ESI The Erwin Schrödinger International Institute for Mathematical Physics

Boltzmanngasse 9
A-1090 Wien, Austria

Semi-classical Limit of Random Walks II

Ursula Porod
Steve Zelditch

Vienna, Preprint ESI 575 (1998)

June 25, 1998

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at

Semi-classical Limit of Random Walks II[1]

Ursula Porod
Steve Zelditch

Institute for Advanced Study, Princeton, NJ 08540
Johns Hopkins University, Baltimore, MD 21218

May 1997
Version 1

Abstract

Let (G, μ) be a symmetric random walk on a compact Lie group G. We will call (G, μ) a *Lagrangean* random walk if the step distribution μ, a probability measure on G, is also a Lagrangean distribution on G with respect to some Lagrangean submanifold $\Lambda \subset T^*G$. In particular, we are interested in the cases where μ is a smooth δ-function δ_C along a 'positively curved hypersurface' C of G or where μ is a sum of δ-functions $\sum_j \delta_{C_j}$ along a finite union of regular conjugacy classes C_j in G. The Markov (transition) operator T_μ of the Lagrangean random walk is then a Fourier integral operator and our purpose is to apply microlocal techniques to study the convolution powers μ^{*k} of μ.

In cases where all convolution powers are 'clean' (such as for δ-functions on positively curved hypersurfaces), classical FIO methods will be used to determine

- the Sobolev smoothing order of T_μ on $W^s(G)$,
- the minimal power $k = k_\mu$ for which $\mu^{*k} \in L^2$,
- the asympotics of the Fourier transform $\hat{\mu}(\rho)$ of μ along rays $L = \mathbb{N}\rho$ of representations.

In general, convolutions of Lagrangean measures are not 'clean' and there can occur a large variety of possible singular behaviour in the convolution powers μ^{*k}. Classical FIO methods are then no longer sufficient to analyze the asymptotic properties of Lagrangean random walks. However, it is sometimes possible to restore the simple 'clean convolution' behaviour by restricting the random walk to a fixed 'ray of representations.' In such cases, classical Toeplitz methods can be used to determine restricted versions of the above features along the ray. We will illustrate with the case of sums of δ-functions along unions of regular conjugacy classes.

Contents

[1]Partially supported by the Miller Institute for Basic Research in Science, partially by the Institute for Advanced Study, and partially by NSF grant #DMS-9404637.

0 Introduction

This paper, a continuation of [P.Z], is concerned with the spectral theory of random walks (G, μ) on a compact Lie group G. Our purpose is to apply microlocal methods to study the spectrum of the Markov transition operator T_μ of a random walk in the case where μ is a *Lagrangean measure* on G. That is, where $\mu \in I^*(G, \Lambda)$ is a Lagrangean distribution on G, with respect to some Lagrangean $\Lambda \subset T^*G - 0$, in addition to being a probability measure on G. In particular, we are interested in the cases where μ is a probability measure of the form

(i) $\mu = \frac{1}{C}(\delta_X + \delta_{X^{-1}})$ with $X \subset G$ a *positively curved* hypersurface; or

(ii) $\mu = \frac{1}{C}\sum_{j=1}^N \delta_{C_{x_j}}$ with C_{x_j} the conjugacy class of x_j.

Here, δ_Y is a generalized δ-function along Y, i.e. $\int_Y f d\sigma_Y$ with $d\sigma_Y$ a smooth density on Y. Our aim is to determine:

- The asymptotics of the Fourier transform of μ along rays of representations $L = \mathbb{N}\rho$ of G;
- The Sobolev smoothing order of T_μ;
- The minimal power so that $\mu^{*k} \in L^2(G)$.

When μ is a Lagrangean measure, T_μ is a Fourier integral operator and in principle the global theory of such operators [Hö, Vols.III-IV] can be used for this purpose. However, there is too large a variety of possible behaviour to make such a general study feasible. Indeed, as μ ranges over all Lagrangean measures, T_μ ranges over many possible types of FIOs (Fourier integral operators), each with its own Sobolev mapping properties and asymptotic behaviour. In many (even 'typical') cases, repeated convolutions μ^{*k} and compositions T_μ^k are 'unclean' and lead to singular FIOs of various kinds [Ph][T.U]. Hence the classical theory of FIOs is rarely sufficient to analyse the spectral theory of continuous Lagrangean random walks. However, for the special classes of random walks (i)–(ii) above we will show that all convolutions and compositions are clean, at least along rays, and hence obtain simple and relatively complete solutions of the problems above.

To introduce and motivate the problems we are considering, let us recall some classical results on the (Euclidean) Fourier analysis of δ-functions on positively curved hypersurfaces. Thus, we suppose that $X \subset \mathbb{R}^n$ is a compact embedded oriented hypersurface and let $\mathcal{G}_+$ (resp. $\mathcal{G}_-$) be the Gauss map corresponding to the outward (resp. inward) unit normal. Recall that X is positively curved if $\mathcal{G}_+ : X \to S^{n-1}$ is a diffeomorphism and that the Gaussian curvature $K(x)$ at $x \in X$ is the density of $\mathcal{G}^*(d\sigma_{S^{n-1}})$ with respect to dS (where dS is the Euclidean surface measure of the hypersurface). The following is a well-known result due to Hlawka:

Theorem 0.0.1 (cf.[Hö, Vol.I Theorem 7.7.14 - 15]) *Suppose that X is a hypersurface. Fix ξ with $|\xi| = 1$. Then:*

$$\hat{\mu}(\tau\xi) = \tau^{-\frac{n-1}{2}} \sum_{x:\mathcal{G}_\pm(x)=\xi} a(x)|K(x)|^{-\frac{1}{2}} e^{-i\tau\langle x,\xi\rangle - i\pi\sigma/4} + O(\tau^{-\frac{n+1}{2}})$$

with σ equal to the excess in the number of positive over negative curvatures. If X is positively curved then $\{x : \mathcal{G}_\pm(x) = \xi\} = \{x_\pm(\xi)\}$ where $x_+(\xi)$ (resp. $x_-(\xi)$) is the unique point of X where the outward (resp. inward) unit normal is in the direction ξ.

Since $\mathbb{R}^n_\xi$ is the unitary dual of $\mathbb{R}^n$, this theorem gives the asymptotics of μ along rays of representations of $\mathbb{R}^n$. ¿From these asymptotics, it follows that T_μ is a Fourier integral operator of order $-\frac{n-1}{2}$. Hence it is Sobolev smoothing of order $\frac{n-1}{2}$. For more details see Section 1.2.

Our first goal is to generalize this model result to a compact semi-simple Lie group G. To do so, we must reformulate the notion of the asymptotics of $\hat{\mu}$ along rays of representations. Recall that, for each irreducible representation (ρ, V_ρ) of G, the Fourier coefficient

$$\hat{\mu}(\rho) = \int_G \rho(g) d\mu(g) : V_\rho \to V_\rho$$

is an operator on the finite dimensional representation space V_ρ. When μ is symmetric, $\mu(g) = \mu(g^{-1})$, then $\hat{\mu}(\rho)$ is self-adjoint. ¿From its eigenvalues $\{\lambda_{\rho,j} : j = 1, \ldots, \dim V_\rho\}$ we form the spectral measure

$$dm^\mu_\rho = \frac{1}{\dim V_\rho} \sum_{j=1}^{\dim V_\rho} \delta(\lambda - \lambda_{\rho,j}) \tag{1}$$

on $\mathbb{R}$. These measures will play the role of the scalar Fourier coefficients of a measure μ in the abelian case.

To explain the notion of asymptotics along rays, we recall that by the Cartan-Weyl theory, the unitary dual $\hat{G}$ of G may be parametrized by integral lattice points ρ in a Weyl chamber $\mathbf{t}^*_+$ of the (dual) Cartan subalgebra. By a ray of representations $L = \mathbb{N}\rho$ we mean the direct sum of the irreducible representations parametrized by the ray of lattice points. Thus, we are interested in the asymptotics of $dm^\mu_{n\rho}$ as $n \to \infty$.

By a basic construction in homogeneous quantization theory [G.S.1], this ray of representations may be concretely realized as the Hardy space $H^2(B_\rho)$ where $B_\rho \to \mathcal{O}_\rho$ is the canonical circle bundle over the coadjoint orbit associated to ρ. Just as the representations $V_{n\rho}$ parametrized by lattice points $n\rho$ along the ray may be concretely realized as the holomorphic sections $\Gamma(\mathcal{O}_\rho, L^{\otimes n})$ of the 'pre-quantum' holomorphic line bundle $L \to \mathcal{O}_\rho$, so the direct sum of these holomorphic sections may be realized as the space $H^2(B_\rho)$ of CR-functions on the principal circle bundle B_ρ associated to L. The (Cauchy-Szego) projector $\Pi_\rho : L^2(B_\rho) \to H^2(B_\rho)$ is a special type of Fourier integral operator (with complex phase) known as a Toeplitz operator. It has the (symplectic) geometric interpretation of restricting things to the cone thru $\mathcal{O}_\rho$ or, more precisely, to the symplectic cone

$$Y = \{(b, r\alpha_b) : r \in \mathbb{R}^+\} \subset T^*(B_\rho)$$

thru the 'contact structure' α on B_ρ; Y is a circle bundle over $\mathbb{R}^+\mathcal{O}_\rho$. In particular, given a random walk μ on G, we can form the Markov transition operator along the ray,

$$T_{\mu,\mathbb{N}\rho} := \Pi_\rho T_\mu \Pi_\rho : H^2(B_\rho) \to H^2(B_\rho), \qquad T_\mu = \int_G T_g d\mu(g)$$

where T_g is the translation operator by g on $L^2(B_\rho)$.

The asymptotics of the spectral measures $dm^\mu_{n\rho}$ can be read off from trace formulae involving powers of $T_{\mu,\mathbb{N}\rho}$ as long as these powers are clean compositions. A key point of this paper is that the composition powers of our two basic random walks (i)–(ii) above are always clean when restricted to rays. This allows us to avoid the serious technical complications involving singular compositions [Gr.U, 2] [Ph], which would grow more and more difficult as one took higher convolution powers.

In the case of a δ-function on a positively curved hypersurface, the Markov operator $T_{\mu,G}$ on $L^2(G)$ is a standard FIO associated to a union of canonical graphs and so the theory of convolutions with such δ-functions is very analogous to the $\mathbb{R}^n$ case. The main difference is that there are many more possible types of asymptotics of $\hat{\mu}(n\rho)$ as X varies over positively curved hypersurfaces of G and ρ varies over irreducibles. To describe the different asymptotics we will need to introduce some notation. For simplicity, assume that $X = X^{-1}$. Given an orbit $\mathcal{O} = \mathcal{O}_\rho$ we then let $X^\pm_{\mathcal{O}} = \mathcal{G}^{-1}_\pm(\mathcal{O})$ so that $\mathcal{G}_\pm : X^\pm_{\mathcal{O}} \to \mathcal{O}$ is a diffeomorphism. The Gauss maps induce contact transformations on B_ρ (or equivalently homogeneous canonical transformations on Y) given by

$$\chi_\pm : B_\rho \to B_\rho, \qquad \chi_\pm(b) = \mathcal{G}_\pm(o) \cdot b$$

where $g \cdot b$ denotes the action of G on B_ρ. We also denote by ϕ^θ the S^1-action of $B_\rho \to \mathcal{O}_\rho$; S^1 acts by contact transformations. Let $Fix(\chi)$ denote the fixed point set of a contact transformation χ on B_ρ. Assuming (as we always will) that it is clean, $Fix(\chi)$ carries a canonical density which we will denote by $d\mu_\chi$. Finally, σ denotes a certain Maslov index (for the sake of simplicity we will not describe it in detail). We then have:

Theorem A *Suppose that $X \subset G$ is a positively curved hypersurface, and let $\mu = a\delta_X$, i.e. $\int_G f(g)d\mu(g) := \int_X f(x)a(x)dS(x)$ where dS is the surface measure on X induced by Haar measure and where $a \in C^\infty(X)$. Then:*

(i) $T_{\mu,G}$ is a Fourier integral operator of order $-\frac{dim G-1}{2}$ associated to the union of canonical graphs

$$\Gamma_{\mu,G} = \{((x,\xi), g_\pm(x,\xi)\cdot(x,\xi)) \in T^*(G\times G) - 0, g_\pm(x,\xi) = \mathcal{G}^{-1}_\pm Ad(x)^*\xi\}.$$

(ii) $T_{\mu,G}$ is Sobolev smoothing of order $\frac{\dim G-1}{2}$, i.e. is bounded from $W^s(G) \to W^{s+\frac{\dim G-1}{2}}(G)$.
(iii) $\mu^{*k} \in L^2(G)$ for $k \geq 3$.
(iv) Along the ray of representations $\mathbb{N}\rho$, the moments of $dm^\mu_{n\rho}$ have the asymptotic expansion

$$M^\mu_{n\rho}(k) \sim n^{\frac{k}{2}(1-\dim G)+\frac{1+e}{2}-\dim B}\frac{1}{vol(B)}\sum_{\pm\pm\cdots\pm}\sum_{\theta_j\in\Theta_{\pm\pm\cdots\pm}} e^{-in\theta_j - i\pi\sigma/4}\int_{Fix(\chi_{\pm\pm\cdots\pm}\circ\phi^{\theta_j})} f_{\pm\pm\cdots\pm}\, d\mu_{\chi_{\pm\pm\cdots\pm}}$$

where

$$f_{\pm\pm\cdots\pm} := \mathcal{G}^{-1*}_\pm(\frac{a}{\sqrt{K}})\cdot(\mathcal{G}^{-1*}_\pm\circ\chi_\pm)^*(\frac{a}{\sqrt{K}})\cdots(\mathcal{G}^{-1*}_\pm\circ\chi_\pm\circ\cdots\circ\chi_\pm)^*(\frac{a}{\sqrt{K}})$$

and where

$$\Theta_{\pm\pm\cdots\pm} := \{\theta_j : e := \dim Fix(\chi_\pm\circ\cdots\circ\chi_\pm\circ\phi^{\theta_j}) \ \ \textit{is maximal}\ \}.$$

The reason why the moment formula is simpler in the Euclidean case is simply that co-adjoint orbits are single points ξ. Hence the analogue of $X^\pm_{\mathcal{O}}$ is just $X^\pm_\xi = \{x_\pm(\xi)\}$. The corresponding circle bundle is simply a circle $B_\xi \cong S^1$ on which $G = \mathbb{R}^n$ acts by the character $e^{i\langle x,\xi\rangle}$. The canonical transformations $\chi_\pm$ act on B_ξ by

$$\chi_\pm(b) = \mathcal{G}^{-1}_\pm(b)\cdot b = e^{i\langle x_\pm(\xi),\xi\rangle}b$$

and hence the integrals over fixed point sets reduce to evaluations of $e^{i\langle x,\xi\rangle}\frac{a}{\sqrt{K}}$ at the points $\{x_\pm(\xi)\}$. Thus we reproduce the Euclean expression modulo Maslov factors.

The moment asympotics above allow for all possible dimensions of fixed point sets from $e = 0$ up to $e = \dim \mathcal{O}$ and indeed as X varies over all hypersufaces, any of the even dimensions can occur. To illustrate this we will look in detail at the case of geodesic spheres $S_s(g)$ centered at different $g \in G$. The nature of the canonical transformations $\chi_\pm$ and particularly the dimensions of the fixed point sets of $\chi_\pm \cdots \chi_\pm$ then turns out to depend on the degree of singularity of g. When $g = e$ the $\chi_\pm$ reduce to the identity on the orbit and all of $\mathcal{O}$ is fixed, while if g is a regular element the set of fixed points on $\mathcal{O}$ is discrete.

The case of a sum of δ-functions along a union of conjugacy classes is more difficult because the Lagrangean $\Gamma_{\mu,G}$ underlying the transition operator $T_{\mu,G}$ for

$$\mu = \frac{1}{C}\sum_{j=1}^{n} \delta_{C_{x_j}} + \delta_{C_{x_j^{-1}}}, \qquad \int_G f(g)\delta_{C_{x_j}} = \int_{C_{x_j}} f(y) d\nu_j(y) \tag{2}$$

is not a local canonical graph in $T^*(G \times G)$. However, when restricted to a ray of representations it does become a Fourier Toeplitz operator associated to local canonical graph on the symplectic cone Y. This simplification occurs because the ray involves just one orbit $\mathcal{O}_\rho$. For simplicity, let us assume that there is just one x. With no loss of generality we may assume x lies in the maximal torus T. Then to each $o = k\rho k^{-1} \in \mathcal{O}$ there corresponds $2|W|$ group elements $g_w^\pm(o) := kwx^\pm w^{-1}k^{-1}$ such that $(g_w^\pm(o), o) \in N^*C_{x^\pm}$. These 'inverse Gauss maps' $g_w^\pm : \mathcal{O} \to C_x$ induce contact transformations $\chi_w^\pm$ on B defined by

$$\chi_w^\pm(o, e^{i\theta}) = (g_w^\pm(o)\cdot o, \chi_\rho(g_w^\pm(o))e^{i\theta}).$$

It is easily seen that $g_w^\pm(o)\cdot o = o$ so that χ_ρ is well-defined on $g_w^\pm(o)$. Since $Y \to B \to \mathcal{O}$ are all bundles, we can (and will) lift $g_w^\pm$ to B or to Y and regard $\chi_w^\pm$ as a homogenous canonical transformation on Y.

