + \dim Y)$ is called the *excess* of the intersection. If e = 0, the intersection is *transversal*.

For the following theorem see, for example, [D.G], Lemma 5.3. **Theorem 1.2.2** Let $\Lambda'_1 \subset M_1 \times M_2$ and $\Lambda'_2 \subset M_2 \times M_3$ be two canonical relations. If the fiber product diagram

$$\begin{array}{cccc} F & \longrightarrow & \Lambda_1' \\ \downarrow & & \downarrow \\ \Lambda_2' & \longrightarrow & M_2 \end{array}$$

is clean, then $\Lambda' := \Lambda'_1 \circ \Lambda'_2$ is an immersed canonical relation in $M_1 \times M_3$ and the map

$$\alpha: F \to \Lambda'$$

is a fiber mapping with compact fiber.

1.3 Lagrangean distributions and FIO's.

These are distributions which can be written as certain types of oscillatory integrals $\int ae^{i\phi}$. We begin by describing the assumptions on the amplitude *a* and the phase ϕ . Let *M* be an *n*-dimensional smooth manifold and $X \subset M$ an open set.

Definition 1.3.1 A function $\phi \in C^{\infty}(X \times (\mathbb{R} \setminus \{0\})^N)$ is called a *phase function* if

1)
$$\mathrm{Im}\phi \geq 0$$

2)
$$\phi(x, r\theta) = r\phi(x, \theta)$$
 for all $r \in \mathbb{R}^+$.

3)
$$d\phi \neq 0$$
.

The phase function ϕ is called *non-degenerate* if from $d_{\theta}\phi(x,\theta) = 0$, $(x,\theta) \in X \times (\mathbb{R} \setminus \{0\})^N$, it follows that $d_{(x,\theta)} \frac{\partial \phi}{\partial \theta_j}(x,\theta)$ are linearly independent for $j = 1, \ldots, N$.

It follows from the implicit function theorem that

(1.3.1)
$$C_{\phi} = \{(x,\theta) | d_{\theta}(x,\theta) = 0\},\$$

called the *critical set of* ϕ , is a conic *n*-dimensional smooth submanifold of $X \times (\mathbb{R} \setminus \{0\})^N$. Its image under the map

(1.3.2)
$$I_{\phi}: C_{\phi} \to T^*(X), \quad I_{\phi}(x, \theta) = (x, d_x \phi(x, \theta))$$

is an immersed, conic, Lagrangean submanifold, Λ_{ϕ} , of $T^*(X)$ (see [D], Section 2.3). It can be shown (see [Sj-G], Prop. 11.4) that vice versa, every closed conic Lagrangean submanifold $\Lambda \subset T^*(X) \setminus \{0\}$, can locally be parametrized in this way. More precisely, for every $(x_0, \theta_0) \in \Lambda$ there exists a non-degenerate phase function ϕ such that $\Lambda = \Lambda_{\phi}$ in a neighbourhood of (x_0, θ_0) .

Definition 1.3.2 Let *m* be a real number. The set $S^m(X \times \mathbb{R}^N)$, called the space of symbols of order *m*, is defined as the space of all $a \in C^{\infty}(X \times \mathbb{R}^N)$ with the property that for all compact $K \subset X$ and all $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^N$, there exists a constant $C_{K,\alpha,\beta}(a)$ such that

$$\left|\partial_x^{\alpha}\partial_{\theta}^{\beta}a(x,\theta)\right| \le C(1+|\theta|)^{m-|\beta|}$$

for all $(x, \theta) \in X \times \mathbb{R}^N$. A symbol $a \in S^m(X \times \mathbb{R}^N)$ has an asymptotic expansion

(1.3.3)
$$a(x,\theta) \sim \sum_{j=0}^{\infty} a_j(x,\theta) \qquad (\theta \to \infty)$$

with $a_j \in S^{m-j}$. The space of *polyhomogeneous* or *classical* symbols of degree m, S^m_{phg} , is the set of all symbols a such that

(1.3.4)
$$a(x,\theta) \sim \sum_{j=0}^{\infty} a_j(x,\theta)$$

where a_j is homogeneous of degree m - j, i.e., $a(x, r\theta) = r^{m-j}a(x, \theta)$ for r > 0.

For a fixed phase function ϕ and a fixed symbol $a \in S^m(X \times \mathbb{R}^N)$, the oscillatory integral $I_{\phi}(a)$ given by

(1.3.5)
$$I_{\phi}(a) = \int e^{i\phi(x,\theta)} a(x,\theta) d\theta$$

defines a distribution, i.e., an element in $\mathcal{D}'(X \times \mathbb{R}^N)$. Such a $u = I_{\phi}(a)$ has its singularities (its "Wave front set") in Λ_{ϕ} .

Definition 1.3.3 Let $\Lambda \subset T^*(X) \setminus \{0\}$ be a closed conic Lagrangean submanifold of $T^*(X)$. The space of Lagrangean distributions of order m associated with Λ , $I^m(X,\Lambda)$, is defined as the space of all $u \in \mathcal{D}'(X)$ with the following properties:

1) The Wave front set $WF(u) \subset \Lambda$.

2) For all $(x_0, \theta_0) \in \Lambda$ and any non-degenerate phase function ϕ , defined on an open cone $V \subset X \times (\mathbb{R} \setminus \{0\})^N$ and parametrizing Λ in a neighbourhood of (x_0, θ_0) , there exists a symbol $a \in S^{m+\frac{n}{4}-\frac{N}{2}}(X \times \mathbb{R}^N)$ with support in a cone $\subset V$ such that $u \equiv I_{\phi}(a)$ in neighbourhood of (x_0, θ_0) .

The local representation of a Lagrangean distribution in terms of an oscillatory integral is not unique. On the other hand, for a given Lagrangean distribution u, the Lagrangean Λ , its principal symbol σ_u , which we will describe below, and its order are uniquely determined. Our calculations will only involve these invariants.

Definition 1.3.4 If u is a Lagrangean distribution on X satisfying u(1) = 1 and $u(f) \ge 0$ for all nonnegative $f \in C_0^{\infty}(X)$, then u is called a Lagrangean probability measure.

Example 6. Discrete measures. The delta function $\delta_{x_0}(x)$ on \mathbb{R}^n at a point $x_0 \in \mathbb{R}^n$ is given by the oscillatory integral

(1.3.6)
$$\delta_{x_0}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-x_0,\theta)} d\theta$$

We have $\Lambda_{\phi} = T_{x_0}^* \mathbb{R}$ and $\delta_{x_0} \in I^{\frac{n}{4}}(\mathbb{R}, T_{x_0}^*\mathbb{R})$. More generally, a discrete measure

(1.3.7)
$$\mu = \frac{1}{m_{\mu}} \sum_{i=1}^{k} c_i \delta_x$$

with $m_{\mu} = \sum_{i=1}^{k} c_i$ on a manifold M of dimension n is a mixture of delta functions, hence can be expressed in local coordinates as a sum of oscillatory integrals of the form (1.3.6). We have $\mu \in I^{\frac{n}{4}}(M, \Lambda)$ with $\Lambda = \bigcup_{x \in \text{supp } \mu} T_x^* M$.

Example 7. Delta functions on submanifolds. Any C^{∞} density u on a smooth submanifold $S \subset \mathbb{R}^n$ can locally be written as an oscillatory integral. Since S is a smooth submanifold, for any given point $x_0 \in S$ we can find k functions ϕ_1, \ldots, ϕ_k with $d\phi_1, \ldots, d\phi_k$ linearly independent at x_0 , such that S is defined in a neighbourhood around x_0 by

$$\phi_1(x) = \ldots = \phi_k(x) = 0 \; .$$

There exists a function $a \in C_0^{\infty}(\mathbb{R}^n)$ with support near x_0 such that

(1.3.8)
$$u(x) = (2\pi)^{-k} \int_{\mathbb{R}^k} a(x) e^{i\phi(x,\theta)} d\theta \qquad \text{where } \phi(x,\theta) = \sum_{j=1}^k \phi_j(x) \theta_j \ .$$

Thus $\Lambda_{\phi} = \{(x, d_x\phi(x, \theta)) | \phi_j = 0, j = 1, \dots, k, x \in \operatorname{supp} u\}$. Note that this is the *conormal bundle* $N^*(S)$ of S restricted to suppu. We have $u \in I^{\frac{k}{2} - \frac{n}{4}}(\mathbb{R}^n, \Lambda_{\phi})$.

Delta functions on submanifolds are examples of *conormal* distributions. They are defined as having a local expression as an oscillatory integral with phase function *linear* in θ .

Example 8. Homogeneous distributions on the circle S^1 . The basic homogeneous distributions on S^1 are the distributions $\chi_k(\theta)$ defined by the Fourier series

$$\chi_k(\theta) = \sum_{r=1}^{\infty} r^k e^{ir\theta}$$

with Fourier coefficients homogeneous of degree k. As Lagrangean distribution, $\chi_k \in I^{k+\frac{1}{4}}(S^1, T_0^{+*}S^1)$, that is, χ_k has only positive Fourier coefficients, has order $(k + \frac{1}{4})$ and has sole singularity at $\theta = 0$. It also has precisely the same singularity as the Fourier integral $(\theta + i0)^{-k-1} := \int_0^\infty \xi^r e^{i\theta\xi} d\xi$, i.e. differs from it in a neighborhood of 0 by a smooth function. (The integral needs to be regularized at $\xi = 0$; we refer to [G.S–3]or to [HöI] for a more detailed discussion). The analogous homogeneous distribution with singularity at $\theta = \theta_0$ instead of $\theta = 0$ is given by the Fourier expansion

(1.3.9)
$$\chi_k(\theta - \theta_0) = \sum_{r=1}^{\infty} r^k e^{ir\theta} e^{-ir\theta_0}$$

Example 9. Lagrangean distributions on S^1 . Since S^1 is 1-dimensional, the homogeneous Lagrangean submanifolds of T^*S^1 must be finite unions of rays $T_{\theta_j}^{\pm *}(S^1) := \{(\theta_j, r\xi) : r\pm 0\}$. Hence a Lagrangean distribution \mathcal{Y} on S^1 can only have a finite number of singular points $\{\theta_1, \ldots, \theta_n\}$. Since $T_{\theta_j}^{\pm *}(S^1)$ is parametrized by the phase function $\phi(\theta, \xi) := (\theta - \theta_j)\xi$, near $\theta_j \ \mathcal{Y}$ must have the form

$$\mathcal{Y}(\theta) = \int_0^\infty e^{(\theta - \theta_j)\xi} a_j(\theta, \xi) d\xi$$

with polyhomogeneous amplitude of the form

$$a_j(\theta,\xi) \sim \sum_{i=0}^{\infty} a_{m_j-i} \xi^{m_j-i}$$

Comparing with Example 8, we see that

(1.3.10)
$$\mathcal{Y}(\theta) \sim \sum_{j=1}^{n} \sum_{i=0}^{\infty} a_{m_j - i} \chi_{m_j - i} (\theta - \theta_j)$$

Thus, \mathcal{Y} is a finite sum of polyhomogeneous distributions.

Let X and Y be two C^{∞} manifolds. A distribution $k \in \mathcal{D}'(X \times Y)$ defines a linear operator

(1.3.11)
$$K: C_0^{\infty}(Y) \to \mathcal{D}'(X)$$

via

(1.3.12)
$$\langle K(f), g \rangle = \langle k, f \otimes g \rangle, \quad f \in C_0^{\infty}(Y), g \in C_0^{\infty}(X) .$$

Conversely, by the Schwartz kernel theorem, for every map K as in (1.3.11) there exists a unique distribution $k \in \mathcal{D}'(X \times Y)$ such that (1.3.12) is valid.

Definition 1.3.5 A linear operator $K : C_0^{\infty}(Y) \to \mathcal{D}'(X)$ is called *Fourier Integral Operator* (FIO) if its Schwartz kernel k is a Lagrangean distribution on $X \times Y$, i.e., if $k \in I^m(X \times Y, \Lambda)$ for some $m \in \mathbb{R}$ and $\Lambda \subset T^*(X \times Y)$ a Lagrangean submanifold. We will sometimes also write in this case, a bit loosely, $K \in I^m(X \times Y, \Lambda)$.

Notation. If $\Lambda \subset T^*(X \times Y)$ is a Lagrangean submanifold, we will denote $\{(x, y; \xi, \eta) | (x, y; \xi, -\eta) \in \Lambda\}$ by Λ' . Recall that Λ' is a Lagrangean submanifold of $T^*(X \times Y)$ with respect to the symplectic form $\omega_X - \omega_Y$ where ω_X and ω_Y are the canonical forms on T^*X and T^*Y , respectively.