To state the results, we will need some further notation from compact Lie groups, which we adopt from [B.tD]. We let $\delta : \mathbf{t} \to \mathbb{C}$ denote the Weyl denominator

$$\delta(H) = \Pi_{\alpha\in R_+}(e(\frac{1}{2}\alpha(H)) - e(-\frac{1}{2}\alpha(H))), \qquad H \in \mathbf{t}$$

where t is the Cartan subalgebra and R_+ denotes the set of positive roots. We put $\rho_+ = \frac{1}{2}\sum \alpha \in R_+\alpha$. We also use the notation ρ for an irreducible (no connection to ρ_+), and denote its highest weight character by χ_ρ. Finally, we denote by $d\dot{x}$ the invariant normalized density on a conjugacy class. :

Theorem B *Let $\mu = \frac{1}{2}\delta_{C_x} + \frac{1}{2}\delta_{C_{x^{-1}}}$ where $\delta_{C_x} = a\, d\dot{x}$. Suppose $x = e^X$ is a regular element ($X \in \mathbf{t}$). Then:*

(i) $T_{\mu,L}$ is a Fourier-Toeplitz integral operator on $H^2(B)$ associated to the union of graphs

$$\Gamma_{\mu,L} = \sqcup_{w\in W,\pm} \quad \mathrm{graph}(\chi_w^\pm).$$

(ii) $T_{\mu,L}$ is Sobolev smoothing of order $\frac{\dim B - 1}{2}$ on $H^2(B)$.
(iii) Assume that G is a classical compact Lie group. Then $T_{\mu,L}$ is a Hilbert-Schmidt operator.
(iv) The asymptotics of the moments of the spectral measures $m^\mu_{n\rho}$ of $T_{\mu,L}$ along L are given by

$$M^\mu_{n\rho}(k) \sim n^{\frac{k}{2}(\dim T - \dim G)} \frac{1}{vol(B)} \sum_{\substack{(j_k, \dots, j_1) \\ (\pm\cdots\pm)}} e(\rho_+(\pm w_{j_k}(X)))\cdots e(\rho_+(\pm w_{j_1}(X))))$$

$$\frac{(-1)^{w_{j_k}}\cdots(-1)^{w_{j_1}}}{\delta(\pm X)\dots\delta(\pm X)}(\chi_\rho(w_{j_k}x^{\pm 1}w_{j_k}^{-1}\cdots w_{j_1}x^{\pm 1}w_{j_1}^{-1}))^n \left[\int_B \left(\prod_{i=1}^{k} a(g^\pm_{w_{j_i}}(b))\right) dvol\right]$$

where the w_j's are the elements of the Weyl group W.

These complete asymptotic expansions of the spectral measures along rays of representations generalize the results of [P.Z] from the case of discrete random walks along rays to the continuous Lagrangean walks of the types (i)-(ii) above.

We end this introduction by relating our methods and results to the usual concerns regarding random walks (G, μ). The main problem is to determine the rate of convergence of the convolution powers μ^{*k} to stationarity (i.e. Haar measure dg). There are several reasonable ways to measure this distance, e.g. the operator norm $||T_\mu^k - E||$, the total variation norm $||\mu^{*k} - dg||_{TV}$, or, when well-defined, the L^2-norm $||\mu^{*k} - dg||_{L^2}$. Here, $E(f) = \int_G f dg$. Although the total variation norm is viewed as primary, in practice it is often estimated from above by use of the Schwartz inequality

$$4||\mu^{*k} - dg||^2_{TV} \leq ||\mu^{*k} - 1||^2_{L^2} = \sum_{\rho \neq 1 \in \hat{G}} ||\hat{\mu}^k||^2$$

when $\mu^{*k} \in L^2$. Hence it is important to know the minimal power k such that $\mu^{*k} \in L^2$ and if possible to measure the rate of decay of $||\mu^{*k} - 1||^2_{L^2}$.

In favorable cases, our methods at least determine the minimal such k and the asymptotics of $||\hat{\mu}^k||^2$ along rays. They do not (as they stand) determine the decay of $||\mu^{*k} - 1||^2_{L^2}$, since this also depends on low-lying eigenvalues (in particular, the spectral gap) and on sparse sets of eigenvalues along rays. Indeed, the results of this paper involve only the principal symbol data of the Markov operator T_μ, and would not change if a finite rank or smoothing operator were added to T_μ, or if instead of δ-functions we considered more general polyhomogeneous distributions conormal to a hypersurface or conjugacy class. This is the price we pay for general asymptotic results.

On the other hand, as far as we know, there are few known estimates of rates of convergence to stationarity of random walks on compact semi-simple Lie groups. The estimates of which we are aware involve random walks where all of the eigenvalues of T_μ can be calculated in closed form [Ro] [Po] [L.P.S], or at least where there is a comparison to such a walk. Moreover even when explicit formulae for the eigenvalues are available, it is not clear what properties of the variety V supporting μ or of the choice of measure μ on V determine the decay rate of $||\mu^{*k} - 1||^2_{L^2}$. Indeed, this paper began as an attempt to understand the L^2 rates of convergence to stationarity of various random walks (especially random reflections) in [Ro][Po1,2]. By explicitly calculating the spectrum of T_μ in these cases, the first author found that the decay rate depended sensitively on the singularities of V and μ. Since microlocal methods are designed to relate singularities of μ to decay in $\hat{\mu}$, it seemed natural to apply them to these and related random walks.

1 Background

In this section we review a number of prior results on convolution operators

$$T_\mu f = \mu * f, \qquad f \in L^2(G), \;\; \mu \in \mathcal{M}(G)$$

for various kinds of measures μ on Lie groups. We will not be using these results, but include them as representing the currently known general results on random walks and convolution operators.

1.1 Convolution of measures on a Lie group.

In the case of non-abelian Lie groups convolution operators $T_\mu f = \mu * f$ are Fourier multipliers

$$\widehat{T_\mu f}(\rho) = \hat{\mu}(\rho) \hat{f}(\rho)$$

where (ρ, V_ρ) runs over the unitary dual $\hat{G}$ of G, where $\hat{f}(\rho)$ is the component of $f \in L^2(G)$ in V_ρ and where

$$\hat{\mu}(\rho) = \int_G \rho(g)d\mu(g) : V_\rho \to V_\rho$$

is the group Fourier transform of μ. For background see [H.R, Vols.I-II].

The simplest case is that of central measures. Recall that a measure μ on G is *central* if it is invariant under conjugation, i.e. $\mu(S) = \mu(xSx^{-1})$ for all $x \in G$ and all Borel sets S. The Fourier transform $\hat{\mu}(\rho) = \int_G \rho(g)d\mu(g)$ of a central measure is a scalar matrix $\hat{\mu}(\rho) = c_{\mu,\rho} I_{d_\rho}$ for every irreducible representation $\rho \in \hat{G}$. The asymptotic behaviour of $\hat{\mu}$ and the Sobolev smoothing properties of T_μ are in many ways analogous to the abelian case. Some general results are the following:

Theorem 1.1.1 ([Ra, Theorem 2.2]) *Let G be a compact simple Lie group of dimension n and let μ_i, $i = 1, ..., n$ be continuous central measures on G. Then the convolution product $\mu = \mu_1 * \cdots * \mu_n$ is absolutely continuous with respect to Haar measure on G.*

Theorem 1.1.2 ([Ra, Corollary 3.5]) *Let G be a compact simple Lie group and μ a central measure on G. Then μ is a continuous measure if and only if*

$$c_{\mu,\rho} \to 0 \qquad \textit{as } \rho \to \infty \quad \textit{in } \hat{G}\,.$$

A number of basic results on more general measures have been proved by Ricci-Stein and Ricci-Travaglini. The following are most relevant to this paper.

Theorem 1.1.3 ([R.S.II]) *Let $V_1, \ldots, V_n$ be connected analytic submanifolds of a unimodular Lie group G and assume that the product $V_1 \ldots V_k$ contains an open set of G. If for each $j = 1, \ldots, k$ we are given measures $d\mu_j = \phi_j d\sigma_j$ where $d\sigma_j$ is surface measure on V_j and where ϕ_j is a smooth function with compact support on V_j, then $d\mu_1 * d\mu_2 * \ldots * d\mu_k$ is absolutely continuous with respect to Haar measure dg and its density ρ satisfies a right L^1-Holder condition.*

By a right L^1- Holder condition of exponent $\delta > 0$ one means that

$$\int_G |\rho(x exp Y) - \rho(x)| dx \leq C||Y||^\delta$$

where $Y \in \mathbf{g}$. Equivalently, $\tilde{\rho} := \rho \circ exp \in L^1(\mathbf{g})$ satisfies

$$\int_{\mathbf{g}} |\tilde{\rho}(X+Y) - \tilde{\rho}(X)| dX \leq C||Y||^\delta.$$

In particular, suppose that V is an analytic submanifold of a compact Lie group G which generates G in the sense that V is not contained in any proper closed subgroup of G. By [R.S.II, Proposition (1.1)], there is a positive integer m such that V^m contains an open subset of G. Hence:

Corollary 1.1.4 *Let μ be a smooth delta function along an analytic submanifold $V \subset G$ which generates G and let m be the least integer such that V^m contains an open subset of G. Then $\mu^{*m} << dg$ and its density ρ_m satisfies*

$$\int_{\mathbf{g}} |\tilde{\rho_m}(X+Y) - \tilde{\rho}(X)| dX \leq C||Y||^\delta.$$

.

The last condition implies that $\tilde{\rho_m} \in L^r(\mathbf{g})$ or equivalently $\mu^{*m} \in L^r(G)$ for some $r > 1$ [R.S.III]. By Young's inequality for convolutions on a compact Lie group [H.R],

$$||\rho_m^{*k}||_q \leq ||\rho_m||_r \dots ||\rho_m||_r, \qquad \frac{k}{r} = k - 1 + \frac{1}{q},$$

it follows that

$$\mu^{*mk} \in L^2(G) \qquad \text{if} \qquad k \geq \frac{r}{2(r-1)}.$$

Thus, a sufficiently high convolution power of μ lies in $L^2(G)$. By this method, the power depends on m, δ, r.

1.2 Sobolev smoothing properties of convolutions.

Definition 1.2.1 A finite measure μ on G is said to be smoothing of order s on Sobolev spaces, or H^s-*improving* if $T_\mu : f \to f * \mu$ is bounded from $H^m(G)$ to $H^{m+s}(G)$ for all $m \in \mathbb{R}$.

Theorem 1.2.2 ([Ph]) *Assume V is an analytic surface with non-vanishing μ-curvature in $\mathbb{R}^n$. Then for any $\epsilon > 0$, the Radon transform with measure $d\sigma_V$ supported on V is smoothing of order $\frac{2}{\mu^{\frac{1}{2}}+1} - \epsilon$ on Sobolev spaces.*

We note that surface is in the literal sense that $\dim V = 2$. When $\dim V = d$ it is natural to conjecture that the order of smoothing is given by $\frac{d}{\mu^{\frac{1}{d}}+1} - \epsilon$. The notion of μ-curvature employed here is defined as follows:

Definition 1.2.3 The analytic submanifold $V \subset \mathbb{R}^n$ is said to have nonvanishing μ-curvature if, for any $\lambda \in \mathbb{R}^n - \{0\}$, the function $t \to \langle \lambda, S(t) \rangle$ on $\mathbb{R}^d$ has multiplicity at most μ at any critical point. Here, $t \to S(t) \in V \subset \mathbb{R}^n$ is a local analytic parametrization of V.

Roughly speaking, V has multiplicity at most μ if μ is the maximum number of points admitting a given direction $\lambda \in \mathbb{R}^n$ among its normals. To be more precise, the multiplicity μ of an analytic function $f : \mathbb{R}^d \to \mathbb{R}$ at an isolated critical point a is defined by $\mu = \dim \mathcal{A}(a)/\mathcal{I}[\partial_1 f, \dots, \partial_d f]$ where $\mathcal{A}(a)$ is the space of germs of analytic functions at a and $\mathcal{I}[\partial_1 f, \dots, \partial_d f]$ is the ideal generated by the germs of $\partial_j f$ at a. Nonvanishing Gauss curvature of a hypersurface V is equivalent to V having $\mu = 1$.

Convolution with a smooth δ-function along a positively curved hypersurface of a compact Lie group G of dimension n should therefore be Sobolev-smoothing of order $\frac{n-1}{2} - \epsilon$ with $n = \dim G$. Indeed, our results show it is smoothing of order $\frac{n-1}{2}$.

2 Markov operators of random walks

We now take up the study of Lagrangean random walks and their Markov operators. The relevant background on homogeneous quantization, rays of representations, Toeplitz operators and so on is contained in [P.Z, Section 2]. We continue here with a study of the Markov operators which arise from convolution with continuous Lagrangean distributions.

Note: Throughout the paper we will use the isomorphisms $TG \cong G \times \mathbf{g}$ and $T^*G \cong G \times \mathbf{g}^*$ via left translation.

2.1 Markov operators and moment Lagrangeans

A random walk is defined by the pair (μ, ρ) where ρ is a representation of G. By definition, the associated Markov operator is given by $T_{\mu,\rho} = \int_G \rho(g) d\mu(g)$. Most often, ρ is taken to be an action G by translations on some homogeneous space G/K. However, all that is needed to get a geometric theory is that ρ is a representation of G by Fourier integral operators. As recalled in Section 0 (see also [G.S.1]), a Fourier integral representation on $L^2(X)$ is the quantization of a Hamiltonian group action on $T^*(X) - 0$. The following describes the (Schwartz) kernels of the Markov operators for random walks of this kind.

Proposition 2.1.1 *Suppose ρ is a Fourier integral representation of G on $L^2(X)$ with moment Lagrangean Γ and suppose $\mu \in I^s(G, \Lambda_\mu)$ is a Lagrangean measure on G. Then, under clean composition hypothesis for $\Lambda_\mu \circ \Gamma$, the Markov operator*

$$T_{\mu,\rho} := \int_G \rho(g) d\mu(g) \in I^k(X \times X, \Gamma_{\mu,\rho})$$

is a Fourier integral operator of order $k \geq s - \frac{\dim G}{4}$ associated to the Lagrangean

$$\Gamma_{\mu,\rho} := \{((x,\xi), g \cdot (x,\xi)) \in T^*(X \times X) - 0 : (g, \Phi(x,\xi)) \in \Lambda_\mu\}.$$

Here Φ denotes the moment map of the lift of the G-action to the cotangent bundle T^*X.

Proof: This follows from the composition theorem for Fourier integral operators [Hö, Vol.IV Theorem 25.2.3]. As proved in [G.S.1] and reviewed in [P.Z], ρ is a Fourier integral operator of order $-\frac{\dim G}{4}$ from $G \times X$ to X associated to the moment Lagrangean

$$\Gamma = \{((g,\gamma), (x,\xi), g \cdot (x,\xi)) : g \in G, x \in X, \xi, \gamma \in \mathbf{g}^* : \Phi(x,\xi) = \gamma\}$$

and μ is associated to Λ_μ. If the composition T_μ is clean, then it is a Fourier integral operator associated to $\Gamma \circ \Lambda_\mu$. To see whether it is clean and to determine the composite Lagrangean and symbol, one forms the fiber diagram

$$\begin{array}{ccc} F & \rightarrow & \Gamma \\ \downarrow & & \downarrow \quad \pi_1 \\ \Lambda_\mu & \underset{i}{\rightarrow} & T^*G \end{array}.$$

Here the fiber product is

$$F = \{((g,\gamma), (x,\xi), g \cdot (x,\xi)) : \Phi(x,\xi) = \gamma, (g,\gamma) \in \Lambda_\mu\}.$$

The cleanliness conditions are (i) that F be a manifold and (ii) that the derived diagram

$$\begin{array}{ccc} TF & \rightarrow & T\Gamma \\ \downarrow & & \downarrow \quad d\pi_1 \\ T\Lambda_\mu & \underset{di}{\rightarrow} & T(T^*G) \end{array}$$

is also a fiber diagram. For condition (i), put

$$\Lambda_\mu^\Phi := \{(g,\gamma) \in \Lambda_\mu : \gamma \in \Phi(T^*X - 0)\}, \qquad \Lambda_\mu^{\mathbf{g}^*} := \{\gamma \in \mathbf{g}^* : \exists g \in G, (g,\gamma) \in \Lambda_\mu^\Phi\}.$$

Then the projection to the first factor defines a map $\pi_1 : F \rightarrow \Lambda_\mu^\Phi$ whose fibers are the sets $\Phi^{-1}(\gamma)$ with $\gamma \in \Lambda_\mu^{\mathbf{g}^*}$. A sometimes useful sufficient condition that F be a manifold is that Λ_μ^Φ be a manifold and that π_1 be a map of constant rank. For instance, in some applications one has $\Phi(T^*G - 0) = \mathbf{g}^*$ and even that $\Phi(T_x^*G - 0) = \mathbf{g}^*$ for each $x \in G$ (see the next proposition). In this case, $F \cong \Lambda_\mu \times G$.

For condition (ii), we need to show additionally that $TF = \{(u,v) \in T\Lambda_\mu \times T\Gamma : di(u) = d\pi_1(v)\}$. For each $(a,b) \in F$, the inclusion $TF_{(a,b)} \subset \{(u,v) \in T_a\Lambda_\mu \times T_b\Gamma : di(u) = d\pi_1(v)\}$ is trivially true. Hence, a necessary and sufficient condition for cleanliness is that $\dim T_{(a,b)}F = \dim\{(u,v) \in T\Lambda_\mu \times T\Gamma : di(u) = d\pi_1(v)\}$. ∎

2.2 The principal symbol of $T_{\mu,\rho}$

To determine moment asymptotics we will need to know special cases of the principal symbol $\sigma(T_{\mu,\rho})$. We will treat it as a 1/2-density on $\Gamma_{\mu,\rho}$, although it is actually a 1/2-density tensor a section of the Maslov line bundle $L \to \Gamma_{\mu,\rho}$. Ignoring the Maslov factors has the consequence that the coefficients of our moment asymptotics contain undetermined powers of i. Often they can be determined by comparison with known traces such as given by the Weyl character formula (see §5.5). They could also be avoided by using the 1/2-form formalism of [G.S.2]. In any case, it would require a technical digression of unwonted length to pin down these powers of i and we have refrained from doing so.

Let us now outline the calculation of the 1/2-density factor in the general case.

Under the cleanliness hypothesis, the derived diagram

$$\begin{array}{ccc} T_f F & \to & T_\gamma \Gamma \\ \downarrow & & \downarrow \quad d\pi_1 \\ T_\lambda \Lambda_\mu & \to & T_\lambda(T^*G) \\ & di & \end{array}$$

is a fiber product diagram. Hence the following sequence of vector spaces is exact:

$$0 \longrightarrow T_f F \xrightarrow{\iota} T_\lambda \Lambda_\mu \oplus T_\gamma \Gamma \xrightarrow{\tau} T_\lambda(T^*G) \longrightarrow \operatorname{coker}\tau \longrightarrow 0 \tag{3}$$

where ι is the inclusion and $\tau(f_1, f_2) = f_1 - f_2$. The excess e of the diagram is the dimension $\dim \operatorname{coker}\tau$ of $T_\lambda(T^*G)/\tau(T_\lambda \Lambda_\mu \oplus T_\gamma)$. When $e = 0$ the composition (or diagram) is called transversal; it will arise often in our applications.

Let $|V|^s$ denote the space of s-densities on a vector space V. The alternating tensor product of 1/2 densities on an exact sequence of vector spaces has a canonical trivialization, and therefore

$$|T_f F|^{\frac{1}{2}} \otimes |\operatorname{coker}\tau|^{-\frac{1}{2}} \cong |T_\lambda \Lambda_\mu \oplus T_\gamma \Gamma|^{\frac{1}{2}} \otimes |T_\lambda(T^*G)|^{-\frac{1}{2}}. \tag{4}$$

Since T^*G carries a canonical symplectic volume density $|dg \wedge d\gamma|$, we can remove the factor $|T_\lambda(T^*G)|^{-\frac{1}{2}}$.

When $e = 0$ the canonical isomorphism further simplifies to

$$|T_f F|^{\frac{1}{2}} \cong |T_\lambda \Lambda_\mu \oplus T_\gamma \Gamma|^{\frac{1}{2}}. \tag{5}$$

Hence in the transversal case we have a natural composition of 1/2-densities

$$(a, b) \in |T\Gamma|^{\frac{1}{2}} \otimes |T\Lambda|^{\frac{1}{2}} \to a \circ b \in |T(\Gamma \circ \Lambda)|^{\frac{1}{2}}. \tag{6}$$

Here $\Gamma \circ \Lambda$ is the composition of the Lagrangeans. In the transversal case, $F \to \Gamma \circ \Lambda$ is a finite cover, so the composite symbol is a sum over the fiber of the pointwise composition.

In the case $e \neq 0$, the projection $F \to \Gamma \circ \Lambda$ is a fibration and the composite symbol is given by

$$a \circ b_\gamma = \int_{F_\gamma} a \times b \tag{7}$$

where $a \times b$ is the density on the fiber F_γ over γ with values in $|T(\Gamma \circ \Lambda)|^{\frac{1}{2}}$ defined in [Hö, Vol.III Theorem 21.6.7] (see also [Hö, Vol.IV Theorem 25.2.3]).

2.3 The Markov operator on $L^2(G)$

We now specialize to the case where G acts on itself by left multiplication L. The resulting representation ρ on $L^2(G)$ is isomorphic to the left regular representation of G. In this case the Markov operator

$$T_\mu : L^2(G) \to L^2(G), \qquad T_\mu f(x) = \int_G f(gx) d\mu(g)$$

associated to a symmetric Lagrangean measure μ is always an FIO.

Proposition 2.3.1 *Let $\mu \in I^s(G, \Lambda_\mu)$ be a Lagrangean measure on G. Then the corresponding Markov operator $T_\mu \in I^k(G\times G, \Gamma_\mu)$ is a Fourier integral operator of order $k = s - \frac{\dim G}{4}$ associated to the Lagrangean*

$$\Gamma_\mu = \{((x,\xi),(gx,\xi))) \in T^*(G\times G) - 0 : (g, Ad^*(x)\xi) \in \Lambda_\mu\} \cong \Lambda_\mu \times G\,.$$

Its principal symbol is given by

$$\sigma_{T_\mu} = \sigma(\mu) \otimes |dx|^{\frac{1}{2}}$$

as 1/2-densities on $\Lambda_\mu \times G$.

Proof: As above, we need to show that the composition $\Lambda_\mu \circ \Gamma$ is clean. In fact, it is always transversal in this case. To prove this, we begin by describing the moment map Φ.

Let us denote the canonical 1-form on T^*G by α. For every $A \in \mathbf{g}$, the lift of the group action to T^*G induces a vector field $A^\sharp$ on T^*G. Since the Lie derivative $D_{A^\sharp}\alpha$ is zero for all $A \in \mathbf{g}$, we have

$$0 = i_{A^\sharp}d\alpha + di_{A^\sharp}\alpha = -i_{A^\sharp}\omega + di_{A^\sharp}\alpha\,,$$

where i means insertion and hence

$$\langle \Phi(x,\xi), A\rangle = i_{A^\sharp}\alpha_{(x,\xi)}\,.$$

Since $A^\sharp_{(x,\xi)} = \frac{d}{dt}|_{t=0}((\exp tA)x, \xi)$, we have

$$\langle \Phi(x,\xi), A\rangle = \xi(Ad(x^{-1})A)$$

and we get

$$\Phi(x,\xi) = Ad^*(x)\xi\,.$$

The moment Lagrangean therefore has the form

$$\begin{aligned}\Gamma &= \{((g, Ad^*(x)\xi),(x,\xi),(gx,\xi)) : g, x \in G, \xi \in \mathbf{g}^*\}\\ &= \{((g,\gamma),(x, Ad^*(x^{-1})\gamma),(gx, Ad^*(x^{-1})\gamma)) : x, g \in G, \gamma \in \mathbf{g}^*\} \cong T^*G \times G\,.\end{aligned}$$

We now claim that the fiber diagram

$$\begin{array}{ccc} F & \rightarrow & \Gamma \\ \downarrow & & \downarrow \ \pi_1 \\ \Lambda_\mu & \underset{i}{\rightarrow} & T^*G \end{array}$$

is transversal. The fiber product equals

$$F = \{((g,\gamma),(g,\gamma),(x, Ad^*(x^{-1})\gamma),(gx, Ad^*(x^{-1})\gamma)) : (g,\gamma) \in \Lambda_\mu\}.$$

We see that $\Phi(T^*_xG - 0) = \mathbf{g}^* - 0$ for any x and hence that

$$F \cong \Lambda_\mu \times G$$

for any Λ_μ. In particular, F is always a manifold of dimension $2\dim G$.

¿From this it follows easily that $TF = \{(u,v) \in T\Lambda_\mu \times T\Gamma : di(u) = d\pi_1(v)\}$ and hence that the derived fiber diagram is clean. Indeed, the dimensions of the two vector spaces are equal and hence the spaces must coincide. It follows that the above fiber diagram is clean with excess $e = \dim F + \dim T^*G - \dim \Lambda_\mu - \dim \Gamma = 0$.

Finally, any $((g,\gamma),x) \in \Lambda_\mu \times G$ determines a unique point $((x, Ad^*(x)^{-1}\gamma),(g\cdot x, Ad^*(x)^{-1}\gamma)) \in \Gamma_\mu$ and conversely any point of Γ_μ determines $((g,\gamma),x) \in \Lambda_\mu \times G$. The first statement of the proposition follows then from Proposition 2.1.1.

Now consider the principal symbol $\sigma(T_\mu)$, a 1/2-density on $\Gamma_\mu \cong \Lambda_\mu \times G$. By the above, it is given by $\sigma(\mu)\circ\sigma(T_G)$ on the composite Lagrangean $\Gamma_\mu \cong F$. We also have that $\sigma(T_G)$ is the canonical volume 1/2-density given by $|dx|^{\frac{1}{2}} \otimes |dg \wedge d\gamma|^{\frac{1}{2}}$ in the parametrization of Γ by $G \times T^*G$. According to the isomorphism above, we divide by the canonical 1/2-density on T^*G, leaving the stated result. ∎

2.4 Random walks on rays of representations

In this section, we specialize to the case of a ray of representations $L = \mathbb{N}\rho$, that is, we restrict the Markov operator $T_{\mu,G}$ from $L^2(G)$ to the direct sum of irreducibles along a ray thru a given irreducible ρ. The *ray Markov operator* is thus:

$$T_{\mu,L} := \bigoplus_{n=1}^{\infty} \int_G (n\rho)(g)d\mu(g) \tag{8}$$

Borrowing from [G.S.1], we realize the ray L as the Hardy space $H^2(B_\rho)$ of CR-functions on the pre-quantum circle bundle $B_\rho \to \mathcal{O}_\rho$ where $\mathcal{O}_\rho$ is the coadjoint orbit of ρ. We often drop the subscript ρ when the ray is understood to be fixed. Thus, $T_{\mu,L} = \int_G \Pi T_{B,g} d\mu(g)$ where $T_{B,g}$ is the left action of G on the homogeneous space B and where Π is the Cauchy-Szego projector (the orthogonal projection $L^2(B) \to H^2(B)$. For the relevant background we refer to [G.S.1] and [P.Z, Section 2].

The ray Markov operator is analogous to but somewhat more complicated than $T_{\mu,G}$ because it is a Fourier-Toeplitz operator rather than a standard FIO. Roughly speaking, a Fourier-Toeplitz operator A is a Fourier integral operator with complex phase which is partly oscillatory and partly Gaussian. The oscillatory part of the phase parametrizes a canonical relation $C \subset Y \times Y$ where $Y \subset T^*B_\rho$ is the symplectic cone generated by the contact form α (cf. §0). In most respects A behaves like an FIO associated to the canonical relation C, except that its symbol is a symplectic spinor rather than a 1/2-density [BdM.G]G: that is,

$$\sigma(A) \in \Omega^{\frac{1}{2}}(C) \otimes Spin(C) \otimes \Gamma(L)$$

where $\Omega^{\frac{1}{2}}(C)$ are the 1/2-densities, where $Spin(C)$ are the symplectic spinors, and $\Gamma(L)$ are the sections of the Maslov bundle, over C.

As mentioned above, we will ignore the Maslov factors. The spinor factors are by comparison too important to omit: As will be seen in §5.5 they are responsible for the presence of the factors $e(\rho_+(w(X)))$ in the terms of the Weyl character formula. However, the only symplectic spinors we need to confront are those which arise as parts of symbols of the Toeplitz operators $\Pi F \Pi$ where F is a Fourier integral operator associated to a local canonical graph.

Recall from [BdM.G] that the symbol $\pi := \sigma(\Pi)$ of the Toeplitz (Cauchy- Szego) projector is the idempotent symplectic spinor $\pi = e_\Lambda \otimes e_\Lambda^*$ equal to the projection operator onto the 'vacuum state' associated to Π. It is a symplectic spinor on the twisted diagonal $Y^\# = \{(y,-y) \in Y \times Y\}$. We refer to [BdM.G, §4, 11] for the definitions and background.

Suppose now that F is a Fourier integral operator associated to a local canonical graph. Since the discussion is local, we may assume it is actually associated to the (twisted) graph of a canonical transformation χ. Then the symbol of $\Pi F \Pi$ is a an element of $\Omega_C^{\frac{1}{2}} \otimes Spin(C)$. Just as $e_\Lambda \otimes e_\Lambda^*$ is the symbol of Π at a point $(y,-y)$, so the symplectic spinor part of $\sigma(\Pi F \Pi)$ has the form $e_\Lambda \otimes \chi_* e_\Lambda$ where χ_* is the map on symplectic spinors at y to symplectic spinors at $\chi(y)$ induced by χ (see [G, §8][BdM.G, §11]). Roughly speaking, $d\chi_y$ takes the symplectic normal bundle T_yY^- at y to that at $\chi(y)$. Choosing a metaplectic frame for each, $d\chi_y$ is identifed with a linear symplectic map. Hence we can apply the metaplectic representation $\mathcal{M}$ to the normal part $d\chi_y^- : T_yY^- \to T_{\chi(y)}Y^-$ to get a map from symplectic spinors at y to those at $\chi(y)$; this is the induced linear map χ_*. For simplicity of notation we will also write $e_\Lambda \otimes \chi_* e_\Lambda$ more simply as $\chi_*\pi$. In cases where χ is the lift of a group element g we will denote the induced maps by $g_* e_\Lambda$ resp.$g_*\pi$. Then we have

$$\sigma(\Pi F \Pi) = a|dy|^{\frac{1}{2}} \otimes \chi_*\pi \tag{9}$$

for some function a on C.

The following proposition gives a general description of ray Markov operators. The notation $pr. : B \to \mathcal{O}$ stands for the natural projection and Γ_Y for the restriction of the moment Lagrangean $\Gamma \subset T^*G \times T^*B \times T^*B$ to $T^*G \times Y \times Y$.

Proposition 2.4.1 *Suppose $\mu \in I^s(G, \Lambda_\mu)$ and let $L = \mathbb{N}\rho$ be a ray of representations. Under the clean composition hypothesis for the composition $\Lambda_\mu \circ \Gamma_Y$,*

$$T_{\mu,L} :\in I^k(B \times B, \Gamma_{\mu,L})$$

is a Fourier-Toeplitz integral operator of order $k \geq s + \frac{1}{2} - \frac{2\dim B + \dim G}{4}$ on $H^2(B)$ associated to the Lagrangean

$$\Gamma_{\mu,L} = \Lambda_\mu \circ \Gamma_Y = \{((b,r),(g\cdot b, r)) : (g, pr.(b)) \in \Lambda_\mu\}.$$

Its symbol is given by

$$(\sigma(T_{g,L}) \circ \sigma(\mu))|_{(y,y')} = \int_{g \in F_{y,y'}} (|dy|^{\frac{1}{2}} \otimes g_*\pi) \times \sigma(\mu)$$

where $F_{y,y'} = \{g \in G : g \cdot y = y', (g, pr(y) \in \Lambda_\mu)$. is the fiber of the composition and where times is the composition law defined in [BdM.G].

Proof: We begin with a microlocal description of the Toeplitz group representation $T_{g,L} := \bigoplus_{n=0}^\infty \int_G (n\rho)(g)$ and then consider its integration against μ.

Thus, let $T_{g,B}$ be the representation of G by translations on $L^2(B)$. The moment Lagrangean of $T_{g,B}$ is given by

$$\Gamma_B = \{((g,\gamma),(b,\beta), g\cdot(b,\beta)) : \Phi_B(b,\beta) = \gamma\}$$

where

$$\langle \Phi_B(b,\beta), X\rangle = \beta(X_b^\#)$$

is the moment map.

Then recall from [P.Z] that Π is a Toeplitz operator of order $-\frac{1}{2}(\dim B - 1)$ corresponding to the identity relation on the symplectic cone $Y \subset T^*B$ associated to B , i.e., $\Pi \in I^{-\frac{1}{2}(\dim B - 1)}(B \times B, \Delta(Y))$. By [BdM.G, Theorem 7.5] composition with Π is always clean. Hence the fiber diagram

$$\begin{array}{ccc} F & \to & \Gamma_B \cong G \times T^*B \\ \downarrow & & \downarrow \\ \Delta(Y) \cong Y & \to & T^*B \end{array}$$

is clean. Since $F = \{((g, \Phi(b,r)),(b,r),(g\cdot b, r) : g \in G, (b,r) \in Y\} \cong G \times Y$, the excess is $e = 0$. Hence

$$T_{g,B} \circ \Pi = T_{g,L} \in I^{\frac{1}{2} - \frac{2\dim B + \dim G}{4}}(G \times B \times B, \Gamma_Y)$$

where

$$\Gamma_Y = \{((g, \Phi(y)), y, g\cdot y) : g \in G, y \in Y\} \subset T^*G \times Y \times Y. \tag{10}$$

Since $T_{\mu,L} = \int_G T_{g,L} d\mu(g)$, we need to compose $\Lambda_\mu \circ \Gamma_Y$. The relevant fiber product diagram is

$$\begin{array}{ccc} F & \rightarrow & \Gamma_Y \\ \downarrow & & \downarrow \quad \pi_1 \\ \Lambda_\mu & \underset{i}{\rightarrow} & T^*G \end{array}$$

Provided the diagram is clean with excess e, we get

$$T_{\mu,L} \in I^{s+\frac{1}{2}-\frac{2\dim B+\dim G}{4}+\frac{e}{2}}(B \times B, \Gamma_{\mu,L})$$

where

$$\Gamma_{\mu,L} = \Lambda_\mu \circ \Gamma_Y = \{((b,r),(g\cdot b,r)) : (b,r) \in Y, (g, ro) \in \Lambda_\mu\}.$$

Here we have used the fact that the moment map Φ_L for the action of G on Y is given by $\Phi_L(b,r) = r \cdot pr.(b) = ro$ (see [G.S.1, Theorem 4.6]).