The following *clean composition theorem* gives a sufficient condition for the composition of two FIO's to be FIO again. See, for example, [HöIV], Theorem 25.2.3.

Theorem 1.3.6 Let X, Y, Z be smooth manifolds and $\Lambda'_1 \subset T^*X \times T^*Y$ and $\Lambda'_2 \subset T^*Y \times T^*Z$ two homogeneous canonical relations. Let K_1 and K_2 be two FIO's whose Schwartz kernels k_1 and k_2 are in $I^{m_1}(X \times Y, \Lambda_1)$ and $I^{m_2}(Y \times Z, \Lambda_2)$, respectively. Assume that the composition $\Lambda = \Lambda'_1 \circ \Lambda_2$ is clean with excess e. Then $K_3 = K_1K_2$ is also an FIO and its Schwartz kernel

$$k_3 \in I^{m_1 + m_2 + e/2} \left(X \times Z, \Lambda \right)$$

With few exceptions (the Szegö projector and the trace operation), all of the Fourier Integral operators (FIO's) we will be considering in this paper come from Fourier Integral representations of Lie Groups. In fact, they will just be translations by group elements on homogeneous spaces. The following gives a summary of the notation and terminology regarding FIO group actions.

1.4 Hamiltonian and Fourier Integral group actions

We begin by describing the simplest Hamiltonian actions on the cotangent bundle T^*M of a compact manifold M which arise from lifts of geometric (translation) actions on the base M.

Thus, let $l : G \times M \to M$ be an action of the Lie group G on M. The action of G on M can be lifted to an action L of G on the cotangent bundle T^*M in a canonical way: For fixed $g \in G$, $L(g;m,\xi) = (l_g(m), (dl_{g^{-1}})^*\xi))$ for all $(m,\xi) \in T^*M$ (we have denoted l(g,m) by $l_g(m)$). This action is always Hamiltonian and we denote its moment map by Φ . It is determined by the following formula:

(1.4.1)
$$\iota(X^{\#})\omega = d(\langle X, \Phi(m,\xi) \rangle) \quad \text{for all } X \in \mathbf{g}, (m,\xi) \in T^*M$$

More concretely, let $\{X_1, \ldots, X_r\}$ denote a basis of the Lie algebra **g** of G and let $X_i^{\#}$ denote the corresponding vector fields on M. They are all Hamiltonian vector fields on T^*M with Hamiltonians $f_i(x,\xi) := \langle (X_i^{\#})_x, \xi \rangle$. The moment map can then be written as

$$\Phi(x,\xi) = (f_1,\ldots,f_r).$$

In particular if $G = S^1$, and if $\frac{\partial}{\partial \theta}$ denotes the generator of the action on M, then the moment map is the Hamiltonian $f(x,\xi) := \langle \frac{\partial}{\partial \theta}, \xi \rangle$.

The action l of G on M induces a unitary representation ρ

(1.4.2)
$$\rho(g) \cdot \eta = l_g^* \eta$$

of G on $L^2(M, \Omega^{\frac{1}{2}})$, the space of half densities on M. Here, a 1/2-density η is the square root $|dvol|^{1/2}$ of a volume density on M. These rather unusual objects give M an intrisnic L^2 - structure since the product of two 1/2-densities is a density which can be integrated over M. They are ubiquitous in microlocal analysis and although all of our manifolds have natural volume densities, we will state things in terms of 1/2-densities to enable us easily to quote relevant microlocal facts from the references.

Since $\rho(g)$ is translation (pull-back) by g, the Schwartz kernel $k_g(x, y)$ of $\rho(g)$ is essentially the deltafunction $\delta(y-l_g(x))$. To make this intuitively clear notion precise, we would have to discuss delta 1/2-densities on a manifold. For the sake of brevity, we refer to [G] for the precise definitions, and only note that as a delta function on Graph $(l) = \{(g, m, l(g, m)) | g \in G, m \in M\}, k(g, x, y) \text{ is Lagrangean distribution on } G \times M \times M$ with associated Lagrangean Γ equal to the conormal bundle of Graph(l), which is easily seen to equal the moment Lagrangian of the Hamiltonian action of G on T^*M , given by

(1.4.3)
$$\Gamma = \{ (g, \Phi(m, \xi); m, \xi; L(g; m, \xi)) | g \in G, (m, \xi) \in T^*M \}.$$

In the case of composition of FIO's coming from translations by (possibly different) group actions, the composition of FIO's is simply the composition of the translations and the composition of the Lagrangeans is simply given by the graph of the composite translation. For future reference, we consider this case in detail.

Cleanness of composition of T_g and $T_{e^{i\theta}}$. Let M be a manifold and $l : G \times M \to M$ and and $k : S^1 \times M \to M$ two group actions with G not necessarily equal to S^1 . For fixed $g \in G$, we denote the corresponding translation operator in $L^2(M, \Omega^{\frac{1}{2}})$ by T_g . Its Schwartz kernel, $k_g(x, y)$, is a Lagrangean distribution (the delta function $\delta(y - l_g(x))$) with corresponding Lagrangean

(1.4.4)
$$\Lambda_g = \{ (x,\xi; l_g(x), -(dl_{g^{-1}})^*\xi) | (x,\xi) \in T^*M \} \cong T^*M$$

For the group action $k: S^1 \times M \to M$ we also consider translation in $L^2(M, \Omega^{\frac{1}{2}})$, denoted by $T_{e^{i\theta}}$. Now we do not hold a specific element in S^1 fixed, but view the resulting representation of S^1 as a distribution on $S^1 \times M \times M$. This distribution is a delta function on

$$\operatorname{Graph}(k) = \{(e^{i\theta}; x; k(e^{i\theta}, x)) | e^{i\theta} \in S^1, x \in M\}$$

whose conormal bundle is given by

(1.4.5)
$$\Gamma = \{ (e^{i\theta}, \Phi(x,\xi); x,\xi; K_{\theta}(x,\xi) | e^{i\theta} \in S^1, (x,\xi) \in T^*M \} \cong S^1 \times T^*M .$$

Here $K_{\theta}(x,\xi) := (k(e^{i\theta},x), dk_{e^{-i\theta}}^*\xi)$ and Φ denotes the moment map of the group action $K: S^1 \times T^*M \to T^*M$.

The Schwartz kernel of the operator $T_{e^{i\theta}g} := T_{e^{i\theta}} \circ T_g$, viewed as a distribution on $S^1 \times B \times B$, is always Lagrangean, since the composition $\Gamma' \circ \Lambda'_g$ is automatically clean (see Theorem 1.3.6). We recall that this means that the fiber diagram

$$\begin{array}{cccc} F & \longrightarrow & \Gamma' \\ \downarrow & & \downarrow & \pi_3 \\ \Lambda'_g & \longrightarrow & T^*M \\ & & \pi_1 \end{array}$$

is clean, where $\pi_3 : \Lambda' \to T^*M$ denotes projection onto the third factor in $\Gamma' \subset T^*S^1 \times T^*M \times T^*M$, $\pi_1 : \Lambda'_a \to T^*M$ is projection onto the first factor, and where

$$F = \{ (e^{i\theta}, \Phi(x,\xi); x,\xi; K_{\theta}(x,\xi); K_{\theta}(x,\xi); L_{g}K_{\theta}(x,\xi)) | e^{i\theta} \in S^{1}, (x,\xi) \in T^{*}M \} \cong S^{1} \times T^{*}M$$

with $L_g(x,\xi) := (l_g(x), (dl_{g^{-1}})^*\xi)$. Since it is a graph, the fiber product F is always a submanifold of $\Gamma' \times \Lambda'_g$ and it is easy to see that the derived diagram

$$\begin{array}{cccc} TF & \longrightarrow & T\Gamma' \\ \downarrow & & \downarrow & d\pi_3 \\ T\Lambda'_g & \longrightarrow & T(T^*M) \\ & & d\pi_1 \end{array}$$

is a fiber diagram with excess e = 0. Denoting the Lagrangean $\Gamma' \circ \Lambda_q$ by Λ we have

(1.4.6)
$$\Lambda = \{ (e^{i\theta}, \Phi(x,\xi); x,\xi; L_g K_\theta(x,\xi)) | e^{i\theta} \in S^1, (x,\xi) \in T^* M \} \cong S^1 \times T^* M$$

and

(1.4.7)
$$T_{e^{i\theta}g} \in I^{-\frac{1}{4}}(S^1 \times M \times M, \Lambda) .$$

1.5 The trace operation

Taking the trace of an operator A can be viewed as composing a certain Fourier integral operator Tr with the Schwartz kernel k_A of A.

Let k_A be a Lagrangean distribution on $G \times M \times M$. Then $\operatorname{Tr} A = \operatorname{Tr} k_A = \int k_A(g, m, m) dm$. It was pointed out in [D.G] that $\operatorname{Tr} A$ can be viewed as the composition

(1.5.1)
$$\operatorname{Tr} A = \pi_* \Delta^* k_A$$

with $\Delta: G \times M \to G \times M \times M$ the diagonal embedding and $\pi: G \times M \to G$ the projection onto the first factor.

Since the operator $\pi_*\Delta^*$ maps distributions on $G \times M \times M$ to distributions on G, its Schwartz kernel $k_{\pi_*\Delta^*}$ is a distribution on $G \times M \times M \times G$. The following lemma can be found in [D.G] (p.66).

Lemma 1.5.1 We identify $G \times M \times M \times G$ with $(G \times M) \times (G \times M)$. The Schwartz kernel $k_{\pi_*\Delta^*}$ is the same as k_{id} , the Schwartz kernel of the identity map. Furthermore, $k_{\pi_*\Delta^*} \in I^0(G \times M \times M \times G, \Gamma)$ with Γ being the conormal bundle to the diagonal in $G \times M \times M \times G$.

1.6 The principal symbol of a Lagrangean distribution.

Roughly speaking, the principal symbol σ_u is a 1/2-density on Λ . (To be completely accurate, it involves additional factors $e^{\pm i\pi}$, called Maslov factors.) To define σ_u , we use a parametrization of Λ .

Let u be an element in $I^m(X, \Lambda)$, and let

$$I_{\phi}(a) = \int e^{i\phi} a d\theta$$

be a local representation of u. We assume that the phase function ϕu is defined in an open conic subset $V \subset X \times (\mathbb{R} \setminus \{0\})^N$ and that the projection of V into X is a domain of local coordinates x_1, \ldots, x_n . We can use $\partial \phi / \partial \theta_i$, $i = 1, \ldots, N$ as local coordinates transversal to C_{ϕ} in $X \times (\mathbb{R} \setminus \{0\})^N$. On C_{ϕ} itself we choose local coordinates c_j , $j = 1, \ldots, n$, that are homogeneous of degree one with respect to θ . Together, the $\partial \phi / \partial \theta_i$ and c_j form a complete coordinate system in V. C_{ϕ} carries a volume form d_{ϕ} defined by

(1.6.1)
$$d_{\phi} = \left| \frac{D(x,\theta)}{D(c,\partial\phi/\partial\theta)} \right| dc_1 \wedge \dots \wedge dc_n .$$

The coefficient on the right hand side denotes the Jacobian determinant of the x_j and θ_i with respect to the c_j and $\partial \phi / \partial \theta_i$, j = 1, ..., n and i = 1, ..., N. Let

$$a \sim a_0 + a_{-1} + \dots$$

be the polyhomogeneous expansion of a with $a_0 \in S^{m+\frac{n}{4}-\frac{N}{2}}$ and with terms decreasing by steps of degree one. Then the local formula for the symbol is given by

(1.6.2)
$$\sigma_u = a_0 \sqrt{d_d}$$

on C_{ϕ} . More precisely, the diffeomorphism I_{ϕ} transfers the 1/2-density $a_0\sqrt{d_{\phi}}$ on C_{ϕ} to a half density $I_{\phi*}a_0\sqrt{d_{\phi}}$ on an open set $U \subset \Lambda$, which is by definition the symbol σ_u on this set. Recall that the c_j are homogeneous of degree one in θ , the $\partial^2 \phi / \partial \theta_i \partial \theta_k$ are of degree -1 in θ , therefore $\frac{D(c,\partial \phi/\partial \theta)}{D(x,\theta)}$ is of degree n-N in θ and the right hand side in (1.6.1) is of degree N in θ . It follows that $\sigma_u'_U \in S^{m+\frac{n}{4}}(U;\Omega^{\frac{1}{2}})$ (here $\Omega^{\frac{1}{2}}$ denotes the half density bundle over Λ).

There is a complex line bundle, \mathbb{IL} , over Λ , called the Keller-Maslov line bundle, which is also involved in piecing together the σ from different coordinate patches in a smooth way. The precise definition of \mathbb{IL} is rather technical. Since it will play a minor role in our computations, we omit further details and refer the reader to [T], Chapters VII and VIII. Here we merely state the following.

Definition 1.6.1 The section $\sigma_u \in S^{m+\frac{n}{4}}(\Lambda; \Omega^{\frac{1}{2}} \otimes \mathbb{L})$ for which

$$\sigma_u'|_{\Lambda_\phi} = I_{\phi*} a_0 \sqrt{d_\phi}$$

is called the *principal symbol* of u. The principal symbol σ_A of a Fourier Integral operator A is the principal symbol of its Schwartz kernel k_A .

Example 10. The principal symbol of T_g . Recall that the Schwartz kernel of T_g , $k_g(x, y)$, is a Lagrangean distribution (the delta function $\delta(y - l_g(x))$) with corresponding Lagrangean

$$\Lambda = \{ (x,\xi; l_g(x), -(dl_{g^{-1}})^* \xi) | (x,\xi) \in T^* M \} \cong T^* M$$

Any symplectic manifold (N, ω) of dimension (2n) carries a canonical volume form $\Omega := (-1)^n \frac{1}{n!} \omega^{\wedge n}$. In the case of T^*M , $\Omega = dx_1 \wedge \cdots \wedge dx_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n$ for which we will write for brevity $dx \wedge d\xi$. Since Λ

is a graph, it carries the volume form $\pi^*\Omega$ where $\pi: \Lambda \to T^*M$ denotes projection onto the first factor. We claim that the principal symbol σ_{T_a} is simply $\pi^*|\Omega|^{\frac{1}{2}}$. Indeed, for any coordinate patch $X \subset M$ we can write

$$k_g(x,y)'|_{X \times X} = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\phi(x,y,\xi)} d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(\xi,y-l_g(x))} d\xi$$

(recall Example 7). Since $\phi(x, y, \xi) = \langle \xi, y - l_g(x) \rangle$,

$$C_{\phi} = \{ (x, l_g(x), \xi | \xi \in \mathbb{R}^n \} \cong X \times \mathbb{R}^n \cong T^* X$$

and a complete coordinate system for $V = X \times X \times \mathbb{R}^n$ satisfying the above conditions is

$$c_i = \xi_i, i = 1, \dots, n; \quad c_j = x_j |\xi|, j = n + 1, \dots, 2n; \quad \partial \phi / \partial \xi_k = y_k - (l_g(x))_k, k = 1, \dots, n$$

The inverse of the Jacobian in (1.6.1) is

$$\left|\frac{D(\xi, x|\xi|, y - l_g(x))}{D(x, y, \xi)}\right| = \left|\begin{array}{ccc} 0 & 0 & I \\ |\xi|I & 0 & (x_i\xi_j/|\xi|)_{1 \le i,j \le n} \\ dg & I & 0 \end{array}\right| = |\xi|^n$$

(all blocks in the second determinant are of size $n \times n$). This yields for d_{ϕ} in (1.6.1),

$$d_{\phi} = |\xi|^{-n} dc_1 \wedge \cdots \wedge dc_{2n} = dx \wedge d\xi = \Omega.$$

Since $a_0 = 1$ and clearly $I_{\phi} = \pi^{-1} |_{T^*X}$, we have $I_{\phi \star} |\Omega|^{\frac{1}{2}} = \pi^* |\Omega|^{\frac{1}{2}}$ which proves that $\sigma_{T_g} = \pi^* |\Omega|^{\frac{1}{2}}$. \Box

Example 11. The canonical 1/2-density on a graph. Generalizing Example 10, we note that the graph of any homogeneous canonical transformation χ carries a canonical 1/2-density. Since the graph of $\chi : T^*X \to T^*X$ is naturally diffeomorphic to T^*X (by the conic diffeomorphism π_1 , see (1.2.1)), one can pull back the symplectic volume 1/2-density $|\Omega|^{\frac{1}{2}}$ to the 1/2-density $\pi_1^*|\Omega|^{\frac{1}{2}} = \pi_1^*|dx \wedge d\xi|^{\frac{1}{2}}$ on $\operatorname{Graph}(\chi)$. We will refer to it as the canonical graph 1/2-density and denote it by $|\Omega_{\chi}|^{\frac{1}{2}}$. The calculation in Example 10 shows that $\sigma_{T_g} = |\Omega_{L_g}|^{\frac{1}{2}}$.

Example 12(a). The principal symbol of a discrete measure $\mu := \sum_{j=1}^{k} c_j \delta_{x_j}$ on \mathbb{R} . Recall from Example 6 that $\mu \in I^{\frac{1}{4}}(\mathbb{R}, \bigcup_{j=1}^{k} T_{x_j}^*\mathbb{R})$. Each cotangent space $T_{x_j}^*\mathbb{R} \cong \mathbb{R}$ carries a natural volume density dt. The principal symbol of μ is the half density σ_{μ} on $\bigcup_{j=1}^{k} T_{x_j}^*\mathbb{R}$ with $\sigma_{\mu} |_{T_{x_j}^*\mathbb{R}} = c_j |dt|^{\frac{1}{2}}$.

Example 12(b). The principal symbol of a transition operator $T := \sum_{j=1}^{k} c_j T_{g_j}$. The Lagrangean corresponding to the Schwartz kernel k_T of T is the union $\Sigma = \bigcup_{j=1}^{k} \Lambda_{g_j}$ where the Λ_{g_j} 's are as in (1.4.4). By Example 10, the principal symbol of T is the half density σ_T on Σ with $\sigma_T |_{\Lambda_{g_j}} = c_j \pi^* |\Omega|^{\frac{1}{2}}$.

Example 13. The principal symbol of a Fourier Integral group action. We return to the example of the moment Lagrangean Γ (1.4.5) of a Hamiltonian S¹-action and to the associated Fourier Integral representation K_{θ} . The Lagrangean has a natural parametrization

$$j: S^1 \times T^*M \to T^*S^1 \times T^*M \times T^*M \qquad (e^{i\theta}, x, \xi) \to (e^{i\theta}, \Phi(x, \xi), x, \xi, K_{\theta}(x, \xi))$$

with image Γ . The parameter space carries the natural 1/2-density $|d\theta|^{\frac{1}{2}} \otimes |\Omega|^{\frac{1}{2}}$ which gets transported to a 1/2-density $|\Omega_k|^{\frac{1}{2}}$ on Γ , which we will refer to as the 'canonical density' on the graph of the action.

Propositon 1.6.2 The principal symbol σ_K of a Fourier Integral S^1 -action is given by: $\sigma_K = |\Omega_k|^{\frac{1}{2}}$.

Proof. (Sketch) Let us denote by D the generator of the S^1 -action, so that $K = e^{i\theta D}$. Then K is the unique solution of the initial value problem

$$\left(\begin{array}{c} \left(\frac{\partial}{\partial\theta} - D\right)K_{\theta} = 0\\ K_{\theta}|_{\theta=0} = Id \end{array}\right)$$

The symbol can be determined by considering the corresponding initial value problem for the symbols. At $\theta = 0$ we have the canonical graph 1/2-density of the identity graph, and by the equation the symbol is invariant under the classical Hamiltonian S^1 action. The 1/2-density given in the statement of the proposition is the unique solution of this initial value problem. For more details, see [D.G].

Example 14. The principal symbol of the trace operation. Recall from Lemma 1.5.1 that the Lagrangean Γ corresponding to the Lagrangean distribution $k_{\pi\star\Delta\star}$ is the conormal bundle

$$\Gamma = \{ (g, \gamma; x, \xi; g, -\gamma; x, -\xi) | (g, \gamma) \in T^*G, (x, \xi) \in T^*M \}$$

of the diagonal $\Delta(G \times M)$. Clearly, $\Gamma \cong T^*G \times T^*M$. The principal symbol of the trace operation is

(1.6.3)
$$\sigma_{Tr} = \tilde{\pi}^* |dg \wedge d\gamma \wedge dx \wedge d\xi|^{\frac{1}{2}}$$

where $\tilde{\pi} : \Gamma \to T^*G \times T^*M$ denotes projection onto the first and second factors. See, for example, Lemma 6.3 in [D.G].

We will also need to review the principal symbol calculus of FIO's. Given a pair of Lagrangeans Λ_1 , respectively Λ_2 , and a pair of 1/2-densities σ_1 , respectively σ_2 , on them we wish to define a composite 1/2-density $\sigma_1 \circ \sigma_2$ on the composite Lagrangean $\Lambda_1 \circ \Lambda_2$. To do so, we begin with the exterior tensor product of the 1/2-densities, that is, the 1/2-density on the product space

$$\sigma_1\otimes\sigma_2\in\Omega^{rac{1}{2}}(\Lambda_1 imes\Lambda_2)$$

given by

$$\sigma_1 \otimes \sigma_2(b_1, b_2) := \sigma_1(b_1)\sigma_2(b_2)$$

where b_i is a basis for $T_{\lambda_i}(\Lambda_i)$. The composition product is a bilinear operation

$$\sigma_1 \otimes \sigma_2 \to \sigma_1 \circ \sigma_2$$

constructing $\sigma_1 \circ \sigma_2$ in a canonical way from $\sigma_1 \otimes \sigma_2$ in the form

(1.6.4)
$$(\sigma_1 \otimes \sigma_2)_{\lambda} = \int_{F_{\lambda}} \sigma_1 \times \sigma_2$$

with $\lambda \in \Lambda_1 \circ \Lambda_2$, with F_{λ} is the fiber $\{(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2 : \lambda_1 \circ \lambda_2 = \lambda\}$ of the composition over λ , and with $\sigma_1 \times \sigma_2$ a certain density along the fiber with values in $\Omega^{\frac{1}{2}}(\Lambda_1 \circ \Lambda_2)$, the space of 1/2-densities on the composite. See [HöIV] for the precise definition. Roughly speaking, if $\sigma_1 = \sqrt{dxdz}$ and if $\sigma_2 = \sqrt{dzdy}$, with dz a volume density on F_{λ} , then $\sigma_1 \times \sigma_2 = \sqrt{dxdy}dz$

We will only need the following special cases of the composition law, which follow from the general composition theorem in [HöIV].

Theorem 1.3.6 (addendum)

(a) Let $\Lambda_i = \operatorname{Graph}(\chi_i)$ be the graph of a homogeneous canonical transformation χ_i on T^*M (i=1,2), and as above let $|\Omega_{\chi_i}|^{\frac{1}{2}}$ be the canonical 1/2-densities on the graphs. Then $\Lambda_1 \circ \Lambda_2 = \operatorname{Graph}(\chi_1 \circ \chi_2)$ and $|\Omega_{\chi_1}|^{\frac{1}{2}} \circ |\Omega_{\chi_2}|^{\frac{1}{2}} = |\Omega_{\chi_1 \circ \chi_2}|^{\frac{1}{2}}$. (b) Let Γ be the moment Lagrangean of a classical Hamiltonian S^1 action k, equipped with the canonical density $|\Omega_k|^{\frac{1}{2}}$ and let $\operatorname{Graph}(\chi)$ be the graph of a homogeneous canonical transformation, equipped with the canonical graph 1/2-density $|\Omega_{\chi}|^{\frac{1}{2}}$. Then the composite Lagrangean Λ (1.4.6) is parametrized by the map

 $j_{\chi}: S^1 \times T^*M \to T^*S^1 \times T^*M \times T^*M \qquad (e^{i\theta}, x, \xi) \to (e^{i\theta}, \Phi(x, \xi), x, \xi, L_g\chi(x, \xi))$

and the composite 1/2-density is again given by $|\Omega_{\chi \circ k}|^{\frac{1}{2}} := |d\theta|^{\frac{1}{2}} \otimes |\Omega|^{\frac{1}{2}}$.

2 Rays of representations

2.1 Geometric quantization

Let G be a compact, semisimple Lie group, \hat{G} its unitary dual, and $I^* \cap \mathbf{t}^*_+$ the set of lattice points in the chosen positive Weyl chamber \mathbf{t}^*_+ . Recall that there is a one-to-one correspondence $\hat{G} \leftrightarrow I^* \cap \mathbf{t}^*_+$ under which $\rho_{\nu} \leftrightarrow \nu$ with ν the highest weight of the irreducible character $\chi_{\rho_{\nu}}$. We call a representation ρ of G a ray of representations through ν if each irreducible $\rho_{n\nu}$, $n \in \mathbb{N}^0$ occurs exactly with multiplicity 1 in ρ .