The fiber product equals

$$F \cong \{((g, pr.(b)), (b,r), g\cdot(b,r) \,|(g, pr.(b)) \in \Lambda_\mu\}. \tag{11}$$

As above, put

$$\Lambda_\mu^{\Phi_L} := \{(g,\gamma) \in \Lambda_\mu : \gamma \in \Phi_L(Y-0) = \mathbb{R}^+\mathcal{O}\}, \qquad \Lambda_\mu^{\mathcal{O}} := \{\gamma \in \mathbb{R}^+\mathcal{O} : \exists g \in G, (g,\gamma) \in \Lambda_\mu\}.$$

Then the projection to the first factor defines a map $\pi_1 : F \to \Lambda_\mu^{\Phi_L}$ whose fibers are the sets $\Phi_L^{-1}(\gamma)$ with $\gamma \in \Lambda_\mu^{\mathcal{O}}$.

Consider now the symbol $\sigma(T_{\mu,L}) = \sigma(T_{g,L}) \circ \sigma(\mu)$. We note first that

$$\sigma(T_{g,L})|_{(g,\Phi(y),y,g\cdot y)} = |dg|^{\frac{1}{2}} \otimes |dy|^{\frac{1}{2}} \otimes g_*\pi \in \Omega^{\frac{1}{2}}(\Gamma_Y) \otimes Spin(\Gamma_Y). \tag{12}$$

This follows from the fact that $\Pi \circ T_{g,B}$ is a transversal composition and from the fact that $\sigma(T_{g,B})$ is the canonical symplectic volume 1/2-density. Also, as described above, the canonical transformation defined by g transforms π to $g_*\pi = e_\Lambda \otimes g_* e_\Lambda$ (cf. [G, p.233], [BdM.G, §4,11].)

The further composition with $\sigma(\mu)$ is given by the composition formula for the Fourier Toeplitz operator $T_{g,L}$ and the Lagrangean distribution μ in [BdM.G, §7]. The fiber of the composition may be identified with a submanifold of G and $\sigma(T_{g,L}) \times \sigma(\mu)|_{(g,y,g\cdot y)}$ is a density along the fiber with values in $\Omega^{\frac{1}{2}}(\Gamma_{\mu,L}) \otimes Spin(\Gamma_{\mu,L})$.
■

3 Generalities on moment asymptotics

As a final preliminary to the proofs of Theorems A and B, we state some generalities on convolution powers of Lagrangean submanifolds, composition powers of Markov Lagrangeans and asymptotics of moments of spectral measures along rays for any Lagrangean measures satisfying appropriate cleanliness conditions. We will show in later sections that our basic examples satisfy these conditions.

3.1 Convolution of Lagrangean submanifolds $\Lambda \star \Gamma$.

Underlying the asymptotics we are interested in is the geometry of convolutions of Lagrangean submanifolds. As usual, we use the identification $T^*G \cong G \times \mathfrak{g}$ by left translation.

Definition 3.1.1 The convolution $\Lambda \star \Gamma$ of two Lagrangeans $\Lambda, \Gamma \subset T^*G$ is defined as

$$\Lambda \circ \Gamma := M_*(\Lambda \times \Gamma)$$

where $M : G \times G \to G$ denotes the the multplication map $(x, y) \to xy$.

Recall that the pushforward $f_*\Lambda$ of a Lagrangean $\Lambda \subset T^*X$ under a smooth map $f : X \to Y$ is given by

$$f_*\Lambda = \{(y, \xi) | \exists x \in X \text{ with } f(x) = y \text{ and } (x, df^*(\xi)) \in \Lambda\} .$$

Hence we have:

Proposition 3.1.2 *The convolution $\Lambda \star \Gamma$ is given by*

$$\Lambda \star \Gamma = \{(xy, \gamma) \in T^*G | (x, Ad^*(y)\gamma) \in \Lambda, (y, \gamma) \in \Gamma\} .$$

Proof: For any $x, y \in G$, $A, B \in \mathbf{g}$, we have

$$(x \exp tA)(y \exp tB) = xy \exp\{t(Ad(y^{-1})A + B) + t^2[Ad(y^{-1})A, B] + ...\} .$$

Hence $dM(A, B)_{xy} = Ad(y^{-1})A + B$ and $dM^*(\gamma) = (Ad^*(\gamma), \gamma)$. ∎

The convolution of Lagrangeans is associative. The n-fold convolution power $\Lambda^{*n} = \Lambda \star \Lambda \star ... \star \Lambda$ (n times) of a Lagrangean manifold is thus given by:

Corollary 3.1.3

$$\Lambda^{*n} = \{(x_1 x_2 \cdots x_n, \gamma) | (x_1, Ad^*(x_2 \cdots x_n)\gamma), (x_2, Ad^*(x_3 \cdots x_n)\gamma), \ldots, (x_{n-1}, Ad^*(x_n)\gamma), (x_n, \gamma) \in \Lambda\} .$$

3.2 Clean compositions of Markov Lagrangeans

As above, we suppose that $\mu \in I^s(G, \Lambda_\mu)$ is a Lagrangean measure. We fix a ray of representations $L = \mathbb{N}\rho$ and consider the ray Markov operator $T_{\mu,L}$. Our purpose here is to prove:

Proposition 3.2.1 *Suppose that $T^k_{\mu,L} = T_{\mu,L} \circ T_{\mu,L} \circ \cdots \circ T_{\mu,L}$ is a clean composition of Fourier-Toeplitz integral operators. Then $T^k_{\mu,L} \in I^{r_k}(B \times B, \Gamma^k_{\mu,L})$, that is, $T^k_{\mu,L}$ is a Fourier-Toeplitz operator on $L^2(B)$ of order $r_k = k(s + \frac{1}{2} - \frac{2 \dim B + \dim G}{4} + \frac{e}{2}) + \frac{e^*_k}{2}$ associated to the isotropic manifold*

$$\Gamma^k_{\mu,L} = \{(y, g_1 g_2 \ldots g_k \cdot y) : y \in Y, (g_1, g_2 \cdots g_k \cdot o), (g_2, g_3 \cdots g_k \cdot o), \ldots, (g_k, o) \in \Lambda_\mu\} \subset T^*B \times T^*B$$

with $o = pr.(y)$.

Proof: Note that $T^k_{\mu,L} = T_{\mu^{*k},L}$ and $\Lambda_{\mu^{*k}} = \Lambda^{*k}_\mu$. Assuming clean composition and using Corollary 3.1.3, we get

$$T^k_{\mu,L} \in I^{k(s+\frac{1}{2}-\frac{2\dim B+\dim G}{4}+\frac{e}{2})+\frac{e^*_k}{2}}(B \times B, \Gamma^k_{\mu,L})$$

where

$$e^*_k = e_2 + ... + e_m$$

with e_j denoting the excess of the composition $\Gamma^{j-1}_{\mu,L} \circ \Gamma_{\mu,L}$ and where

$$\Gamma^k_{\mu,L} = \Gamma_{\mu,L} \circ \Gamma_{\mu,L} \circ \ldots \circ \Gamma_{\mu,L} = \Lambda^{*k}_\mu \circ \Gamma_Y = \{((b, r), (g \cdot b, r)) : (b, r) \in Y, (g, o) \in \Lambda^{*k}_\mu\} =$$

$$\{((b, r), (g_1 g_2 \cdots g_k \cdot b, r)) : (b, r) \in Y, (g_1, g_2 \cdots g_k \cdot o), (g_2, g_3 \cdots g_k \cdot o), \ldots (g_{k-1}, g_k \cdot o), (g_k, o) \in \Lambda_\mu\} .$$ ∎

3.3 General formula for the limit spectral measure of a random walk along a ray

Our next object is to give a schematic formula for the asymptotics of the spectral measures $m^{\mu}_{n\rho}$ as defined in (1) along rays of representations. As in [P.Z], we do this by using the moment method. We begin by stating a general result under a clean composition hypothesis. This hypothesis is hard to check in practice and frequently fails to be true, so we do not put much emphasis on the general result. Its main purpose is to formulate the general shape of things and to pave the way for the special random walks where we have much more precise results.

We fix an interior weight $\rho \in int(\mathbf{t}^*_+)$. The following theorem gives a formula for the asymptotics of the k^{th} moments

$$M^{\mu}_{n\rho}(k) := \int_{\mathbb{R}} x^k dm^{\mu}_{n\rho}(x) = \frac{1}{\dim V_{n\rho}} TrT^k_{\mu}|_{V_{n\rho}}$$

of the spectral measures $m^{\mu}_{n\rho}$ along the ray $\mathbb{N}\rho$ which in turn determine the asymptotics of the $m^{\mu}_{n\rho}$. The notation D refers to the operator satisfying $D|_{V_{n\rho}} = n$.

Proposition 3.3.1 *Assume that for all $k \in \mathbb{N}$, the k^{th} powers $T^k_{\mu,L}$ and the trace operation for the operator $e^{i\theta D} \circ T^k_{\mu,L}$ are clean compositions. Then the asymptotics of the k^{th} moments of the spectral measures along the ray $L = \mathbb{N}\rho$ are given by*

$$M^{\mu}_{n\rho}(k) \sim n^{r_k + \frac{e - \dim B}{2}} \sum_{j \in \mathcal{K}} \frac{c^k_j}{Cvol(B)} e^{-in\theta_j}$$

where:
(i) r_k denotes the order of $T^k_{\mu,L}$ (as in Proposition 3.2.1),
(ii) $e = \max \dim((\Gamma^k_{\mu,L} \circ \Gamma_\theta) \cap \Delta(T^*B)) + 2\dim B - 1 - \dim \Gamma^k_{\mu,L}$ with Γ_θ being the Lagrangean corresponding to the operator $e^{i\theta D}$,
(iii) C is a universal constant,
(iv) $e^{i\theta_j}$, $j \in \mathcal{K}$, are those circle elements at which the projection $p : (\Gamma^k_{\mu,L} \circ \Gamma_\theta) \cap \Delta(T^*B) \to T^*S^1$ has maximal fiber dimension, and
(v)

$$c^k_j = \int_{((\Gamma^k_{\mu,L} \circ \Gamma_\theta) \cap \Delta(T^*B))_{\theta_j}} tr\sigma(T^k_{\mu,L} \circ \Gamma_\theta)$$

where $(\Gamma^k_{\mu,L} \circ \Gamma_\theta \cap \Delta(T^*B))_{\theta_j}$ denotes the fiber of p above $e^{i\theta_j}$ and where $tr\sigma(T^k_{\mu,L} \circ \Gamma_\theta)$ denotes the trace of the symbol, i.e. the composition $\sigma(T^k_{\mu,L} \circ \Gamma_\theta) \times \sigma(Tr)$.

Proof: We form the generating function

$$\Upsilon_k(\theta) := \sum_{n=1}^{\infty} e^{in\theta} TrT^k_{\mu}|_{V_{k\rho}} = \sum_{n=1}^{\infty} e^{in\theta} \dim V_{n\rho} M^{\mu}_{n\rho}(k) \tag{13}$$

which can be rewritten as

$$\Upsilon_k(\theta) = Tr(e^{i\theta D} \circ T^k_{\mu,L}) = Tr(T_{e^{i\theta}} \circ T^k_{\mu,L}) \tag{14}$$

where $T_{e^{i\theta}}$ is translation by $e^{i\theta}$ in $L^2(B)$. Under clean composition hypothesis, the operator $e^{i\theta D} \circ T^k_{\mu,L}$ is a Fourier-Toeplitz operator and its trace $\Upsilon_k(\theta)$ is a Hardy-Lagrangean distribution on the unit circle S^1. The main point is to determine the isotropic relation, the order, and the principal symbol of $\Upsilon_k(\theta)$. This will allow us to read off the asymptotics of the moments.

Recall from Proposition 3.2.1 that $T^k_{\mu,L} \in I^{r_k}(B \times B, \Gamma^k_{\mu,L})$ with

$$\Gamma^k_{\mu,L} = \{(y, g\cdot y) : y \in Y, (g, pr.(y)) \in \Lambda^{*k}_\mu\}\,.$$

The operator $T_{e^{i\theta}}$ is a Fourier integral representation of S^1 on $L^2(B)$. Its kernel is the δ-function $\delta_{b'-e^{i\theta}\cdot b} \in I^{-\frac{1}{4}}(S^1 \times B \times B, \Gamma_\theta)$ with

$$\Gamma_\theta = \{((e^{i\theta}, \Phi(b,\xi)), (b,\xi), e^{i\theta}(b,\xi)) : e^{i\theta} \in S^1, (b,\xi) \in T^*B\}\,.$$

The composition $\Gamma_\theta \circ \Gamma^k_{\mu,L}$ is always clean with excess $e = 0$. Hence $T_{e^{i\theta}} \circ T^k_{\mu,L} \in I^{r_k - \frac{1}{4}}(S^1 \times B \times B, \Gamma^k_{\theta,\mu,L})$ with

$$\Gamma^k_{\theta,\mu,L} = \{((e^{i\theta}, \Phi(y)), y, e^{i\theta} g\cdot y) : y \in Y, e^{i\theta} \in S^1, (g, pr.(y)) \in \Lambda^{*k}_\mu\}\,.$$

Recall from [P.Z] that the trace operation Tr is an FIO of order 0 with corresponding Lagrangean $\Delta(T^*B)$ (the diagonal in T^*B). We assume that the fiber diagram

$$\begin{array}{ccc} F & \rightarrow & \Gamma^k_{\theta,\mu,L} \\ \downarrow & & \downarrow \quad \pi_{2,3} \\ \Delta(T^*B) & \underset{i}{\rightarrow} & T^*B \times T^*B \end{array}$$

is clean with excess e. The composition

$$\Gamma^k_{\theta,\mu,L} \circ \Delta(T^*B) = \{(e^{i\theta}, \Phi(y)) : \exists g \in G \text{ with } e^{i\theta} g \cdot y = y, (g, pr.(y)) \in \Lambda^{*k}_\mu\}$$

is then a Hardy-Lagrangean subspace of T^*S^1 and thus a finite union $\bigcup_{j=1}^K T^{+*}_{\theta_j}$ of positive half spaces. Furthermore, $\Upsilon_k(\theta) \in I^p(S^1, \bigcup_{j=1}^K T^{+*}_{\theta_j} S^1)$ with $p = r_k - \frac{1}{4} + \frac{e}{2}$ is a Hardy-Lagrangean distribution on the circle S^1.

As in [P.Z], we now use the fact that Hardy-Lagrangean distributions on the circle S^1 are polyhomogeneous and can be written as a sum of the basic homogeneous distributions $\chi_q(\theta - \theta_0) \in I^{q+\frac{1}{4}}(S^1, T^{+*}_{\theta_0} S^1)$. Such a basic homogeneous distribution $\chi_q(\theta - \theta_0)$ has principal symbol

$$\sigma(\chi_q(\theta - \theta_0)) = \xi^q |d\xi|^{\frac{1}{2}} \qquad \text{on } T^{+*}_{\theta_0} S^1$$

and Fourier series expansion

$$\chi_q(\theta - \theta_0) = \sum_{n=1}^\infty n^q e^{in\theta} e^{-in\theta_0}\,. \tag{15}$$

We can thus write

$$\Upsilon_k(\theta) = \sum_{j=1}^K \sum_{r=0}^\infty a_{j,r} \chi_{p-\frac{1}{4}-r}(\theta - \theta_j) \tag{16}$$

and by comparison with the principal symbol $\sigma(\Upsilon_k)$ of Υ_k we get

$$a_{j,0} = c^k_j = \int_{((\Gamma^k_{\mu,L} \circ \Gamma_\theta) \cap \Delta(T^*B))_{\theta_j}} \sigma(T^k_{\mu,L} \circ \Gamma_\theta) \times \sigma(Tr)$$

for all $j \in \mathcal{K}$ and $a_{j,0} = 0$ for all $j \in \mathcal{K}^c$. By plugging in the Fourier series expansions (15) for the $\sigma(\chi_{p-\frac{1}{4}-r}(\theta - \theta_j))$ on the right hand side in (16) and comparing coefficients with (13) we then get

$$\dim V_{n\rho} M^\mu_{n\rho}(k) = n^{p-\frac{1}{4}} \sum_{j \in \mathcal{K}} c^k_j e^{-in\theta_j} + \qquad \text{lower order terms in } n\,.$$

But

$$\dim V_{n\rho} \sim n^{\frac{\dim G - \dim T}{2}} C vol(B)\,.$$

Thus dividing yields

$$M^{\mu}_{n\rho}(k) \sim n^{r_k + \frac{e - \dim B}{2}} \sum_{j \in \mathcal{K}} \frac{c^k_j}{C vol(B)} e^{-ni\theta_j}\,. \qquad \blacksquare$$

3.4 The canonical graph case

The main simplification to Proposition 3.3.1 for our two set of examples discussed below is that the ray Markov Lagrangean $\Gamma_{\mu,L}$ is a local canonical graph. Indeed, it is a union of graphs of global canonical transformations $\chi_i : Y \to Y$. In this case we can be more specific about the 'traces' c^k_j appearing in Proposition (3.3.1). Basically, they are what might be called 'symplectic - spinor traces' $\tau(\chi_i)$ of the symplectic maps $\chi_i \circ \phi^\theta$ where $\phi^\theta : Y \to Y,\ y \mapsto e^{i\theta} \cdot y$. This is the Toeplitz analogue of the 'symplectic trace' of a symplectic map discussed in [G.U]. Let us first briefly recall the symplectic trace and then, also briefly, indicate its extension to symplectic spinors.

Let (V, ω) be a symplectic vector space and let $T \in Sp(V)$ be a symplectic linear map. The fixed point set of T is the subspace $ker(I - T)$. According to [D.G, Lemma 5.1] it possesses an intrinsic density Ω_T which depends on the symplectic nature of $ker(I - T)$. Two special cases discussed in [G.U] are:

• $ker(I - T)$ is a symplectic subspace. Let $W = ker(I - T)^-$ be its symplectic orthogonal complement and let T^- equal $T|_W$. Then: $\Omega_T = \frac{\nu}{|det(I - T^-)|^{\frac{1}{2}}}$ where ν is the symplectic volume density on W.

• $ker(I - T)$ is a Lagrangean subspace L. Then there exists a dual Lagrangean subspace L^* such that $V \cong L \oplus L^*$ and such that $(I - T) : L^* \to L$ is an isomorphism. The symplectic volume form Ω on V may be factored as $d\ell \wedge d\ell^*$ where $d\ell, d\ell^*$ are volume forms on L, L^*. Define $det(I - T)$ by $(I - T)^* d\ell = det(I - T) d\ell^*$. This determinant depends on the choice of $d\ell$ but the quotient

$$\Omega_T := \frac{d\ell}{|det(I - T)|^{\frac{1}{2}}}$$

does not.

Now suppose that χ is a symplectic map on a *compact* symplectic manifold with clean fixed point set $Fix(\chi)$, i.e. such that $Fix(\chi)$ is a submanifold satisfying $Fix(d\chi_y) = T_y Fix(\chi)$ for all $y \in Fix(\chi)$. By the above, $Fix(\chi)$ carries a natural density Ω_χ. The symplectic trace of χ is then defined by

$$ST(\chi) = \int_{Fix(\chi)} \Omega_\chi. \tag{17}$$

This formula does not quite apply to our situation since we are dealing with homogenous canonical transformations. However the only necessary modification is to break the $\mathbb{R}^+$-action, as follows: Let $\chi : T^*B - 0 \to T^*B - 0$ be a homogeneous canonical transformation which preserves the cone $Y \subset T^*B - 0$ and let $\chi_Y : (Y, \omega) \to (Y, \omega)$ be its restriction to Y. It is of course a homogeneous canonical transformation on $Y - 0$. Since $T_y Fix(\chi)$ possesses a canonical density, so does $Fix(\chi)$; we denote it by Ω_χ. Further, let $\mathcal{R}$ denote the radial vector field on Y, i.e. the generator of the $\mathbb{R}^+$-action and define the Liouville density

$$\mu_\chi = i_{\mathcal{R}} \Omega_\chi$$

(where i means insertion) on the base $SFix(\chi)$ of the cone $Fix(\chi)$. Then the homogeneous analogue of the symplectic trace is given by

$$ST(\chi) = \int_{SFix(\chi)} d\mu_\chi. \tag{18}$$

In our applications, we will be concerned with traces of the form

$$\Upsilon_\chi(\theta) = Tr(e^{i\theta D} \circ T_\chi \circ \Pi)$$

where T_χ is a Fourier integral operator associated to the graph of a homogeneous symplectic map χ restricted to the symplectic cone Y generated by the contact form on the base B. The 1/2-density factor of the trace is calculated as for a standard FIO and is similar to the symplectic trace of $\chi|_Y$; it is only necessary to include the symbol as a coefficient of $d\mu_\chi$. Since χ commutes with the S^1-action on Y, it is the lift of a symplectic map $\chi_{\mathcal{O}}$ on $\mathcal{O}$ and the 1/2-density part of the trace basically comes down to the symplectic trace on the compact symplectic manifold $\mathcal{O}$.

However, we still need one further ingredient to describe the coefficients c_j^k, namely the symplectic spinor factor. As discussed in §2.4, this factor is due to the action of $\chi : T^*B \to T^*B$ on directions symplectically normal to Y, i.e. to $d\chi_y^-$ on $T_yY^- \subset T(T^*B)$. The spinor part of the symbol has the form $e_\Lambda \otimes \chi_* e_\Lambda$ where after a choice of metaplectic frames $\chi_* = \mathcal{M}(d\chi_y^-)$, the metaplectic representation applied to the symplectic normal part of $d\chi_y$. Hence the spinor contribution to the trace is

$$Tr\mathcal{M}(d\chi_y^-) \circ \pi = \langle \chi_* e_\Lambda, e_\Lambda \rangle.$$

Therefore, given a homogeneous canonical transformation preserving a symplectic cone Y, the appropriate definition of the symplectic spinor trace is:

$$SSTr(\chi) = \int_{SFix(\chi|_Y)} \langle \chi_* e_\Lambda, e_\Lambda \rangle d\mu_{\chi_Y}. \tag{19}$$

Let us evaluate the expression $\chi_* e_\Lambda, e_\Lambda$ in the case which will concern us, namely when χ is the lift to T^*B of the action of a group element g on B. Suppose then that $g\phi^\theta y = y$. We would like to determine the action of g_* on symplectic spinors at y. This operates on the symplectic orthogonal to $TY \cong T\mathcal{O} \times T(S^1 \times \mathbb{R}^+)$ in T^*B. Note that these two factors are symplectically orthogonal to each other and that g_* operates by scalar multication on the $T(S^1 \times \mathbb{R}^+)$ factor. Hence the non-trivial part is the symplectic orthogonal to $T_o\mathcal{O}$ in $T^*\mathcal{O}$. Since g fixes $pr(y) = o$, we get an induced map g_* on symplectic spinors at o. Since o is a regular element (by assumption), its stabilizer is a maximal torus and without loss of generality we may assume it is the maximal torus T. Now, g_* is obtained by applying $\mathcal{M}$, the metaplectic representation, to the normal part dT_g^- of the derivative. Under the identification $\mathcal{O} \cong G/T$, $T^*\mathcal{O}$ gets identified with $G/T \times \mathbf{t}^{*-}$, $T_o\mathcal{O}^-$ gets identified with $\mathbf{t}^{*-}$ and dT_g^- gets identified with $Ad^*(g)$ on $\mathbf{t}^{*-}$. Its eigenvalues are given by the global roots $e(\alpha)$ evaluated at g. Since $Ad^*(g)$ is a sum of 2-plane rotations, its image in the metaplectic representation is given by a sum of one-dimensional harmonic oscillators. Precisely, we have (with $g = e^X$)

$$\mathcal{M}(Ad^*(g)) = \Pi_{\alpha \in R_+} exp(\langle \alpha, X \rangle \hat{I}_\alpha \tag{20}$$

where $\hat{I}_\alpha$ is the harmonic oscillator $D^2 + u^2$ ($D = \frac{1}{i}\frac{d}{du}$). Since the ground state e_Λ is an eigenfunction of $\hat{I}_\alpha$ with eigenvalue 1/2 it follows that the diagonal matrix element

$$\langle \mathcal{M}(Ad^*(g) e_\Lambda, e_\Lambda \rangle = e(\frac{1}{2}\rho_+(X)). \tag{21}$$

We summarize in the following proposition, whose proof consists of adding the above observations to Proposition (3.3.1).

Lemma 3.4.1 *With the above assumptions,*

$$\Upsilon_\chi(\theta) \in I^{r_\chi - \frac{1}{4} + \frac{e}{2}}(S^1, \bigcup_{j=1}^K T_{\theta_j}^{*+} S^1)$$

where:

(i) r_χ *is the order of* T_χ.

(ii) The angles θ_j *are the ones for which* $Fix(\chi \circ \phi^\theta) \neq \emptyset$ *and the ones which show up in the principal term of the asymptotics are those for which* $\dim Fix(\chi \circ \phi^{\theta_j})$ *is maximal.*

(iii) The principal coefficients of the singularity expansion are given by:

$$\sigma(\Upsilon_\chi(\theta))|_{\theta=\theta_j} = \int_{SFix(\chi\circ\phi^{\theta_j})} f\langle \chi_* e_\Lambda, e_\Lambda\rangle d\mu_{\chi\circ\phi^{\theta_j}}$$

where $\sigma(T_\chi) = f\sqrt{dvol}$ *with* $\sqrt{dvol}$ *is the canonical graph 1/2-density on the graph of* χ.

(iv) In the case where $\chi = T_g$, $g = e^X \in T$, $\langle \chi_* e_\Lambda, e_\Lambda\rangle = e(\frac{1}{2}\rho_+(X))$.

4 δ-functions on positively curved hypersurfaces: Proof of Theorem A

In this section we focus on the class of conormal Markov operators whose underlying probability distributions are δ-functions δ_X on *positively curved* hypersurfaces X of G. We begin with the precise definition.

Definition 4.0.2 Let $X \subset G$ be a smooth compact oriented embedded hypersurface. Denote by $N(X) = N_+(X) \cup N_-(X)$ the inward/outward components of its normal bundle and those of its spherical normal bundle by $SN(X) = S_+N(X) \cup SN_-(X)$. The spherical Gauss maps of X are defined by

$$\mathcal{G}_{\pm,X} : SN_\pm(X) \to S\mathbf{g}, \qquad \mathcal{G}_{\pm,X}(x,v) = dL_{x^{-1}}v$$

where L denotes left translation on G and where $S\mathbf{g}$ denotes the unit sphere of $\mathbf{g}$. The Gaussian curvature of X is defined by

$$\mathcal{G}^*_{\pm,X} d\omega = K dS$$

where $d\omega$ denotes the Euclidean surface measure on $S\mathbf{g}$ induced by the Killing metric. X is called *positively curved* if $K > 0$, i.e. if $\mathcal{G}^*_{+,X}$ is a diffeomorphism. The homogeneous extension of $\mathcal{G}_{\pm,X}$ to the entire normal bundle is also referred to as the Gauss map.

Remark We obviously have $\mathcal{G}_{-,X}(x,-\xi) = -\mathcal{G}_{+,X}(x,\xi)$ for all $(x,\xi) \in N_+(X)$. Since the antipodal map is an isometry, the definition of K is unambiguous.

To prepare for the statement of Theorem A, we introduce a number of notions and notations.

Symmetric δ-functions.

In order to deal with self-adjoint Markov operators we require that the underlying measure be symmetric, i.e. invariant under the inversion map $inv : G \to G$, $g \mapsto g^{-1}$. Since X need not be invariant under inversion, we have to consider the union $X \cup X^{-1}$.

As above, we denote by dS the surface measure on X induced by the Haar volume form dg. By a δ-function δ_X on X is meant a measure of the form

$$\delta_X(f) = \int_X f a dS$$

where $f \in C(G)$ and $a \in C^\infty(G)$. To make it symmetric we average it with respect to the inversion map, i.e. put

$$\mu = \frac{1}{2}(\delta_X + \delta_{X^{-1}}) := \frac{1}{2}(\delta_X + inv_*\delta_X)$$

where inv_* denotes pushforward under inv. To make sure that convolution powers of μ_X are clean we require that $X \cup X^{-1}$ is a transversal intersection. Since inv is an isometry, X^{-1} is positively curved as long as X is.

Inverse Gauss maps The canonical relation underlying convolution with $\frac{1}{2}(\delta_X + \delta_{X^{-1}})$ will involve the inverses of the Gauss maps on the inward/outward normal bundles $N^{\pm}(X)$. They are the maps from **g** to G defined by

$$\mathcal{F}_1 := \pi(\mathcal{G}_{+,X})^{-1}, \quad \mathcal{F}_2 := \pi(\mathcal{G}_{-,X})^{-1}, \quad \mathcal{F}_3 := \pi(\mathcal{G}_{+,X^{-1}})^{-1}, \quad \mathcal{F}_4 := \pi(\mathcal{G}_{-,X^{-1}})^{-1}$$

where $\pi : N(X) \to X$ is the natural projection.

4.1 The Markov operator on $L^2(G)$: Proofs of Theorem A(i)-(iii)

The following is a more general and precise statement of Theorem A(i):

Theorem A(i) *Let X be a positively curved orientable hypersurface of G and assume that either (i) $X = X^{-1}$ or (ii) the intersection $X \cap X^{-1}$ is transversal. Let $\mu = \frac{1}{2}(\delta_X + \delta_{X^{-1}})$ with $\delta_X = adS$ for some $a \in C^\infty(X)$. Then $T_{\mu,G} f = \mu * f$ is a Fourier integral operator of order $-\frac{\dim G - 1}{2}$ on $L^2(G)$ associated to the disjoint union of canonical graphs*

$$\Gamma_{\mu,G} = \bigsqcup_{i=1}^{4} Graph(\chi_i) \quad \textit{with} \quad \chi_i : T^*G \to T^*G\,, \;\; (x,\xi) \mapsto ((\mathcal{F}_i(Ad^*(x)\xi))x, \xi)\,, \;\; i = 1, ..., 4\,. \tag{22}$$

Proof: We have $\mu \in I^{\frac{1}{2} - \frac{\dim G}{4}}(G, \Lambda_\mu)$ where in case (i), $\Lambda_\mu = N^*X$ and in case (ii), $\Lambda_\mu = N^*X \cup N^*X^{-1}$. By Proposition 2.3.1, $T_{\mu,G}$ is an FIO of order $-\frac{\dim G - 1}{2}$ corresponding to

$$\Gamma_{\mu,G} = \{((x,\xi),(gx,\xi))) \in T^*(G \times G) - 0 : (g, Ad^*(x)\xi) \in \Lambda_\mu\}\,.$$

But

$$(g, Ad^*(x)\xi) \in \Lambda_\mu \Leftrightarrow g = \mathcal{F}_i(Ad^*(x)\xi) \qquad \text{for some } i \in \{1, ..., 4\}\,.$$

Since X and X^{-1} are positively curved and because of the transversal intersection hypothesis, for each $(x,\xi) \in T^*G$, there exist exactly two, resp. four (in case (i), resp. case (ii)) distinct elements $g_i = \mathcal{F}_i(Ad^*(x)\xi) \in G$, such that $(g_i, Ad^*(x)\xi) \in \Lambda_\mu$, $i = 1, ..., 4$. This shows that $\Gamma_{\mu,G}$ consists of the disjoint union of the two (four) canonical graphs in (22). ■

Proof of Theorem A(ii): This is a corollary of Thereom A(i). Since $T_{\mu,G}$ is a Fourier integral operator of order $-\frac{\dim G - 1}{2}$ associated to a local canonical graph, it follows by [Hö, Vol.IV Corollary 25.3.2] that $T_{\mu,G}$ defines a bounded operator from $W^s(G) \to W^{s + \frac{\dim G - 1}{2}}(G)$. ■

Proof of Theorem A(iii): A more precise statement of the result is

$$\mu^{*k} \in L^2(G) \qquad \text{for} \quad \begin{cases} k \geq 2 \;\text{ and } \;\; \dim G \geq 3 \\ k \geq 3 \;\text{ and } \;\; \dim G = 2 \end{cases}\,.$$

First, the case $\dim G \geq 3$. We will show that $T^k_{\mu,G}$ is Hilbert-Schmidt for $k \geq 2$. Set $A := (T^{*k}_{\mu,G} T^k_{\mu,G})$ and write A as $\sqrt{\Delta}^{-k(\dim G-1)} \sqrt{\Delta}^{k(\dim G-1)} A$. Since A is an FIO of order $-k(\dim G - 1)$ associated to a union of canonical graphs, $\sqrt{\Delta}^{k(\dim G-1)} A$ is bounded on $L^2(G)$ and it suffices to show that $\sqrt{\Delta}^{-k(\dim G-1)}$ is of trace class, i.e., that

$$Tr \sqrt{\Delta}^{-k(\dim G-1)} = \int \lambda^{-\frac{k}{2}(\dim G-1)} dN(\lambda) < \infty \tag{23}$$

where $N(\lambda) := \#\{j : \lambda_j \leq \lambda\}$ for the eigenvalues λ_j of the Laplacian Δ. But by Weyl's law $N(\lambda) \sim \lambda^{\frac{\dim G}{2}}$. Hence, by integrating by parts, the right hand side in (23) is

$$-\frac{k}{2}(\dim G - 1)\int N(\lambda)\lambda^{-\frac{k}{2}(\dim G-1)-1}d\lambda \sim \int \lambda^{\frac{\dim G}{2}}\lambda^{-\frac{k}{2}(\dim G-1)-1}d\lambda$$

which is finite for $\dim G \geq 3$ and $k \geq 2$. It follows that $(T^{*k}_{\mu,G}T^k_{\mu,G})$ is of trace class for $k \geq 2$. Similarly, for the case of $\dim G = 2$, we can show that $(T^{*k}_{\mu,G}T^k_{\mu,G})$ is of trace class for $k \geq 3$. ∎

For the asymptotic expansions of Theorem A(iv), we will need to to know the principal symbol $\sigma(T_{\mu,G})$ or more generally the principal symbol $\sigma(\delta_X *)$ of convolution with a δ-function δ_X along a hypersurface. To quote easily from the literature we will view δ_X as a 1/2-density rather than as a density, i.e. as acting on smooth 1/2-densities u on G by

$$\langle \delta_X, u\rangle = \int_X a\sqrt{dS}\cdot i_\nu u.$$

Here, $i_\nu u$ is the 1/2-density on X by inserting the outward unit normal ν into u, so that $\sqrt{dS}\cdot i_\nu u$ is a density, on X.