Let \mathcal{O}_{ν} be the coadjoint orbit corresponding to the integral form $\nu \in I^* \cap \mathbf{t}^*_+$ (see Example 2 above). For any $\xi \in \mathcal{O}_{\nu}$, we denote the stabilizer group of ξ with respect to the *G*-action by G_{ξ} . We have $\mathcal{O}_{\nu} \cong G/G_{\nu}$. Since ν is an integral point, the map $\chi_{\nu} : G_{\nu} \to S^1$, $g \to e^{2\pi i \langle \nu, X_g \rangle}$ where $g = \exp X_g$ is a homomorphism (see [G,S-1], (2.11)).

By the classical Borel-Weil-Bott theorem, there exists a holomorphic structure on the orbit \mathcal{O}_{ν} and a complex line bundle E over \mathcal{O}_{ν} whose holomorphic sections $\Gamma(E)$ form the representation space of the irreducible representation of highest weight ν . Let us explain the situation in more detail. The holomorphic line bundle E is the quotient space of $G \times \mathbb{C}$ with respect to the equivalence relation

(2.1.1)
$$(gh, z) \sim (g, \chi_{\nu}(h)z) \quad g \in G, h \in G_{\nu}, z \in \mathbb{C},$$

i.e., the associated complex line bundle $G \times_{G_{\nu}} \mathbb{C}$. A holomorphic section s of E is a holomorphic map $s: \mathcal{O}_{\nu} \to E$ with $\pi \circ s(\xi) = \xi$ for all $\xi \in \mathcal{O}_{\nu}$ (here $\pi: E \to \mathcal{O}_{\nu}$ is the natural projection). Equivalently, s can be viewed as a map $\tilde{s}: G \to \mathbb{C}$ for which $\tilde{s}(gh) = \chi_{\nu}(h)^{-1}\tilde{s}(g)$ for all $g \in G$ and $h \in G_{\nu}$. For each $\nu \in \mathfrak{t}^*_+$, the set of holomorphic sections $\Gamma(E)$ of the line bundle E over \mathcal{O}_{ν} is finite. The action of G on $\Gamma(E)$ is given by

(2.1.2)
$$g \cdot \tilde{s}(p) = \tilde{s}(g^{-1}p)$$
 for all $g \in G, s \in \Gamma(E)$.

Any other irreducible $\rho_{n\nu}$ of G can of course be constructed in the very same way. Note that the line bundle $E^{(n)}$ over $\mathcal{O}_{n\nu} \cong \mathcal{O}_{\nu}$ can also be viewed as the quotient space of $G \times \mathbb{C}^{\otimes n}$ with respect to the equivalence relation

(2.1.3)
$$(gh, z) \sim (g, \chi_{\nu}^{(n)}(h)z) \quad g \in G, h \in G_{\nu}, z \in \mathbb{C}^{\otimes n}$$

Here $\chi_{\nu}^{(n)}$ denotes the character χ_{ν} extended to the n^{th} tensor product $\mathbb{C}^{\otimes n}$ of \mathbb{C} .

2.2 The Hardy space $H^2(B)$

For our computations we need to have for each ray of representations *one* unified function space which is exactly the representation space and on which the representations act as Fourier Integral Operators. Here we describe the construction of such a ray of representations due to [G.S-1].

We denote the E^* be the dual bundle of E, i.e., the quotient space of $G \times \mathbb{C}$ with respect to the equivalence relation

(2.2.1)
$$(gh, z) \sim (g, \chi_{\nu}^{-1}(h)z) \quad g \in G, h \in G_{\nu}, z \in \mathbb{C}$$
.

Its disk bundle D is defined by

(2.2.2)
$$D = \{(\xi, z) | \xi \in \mathcal{O}_{\nu}, z \in E_{\mathcal{E}}^{\star}, |z| \le 1\}$$

and its circle bundle B by

(2.2.3)
$$B = \{(\xi, z) | \xi \in \mathcal{O}_{\nu}, z \in E_{\xi}^{\star}, |z| = 1\}$$

D is a compact complex domain whose boundary is B. Furthermore, G and the circle S^1 act on B and their actions commute. The Hardy space of B, $H^2(B)$, is the space of Cauchy-Riemann (CR) functions on B, i.e., the space of L^2 -functions on B that can be extended to holomorphic functions on the whole disk bundle D. Holomorphic sections of E correspond to S^1 -equivariant CR functions on B in a one-to-one way: Let $s \in \Gamma(E^*)$ with $s(\xi) = [g, z], \xi \sim gG_{\nu}$. We then define the corresponding $\hat{s} \in H^2(B)$ by

(2.2.4)
$$\hat{s}([g, e^{i\phi}]) = ze^{i\phi}, \quad \text{for all } g \in G, e^{i\phi} \in S^1$$

It is easy to check that \hat{s} is in fact well defined on $B \sim G \times_{G_{\nu}} S^1$ and that $\hat{s}(e^{i\theta} \cdot b) = e^{i\theta} \hat{s}(b)$ for all $e^{i\theta} \in S^1$ and $b \in B$. We denote the set of CR functions \hat{s} on B that can be derived from holomorphic sections in this way by $H_1^2(B) = \Gamma(E)$. Since G acts on B and this action commutes with the action of S^1 , G leaves $H_1^2(B)$ invariant. In fact, $H_1^2(B)$ is a representation space of the irreducible representation ρ_{ν} of G. By redefining the action of S^1 on B successively for each $n \in \mathbb{N}$ as

$$(2.2.5) e^{i\theta} \cdot [g, e^{i\phi}] = [g, e^{in\theta}e^{i\phi}],$$

we arrive at the spaces $H_n^2(B) = \{f \in H^2(B) | f(e^{i\theta} \cdot b) = e^{in\theta}f(b) \text{ for all } b \in B, e^{in\theta} \in S^1\}$ which are each representation spaces of the irreducibles $\rho_{n\nu}$. Since

(2.2.6)
$$H^{2}(B) = \bigoplus_{n=1}^{\infty} H^{2}_{n}(B) ,$$

this Hardy space is exactly a representation space for the ray through ρ_{ν} .

2.3 The Szegö projector and Toeplitz operators

The Szegö projector, i.e. the orthogonal projection

$$(2.3.1) \qquad \Pi: L^2(B) \to H^2(B)$$

is a special kind of FIO (of complex type) called a Toeplitz operator. A Toeplitz operator is like an FIO associated to a canonical transformation on a symplectic sub-cone $Y \subset T^*M$ rather than on the full cotangent bundle. The canonical transformation associated to Π is simply the identity transformation. Hence the compressions $\Pi \rho(g) \Pi$ correspond to the classical notion of restricting the group action to the subcone.

The relevant subcone in the case of Π is the cone through the *prequantum contact structure* of B. Here, a 1-form α on a (2n-1)-dimensional manifold B is called a contact form if $\alpha \wedge (d\alpha)^{n-1}$ is a volume form. The halfline subbundle generated by α ,

(2.3.2)
$$Y = \{(x, r\alpha_x) | x \in B, r \in \mathbb{R}^+\} \subset T^*B,$$

is then symplectic as a submanifold of T^*B . In the case of our circle bundle B, α is the connection 1-form with curvature ω_{ν} for the S^1 -bundle $\pi : B \to \mathcal{O}_{\nu}$ (see [G.S-2]). It is the unique 1-form invariant under the S^1 action for which $\iota(\partial/\partial\theta)\alpha = 1$ and $\iota(\partial/\partial\theta)d\alpha = 0$ (here, $\partial/\partial\theta$ denotes the infinitesimal generator of the S^1 action on B) The cone Y may also be described as the subspace of T^*B given by $\{(x,\xi) : \langle v,\xi \rangle = 0\}$ for all holomorphic tangent vectors $v \in T_x B$ (see [GS-1, p.354].

We then have:

Proposition 2.3.1 ([G.S-1], Proposition 6.5) For all $\nu \in I^* \cap t^*_+$, the orthogonal projection $\Pi : L^2(B) \to H^2(B)$, called the Szegö projector, is a Toeplitz operator (Fourier integral operator of complex type) associated to the identity canonical relation in $Y \times Y$.

The symbol of Π ,

(2.3.3) $\sigma_{\Pi} = |dy|^{\frac{1}{2}} \otimes \pi_Y,$

is a 1/2-density along Y times a "symplectic spinor" π_Y in the normal direction Y⁻. In fact, π_Y is a certain rank one projection onto a ground state (Gaussian) associated to Y⁻. In the case of discrete random walks, the asymptotics of the spectral measures are determined by traces which kill the symplectic spinor part. The basic reason is that in all the principal terms, the canonical relations will fix π_Y and then the trace will remove it (since $\operatorname{Tr} \pi_Y = 1$). Hence in the case of discrete random walks it is not necessary to discuss symbols of Toeplitz operators in detail, and we will concentrate only on the 1/2-density $|dy|^{\frac{1}{2}}$. In the case of continuous Lagrangean random walks, the spinor factor is not killed in the trace and plays an important role (see [P.Z]).

In the proof of our Main Theorem we will need to consider the two compositions of FIO's and Toeplitz operators: namely, $T_{ge^{i\theta}} \circ \Pi$ and $Tr \circ T_{ge^{i\theta}} \circ \Pi$. According to [BdM.G, Theorem 9.4], under a clean composition hypothesis both compositions produce Toeplitz operators associated to the composite isotropic relations. The general theorem is as follows:

Theorem 2.3.2 Let $K \in I^r(X \times Z, \Lambda)$ be a Toeplitz operator or an FIO and $T \in I^s(Z \times Y, \Sigma)$ be a Toeplitz operator. We assume that

(i) the composition $\Lambda' \circ \Sigma$ is clean (cf. Definition 1.2.1);

(ii) the natural projection $\pi : F \to \Lambda' \circ \Sigma$ has constant rank on each connected component (F denotes the fiber product of Λ' and Σ).

Then the composition is a Toeplitz operator associated to the isotropic relation $\Lambda' \circ \Sigma$, that is

$$K \circ T \in I^{r+s+\frac{1}{2}e}(X \times Z, \Lambda' \circ \Sigma)$$

Example 15. The order of the Szegö projector Π . Theorem 2.3.2 allows us to compute the order of Π : The composition $\Pi \circ \Pi$ can easily seen to be clean. Its excess is $e = \dim B - 1$. From this and the fact that $\Pi^2 = \Pi$ we get that $\Pi \in I^{-\frac{1}{2}(\dim B-1)}(B \times B, \Delta(Y)')$.

Example 16. The composition $T_{e^{i\theta}g} \circ \Pi$. Recall that $T_{e^{i\theta}g} \in I^{-\frac{1}{4}}(S^1 \times B \times B, \Lambda)$ with $\Lambda \cong S^1 \times T^*B$ (see (1.4.6-7)) and Π is the Szegö projector $\Pi \in I^{-\frac{1}{2}}(\dim B^{-1})(B \times B, \Delta(Y)')$. We thus have the fiber diagram

with fiber product

$$F = \{ (e^{i\theta}, -r; b, r\alpha; e^{i\theta}g \cdot b, r\alpha; e^{i\theta}g \cdot b, r\alpha; e^{i\theta}g \cdot b, r\alpha; e^{i\theta}g \cdot b, r\alpha) | r \in \mathbb{R}^+, b \in B, e^{i\theta} \in S^1 \} \cong S^1 \times Y$$

Since $dK_{\theta}dL_{q}(v) \in TY \Leftrightarrow v \in TY$, the following diagram

$$\begin{array}{cccc} TF & \longrightarrow & T\Lambda' \cong TS^1 \times T(T^*B) \\ \downarrow & & \downarrow \\ T\Delta(Y) \cong TY & \longrightarrow & T(T^*B) \end{array}$$

is a fiber diagram with excess e = 0. We have

$$\Lambda_Y := \Lambda' \circ \Delta(Y)' = \{ (e^{i\theta}, -r; b, r\alpha; e^{i\theta}g \cdot b, r\alpha) | r \in \mathbb{R}^+, b \in B, e^{i\theta} \in S^1 \} \cong S^1 \times Y$$

and it is obvious that $\pi: F \to \Lambda_Y$ is of constant rank. By Theorem 2.3.2,

(2.3.4)
$$T_{e^{i\theta}g} \circ \Pi \in I^{\frac{1}{4} - \frac{1}{2}\dim B}(S^1 \times B \times B, \Lambda_Y) .$$

We will also need to discuss the composition principal symbols of FIO's and Toeplitz operators. To avoid unnecessary technicalities, we will only state the composition law for symbols in the special cases that arise in the proof of our Main Theorem.

As in the case of symbol composition for FIO's the symbols of our operators are best described using parametrizations of our Lagrangeans and isotropic relations. As above, the canonical relation Λ underlying the FIO $T_{ge^{i\theta}} \in I^{-\frac{1}{4}}(S^1 \times B \times B, \Lambda)$ can be parametrized by $S^1 \times T^*B$. Also, the isotropic cone underlying Π is the graph of the identity on Y, Graph(id'|_Y). Hence the composition Toeplitz operator has underlying isotropic relation parametrized by the map

$$j_Y: S^1 \times Y \to T^*S^1 \times T^*B \times T^*B \qquad (e^{i\theta}, y) \to (e^{i\theta}, \Phi(y); y; L_g K_{\theta} y).$$

We recall that $\sigma_{T_{ac}i\theta} = |d\theta|^{\frac{1}{2}} \otimes |\Omega|^{\frac{1}{2}}$ and that $\sigma_{\Pi} = |dy|^{\frac{1}{2}} \otimes \pi_{Y}$. We have:

Proposition 2.3.3 With the above notation and assumptions: If the maps $L_g K_\theta$ preserve Y and if $(dL_g dK_\theta)^*(\sigma_{\Pi}) = \sigma_{\Pi}$, then

(2.3.5)
$$\sigma_{T_{uvi\theta}} \circ \sigma_{\Pi} = |d\theta|^{\frac{1}{2}} \otimes |dy|^{\frac{1}{2}} \otimes \pi_{Y}$$

as a symplectic spinor-valued 1/2-density on $S^1 \times Y$.

Proposition 2.3.4 With the above notation and assumptions: If $\operatorname{Tr}\circ(\Pi \circ T_{ge^{i\theta}})$ is a clean composition of the FIO Tr and the Toeplitz operator $\Pi \circ T_{ge^{i\theta}}$, then the composite isotropic relation is a union $\bigcup_{j=1}^{m} T_{\theta_j}^{+*}(S^1)$ of cotangent rays, and the principal symbol of the composition is given on the component $T_{\theta_i}^{+*}(S^1)$ by

(2.3.6) $\sigma_{Tro(\Pi \circ T_{roi}\theta)} = vol(B)\xi^r |d\xi|^{\frac{1}{2}}$

where $r = \frac{1}{2}(e - \dim B)$.

3 Proof of Main Theorem

We will determine the asymptotics of the measures by calculating the moments.

The p^{th} moment $M^{\mu,p}_{r\rho} = \int x^p dm^{\mu}_{r\rho}$ of the spectral measures $m^{\mu}_{r\rho}$, $r \in \mathbb{N}$, defined in (0.4) can be written as

$$M_{r\rho}^{\mu,p} = (\dim V_{r\rho})^{-1} \operatorname{Tr} T_{\mu}^{p} |_{H_{r}^{2}(B)} = (\dim V_{r\rho})^{-1} \operatorname{Tr} T_{\mu^{\star p}} |_{H_{r}^{2}(B)}.$$

To determine their asympotics, we form the generating functions

(3.1)
$$\mathcal{Y}_p(\theta) = \sum_{r=1}^{\infty} e^{ir\theta} \operatorname{Tr} T_{\mu^{\star p}} \big|_{H^2_r(B)}, \quad p \in \mathbb{N}$$

which equal

(3.1a)
$$\mathcal{Y}_{p}(\theta) = \operatorname{Tr}\left(T_{e^{i\theta}} \circ T_{\mu^{\star p}} \circ \Pi\right)$$

with $T_{e^{i\theta}}$ the translation by $e^{i\theta}$ in $L^2(B)$. This follows since the differential operator $\frac{1}{i}\frac{\partial}{\partial\theta}$ is equal to rI on $H^2_r(B)$. The main part of the proof of our theorem will be to show that the $\mathcal{Y}_p(\theta)$ are Lagrangean distributions on the circle S^1 . Recalling that Lagrangean distributions on S^1 have a very specific type of Fourier expansion (see Example 9), this will allow us to determine the asymptotics of the moments.

Recall that μ is a discrete measure on G, hence $\mu^{\star p}$ is discrete for all $p \in \mathbb{N}$. For fixed p, we can write $\mu^{\star p} = \sum_{g \in \mathrm{supp}\,\mu^{\star p}} \mu^{\star p}(g) \delta_g$. Hence

$$T_{\mu^{\star p}} = \sum_{g \in \operatorname{supp} \mu^{\star p}} \mu^{\star p}(g) T_g .$$

In order to show that $\mathcal{Y}_p(\theta)$ is a Lagrangean distribution, it therefore suffices to show this for the case $\mu^{\star p} = \delta_q$ for some $g \in G$.

We will now verify that the clean composition hypotheses are satisfied in (3.1a) in each composition. The first step is the composition $T_{e^{i\theta}} \circ T_g$. The cleanness of this composition has already been proved in Section 1.4. Recall from (1.4.7) that

$$T_{e^{i\theta}} \circ T_q \in I^{-\frac{1}{4}}(S^1 \times B \times B, \Lambda)$$

Next, we have the composition of $T_{e^{i\theta}} \circ T_g$ with the Szegö projector $\Pi \in I^{-\frac{1}{2}(\dim B-1)}(B \times B, \Delta(Y)')$. By the discussion in Example 16, it is clean with excess 0 and we have

(3.2)
$$T_{e^{i\theta}} \circ T_q \circ \Pi \in I^{\frac{1}{4} - \frac{1}{2} \dim B}(S^1 \times B \times B, \Lambda_Y) .$$

Third, we take the trace of the Toeplitz operator $T_{e^{i\theta}} \circ T_g \circ \Pi$. As explained in Section 1.5, the operation of taking the trace is itself a composition of Fourier Integrals and hence the trace will be a Lagrangean distribution if the composition is clean. We now show that cleanliness of the trace is equivalent to cleanliness of the fixed point sets of T_q on \mathcal{O}_{ν} in the sense of Bott (see below for the definition).

To prove this, we first recall (Lemma 1.5.1) that taking the trace corresponds to the compositon of Λ'_Y with the diagonal $\Delta(T^*B)$. To apply Theorem 2.3.2, we must determine when the fiber diagram

$$(3.3) \qquad \begin{array}{ccc} F & \longrightarrow & \Lambda'_{Y} \\ \downarrow & & \downarrow \\ \Delta(T^*B) & \longrightarrow & T^*B \times T^*B \\ & i \end{array}$$

is clean and when $\pi: F \to \Lambda'_Y \circ \Delta(T^*B)$ is a map of constant rank on the connected components of F.

The fiber product F in the above diagram is clearly

$$(3.4) \quad F = \{(e^{i\phi}, -r; b, r\alpha; b, r\alpha; b, r\alpha; b, r\alpha) | e^{i\phi}g \cdot b = b, r \in \mathbb{R}\} \cong \{(b, r\alpha) | \exists e^{i\phi} \text{ s.t. } e^{i\phi}g \cdot b = b, r \in \mathbb{R}\}.$$

Recall that B is an S^1 bundle over the coadjoint orbit \mathcal{O}_{ν} and that G and S^1 act on B via

$$e^{i\phi}g \cdot b = e^{i\phi}g \cdot [h, e^{i\theta}] = [gh, e^{i\phi+\theta}], \qquad e^{i\phi} \in S^1, g \in G, b = [h, e^{i\theta}] \in B$$

From this we see that

$$(3.5) (b, r\alpha) \in F \Leftrightarrow \underline{l}_{q}(\pi(b)) = \pi(b)$$

where \underline{l}_g denotes the action of g on \mathcal{O}_{ν} and π is the natural projection $\pi: B \to \mathcal{O}_{\nu}$. Let us denote the fixed point set of \underline{l}_g by $\operatorname{Fix}(\underline{l}_g)$. Thus we have

(3.6)
$$F \cong \{(b, r\alpha) | b \in \pi^{-1}(\operatorname{Fix}(\underline{l}_g)), r \in \mathbb{R}\} .$$

hence F is a manifold if $Fix(\underline{l}_q)$ is a manifold. We also need to determine when the diagram

(3.7)
$$\begin{array}{cccc} TF & \longrightarrow & T\Lambda'_{Y} \\ \downarrow & & \downarrow & \\ T(\Delta(T^*B \times T^*B)) & \longrightarrow & T(T^*B \times T^*B) \\ di & & \\ \end{array}$$

is a fiber diagram. Now, for any $v \in T\Lambda'_Y \cong T(S^1 \times Y)$,

$$d\pi_{2,3}(v) \in T(\Delta(T^*B \times T^*B)) \Leftrightarrow dl_{e^{i\phi}} dl_q(w) = w ,$$

w being the orthogonal projection of v onto TB. The contact form α is a connection one form on TB, i.e., at each $b \in B$, the kernel of α defines a horizontal subspace $\operatorname{Hor}_b \subset T_b B$ with $\operatorname{Hor}_b \cong T_{\pi(b)}\mathcal{O}_{\nu}$. Furthermore, $T_b B = \operatorname{Hor}_b \oplus V_b$ where V_b is the one dimensional tangent space along the fiber S^1 ,

$$\begin{split} dl_g(\mathrm{Hor}_b) &= \mathrm{Hor}_{g \cdot b}, \qquad dl_g(\mathrm{V}_b) = \mathrm{V}_{g \cdot b} \quad \text{and} \\ dl_{e^{i\phi}}(\mathrm{Hor}_b) &= \mathrm{Hor}_{e^{i\phi} \cdot b}, \qquad dl_{e^{i\phi}}(\mathrm{V}_b) = \mathrm{V}_{e^{i\phi} \cdot b} \;. \end{split}$$

; From all this it follows that (3.7) is a clean fiber diagram if

1) $\operatorname{Fix}(\underline{l}_{a})$ is a manifold and

2) for all $xG_{\nu} \in \operatorname{Fix}(\underline{l}_g), T_{xG_{\nu}}(\operatorname{Fix}(\underline{l}_g)) = \operatorname{Fix}(d\underline{l}_g)_{xG_{\nu}}$.

This is precisely that l_g has clean fixed point manifolds.

The next step is to show that for all $g \in G$, \underline{l}_g has clean fixed point manifolds on \mathcal{O}_{ν} . Note that it suffices to show this for all $g \in T$, where T is the chosen maximal torus. Indeed, each element $h \in G$ is conjugate to an element $g \in T$, i.e., $h = kgk^{-1}$ for some $k \in G$. This implies that $\operatorname{Fix}(\underline{l}_h) = k \cdot \operatorname{Fix}(\underline{l}_g)$ and Condition 2 holds for \underline{l}_g if and only if it holds for \underline{l}_h .

At times we will distinguish two types of elements in T, regular elements and singular elements. An element g is called regular, if its powers generate T. Almost all (with respect to Haar measure) elements in T are regular. On the other hand, there is the set of singular elements defined as the union of the kernels of the positive roots (see [B.tD], Section V.2). A singular element g generates a closed subgroup of T of dimension strictly smaller than dim T.

The case $\nu \in \text{interior}I^* \cap \mathbf{t}^*_+$.

In case $\nu \in \text{interior } I^* \cap \mathbf{t}^*_+$, the interior of the positive Weyl chamber, we have $G_{\nu} = T$ and thus $\mathcal{O}_{\nu} \cong G/T$. For all $yT \in \mathcal{O}_{\nu}$ we have

(3.8)
$$gyT = yT \iff y^{-1}gy \in T \iff y = zw \text{ with } w \in W_G \text{ and } z \in Z(g)$$

where W_G denotes the Weyl group of G and Z(g) denotes the centralizer of g. The centralizer Z(g) is always a closed subgroup of G. Note that

$$g$$
 regular \Leftrightarrow $Z(g) = T$

This shows that the fixed point set is given by $\operatorname{Fix}(\underline{l}_g) = \{zwT|z \in Z(g), w \in W_G\}$, hence is a closed submanifold of \mathcal{O}_{ν} . (In case g is regular, $\operatorname{Fix}(\underline{l}_g) = \{wT|z \in Z(g), w \in W_G\} \cong W_G$ is a discrete set and independent of g.) This proves Condition 1.