We recall ([Hö, Vol.III]) that in general the principal symbol of a conormal distribution (1/2-density) locally represented by

$$\int e^{i\langle x',\xi''\rangle}a(x'',\xi')d\xi'$$

is given by

$$a(x'',\xi')|dx''|^{\frac{1}{2}}|d\xi'|^{\frac{1}{2}}$$

in coordinates (x', x'') such that locally $X = \{x' = 0\}$. For concreteness, we choose to use Fermi normal coordinates along X, so that x' is the signed distance to X. Then the principal symbol $\sigma(\delta_X)$ equals

$$a(x'')|S'(x'')|^{\frac{1}{2}}|dx''|^{\frac{1}{2}}|d\xi'|^{\frac{1}{2}} = a(x'')|dS|^{\frac{1}{2}}|d\xi'|^{\frac{1}{2}}.$$

Now consider the principal symbol of $\delta_X *$. By Proposition 2.3.1, we can identify the Lagrangean $\Gamma_\mu \cong \Lambda_\mu \times G$ and then $\sigma(\delta_X *) = \sigma(\delta_X)\otimes|dx|^{\frac{1}{2}}$. Thus we have

$$\sigma(\delta_X *) = a(x'')|dS|^{\frac{1}{2}}|d\xi'|^{\frac{1}{2}}\otimes|dx|^{\frac{1}{2}}.$$

In the case where X is a positively curved hypersurface, $\Gamma_{\mu,G}$ was just shown to be a local canonical graph. Since each component $Graph(\chi_i)$, $i = 1, ..., 4$, possesses a canonical graph 1/2-density $|dg\wedge d\gamma|^{\frac{1}{2}}$ (in coordinates coming from its projection to $T^*G \sim G\times \mathbf{g}^*$), each restriction $\sigma(T_{\mu,G})_i$ of $\sigma(T_{\mu,G})$ to $Graph(\chi_i)$, $i = 1, ..., 4$, can also be written as a scalar multiple of $|dg\wedge d\gamma|^{\frac{1}{2}}$. With this identification the symbol is given by:

Proposition 4.1.1

$$\sigma(T_{\mu,G})_i = |\gamma|^{-\frac{1}{2}(dim G-1)}\mathcal{G}_i^{-1*}\left(\frac{a}{\sqrt{K}}\right)|dg\wedge d\gamma|^{\frac{1}{2}} \qquad for\ i = 1, ..., 4.$$

where the Gauss maps $\mathcal{G}_i$ correspond to the inward/outward components $N^*_\pm$ of $N^*(X)$ and $N^*(X^{-1})$ in the same way as the $\mathcal{F}_i$ defined earlier.

Proof: We just consider $N^*_+(X)$ since the calculations for the remaining components are essentially the same. For simplicity we will write $\mathcal{G}$ instead of $\mathcal{G}_1$.

Under the identification $T^*G \sim G\times\mathbf{g}^*$ by left translation, the Gauss map $\mathcal{G} : N^*_+(X) \to \mathbf{g}^*$ is simply the inclusion map followed by projection to $\mathbf{g}^*$. By assumption, it is a diffeomorphism to $\mathbb{R}^+ \times S\mathbf{g}^*$. Write the Euclidean density $|d\gamma|$ on $\mathbf{g}^*$ in polar coordinates (r,ω) coming from the identification $\mathbf{g}^* \sim \mathbb{R}^+ \times S\mathbf{g}^*$: $|d\gamma| = r^{dimG-1} dr \wedge |d\omega|$. Also use polar coordinates $N^*_+(X) \sim \mathbb{R}^+ \times X$. Then $\mathcal{G}^* r^{dimG-1} dr \wedge |d\omega| = K r^{dimG-1} dr \wedge dS$. Under the inverse of the Gauss map, the principal symbol $\sigma(\mu)$ goes over to

$$\mathcal{G}^{-1*}(a|dS|^{\frac{1}{2}}|d\xi'|^{\frac{1}{2}}) = \mathcal{G}^{-1*}(\frac{a}{\sqrt{K}})|dr \wedge d\omega|^{\frac{1}{2}}$$

on $N^*_+(X)$. It follows by Proposition 2.3.1 that $\sigma(T_{\mu,G})$, viewed as a 1/2-density on $G \times \mathbf{g}^*$, is the tensor product of this 1/2-density with $|dg|^{\frac{1}{2}}$. This yields the above formula. ∎

Remark As a check on the order of the coefficient, we note that since $T_{\mu,G}$ has order $-\frac{1}{2}(dimG - 1)$ its symbol must have order $-\frac{1}{2}(dimG - 1) + \frac{1}{2}dimG = \frac{1}{2}$ as a 1/2-density (cf. [Ho IV, Theorem 25.1.9). This indeed is the order of $\mathcal{G}^{-1*}(\frac{a}{\sqrt{K}})|dr \wedge d\omega|^{\frac{1}{2}} \otimes |dg|^{\frac{1}{2}}$. Expressed in terms of the coordinates (g,γ), $|dr \wedge d\omega|^{\frac{1}{2}} \otimes |dg|^{\frac{1}{2}} = |\gamma|^{-\frac{1}{2}(dimG-1)}|dg \wedge d\gamma|^{\frac{1}{2}}$.

4.2 Spectral asymptotics along rays: Proof of Theorem A(iv)

The proof of Theorem A(iv) will require a precise description of the ray Markov operator $T_{\mu,L}$ of a δ-function along a positively curved hypersurface.

4.2.1 The ray Markov operator

Proposition 4.2.1 *Let X be a positively curved orientable hypersurface of G and μ as in Theorem A(i). Consider the ray of representations $L = \mathbb{N}\rho$. Then*

$$T_{\mu,L} \in I^{1-\frac{\dim B + \dim G}{2}}(B \times B, \Gamma_{\mu,L})$$

is a Fourier-Toeplitz operator of order $1 - \frac{\dim B + \dim G}{2}$ and the associated isotropic submanifold $\Gamma_{\mu,L}$ is the union of graphs of two (four) canonical transformations χ_i of Y, i.e.,

$$\Gamma_{\mu,L} = \bigcup_{i=1}^{4} Graph(\chi_i) \qquad with \quad \chi_i : Y \to Y\,, \quad y \mapsto \mathcal{F}_i(o) \cdot o\,, \quad o = pr.(b)\}\,, \quad i = 1, ..., 4\,.$$

Proof: Recall Proposition 2.4.1. The main step is to show that the fiber diagram

$$\begin{array}{ccc} F & \to & \Gamma_Y \\ \downarrow & & \downarrow \quad \pi_1 \\ \Lambda_\mu & \underset{i}{\to} & T^*G \end{array}$$

with fiber product $F = \{((g,ro),(g,ro),(b,r),(g\cdot b,r)) : (g,ro) \in \Lambda_\mu\}$ is clean with excess $e = 0$. Note that

$$(g,ro) \in \Lambda_\mu \Leftrightarrow g_i = \mathcal{F}_i(o) \qquad \text{for some } i \in \{1,...,4\}\,.$$

Thus F is the union of two (four) graphs of two (four) distinct smooth maps from Y to $T^*G \times T^*G \times Y$ and hence is a manifold of dimension $\dim Y$.

We now show that the derived diagram

$$\begin{array}{ccc} TF & \to & T\Gamma_Y \\ \downarrow & & \downarrow \quad d\pi_1 \\ T\Lambda_\mu & \underset{di}{\to} & T(T^*G) \end{array}$$

is also a fiber diagram. For this we need to show that for all $(c,d) \in F \subset \Lambda_\mu \times \Gamma_Y$, $\dim\{(u,v) \in T_c\Lambda_\mu \times T_d\Gamma_Y : di(u) = d\pi_1(v)\} = \dim Y$. Fix $(c,d) \in F$. Since di is an injection, each $v \in T_d\Gamma_Y$ can only have at most one matching vector $u \in T_c\Lambda_\mu$ such that $di(u) = d\pi_1(v)$. On the other hand, every vector tangential to the circle which runs through d, and only such vectors, apart from the zero vector itself, are mapped onto the zero vector under $d\pi_1$. In $T_c\Lambda_\mu$ we can distinguish horizontal and vertical vectors. Vertical vectors are vectors along the fiber above $\pi(c)$ (π is the natural projection $\pi : T^*G \to G$) and vertical vectors are deritaves of curves thru c in Λ_μ for which the length of the conormal vector stays constant. Clearly, every vertical vector can be matched with a vector $v \in T_d\Gamma_Y$ of the form $v = \frac{d}{dt}|_{t=0}q(t)$ with $q(t) = ((g,(r+t)pr.(b)),(b,r+t),(g\cdot b, r+t))$. By construction of Γ_Y, horizontal vectors $u = (u_1,u_2)$ have a match v if and only if there is a curve in $\mathcal{O}_\rho$ whose derivative produces u_2. Altogether, this proves that $\dim\{(u,v) \in T_c\Lambda_\mu \times T_d\Gamma_Y : di(u) = d\pi_1(v)\}$ is precisely $\dim \mathcal{O}_\rho + 2 = \dim Y$. Thus the composition is clean and the excess is $e = \dim Y + 2\dim G - \dim G - (\dim G + \dim Y) = 0$. The statement now follows from Proposition 2.4.1. ■

Proposition 4.2.1 and Proposition 3.2.1 yield the following

Corollary 4.2.2 *All the composition powers of $T_{\mu,L}$ are clean. For all $k \geq 1$,*

$$T^k_{\mu,L} \in I^{\frac{k}{2}(1-\dim G)+\frac{1}{2}(1-\dim B)}(B \times B, \Gamma^k_{\mu,L})$$

where

$$\begin{aligned}\Gamma^k_{\mu,L} &= \textstyle\bigcup_{(i_k,\ldots,i_1)\in\{1,\ldots,4\}^k} Graph(\chi_{i_k} \circ \cdots \circ \chi_{i_1}) \\ &= \{((b,r),(g_k\cdots g_1 b, r)) : \exists (i_k,...,i_1) \in \{1,...,4\}^k \ \ s.t. \ \ g_1 = \mathcal{F}_{i_1}(o),\ldots, g_k = \mathcal{F}_{i_k}(g_{k-1}\cdots g_1 o)\}\,.\end{aligned}$$

4.2.2 The principal symbol $\sigma(T_{\mu,L})$

In Proposition 4.1.1 we showed that the components of the symbol $\sigma(T_{\mu,G})$ are $\sigma(T_{\mu,G})_i = \mathcal{G}_i^{-1*}(\frac{a}{\sqrt{K}})$ times the canonical graph 1/2-density. The situation is very similar for the Markov ray operator. We begin by redoing the argument with G replaced by the homogeneous space B and then restrict to Y.

The Markov Lagrangean for the random walk $T_{\mu,B}$ on all of B is given by

$$\Gamma_{\mu,B} := \{(b,\beta), g\cdot(b,\beta) : (g, \Phi_B(b,\beta)) \in N^*(X) \cup N^*(X^{-1})\}$$

where Φ_B is the moment map. Since for each $(b,\beta) \in T^*B$ there are exactly two (four) distinct points $g_i = \mathcal{F}_i(\Phi_B(b,\beta)) \in X$ with normal $\Phi_B(b,\beta)$, the Lagrangean $\Gamma_{\mu,B} \subset T^*(B\times B) - 0$ is here again the union of two (four) canonical graphs, each diffeomorphic to T^*B. The components of the symbol of the Markov operator $T_{\mu,B}$ may therefore be identified with 1/2-densities on T^*B. To determine them, we use the fact that there are two (four) maps $F_i : T^*B \to \Lambda_\mu = N^*(X) \cup N^*(X^{-1})$, namely $F_i(b,\beta) = (\mathcal{F}_i(\Phi_B(b,\beta)), \Phi_B(b,\beta))$. We claim:

Proposition 4.2.3 *The components $\sigma(T_{\mu,B})_i$ of the principal symbol $\sigma(T_{\mu,B})$ of the Markov operator on B are given by:*

$$\sigma(T_{\mu,B})_i = F_i^*(\sigma_\mu) = F_i^*(\frac{a}{\sqrt{K}})|\beta|^{-\frac{1}{2}(dimB-1)}|db\wedge d\beta|^{\frac{1}{2}}\,, \qquad i = 1,...,4\,.$$

Proof: The argument is similar to Proposition 4.1.1. The moment Lagrangean Γ_B for the action on B may be parametrized by $G \times T^*B$ and the principal symbol of the action is the canonical volume 1/2-density $|dg|^{\frac{1}{2}} \otimes |db\wedge d\beta|^{\frac{1}{2}}$. The relevant composition (fiber product) diagram is:

$$\begin{array}{ccc} F & \rightarrow & \Gamma_B \\ \downarrow & & \downarrow \quad \pi_1 \\ \Lambda_\mu & \underset{i}{\rightarrow} & T^*G \end{array}$$

where $\pi_1(g,(b,\beta)) = (g, \Phi_B(b,\beta))$. As above, the fiber diagram is transversal. ∎

Proposition 4.2.4 *The components $\sigma(T_{\mu,L})_i$ of the principal symbol $\sigma(T_{\mu,L})$ of the Markov operator $T_{\mu,L}$ on $H^2(B)$ are given by:*

$$\sigma(T_{\mu,L})_i = F_i^*(\frac{a}{\sqrt{K}})|y|^{-\frac{1}{2}(\frac{dim Y}{2}-1)}|dy|^{\frac{1}{2}} \otimes g_{i*}\pi \qquad i = 1, ..., 4\,.$$

Proof: Except for the symplectic spinor aspect, the proof is essentially to restrict the formula above to Y. The moment Lagrangean Γ_Y may be parametrized by $G \times Y$ and the relevant composition diagram is:

$$\begin{array}{ccc} F & \rightarrow & \Gamma_Y \\ \downarrow & & \downarrow \;\; \pi_1 \\ \Lambda_\mu & \underset{i}{\rightarrow} & T^*G \end{array}$$

where $\pi_1(g,y) = (g, \Phi_Y(y))$. Here the moment map Φ_Y is the projection $Y \to \mathcal{O}$, $(b,r) \mapsto pr.(b) = o$, followed by the inclusion $\mathcal{O} \subset \mathbf{g}^*$. As in the previous case, the diagram is transversal and the canonical densities compose to give the canonical volume 1/2-density on Y. We compute the scalar coefficients for the components of the principal symbol by pulling back $\sigma(\mu)$ to Y by the two (four) maps

$$F_i : Y \to \Lambda_\mu, \quad (b,r) \mapsto (\mathcal{F}_i(o), r)$$

where we have used the isomorphisms $N^*_\pm(X) \cong X \times \mathrm{I\!R}^+$ and $N^*_\pm(X^{-1}) \cong X^{-1} \times \mathrm{I\!R}^+$. The fiber $F_{y,y'}$ consists of one point so the symplectic spinor part is just the integrand of Proposition 2.4.1. ∎

Corollary 4.2.5 *The components $\sigma(T^k_{\mu,L})_{(i_k,...,i_1)}$, on $Graph(\chi_{i_k} \circ \cdots \circ \chi_{i_1})$, $(i_k,...,i_1) \in \{1,...,4\}^k$, of the principal symbol $\sigma(T^k_{\mu,L})$ are given by*

$$\sigma(T^k_{\mu,L})_{(i_k,...,i_1)} = f_{(i_k,...,i_1)}\, |y|^{-\frac{k}{2}(\frac{dim Y}{2}-1)}|dy|^{\frac{1}{2}} \otimes (g_{1_k} \cdots g_{i_1})_*\pi\;.$$

with

$$f_{(i_k,...,i_1)} = F^*_{i_1}(\frac{a}{\sqrt{K}}) \cdot (F_{i_2} \circ \chi_{i_1})^*(\frac{a}{\sqrt{K}}) \cdots (F_{i_k} \circ \chi_{i_{k-1}} \circ \cdots \circ \chi_{i_1})^*(\frac{a}{\sqrt{K}})\,.$$

4.2.3 The limit spectral measure: Completion of the proof of Theorem A(iv)

We now determine the asymptotics of the moments $M^\mu_{n\rho}(k)$ of the spectral measures $m^\mu_{n\rho}$ along the ray $L = \mathrm{I\!N}\rho$. The outline of the calculation was given in Proposition 3.3.1, so the proof of Theorem A(iv) is basically a matter of filling in the blanks.

Theorem A(iv) *Let X be a positively curved orientable hypersurface of G, $L = \mathrm{I\!N}\rho$ an interior ray of representations of G, and μ as defined above. We assume that the trace operation for the operator $e^{i\theta D} \circ T^k_{\mu,L}$ is a clean composition for all $k \geq 1$. Then the k^{th} moments of the spectral measures $m^\mu_{n\rho}$ are given asymptotically by*

$$M^\mu_{n\rho}(k) \sim n^{\frac{k}{2}(1-\dim G)+\frac{1+e}{2}-\dim B}\frac{1}{Cvol(B)} \sum_{\substack{(i_k,...,i_1) \in \\ \{1,...,4\}^k}} \sum_{\substack{\theta_j \in \\ \Theta_{i_k,...,i_1}}} e^{-in\theta_j + i\pi\sigma/4}$$

$$\int_{SFix(\chi_{i_k}\circ\cdots\circ\chi_{i_1}\circ\phi^{\theta_j})} \langle e_\Lambda, (g_{i_k}\cdots g_{i_1})_* e_\Lambda\rangle f_{(i_k,...,i_1)}\, d\mu_{\chi_{i_k}\circ\cdots\circ\chi_{i_1}\circ\phi^{\theta_j}}$$

where the $f_{(i_k,\ldots,i_1)}$'s are as in Corollary 4.2.5 and

$$\Theta_{i_k,\ldots,i_1} := \{\theta_j : \dim Fix(\chi_{i_k}\circ\cdots\circ\chi_{i_1}\circ\phi^{\theta_j}) \;\; \textit{is maximal}\;\}\,.$$

The factor $\langle e_\Lambda,(g_{i_k}\cdots g_{i_1})_* e_\Lambda\rangle$ is described in Proposition (3.4.1 (iv)).

Proof: Recall from Proposition 3.3.1 the general formula for $M^\mu_{n\rho}(k)$. We need to fill in
(i) the order r_k of $T^k_{\mu,L}$,
(ii) the highest order coefficients c^k_j of the symbol of $\Upsilon_k(\theta) = Tr(e^{i\theta D}\circ T^k_{\mu,L})$, and
(iii) the highest order singular angles θ_j of $\Upsilon_k(\theta)$.

By Corollary 4.2.2, $r_k = \frac{k}{2}(1-\dim G) + \frac{1}{2}(1-\dim B)$ and the corresponding canonical relation is the union of canonical graphs

$$\Gamma^k_{\mu,L} = \bigcup_{(i_k,\ldots,i_1)\in\{1,\ldots,4\}^4} Graph(\chi_{i_k}\circ\cdots\circ\chi_{i_1})\,.$$

The cleanliness assumption for the trace operation $Tr(e^{i\theta D}\circ T^k_{\mu,L})$ implies that for each $\theta\in S^1$ and $(i_k,\ldots,i_1)\in\{1,\ldots,4\}^4$, the map $\chi_{i_k}\circ\cdots\circ\chi_{i_1}\circ\phi^\theta$ has clean fixed point set $Fix(\chi_{i_k}\circ\cdots\circ\chi_{i_1}\circ\phi^\theta)$. But by Lemma 3.4.1,

$$\sigma(\Upsilon_k(\theta))|_{\theta=\theta_j} = \sum_{(i_k,\ldots,i_1)\in\{1,\ldots,4\}^4} \int_{SFix(\chi_{i_k}\circ\cdots\circ\chi_{i_1}\circ\phi^{\theta_j})} \langle e_\Lambda,(g_{i_k}\cdots g_{i_1})_* e_\Lambda\rangle f_{(i_k,\ldots,i_1)}\, d\mu_{\chi_{i_k}\circ\cdots\circ\chi_{i_1}\circ\phi^{\theta_j}}$$

where $d\mu_{\chi_{i_k}\circ\cdots\circ\chi_{i_1}\circ\phi^{\theta_j}}$ is the canonical density on the projection of $Fix(\chi_{i_k}\circ\cdots\circ\chi_{i_1}\circ\phi^{\theta_j})$ to B as described in Section 3.4 and furthermore, the $f_{(i_k,\ldots,i_1)}$'s are the scalar factors appearing in the principal symbol $\sigma(T^k_{\mu,L})$ as described in Corollary 4.2.5. Altogether, this implies the stated formula for the moments $M^\mu_{n\rho}(k)$. ∎

Remark: The fixed point set of the trace operation and hence the excess e appearing in the asymptotic formula for the moments depends on the underlying hypersurface X. This is illustrated in the examples of the spherical means operator and its translates below.

4.3 Example: the spherical means operator and its translates

4.3.1 The spherical means operator

Let $S_s(0)$ be the sphere of radius s in $\mathbf{g}$. Here we consider symmetric probability measures μ whose underlying hypersurface is $X = \mathcal{S}_s(e) := \exp S_s(0)$, the geodesic sphere of radius s centered at e. Note that $X = X^{-1}$. The hypersurface $X = \mathcal{S}_s(e)$ is positively curved for any $s < i$ where i is the injectivity radius of G. The corresponding Markov operator $T_\mu = M_s$ is called the *spherical means operator.* In this case, the underlying isotropic relation and the principal symbol take on an especially simple form:

Lemma 4.3.1 *Let μ be a δ-function on $\mathcal{S}_s(e)$ and $L = \mathbb{N}\rho$ a ray of representations. Consider the k^{th} power $T^k_{\mu,L}$ of the ray Markov operator $T_{\mu,L}$. Then for all $k\geq 1$:*
(a) The underlying isotropic relation $\Gamma^k_{\mu,L}$ has the form

$$\Gamma^k_{\mu,L} = \bigcup_{j=0}^{k} Graph(\phi^{\theta_0(-k+2j)}) \quad \textit{with } \theta_0 = 2\pi s|\rho|\,.$$

(b) The components $\sigma(T^k_{\mu,L})_j$ on $Graph(\phi^{\theta_0(-k+2j)})$ of the principal symbol $\sigma(T^k_{\mu,L})$ are given by

$$\sigma(T^k_{\mu,L})_j = \binom{k}{j} f^k\,|y|^{-\frac{k}{2}(\frac{dim Y}{2}-1)}|dy|^{\frac{1}{2}}\otimes(exp((k-2j)s\frac{o}{|o|})_*\pi \quad \textit{with} \quad f = F_1^*(\frac{a}{\sqrt{K}}) = F_2^*(\frac{a}{\sqrt{K}})\,.$$

Proof: (a) Recall Proposition 4.2.1. For a given $b \in B$, $o = pr.(b) = x\rho x^{-1}$ for some $x \in G$. We use the fact that for any $X \in \mathbf{g}$, the geodesic $\exp tX$ is perpendicular to $\mathcal{S}_s(e)$ at the points of interesection $\exp \pm s\frac{X}{|X|}$. It follows that, in the notation of Proposition 4.2.1, $\mathcal{F}_1(o) = \exp s\frac{o}{|o|}$ and $\mathcal{F}_2(o) = \exp -s\frac{o}{|o|}$. But $\exp \pm s\frac{o}{|o|} = x(\exp \pm s\frac{\rho}{|\rho|})x^{-1}$ and $x(\exp \pm s\frac{\rho}{|\rho|})x^{-1} \cdot b = \chi_\rho(\exp \pm s\frac{\rho}{|\rho|}) \cdot b = e^{\pm 2\pi i s \frac{\langle \rho,\rho\rangle}{|\rho|}} \cdot b$. Setting $\theta_0 = 2\pi s|\rho|$, we thus get

$$\Gamma_{\mu,L} = \{((b,r),(e^{\pm i\theta_0} \cdot b, r))\}$$

and from this $\Gamma^k_{\mu,L}$ for all $k \geq 1$ as claimed.

(b) Recall Corollary 4.2.5. Here $(i_k,...,i_1) \in \{1,2\}^k$ and $\chi_1 = \phi^{\theta_0}$ and $\chi_2 = \phi^{-\theta_0}$. Because of the symmetry of μ we have $F_1^*(\frac{a}{\sqrt{K}}) = F_2^*(\frac{a}{\sqrt{K}})$ which we call f. And finally, by construction of the F_i, $i = 1,2$, we have $\chi_1^*(f) = \chi_2^*(f) = f$. There are $\binom{k}{j}$ k-tuples $(i_k,...,i_1) \in \{1,2\}^k$ for which $\chi_{i_k} \circ \cdots \circ \chi_{i_1} = \phi^{\theta_0(-k+2j)}$. The group element $g_{i_k} \cdots g_{i_1}$ is a product of j factors of $exp - s\frac{o}{|o|}$ and $k-j$ factors of $exps\frac{o}{|o|}$ which equals $exp(k-2j)s\frac{o}{|o|}$. ∎

The asymptotics of the moments of the spectral measures are given by the following:

Proposition 4.3.2 *Let μ be a symmetric δ-function on $\mathcal{S}_s(e)$ and $L = \mathbb{N}\rho$ an interior ray of representations. The k^{th} moments of the spectral measures $m^\mu_{n\rho}$ of the spherical means operator $T_{\mu,L} = M_s$ are given asymptotically by*

$$M^\mu_{n\rho}(k) \sim n^{\frac{k}{2}(1-\dim G)} \frac{1}{Cvol(B)} \int_B f^k\, dvol \cdot \begin{cases} \sum_{j=0}^{\frac{k-1}{2}} 2\binom{k}{j} \cos(n(k-2j)2\pi s(|\rho| + \rho_+(\frac{\rho}{|\rho|}) & \text{for } k \text{ odd} \\ \sum_{j=0}^{\frac{k-2}{2}} 2\binom{k}{j} \cos(n(k-2j)2\pi s(|\rho| + \rho_+(\frac{\rho}{|\rho|}) + \binom{k}{\frac{k}{2}} & \text{for } k \text{ even}. \end{cases}$$

Proof: Recall Proposition 3.3.1 and Theorem A(iv). Here the fiber diagram for the trace operation $Tr(e^{i\theta D} \circ T^k_{\mu,L})$ is

$$\begin{array}{ccc} F & \to & \Gamma^k_{\theta,\mu,L} \\ \downarrow & & \downarrow \\ \Delta(T^*B) & \to & T^*B \times T^*B \end{array}$$

with

$$\Gamma^k_{\theta,\mu,L} = \bigcup_{j=0}^{k} \{((e^{i\theta}, -r),(b,r), e^{i(\theta+\theta_0(-k+2j))} \cdot b, r))\} \cong \bigcup_{j=0}^{k} (S^1 \times Y)_j$$

($k+1$ copies of $S^1 \times Y$) and

$$F \cong \bigcup_{j=0}^{k} \{((e^{-i\theta}, -r),(b,r))\} \cong \bigcup_{j=0}^{k} Y_j$$

($k+1$ copies of Y). Clearly this fiber diagram is clean. Its excess is $e = \dim Y + 4\dim B - 2\dim B - (1 + \dim Y) = 2\dim B - 1$ and the composition is

$$\Gamma^k_{\theta,\mu,L} \circ \Delta(T^*B) = \bigcup_{j=0}^{k} T^{*+}_{\theta_0(-k+2j)}(S^1)\,.$$

The fixed point set corresponding to each singular angle $\theta_0(-k+2j)$, $j = 0,...,k$ is all of Y. The moment asymptotics formula then follows by plugging the formula for the principal symbol $\sigma(T^k_{\mu,L})$ in Lemma 4.3.1(b) into the general formula from Theorem A(iv) and using the formula in proposition 3.4.1 (iv) for the symplectic spinor factor, which comes to $e((k-2j)s\rho_+(\frac{\rho}{|\rho|}))$.

∎

4.3.2 Translates of the spherical means operator

We now translate the previous measure by a group element g to get $\mu = \frac{1}{2}\delta_{g\mathcal{S}_s(e)} + \frac{1}{2}\delta_{g^{-1}\mathcal{S}_s(e)}$. Without loss of generality we will assume that $g \in T$. The moment asymptotics depend on the degree of singularity $ds(g) = \sharp\{\text{roots } \alpha : \alpha(g) = 1\}$ of g and illustrate a variety of cases of Theorem B.

We have the following analogue to Lemma 4.3.1:

Lemma 4.3.3 *Let μ be as described above and let $L = \mathbb{N}\rho$ be a ray of representations. Consider the k^{th} power $T^k_{\mu,L}$ of the ray Markov operator $T_{\mu,L}$. Then for all $k \geq 1$:*
(a) The underlying isotropic relation $\Gamma^k_{\mu,L}$ has the form

$$\Gamma^k_{\mu,L} = \bigcup_{l=0}^{k} \bigcup_{j=0}^{k} Graph(\chi^{g^{-k+2l}} \circ \phi^{\theta_0(-k+2j)})$$

where $\chi^h : Y \to Y,\ y \mapsto h \cdot y,\ \forall h \in G$ and θ_0 is as in Lemma 4.3.1.
(b) The components $\sigma(T^k_{\mu,L})_{l,j}$ on $Graph(\chi^{g^{-k+2l}} \circ \phi^{\theta_0(-k+2j)})$ of the principal symbol $\sigma(T^k_{\mu,L})$ are given by

$$\sigma(T^k_{\mu,L})_{l,j} = f_{S_{l,j}}\, |y|^{-\frac{k}{2}(\frac{dim Y}{2}-1)} |dy|^{\frac{1}{2}} \otimes \chi^{g^{-k+2l}}_* \pi = \sum_{(i_k,\ldots,i_1)\in S_{l,j}} f_{(i_k,\ldots,i_1)}\, |y|^{-\frac{k}{2}(\frac{dim Y}{2}-1)} |dy|^{\frac{1}{2}} \otimes \chi^{g^{-k+2l}}_* \pi$$

where

$$S_{l,j} := \left\{ (i_k, ..., i_1) \in \{(1,1),(1,-1),(-1,1),(-1,-1)\}^k : \sum_{n=1}^{k} i_n = (-k+2l, -k+2j) \right\}$$

and the $f_{(i_k,\ldots,i_1)}$'s are as in Corollary 4.2.5 with

$$\chi_{(\pm1,\pm1)} = \chi^{g^{\pm1}} \circ \phi^{\pm\theta_0} \qquad \textit{and} \qquad F_{(\pm1,\pm1)}(y) = (g^{\pm1} \exp \pm s\frac{o}{|o|}, o)\ \ \forall\ y \in Y\,.$$

Proof: (a) Recall Proposition 4.2.1. For a given $b \in B$, $pr.(b) = o = x\rho x^{-1}$ for some $x \in G$. We have $\mathcal{F}_1(o) = g \exp s\frac{o}{|o|}$, $\mathcal{F}_2(o) = g \exp -s\frac{o}{|o|}$, $\mathcal{F}_3(o) = g^{-1} \exp s\frac{o}{|o|}$, $\mathcal{F}_4(o) = g^{-1} \exp -s\frac{o}{|o|}$. But $g \exp \pm s\frac{o}{|o|} = gx(\exp \pm s\frac{\rho}{|\rho|})x^{-1}$ and $gx(\exp \pm s\frac{\rho}{|\rho|})x^{-1} \cdot b = g\chi_\rho(\exp \pm s\frac{\rho}{|\rho|}) \cdot b = ge^{\pm 2\pi i s \frac{\langle\rho,\rho\rangle}{|\rho|}} \cdot b$. Setting $\theta_0 = 2\pi s|\rho|$, we thus get

$$\Gamma_{\mu,L} = \bigcup_{\pm,\pm} Graph(\chi^{g^{\pm1}} \circ \phi^{\pm\theta_0})$$

and from this, since the actions of G and S^1 commute, $\Gamma^k_{\mu,L}$ for all $k \geq 1$ as claimed.
(b) Recall Corollary 4.2.5. Since

$$Graph(\chi_{i_k} \circ \cdots \circ \chi_{i_1}) = Graph(\chi^{g^{-k+2l}} \circ \phi^{\theta_0(-k+2j)}) \quad \forall \quad (i_k, ..., i_1) \in S_{l,j}\,,$$

the statement follows. ∎

The asymptotics of the moments of the spectral measures are given by the following:

Proposition 4.3.4 *Let $T_{\mu,L}$ be a translate of the spherical means operator as defined above with $L = \mathbb{N}\rho$ an interior ray of representations. The k^{th} moments of the spectral measures $m^\mu_{n\rho}$ of $T_{\mu,L}$ are given asymptotically by*

$$M^\mu_{n\rho}(k) \sim \begin{cases} n^{\frac{k}{2}(1-\dim G)} \frac{1}{Cvol(B)} \sum_{j=0}^{k} C_j\, e^{2\pi i(-k+2j)s(n|\rho|+\rho_+(\frac{\rho}{|\rho|})} & \textit{for} \quad k \quad \textit{even} \\ n^{\frac{k}{2}(1-\dim G)+\frac{ds(g)}{2}-\frac{\dim G - \dim T}{2}} \frac{1}{Cvol(B)} \sum_{l=0}^{k} \sum_{j=0}^{k} C_{l,j}\, \chi_\rho(g^{n(-k+2l)}) \times e(\rho_+(g^{n(-k+2l)}))(e^{-i2\pi(-k+2j)s(n|\rho|+\rho_+(\frac{\rho}{|\rho|})} & \end{cases}$$

where

$$C_j := \int_B f_{S_{\frac{k}{2},j}}\, dvol\,, \qquad C_{l,j} := \int_{\tilde{Fix}(L_{g^{-k+2l}})} f_{S_{l,j}}\, d\mu_l\,.$$

Here we have used the following notation:
(i) $\tilde{Fix}(L_{g^{-k+2l}})$ *denotes the lift to* B *of the fixed point set of* $L_{g^{-k+2l}} : \mathcal{O} \to \mathcal{O},\ o \mapsto g^{-k+2l} \cdot o$,
(ii) $d\mu_l$ *is the induced density on* $\tilde{Fix}(L_{g^{-k+2l}})$,
(iii) χ_ρ *denotes the highest weight character of* ρ.

Proof: Recall Proposition 3.3.1 and Theorem A(iv). Here the fiber diagram for the trace operation $Tr(e^{i\theta D} \circ T^k_{\mu,L})$ is

$$\begin{array}{ccc} F & \to & \Gamma^k_{\theta,\mu,L} \\ \downarrow & & \downarrow \\ \Delta(T^*B) & \to & T^*B \times T^*B \end{array}$$

with

$$\Gamma^k_{\theta,\mu,L} = \bigcup_{l=0}^{k} \bigcup_{j=0}^{k} \{((e^{i\theta}, -r), (b,r), g^{-k+2l} e^{i(\theta+\theta_0(-k+2j))} \cdot b, r))\} \cong \bigcup_{l=0}^{k} \bigcup_{j=0}^{k} (S^1 \times Y)_{l,j}$$

($(l+1)(k+1)$ copies of $S^1 \times Y$) and

$$F \cong \bigcup_{l=0}^{k} \bigcup_{j=0}^{k} \{((\chi_\rho(g^{k-2l}) e^{-i\theta_0(-k+2j)}, -r), (b,r)) : b \in \tilde{Fix}(L_{g^{-k+2l}})\}\,.$$

The cleanliness condition for this diagram is equivalent to the condition that all of the maps $L_{g^{-k+2l}}$, $l = 0, ..., k$, have clean fixed point set in $\mathcal{O}$. But it was shown in [P.Z, Section 3] that for any $g \in T$, $L_g : \mathcal{O} \to \mathcal{O}$ has clean fixed point set which is a submanifold of dimension $d = ds(g)$. Since $ds(g^m) = ds(g)$ $\forall m \neq 0$ and $ds(e) = \dim \mathcal{O}$ we have

$$\max\dim F = \begin{cases} ds(g) + 2 & \text{for } k \text{ odd} \\ \dim Y & \text{for } k \text{ even} \end{cases} \qquad \text{and hence} \qquad e = \begin{cases} ds(g) + \dim B & \text{for } k \text{ odd} \\ 2\dim B - 1 & \text{for } k \text{ even} \end{cases}.$$

Using the formula for the principal symbol $\sigma(T^k_{\mu,L})$ from Lemma 4.3.3(b) and plugging into the general formula from Theorem A(iv) yields the stated formula for the moment asymptotics. ∎

5 δ-functions on finite unions of regular conjugacy classes: Proof of Theorem B

Here the symmetric probability measure μ is a δ-function on the submanifold $X = \bigcup_{i=1}^n C_{x_i} \cup C_{x_i^{-1}}$ where the x_i's are regular elements of the maximal torus T and C_x denotes the conjugacy class of x. To be more precise, it is a δ 1/2-density

$$\mu = a\sqrt{d\dot{x}}$$

where $d\dot{x}$ is the normalized invariant density and where $a \in C^\infty(X)$ is a positive smooth coefficient satisfying $\int_X a d\dot{x} = 1$.