Let us now prove Condition 2. First, we show that Condition 2 holds at $eT \in \operatorname{Fix}(\underline{l}_g)$. The tangent space $T_{eT}\mathcal{O}_{\nu}$ is isomorphic to the subspace $L(G/T) := \bigoplus_{\beta \in R^+} M_{\beta}$ of the Lie algebra \mathbf{g} . Here R^+ denotes the set of positive roots and M_{β} denotes the 2-dimensional rootspace corresponding to the root β . Furthermore, the Lie algebra L(Z(g)) of the centralizer Z(g) is the direct sum

(3.9)
$$L(Z(g)) = \mathbf{t} \oplus \bigoplus_{\beta \in N} M_{\beta}$$

where **t** denotes the Lie algebra of T and N denotes the set of positive roots β whose kernel contains g (N is empty for g regular). See, for example, [B.tD], Proposition V.2.3. The tangent space of $\operatorname{Fix}(\underline{l}_g)$ at eT is therefore isomorphic to $\bigoplus_{\beta \in N} M_{\beta}$. Under the identification of $T_{eT}\mathcal{O}_{\nu}$ with $L(G/T) = \bigoplus_{\beta \in R^+} M_{\beta}$, the differential $d\underline{l}_g : T_{eT}\mathcal{O}_{\nu} \to T_{eT}\mathcal{O}_{\nu}$ is the adjoint representation $\operatorname{Ad}(g)$ at g, restricted to $\bigoplus_{\beta \in R^+} M_{\beta}$. But by definition of N, the subspace consisting of fixed points of $\operatorname{Ad}(g)$ is exactly the space $\bigoplus_{\beta \in N} M_{\beta}$. This proves that $\operatorname{Fix}(d\underline{l}_g)_{eT} = T_{eT}\operatorname{Fix}(\underline{l}_g)$.

We now show that Condition 2 holds at any other point in $\operatorname{Fix}(\underline{l}_a)$. Observe that for any $zwT \in \operatorname{Fix}(\underline{l}_a)$,

(3.10)
$$T_{zwT}\mathcal{O}_{\nu} \cong T_{e/T}\mathcal{O}_{\nu} \quad \text{via} \quad v \in T_{eT}\mathcal{O}_{\nu} \quad \leftrightarrow \quad d\underline{l}_{z}d\underline{r}_{w}(v) \in T_{zwT}\mathcal{O}_{\nu}$$

where \underline{r}_w denotes the map $\underline{r}_w : \mathcal{O}_\nu \to \mathcal{O}_\nu, \underline{r}_w(xT) = xwT$. Since \underline{r}_w and \underline{l}_z commute with \underline{l}_g ,

(3.11)
$$v \in \operatorname{Fix}(d\underline{l}_g)_{eT} \Leftrightarrow d\underline{l}_z d\underline{r}_w(v) \in \operatorname{Fix}(d\underline{l}_g)_{zwT}$$

and in particular

$$\dim\left(\operatorname{Fix}(d\underline{l}_g)_{eT}\right) = \dim\left(\operatorname{Fix}(d\underline{l}_g)_{zwT}\right)$$

which proves 2) for all g.

The case $\nu \notin \text{interior} I^* \cap \mathbf{t}_+^*$.

In this case ν lies in at least one wall of the fundamental Weyl chamber, i.e., there exist $\alpha \in R^+$, the set of positive roots, such that $\alpha(\mu) = 0$. (We have identified \mathbf{t}^* with \mathbf{t} via an inner product on \mathbf{t} .) The Lie algebra $L(G_{\nu})$ of the stabilizer group G_{ν} is

$$L(G_{\nu}) = \mathbf{t} \oplus \bigoplus_{\beta \in K} M_{\beta}$$

where $K := \{ \alpha \in R^+ | \alpha(\mu) = 0 \}$. This follows from the fact that $G_{\mu} = Z(s)$ for all singular elements $s = \exp X \in T$ for which $\alpha(X) = 0$ for all $\alpha \in K$ and $\alpha(X) \neq 0$ for all $\alpha \in K^c := R^+ \setminus K$. We therefore have

$$T_{eG_{\nu}}\mathcal{O}_{\nu} \cong \bigoplus_{\beta \in K^c} M_{\beta}$$

For any $g \in T$ and any $yG_{\nu} \in \mathcal{O}_{\nu}$ we claim that the following holds:

(3.12)
$$gyG_{\nu} = yG_{\nu} \iff y^{-1}gy \in G_{\nu} \iff y = zwu \text{ with } z \in Z(g), w \in WG, \text{ and } u \in G_{\nu}$$

Indeed, $y^{-1}gy \in G_{\nu}$ implies that there exists a $u \in G_{\nu}$ such that $uy^{-1}gyu^{-1} \in T$ (since T is also a maximal torus of G_{ν}). Furthermore, there exists a $w \in W_G$ such that $wuy^{-1}gyu^{-1}w^{-1} = g$. But this implies that $wuy^{-1} \in Z(g)$, from which the claim follows.

So far we have shown that the fixed point set is $\operatorname{Fix}(\underline{L}_g) = \{zwG_\nu | z \in Z(g), w \in WG\}$ and hence is a closed submanifold of \mathcal{O}_{ν} . In order to prove that Condition 2 holds, we can apply the same ideas as in the previous case. Again, we consider the decomposition of L(Z(g)) as in (3.9): $L(Z(g)) = \mathbf{t} \oplus \bigoplus_{\beta \in N} M_{\beta}$. This provides us with the isomorphism

(3.13)
$$T_{eG_{\nu}}\operatorname{Fix}(\underline{l}_{g}) \cong \bigoplus_{\beta \in N \setminus K} M_{\beta} .$$

Under this isomorphism, the differential $d\underline{l}_g: T_{eG_\nu}\mathcal{O}_\nu \to T_{eG_\nu}\mathcal{O}_\nu$ is the adjoint representation $\operatorname{Ad}(g)$ at g, restricted to $\bigoplus_{\beta \in K^c} M_\beta$. But the fixed point set of $\operatorname{Ad}g$ restricted to T_{eG_ν} is exactly $\bigoplus_{\beta \in N \setminus K} M_\beta$. This proves that $\operatorname{Fix}(d\underline{l}_g)_{eG_\nu} = T_{eG_\nu}\operatorname{Fix}(\underline{l}_g)$. The rest is completely analogous to the case $\nu \in \operatorname{interior} I^* \cap \mathfrak{t}^*_+$.

The rank of the map $\pi : F \to \Lambda'_Y \circ \Delta(T^*B)$ is constant and equal to 1. This follows easily from the fact that the action of S^1 and the action of G on Y are both homogeneous in \mathbb{R} (recall (3.4)).

We have now proved that the trace operation is clean and thus that $\mathcal{Y}_p(\theta)$ is a Lagrangean distribution on S^1 . We still need to compute the excess of this composition. From (3.3) and the definition of the excess e we have

$$(3.14) e = \dim F + \dim(T^*B \times T^*B) - (\dim(\Delta(T^*B) - \dim\Lambda'_Y)).$$

We know that a Lagrangean distribution on the circle S^1 has only a finite number of singular points θ_j , j = 1, ..., K. Among those singular θ'_j s, the ones for which $e^{i\theta_j}g$ fixes the *whole* circle bundle *B* produce the connected components of highest possible dimension, namely dimY, of *F*. For $b = [h, e^{i\phi}] \in B$ we have

$$(3.15) \qquad e^{i\theta}g \cdot [h, e^{i\phi}] = [gh, e^{i(\theta+\phi)}] = [h, e^{i\phi}] \quad \Leftrightarrow \quad \exists t \in G_{\nu} \quad \text{such that} \quad gh = ht \text{ and } e^{i\theta} = \chi_{\nu}(t) \quad dt \in G_{\nu}$$

Thus, if the fixed point set $Fix(e^{i\theta}g) = B$ it follows that

$$g \in Z_{\nu} := \bigcap_{h \in G} h \, G_{\nu} \, h^{-1}$$

Hence we have,

$$\operatorname{Fix}(e^{i\theta}g) = B$$
 iff $\nu \in Z_{\nu}$ and $e^{i\theta} = \chi_{\nu}(g)$

Moreover, $Z_{\nu} = Z_G$ if ν is an interior point, since G_{ν} is then a maximal torus and the intersection of all maximal torii is the center. Provided that $\operatorname{Fix}(e^{i\theta}g) = B$, for at least one $(e^{i\theta}, g)$, the excess of the trace operation is

(3.16)
$$e = \dim Y + 4\dim B - 2\dim B - (1 + \dim Y) = 2\dim B - 1.$$

; From this, (3.2), and the fact that the trace operation $\text{Tr} \in I^0(S^1 \times B \times B \times S^1, \Delta(T^*S^1 \times T^*B))$ we get

(3.17)
$$\mathcal{Y}_p(\theta) \in I^{\frac{1}{2}\dim B - \frac{1}{4}}(S^1, \bigcup_{j=1}^K T_{\theta_j}^{+*}S^1) .$$

Remark We should note that often $Z_{\nu} = Z_G$ even when ν is a wall weight. For instance, in the case of G = U(n) this happens as long as ν is not a scalar weight (i.e. a weight of the form (r, r, \ldots, r)). Indeed, $g \in \bigcap_{h \in G} h G_{\nu} h^{-1}$ implies that for all $h \in G$, g commutes with $h\nu h^{-1}$ and thus with any $h\pi h^{-1}$ where π denotes any spectral projection of ν . Here, we are regarding elements of G, of \mathbf{g} , thus also of \mathbf{t}^* , as

endomorphisms of \mathbb{C}^n . If ν is not a scalar, there exists an eigenvalue a of ν whose eigenspace K (the range the spectral projection π_a) has dimension k with k < n. Since for all $h \in G$, g commutes with $h\pi_a h^{-1}$, gleaves each k dimensional subspace h(K), $h \in G$, invariant. But this implies that g leaves every k dimensional subspace invariant (G acts transitively on the Grassman manifolds). We claim this is only possible if g is a scalar, i.e. if $g \in Z_G$. Indeed suppose g is not a scalar. Then there is at least one vector $x \in \mathbb{R}^n$ (\mathbb{C}^n , \mathbb{H}^n) for which g(x) = cx + y with $y - x, y \neq 0$. We then can find a k dimensional subspace W containing x but not containing y, which is a contradiction to $g(W) \subset W$.

The situation for the other classical groups SO(n), Sp(n) is more or less analogous (with almost complex structures J playing the role of scalar weights).

Recall now Example 9. Since $\mathcal{Y}_p(\theta)$ is polyhomogeneous and a Hardy distribution (the wave front set is contained in a union of positive half spaces), it can be written as a sum

(3.18)
$$\mathcal{Y}_p(\theta) = \sum_{j=0}^{\infty} \sum_{l=1}^{K} a_{j,l} \chi_{m-j}(\theta - \theta_l) \,.$$

of the basic homogeneous distributions χ_r discussed in Example 8. Provided that $a_{0,l} \neq 0$ for at least one l, this means that $\mathcal{Y}_p(\theta) \in I^{m+\frac{1}{4}}(S^1, \bigcup_{l=1}^K T_{\theta_l}^{+*}S^1)$. We easily find that $m = \frac{1}{2}(\dim B - 1)$ with the use of (3.17). From (3.18) we get

$$\operatorname{Tr} T_{\mu^{\star p}} \big|_{H^2_r(B)} = r^m \sum_{l=1}^K a_{0,l} e^{-ir\theta_l} + r^{m-1} \sum_{l=1}^K a_{1,l} e^{-ir\theta_l} + \quad \text{lower order terms} \;.$$

On the other hand by Proposition 2.3.4 of Section 2.3,

$$\sigma(\mathcal{Y}_p) = \sum_{(i_1,\dots,i_p)} c_{i_1}\dots c_{i_p} \sigma_{Tr \prod T_{g_{i_1}\dots g_{i_p}} e^{i\theta}}$$
$$= \sum_{(i_1,\dots,i_p)} vol(B)c_{i_1}\dots c_{i_p} \sigma(\chi_m)(\theta_{g_{i_1}\dots g_{i_p}}) .$$

Since a Lagrangean distribution of order m with vanishing principal symbol actually has order m-1, we conclude that

$$\mathcal{Y}_p = \sum_{\substack{(i_1, \dots, i_p):\\g_{i_1} \dots g_{i_p} \in \mathbb{Z}_p}} vol(B)c_{i_1} \dots c_{i_p}\chi_m(\theta - \theta_{g_{i_1} \dots g_{i_p}}) + smoother$$

and hence by (3.18) we have

$$a_{0,l} = c_{i_1} \dots c_{i_p} vol(B).$$

By comparing Fourier series, we find that

$$\operatorname{Tr} T_{\mu^{*p}} |_{H^2_r(B)} \sim Cvol(B) r^m \sum_{\substack{(i_1, \dots, i_p):\\g_{i_1} \dots g_{i_p} \in \mathbb{Z}_p}} c_{i_1} \dots c_{i_p} e^{-ir\theta_{g_{i_1} \dots g_{i_p}}} + O(r^{m-1})$$

with $e^{-ir\theta_{g_{i_1}\dots g_{i_p}}} = \chi_{\rho}(g_{i_1}\dots g_{i_p})$ and with C a universal constant. After normalizing by taking the quotient with $\dim H^2_r(B) = \dim V_{r\rho}$ the coefficient $r^m Cvol(B)$ cancels and we get

$$\frac{\operatorname{Tr} T_{\mu^{\star p}}|_{H^2_r(B)}}{\dim H^2_r(B)} = \sum_{\substack{(i_1,\dots,i_p):\\g_{i_1}\dots g_{i_p}\in \mathbb{Z}_p}} c_{i_1}\dots c_{i_p} e^{-ir\theta_{g_{i_1}\dots g_{i_p}}} + o(1)$$

completing the proof of the Main Theorem.

The proofs of Corollaries 1-2 were given in the introduction. The proof of Corollary 3 is essentially the combination of Corollary 1 with known results on the spectrum of $\lambda(T_{\mu})$ (the spectrum of the induced random walk on $l^2(\Gamma)$.) For the sake of completeness, we include the proof:

Proof of Corollary 3. (i) Kesten's amenability criterion for countable groups Γ states that

 Γ is an amenable group \Leftrightarrow sup supp $\gamma = 1$

where γ is the spectral measure of the random walk (Γ, μ) (see [Ke][K.V., Section 5.1]). Suppose that $\beta_{\rho}^* < 1$, $\rho \neq$ scalar, and $\chi_{\rho}(z) \neq 1$ for $z \in Z_G \cap \Gamma \setminus \{id\}$. Consider an arbitrary continuous function f with support in the interval $[\beta_{\rho}^*, 1]$. It follows that

$$\int f dm^{\mu}_{r\rho} = 0 \quad \forall r \in \mathbb{N} \; .$$

On the other hand, by (0.5),

(3.19)
$$\int f dm_{r\rho}^{\mu} = \sum_{z \in Z_G \cap \Gamma} \chi_{\rho}(z)^r \int f dm^{\mu, z} + O(r^{-1})$$

But $\int f dm^{\mu,e} > 0$ $(m^{\mu,e}$ is Kesten's spectral measure for (Γ,μ) , so, by our assumptions on the $\chi_{\rho}(z)$, the right hand side in (3.19) is unequal to zero, which is a contradiction.

(ii) Let Γ be a free countable group and μ a symmetric probability measure on Γ supported on the set of generators $\{a_i^{\pm 1}, i = 1, ..., n\}$ with $\{a_i, i = 1, ..., n\}$ free. A theorem by Kesten ([Ke, Theorem 3]) states that

$$\sigma(\Gamma, \mu) = \left[-\frac{\sqrt{2n-1}}{n}, \frac{\sqrt{2n-1}}{n}\right]$$

Suppose now that $\left[-\frac{\sqrt{2n-1}}{n}, \frac{\sqrt{2n-1}}{n}\right] \not\subset \overline{\sigma(T_{\mu}|_{\mathbb{N}\rho})}$. It follows that there exists an interval $[c, d] \subset \left[-\frac{\sqrt{2n-1}}{n}, \frac{\sqrt{2n-1}}{n}\right]$ with $[c, d] \cap \overline{\sigma(T_{\mu}|_{\mathbb{N}\rho})} = \emptyset$. Consider an arbitrary continuous function f with support in [c, d]. The same argument as in (i) shows that integrating both sides in (0.5) against f leads to a contradiction.

4 Examples

4.1 Random walk on the torus $T^k = (S^1)^k$. Let $G = T^k$ and $\mu = \frac{1}{2} (\delta_{(e^{i\alpha_1}, \dots, e^{i\alpha_k})} + \delta_{(e^{-i\alpha_1}, \dots, e^{-i\alpha_k})})$ with $(\alpha_1, \dots, \alpha_k) \notin (2\pi \mathbb{Q})^k$. All irreducible representations of T^k are one dimensional. They correspond to the lattice points $(\mathbb{Z})^k \subset \mathbb{R}^k$. For a given $\rho = (n_1, \dots, n_k)$, the eigenvalues of T_{μ} along the ray $\mathbb{N}\rho$ can easily be computed as $\lambda_{r\rho} = \cos(r(n_1\alpha_1 + \dots + n_k\alpha_k))$ $(r \in \mathbb{N})$. By the Main Theorem, we have

$$dm_{r\rho}^{\mu} = \delta_{\cos(r(n_{1}\alpha_{1}+...+n_{k}\alpha_{k}))} \sim \sum_{z \in T^{k}} \chi_{\rho}(z)^{r} dm_{\infty\rho}^{\mu,z} + O(r^{-1})$$

with $\chi_{\rho}(z)^r = e^{ri(n_1z_1 + \ldots + n_kz_k)}$ for $z = (e^{iz_1}, \ldots, e^{iz_k})$. The $dm_{\infty\rho}^{\mu,z}$ are determined by the moments

$$\int_{\mathrm{I\!R}} x^p dm_{\infty\rho}^{\mu,z} = \mu^{*p}(\{z\})$$

and are zero unless $z = (e^{il\alpha_1}, \ldots, e^{il\alpha_k})$ for some $l \in \mathbb{Z}$. Note that for fixed $z = (e^{il\alpha_1}, \ldots, e^{il\alpha_k})$,

$$\mu^{*p}(z) = \begin{cases} \left(\frac{p}{p-|l|}\right)\left(\frac{1}{2}\right)^p & \text{for} \quad |l| \le p \text{ and } (p-|l|) \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

No weak limit exists for $m_{r\rho}^{\mu}$ as $r \to \infty$. Since T^k is an abelian group, Γ is amenable, and by Corollary 2(i), the spectral radius $\beta_{\rho}^* = 1$. (This can, of course, also be seen directly from the fact that $\{e^{r(n_1\alpha_1+\ldots+n_k\alpha_k)} | r \in \mathbb{N}\}$ is dense in S^1 .)

4.2 Random walk on the sphere. The special orthogonal group SO(n + 1) acts transitively on the *n*-dimensional unit sphere $S^n \cong SO(n + 1)/SO(n)$ in \mathbb{R}^{n+1} . Through this action, we get a representation of SO(n + 1) on the Hilbert space $L^2(S^n)$ which is exactly a ray, namely

$$L^2(S^n) = \mathbb{N}\rho_1$$

where ρ_1 is the (n+1)-dimensional natural representation. This follows since $L^2(S^n) \cong L^2(SO(n+1)/SO(n))$ is an induced representation, namely the trivial representation $\tilde{\rho_0}$ of SO(n) induced up to SO(n+1). By Frobenius reciprocity,

multiplicity of
$$\rho \in L^2(S^n)$$
 = multiplicity of $\tilde{\rho_0} \in \operatorname{res}_{SO(n)}^{SO(n+1)} \rho$

By the familiar branching rules for the restiction of an irreducible of SO(n + 1) to SO(n) (see, for example, [F.H, §25.3]), we have

multiplicity of
$$\tilde{\rho_0} \in \operatorname{res}_{SO(n)}^{SO(n+1)} \rho \leq 1$$

and

$$\tilde{\rho_0} \in \operatorname{res}_{SO(n)}^{SO(n+1)} \rho \quad \Leftrightarrow \quad \rho = r\rho_1, r \in \mathbb{N}$$

(Although $L^2(S^n)$ is a ray of representations, it is not presented as the Hardy space of a prequantum circle bundle B over an orbit \mathcal{O} . Nevertheless, the methods of this paper apply to it, and in fact in a more elementary form: there is no need to introduce a Toeplitz projector. Otherwise, all of the previous calculations remain valid.)

4.3 A few rank 2 groups. (a) G = U(2). The maximal torus of the unitary group U(2) is the subgroup of diagonal matrices $T = \{ \text{diag}(e^{i\phi_1}, e^{i\phi_2}) | \phi_1, \phi_2 \in [0, 2\pi) \} \cong U(1) \times U(1)$ and its center is the subgroup of scalar matrices $Z_{U(2)} = \{ \text{diag}(e^{i\phi}, e^{i\phi}) | \phi \in [0, 2\pi) \} \cong U(1)$. There is only one positive root, $(e_2 - e_1) (e_i \text{ denotes the } i\text{-th standard basis vector of } \mathbb{R}^2$. The positive Weyl chamber is $\mathbf{t}^*_+ = \{(x_1, x_2) : x_2 \ge x_1\} \subset \mathbb{R}^2$; it has a unique wall $x_2 = x_1$. Below is a picture of \mathbf{t}^*_+ and a few rays.



The orbit corresponding to an interior lattice point ν is $\mathcal{O}_{\nu} = U(2)/T = U(2)/U(1) \times U(1) = SU(2)/U(1)$ which is the 2-sphere S^2 (equivalently, complex projective space $\mathbb{C}P^1$). The prequantum S^1 bundle B is then SU(2) itself and the projection to S^2 is the usual Hopf projection from $S^3 \to S^2$.

For a lattice point ψ in the wall, we have $G_{\psi} = U(2)$. Thus the orbit \mathcal{O}_{ψ} is just one point. It follows that $B = S^1$, so the Hardy space is just the classical Hardy space of the unit circle, i.e., $H^2(B) = \{\sum_{r=1}^{\infty} a_r e^{ir\theta} \mid \sum_{r=1}^{\infty} a_r^2 < \infty\}$.

(b) G = SO(4). The maximal torus is the subgroup of block diagonal matrices

$$T = \{ \operatorname{diag} \left(\begin{pmatrix} \cos \phi_1 & -\sin \phi_1 \\ \sin \phi_1 & \cos \phi_1 \end{pmatrix} \times \begin{pmatrix} \cos \phi_2 & -\sin \phi_2 \\ \sin \phi_2 & \cos \phi_2 \end{pmatrix} \} \cong SO(2) \times SO(2)$$

and the center is $Z_{SO(4)} = \{\pm I\} \cong Z_2$ $(Z_{SO(n)} \cong Z_2$ for any even n). A positive root system is $R^+ = \{(e_2 - e_1), (e_2 + e_1)\}$; this gives rise to the positive Weyl chamber $\mathbf{t}^*_+ = \{(x_1, x_2) : x_2 \ge |x_1|\}$. Picture (b) below shows \mathbf{t}^*_+ and a few rays. The lattice point corresponding to the natural representation ρ_1 is (0, 1) and lies in the interior. The coadjoint orbit \mathcal{O}_{ν} for any interior ν is the Grassmann manifold $G_{2,4}(\mathbb{R}) = SO(4)/SO(2) \times SO(2)$.

(c) G = SO(5). The maximal torus is the subgroup of block diagonal matrices

$$T = \{ \operatorname{diag} \left(1 \times \left(\begin{array}{c} \cos \phi_1 & -\sin \phi_1 \\ \sin \phi_1 & \cos \phi_1 \end{array} \right) \times \left(\begin{array}{c} \cos \phi_2 & -\sin \phi_2 \\ \sin \phi_2 & \cos \phi_2 \end{array} \right) \right) \} \cong SO(2) \times SO(2)$$

and the center is trivial $(Z_{SO(n)})$ is always trivial for n odd). A positive root system is $R^+ = \{(e_2 - e_1), (e_2 + e_1), e_2, e_1\}$; this gives rise to the positive Weylchamber $\mathbf{t}^*_+ = \{(x_1, x_2) : x_2 \ge x_1 \ge 0\}$ (see Picture (c) below). Here the lattice point (0, 1) corresponding to the natural representation ρ_1 lies in the wall $x_1 = 0$. The stabilizer group G_{ρ_1} is $SO(3) \times SO(2)$. It follows that the coadjoint orbit \mathcal{O}_{ρ_1} is the Grassmann manifold $G_{3,5}(\mathbb{R}) = SO(5)/SO(3) \times SO(2)$.



4.4 Pseudo-Harper operators. As mentioned in the introduction, an interesting example of a random walk on the unitary group U(q) is provided by the image of a rational Harper operator H_{θ} $\left(\theta = \frac{p}{q}\right)$ under the distinguished irreducible unitary representation of the rotation algebra A_{θ} taking U to $u = \text{diag}(1, e^{i\theta}, \ldots, e^{i(q-1)\theta})$ and V to the cyclic permutation $v \in U(q)$ of the standard (eigenvector) basis e_1, \ldots, e_q . We recall that the rotation algebra A_{θ} is the algebra generated by unitaries U, V satisfying $UV = e^{2\pi i\theta}VU$. The elements u, vof U(q) determine the probability measure

$$\mu = \frac{1}{4} (\delta_u + \delta_{u^*} + \delta_v + \delta_{v^*})$$

on U(q) and hence the transition operator $H^2(B_{\rho})$ given by

$$T_{\mu} = \frac{1}{4} (T_u + T_{u^*} + T_v + T_{v^*})$$

with T_g the translation operator by g on $H^2(B_{\rho})$.

We recall from the introduction that

$$(4.4.1) T_u T_v = T_{z_o} T_v T_u$$

where $z_o = e^{2\pi i\theta}$ is in the center of U(q). Since this is not the commutation relation of A_{θ} we cannot expect T_{μ} to have a subspectrum of H_{θ} . Nevertheless, the spectrum of T_{μ} can be determined along any ray of representations of U(q) from the spectral theory of Harper operators. In the following Sp denotes the spectrum in the usual sense as a closed set.

Proposition 4.4.2 With the notations above: Along any ray of representations $\mathbb{N}\rho$, $Sp(T_{\mu}'|_{H^2(B_{\rho})}) = [-1, 1]$ as a set. In fact, it is the union of the spectra of q Harper operators, including the commutative Harper operator.

Proof: We begin by relating the commutation relation (4.4.1) along subrays to those of certain rotation algebras. Observe, then, that at the *r*th representation, say $V_{r\rho}$, along the ray $\mathbb{N}\rho$ the center $Z_{U(q)} = \{zI : z \in U(1)\}$ acts by the scalar $\chi_{\rho}(z)^r$. Indeed, if we let f(g, w) denote a function on $G \times U(1)$ which is invariant under the action of G_{ρ} defining $B_{\rho} := (G \times U(1))/G_{\rho}$, and equivariant of degree *r* under the action of U(1)on B_{ρ} , then

$$T_z f(g, w) = f(zg, w) = f(g, \chi_\rho(z)w) = \chi_\rho^r(z)f(g, w)$$

Since $V_{r\rho}$ consists of the degree r CR-functions on B_{ρ} (under the U(1)-action), we see that on $V_{r\rho}$ the commutation relations for T_u, T_v read:

(4.4.3)
$$T_u T_v = \chi_\rho(z_o)^r T_v T_u \qquad z_o = e^{2\pi i\theta}$$

Here $\chi_{\rho}(z_o)^r$ denotes multiplication by this scalar. We note that if $\rho = (\lambda_1, \ldots, \lambda_q)$ then $\chi_{\rho}(e^{2\pi i\theta}) = e^{2\pi i\theta\overline{\rho}}$ with $\overline{\rho} = \lambda_1 + \cdots + \lambda_q$. Thus, T_u, T_v generate the rotation algebra $A_{r\theta\overline{\rho}}$ on $V_{r\rho}$.

We further observe that when $\theta = \frac{p}{q}$, the multiplier $e^{2\pi i r \theta \overline{\rho}}$ is periodic of period q in r. Of course it might be periodic of a smaller period if $\overline{\rho}$ is not relatively prime to q. However we always have that

(4.4.4*a*)
$$H^{2}(B_{\rho}) = \bigoplus_{m=0}^{q-1} H_{m}^{2}$$

with

and with u, v satisfying the commutation relation

$$uv = e^{2\pi i m\theta \overline{\rho}} vu$$

on H_m^2 .

Let us henceforth assume for the sake of simplicity that $(\overline{\rho}p, q) = 1$. We will call such a ray a "relatively prime ray" for T_{μ} . Also, let $T_{\mu,m}$ denote the restriction of the random walk to the 'sub-ray' H_m^2 . The main step in the proof of the Proposition is contained in the following **Lemma 4.4.5** $SpT_{\mu,m} = SpH_{\frac{p'}{q'}}$ where $\frac{p'}{q'} = \frac{mp\overline{p}}{q}$.

Proof: We first recall that $SpH_{\frac{p'}{q'}}$, the spectrum of $H_{\frac{p'}{q'}}$ as an element of $A_{\frac{p'}{q'}}$, is the union of the spectra of the $q' \times q'$ matrices $\pi_{z_1,z_2}(H_{\frac{p'}{q'}})$ which arise in the irreducible representations of $A_{\frac{p'}{q'}}$. We recall that for $\theta = \frac{p'}{q'}, (p',q') = 1$, the irreducible representations of A_{θ} are all of dimension q' and are parametrized, with some redundancies, by the 2-torus T^2 (see [C.E.Y., p. 227]). One irreducible representation π is defined by $\pi(U) = \text{diag}(1,\eta,\eta^2,\ldots,\eta^{q-1})$ where $\eta = e^{2\pi i\theta}$ and by putting $\pi(V)$ equal to the cyclic permutation operator on the eigenvector basis e_1,\ldots,e_q . The other representations π_{z_1,z_2} are obtained by composing π with the automorphism α_{z_1,z_2} defined by $U \to z_1U$, $V \to z_2V$. Thus, the spectrum of $H_{\frac{p'}{q'}}$ is simply a union of spectra of finite matrices.

On the other hand, the spectrum of $T_{\mu,m}$ is also a union of spectra of finite matrices, namely the matrices $T_{\mu}'|_{V_{r\rho}}$ occuring along the points of the subray. These matrices are the images of T_{μ} in the irreducible representations of $A_{\frac{p'}{q'}}$ occuring in the subray. Hence $Sp(T_{\mu,m})$ is the union of the spectra of these matrices, taken over all the irreducibles occurring along the ray. In particular, $Sp(T_{\mu,m}) \subset Sp(H_{\frac{p'}{q'}})$.

To prove equality, it would suffice to show that the set of irreducibles of $A_{\frac{p'}{q'}}$ occurring along the sub-ray is dense in the set T^2 of all irreducibles. This is not implausible, since the dimensions $d_r = \dim V_{r\rho}$ of the irreducibles of U(q) along the ray are growing at a polynomial rate, while all the irreducibles of $A_{\frac{p'}{q'}}$ are of dimension q'. Hence, $V_{(m+nq)\rho}$ are highly reducible as representations of $A_{\frac{p'}{q'}}$, at least for n >> 1. The set of irreducibles of $A_{\frac{p'}{q'}}$ occurring in H^2_m thus determines a countably infinite subset $\hat{A}_{\frac{p'}{q'},\rho,m}$ of the torus T^2 parametrizing all of the irreducibles, and the closure of the union of all the eigenvalues of $H_{\frac{p'}{q'}}$ (as we range over $\hat{A}_{\frac{p'}{T},\rho,m}$) gives the spectrum of the random walk along the *m*th subray.

However, we can argue more directly, using the Main Theorem. To this end, we reconsider the asymptotic formulae for the spectral measures $m_{r\rho}^{\mu}$ as a sum of measures $m^{\mu,z}$ with the oscillatory coefficients $\chi_{\rho}(z)^r$, where z ranges over $\Gamma \cap Z_{U(q)}$. We note that $uvu^*v^* = z_0 := e^{2\pi i \frac{p}{q}}$ lies in $\Gamma \cap Z_{U(q)}$ and that all other intersection points are powers of z_0 . Indeed, to lie in the center, a word in the letters u, v must consist of a string of u, u^*, v, v^* with an equal number of u, u^* 's and v, v^* 's. Cancelling out we see that all the central elements lie in the cyclic group generated by z_0 . We can therefore rewrite the asymptotic formula for $m_{(m+nq)\rho}^{\mu}$ in the form

(4.4.6*a*)
$$m^{\mu}_{(m+nq)\rho} \sim \sum_{a \in \mathbb{N}} \chi_{\rho}(z_0^a)^m m^{\mu, z_0^a} \qquad (n \to \infty)$$

in the sense that

(4.4.6(m,k))
$$\int_{\mathbb{R}} x^k dm^{\mu}_{(m+nq)\rho} = \sum_{a \in \mathbb{N}} e^{2\pi i m a \overline{\rho} \frac{p}{q}} \mu^{*k} (\{z_0^a\}) + o(n).$$

We observe that the principal term on the right hand side is independent of n and hence gives a well-defined (non-oscillatory) asymptote along the each of the subrays.

Moreover, the asymptote is identifiable as the spectral measure for the Harper operator in its "magnetic translation representation" [B][B.V.Z] [Su]. In this representation U, V get taken to the unitary operators on $l^2(\mathbb{Z}^2)$ defined by

$$(U\phi)(m,n) = e^{-2\pi i \alpha_2 m} \phi(m,n+1) \qquad (V\phi)(m,n) = e^{-2\pi i \alpha_1 n} \phi(m+1,n)$$

where $\theta = \alpha_2 - \alpha_1$. In the physical interpretation, the pair (α_1, α_2) define the vector potential $\alpha_1 y dx + \alpha_2 x dy$ of a magnetic field $B = (\alpha_2 - \alpha_1) dx \wedge dy$ whose flux thru a period cell of the lattice is θ . The Harper operator $H_{\theta} = \frac{1}{4}(U + U^* + V + V^*)$ has then a spectrum as an operator on $l^2(\mathbb{Z}^2)$, and it coincides with the spectrum of this operator as an element of the rotation algebra A_{θ} .

Recall now that there exists a unique trace τ on A_{θ} , namely

$$\tau \sum a_{m,n} U^m V^n = a_{o,o} = \langle \sum a_{m,n} U^m V^n \delta_0, \delta_0 \rangle$$

where 0 denotes the orgin of \mathbb{Z}^2 (cf. [B][Su]). We can thus define a spectral measure for H_{θ} via the formula

(4.4.7)
$$\mu_{\theta}(f) := \tau(f(H_{\theta})) \qquad f \in C(Sp(H_{\theta}))$$

It is closely related to the spectral measure for the discrete random walk defined by H_{θ} on the Heisenberg group (cf. [B.V.Z])

We then observe that dm_{θ} is essentially the same measure as the asymptote to $dm^{\mu}_{(m+nq)\rho}$ in (4.4.6a) when $\theta = \frac{p'}{q'}$ as above. To see this, we compare moments. In the case of dm_{θ} we can compute these by associating to any monomial in $(U + V + U^* + V^*)^k$ a path in the lattice \mathbb{Z}^2 starting at the origin and then determined by the following traffic directions: reading from right to left in the monomial, go right at U, left at U^* , up at V, and down at V^* . We then observe that only monomials giving rise to loops (i.e., closed paths starting at ending at the origin) contribute to the moments. Hence only even moments are non-zero. Using $UV = e^{2\pi i\theta}VU$, we see that the contribution of such "loop-monomials" to the 2kth moment has the form

$$\sum_{a \in \mathbb{N}} \sum_{\text{loops} \in \Gamma_{2k}} \chi_{\rho}(z_o^a)^m \# \{2k \text{-loops} : z_{\text{loop}} = z_o^a\}$$

Here, Γ_{2k} is the set of elements of word-length 2k and for a word representing a "loop", z_{loop} is the element of the center to which the word reduces. We have also used the definition of $\frac{p'}{q'}$. Comparing to the leading asymptote of $dm^{\mu}_{(m+nq)\rho}$ we see that both spectral measures have precisely the same moments. It follows that they have the same supports.

Since the spectrum of H_{θ} equals the support of m_{θ} , we conclude that the spectrum of $T_{\mu,m}$ contains the spectrum of H_{θ} , where as above $\theta = \frac{p'}{q'}$. Combined with the reverse inclusion noted above, the proof of the lemma is complete.

To complete the proof of the proposition, we note that

$$Sp(T_{\mu}'|_{H^{2}_{\rho}(B)}) = \bigcup_{m=0}^{q-1} Sp(T_{\mu}'|_{H^{2}_{m,\rho}(B)}).$$

In the case of m = 0, $T_{\mu} / _{H^{2}_{m,\rho}(B)}$ is Harper operator corresponding to the the commutative Harper algebra, and its spectrum is that of the multiplication operator $\frac{1}{2}(\cos x + \cos y)$, namely [-1, 1]. For the other values of m, $Sp(T_{\mu} / _{H^{2}_{m,\rho}(B)})$ equals the spectrum of the Harper operator $H_{\frac{m_{p}\overline{\rho}}{q}}$ and hence is the union of bands separated by gaps as described in [B][C.E.Y]. It follows that $Sp(T_{\mu} / _{H^{2}_{\rho}(B)})$ is a dense pure point spectrum in [-1, 1] whose closure has a well-defined multiplicity function obtained from the comparison with the q Harper operators.

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