For simplicity, we restrict our discussion to the case where X is just the union of *two* conjugacy classes C_x and $C_{x^{-1}}$ for some regular element $x \in T$. (Our results can easily be generalized to any finite union of regular conjugacy classes.) This implies that

$$\mu \in I^{\frac{\dim T}{2} - \frac{\dim G}{4}}(G, \Lambda_\mu) \qquad \text{with} \qquad \Lambda_\mu = N^*C_x \cup N^*C_{x^{-1}}$$

where $N_x^* C_x = \mathbf{t}^*$ and $N_{yxy^{-1}}^* C_x = Ad^*(y)\mathbf{t}^*$ for all $y \in G$.

Contrary to the case of δ-functions on positively curved hypersurfaces, the Lagrangean Γ_μ associated with the Markov operator T_μ on $L^2(G)$ is not a canonical graph. This of course complicates the study of convolution powers. What saves the day is that the isotropic relation $\Gamma_{\mu,L}$ underlying the ray Markov operator $T_{\mu,L}$ is a local canonical graph on Y.

5.1 The ray Markov operator: Proof of Theorem B(i)

We begin by giving a more precise statement of the result:

Theorem B(i) *Let μ be a δ-function on $C_x \cup C_{x^{-1}}$ and let $L = \mathbb{N}_0 \rho$ be the ray of representations determined by the interior highest weight ρ. Then:*

(a) the ray Markov operator

$$T_{\mu,L} \in I^{\dim T - \dim G}(B \times B, \Gamma_{\mu,L})$$

is a Fourier-Toeplitz operator of order $\dim T - \dim G$ whose associated isotropic submanifold $\Gamma_{\mu,L}$ is a disjoint union

$$\mathrm{Graph}(\chi^{\pm}_{\mathrm{w_j}})$$

of graphs of canonical transformations $\chi^{\pm}_{w_j}$ of Y. The components are (possibly redundantly) indexed by the elements $w_j x w_j^{-1}$; the jth and kth coincide if $\chi_\rho(w_j x w_j^{-1}) = \chi_\rho(w_k x w_k^{-1})$, where as usual χ_ρ denotes the highest weight character.

(b) Each canonical transformation $\chi^{\pm}_{w_j}$ is simply multiplication by one of the circle elements in $\Theta := \{\chi_\rho(w x^{\pm 1} w^{-1}) : w \in W\}$.

(c) The principal symbol of $T_{\mu,L}$ on the $w_j x^{\pm 1} w_j^{-1}$ component is given by

$$\sigma(T_{\mu,L})^{\pm}_{w_j})|_{(y, g^{\pm}_{w_j}(y)\cdot y)} = \frac{1}{\delta(g^{\pm}_{w_j}(y))} a(g^{\pm}_{w_j}(y))|y|^{-\frac{1}{2}(\frac{dim Y}{2}-1)}|dy|^{\frac{1}{2}} \otimes (g^{\pm}_{w_j})_* \pi$$

where $g^{\pm}_w(y) \in C_{x^{\pm}}$ is given by $g^{\pm}_w(y) = k(y) w x^{\pm 1} w^{-1} k(y)^{-1}$ where $pr.(y) = o = k(y)\rho k(y)^{-1}$. Here, we assume that no two $g^{\pm}_{w_j}(y)$ are equivalent in the sense of $\sim$ above. In general, one sums over the elements of the equivalence class.

Proof of (a) : We first show that the fiber product diagram

$$\begin{array}{ccc} F & \rightarrow & \Gamma_Y \cong G \times Y \\ \downarrow & & \downarrow \pi_1 \\ \Lambda_\mu & \underset{i}{\rightarrow} & T^*G \end{array}$$

is clean. Here i denotes inclusion and π_1 is projection onto the first factor. We claim that:

- $F \cong \{((g, ro), (b, r)) : pr.(b) = o, (g, ro) \in \Lambda_\mu\}$ is a submanifold of $\Lambda_\mu \times \Gamma_Y$.
- The natural projection $F \to Y$ given by $((g, ro), y) \to y$ is a trivial $2|W|$-sheeted cover, where $|W|$ denotes the order of the Weyl group..
- The natural projection $p : F \to \Gamma_{\mu,L}, ((g, ro), y) \to (y, g \cdot y)$ is a finite covering map of the trivial covers

$$Y \times \{w_j x w_j^{-1}, w_k x^{-1} x_k^{-1}\}_{j,l} \to Y \times \left(\{w_j x w_j^{-1}, w_k x^{-1} w_k^{-1}\}_{j,k} / \cong\right) ,$$

where $a \cong b$ iff $\chi_\rho(a) = \chi_\rho(b)$.

- The derived diagram is a fiber diagram.

To verify these properties, suppose first that $o = \rho$. We need to find all $g \in G$ for which $(g, \rho) \in \Lambda_\mu$. Recall that $\Lambda_\mu = N^*C_x \cup N^*C_{x^{-1}} = \{(yx^{\pm 1}y^{-1}, Ad^*(y)\xi) : y \in G, \xi \in \mathbf{t}^*\}$. Thus, equivalently, we need to find all $g \in G$ for which $Ad^*(g)\rho \in \mathbf{t}^*$. Clearly, this is true for all $g \in T$ and all $y \in W$. Hence we have

$$(wx^{\pm 1}w^{-1}, \rho) \in \Lambda_\mu \qquad \text{for all} \qquad w \in W\,.$$

Since ρ is an interior weight there cannot be any other g with this property: Indeed, $g\rho g^{-1} \in \mathbf{t}^*$ implies that $g \exp \rho g^{-1} \in T$ and hence, since $\exp \rho$ is regular, $gTg^{-1} \subset T$ and $g \in W$.

In general we have $o = k\rho k^{-1}$. Then

$$(kwx^{\pm 1}w^{-1}k^{-1}, k\rho k^{-1}) \in \Lambda_\mu \qquad \text{for all} \qquad w \in W\,.$$

On the other hand, assume $(l, k\rho k^{-1}) \in \Lambda_\mu$. This implies that $(l, k\rho k^{-1}) = (nxn^{-1}, n\xi n^{-1})$ or $(l, k\pi k^{-1}) = (nx^{-1}n^{-1}, n\xi n^{-1})$ for some $n \in G$ and for some $\xi \in \mathbf{t}^*$. It follows that $\rho = k^{-1}n\xi n^{-1}k$ and $n \in kW$. Hence $l = kyxy^{-1}k^{-1}$ or $l = kyx^{-1}y^{-1}k^{-1}$ for some $y \in W$.

It follows that F consists of $2|W|$ copies of Y, indexed by the elements $wxw^{-1}, wx^{-1}w^{-1}$. To be precise, given $y \in Y$ the corresponding points of F are given by the elements $\{((g^{\pm}_{w_j}(y), ro), y)\}$ with $g^{\pm}_{w_j}(y)$ defined above.

Now we show that the derived diagram

$$\begin{array}{ccc} TF & \rightarrow & T\Gamma_Y \\ \downarrow & & \downarrow d\pi_1 \\ T\Lambda_\mu & \rightarrow & T(T^*G) \\ & di & \end{array}$$

is also a fiber diagram, i.e., that $TF = \{(u, v) \in T\Lambda_\mu \times T\Gamma_Y : d\pi_1(v) = di(u)\}$. The inclusion $TF_{(a,b)} \subset \{(u, v) \in T_a\Lambda_\mu \times T_b\Gamma_Y : d\pi_1(v) = di(u)\}$ is trivially true. The reverse inclusion will follow from equality of dimensions of the two vector spaces.

First we take a closer look at the tangent bundle $T\Lambda_\mu$. The Lagrangean Λ_μ itself is a fiber bundle. We distinguish horizontal and vertical tangent vectors to Λ_μ as follows: a curve $c(t)$ with $c(0) = (z, \xi)$ in Λ_μ is horizontal if $c(t) = (g(t)zg(t)^{-1}, g(t)\xi g(t)^{-1})$ for some curve $g(t)$ in G and vertical if $c(t) = (z, \xi(t))$ for some curve $\xi(t)$ in the fiber over z.

Clearly, for every horizontal vector $u \in T\Lambda_\mu$ there is a matching $v \in T\Gamma_Y$ for which $d\pi_1(v) = di(u)$. Any vertical $u \in T_{(z,\xi)}\Lambda_\mu$ has the form $(0, u_2)$ with $u_2 \in T(Ad^*(y)\mathbf{t}^*)$ if $z = yxy^{-1}$. In case $u = \frac{d}{dt}|_{t=0}(z, t\xi)$, there is a matching $v \in T\Gamma_Y$ such that $di(u) = d\pi_1(v)$, namely $v = \frac{d}{dt}|_{t=0}((z, (t+s)o), (b, t+s), (z \cdot b, t+s))$ with $pr.(b) = o = \xi/s$. In case $u = (0, 0)$, there is an exactly one dimensional space of matching nontrivial v's, namely the vectors tangential to the circle in the circle bundle B. Furthermore, by construction of Γ_Y, for every other $v \in T\Gamma_Y$, $d\pi_1(v) = (v_1, v_2)$ where v_2 has a component in $T\mathcal{O}_\rho$, hence in the horizontal direction for $T\Lambda_\mu$ and cannot be matched with a vertical $u \in T\Lambda_\mu$.

This shows that for all $(a, b) \in F$, $\dim\{(u, v) \in T_a\Lambda_\mu \times T_b\Gamma_Y : d\pi_1(v) = di(u)\} = \dim Y = \dim TF_{(a,b)}$ and the above fiber diagram is clean with excess $e = 0$.

Thus, we have verified all of the claimed properties. By Proposition 2.4.1, the operator $T_{\mu,L}$ is a Fourier-Toeplitz operator of order $\dim T - \dim G$ associated to the isotropic submanifold

$$\Gamma_{\mu,L} = \{((b, r), (g \cdot b, r)) : g = kwx^{\pm 1}w^{-1}k^{-1},\ w \in W_G,\ pr.(b) = k\rho k^{-1}\}$$

which completes the proof of (a).

Proof of (b): For $pr.(b) = k\rho k^{-1}$, we have

$$(kwx^{\pm 1}w^{-1}k^{-1}) \cdot b = \chi_\rho(wx^{\pm 1}w^{-1}) \cdot b.$$

Here we have used the assumption (with no loss of generality) that $x \in T$; thus $wxw^{-1} \in T$ and conjugation with $kwx^{\pm 1}w^{-1}k^{-1}$ fixes $o = pr.(b)$. Therefore,

$$\Gamma_{\mu,L} = \bigcup_{\pm,j=1}^{J} \mathrm{Graph}(\chi_j^{\pm})$$

where

$$\chi_j^{\pm}(y) = e^{\pm i\theta_j} \cdot y\,, \qquad e^{\pm i\theta_j} := \chi_\rho(wx^{\pm 1}w^{-1}) \quad \text{for some} \quad w \in W_G\,.$$

Proof of (c): By the lemma above, the symbol splits up into a collection of symbols on the components of $\Gamma_{\mu,L}$. We pull them back to F and consider the fiber diagram:

$$\begin{array}{ccc} F & \rightarrow & \Gamma_Y \\ \downarrow & & \downarrow \pi_1 \\ \Lambda_\mu & \rightarrow & T^*G \\ & i & \end{array}$$

which simplifies to

$$\begin{array}{ccc} F & \rightarrow & C_x \times Y \\ \downarrow & & \downarrow \pi_1 \\ N^*(C_x) & \rightarrow & C_x \times \mathbf{g}^* \\ & i & \end{array}.$$

Since the diagram is clean and of excess zero we have the exact sequences on the tangent level:

$$0 \longrightarrow TF \longrightarrow T(C_x \times Y \times N^*(C_x)) \longrightarrow T(C_x \times \mathbf{g}^*) \longrightarrow 0$$

and hence

$$|TF|^{\frac{1}{2}} \cong |T(C_x \times Y)|^{\frac{1}{2}} \otimes |TN^*(C_x)|^{\frac{1}{2}} \otimes |T(C_x \times \mathbf{g}^*)|^{-\frac{1}{2}} \cong |Y|^{\frac{1}{2}} \otimes |T(N^*(C_x))|^{\frac{1}{2}} \otimes |\mathbf{g}^*|^{-\frac{1}{2}}.$$

To determine $\sigma(\mu)$, let us choose local coordinates (x', x'') so that locally $C_x = \{x' = 0\}$. Then locally we have

$$\delta_{C_x} = a(x'') \int e^{i\langle x', \xi''\rangle} d\xi' d\dot{x}$$

so that

$$\sigma(\delta_{C_x}) = a(x'')\sqrt{d\dot{x}}\sqrt{d\xi'}.$$

The 1/2-density $\sqrt{d\dot{x}}\sqrt{d\xi'}$ is invariantly defined on N^*C_x as the principal symbol of the invariant convolution operator. The principal symbol of the convolution with $a\sqrt{d\dot{x}}$ is just a times this canonical one.

By part (a), $\Gamma_{\mu,L}$ is diffeomorphic to a disjoint union of copies of Y. So we may express $\sigma(T_{\mu,L})$ as a collection of 1/2-densities on Y, each one given as a smooth coefficient times the canonical volume 1/2-density $|dy|^{\frac{1}{2}}$. To determine the coefficients, we first note by regularity of x, $C_x \cong G/T$ where the action of G is by conjugation. The derivative of the action, $Ad^*(g)$, trivializes the normal bundle, i.e. gives the isomorphism $N^*(C_x) \cong G/T \times \mathbf{t}^*$. That is,

$$T_g N^*(C_x) \cong (g^{-1}\mathbf{t}g)^- \oplus g^{-1}\mathbf{t}^* g \cong \mathbf{g}^*$$

so that

$$|T_g(N^*(C_x))|^{\frac{1}{2}} \otimes |\mathbf{g}^*|^{-\frac{1}{2}} \cong 1.$$

Now $\mathbf{g}$ carries the natural bi-invariant volume density dg of the Killing metric. However, the isomorphism above to $T_g N^*(C_x)$ is the same as occurs in the Weyl integration formula to express the Haar density dg as a density on $G/T \times T$ ([B.tD IV]). Namely if $q : G/T \times T \to G$ is the map $(g,t) \to gtg^{-1}$ then $q^*(dg) = det(Ad_{G/T}(t^{-1}) - E_{G/T}) d\dot{x} dt$ where $E_{G/T}$ is the identity on $T_e(G/T)$. It follows that the canonical 1/2-density on $N^*(C_x)$ is given by

$$\sqrt{d\dot{x}}\sqrt{d\xi'} = \sqrt{d\dot{x}}\sqrt{dt} = \sqrt{\frac{q^*(dg)}{det(Ad_{G/T}(t^{-1}) - E_{G/T})}}. \tag{24}$$

Since q is used to identify the 1/2-densities on $N^*(C_x)$ with those on $\mathbf{g}^*$, the ratio with the bi-invariant 1/2-density on $\mathbf{g}^*$ is $\frac{1}{\sqrt{det(Ad_{G/T}(t^{-1}) - E_{G/T})}}$. Hence the 1/2-density factor of the symbol on the $g_j^{\pm}(y)$ -component of $\sigma(T_{\mu,L})$ equals $\frac{a(g_j^{\pm}(y))}{\sqrt{det(Ad_{G/T}(t^{-1}) - E_{G/T})}}$. Since the fiber of $F \to \Gamma_{\mu,L}$ is discrete (and usually a single point), we get the same (kind of) 1/2-density on $\Gamma_{\mu,L}$. As in the proof of the Weyl character formula (see the discussion in §5.5), the denominator $\sqrt{det(Ad_{G/T}(t^{-1}) - E_{G/T})}$ gives (at least, up to sign) the the Weyl denominator in the symbol described in (c).

This concludes the proof of (c) and hence of the theorem. ∎

Theorem B(i) and Proposition 3.2.1 yield the following

Corollary 5.1.1 *(a) For all $k \geq 1$, the composition power $T^k_{\mu,L}$ is clean and*

$$T^k_{\mu,L} \in I^{\frac{k+1}{2}(\dim T - \dim G)}(B \times B, \Gamma^k_{\mu,L})$$

with $\Gamma^k_{\mu,L} = Graph(\chi^{\pm\cdots\pm}_{k,J})$ where $J = (j_k, \ldots, j_1)$ and where

$$\chi^{\pm\cdots\pm}_{k,J}(y) = \chi_\rho(w_{j_k} x^{\pm 1} w^{-1}_{j_k} \cdots w_{j_1} x^{\pm 1} w^{-1}_{j_1}) \cdot y.$$

(b) The component $\sigma(T^k_{\mu,L})^{\pm\cdots\pm}_J$ of the principal symbol $\sigma(T^k_{\mu,L})$ on $Graph(\chi^{\pm\cdots\pm}_{k,J})$ is given by

$$\sigma(T^k_{\mu,L,j})^{\pm\cdots\pm}_J|_{(y, g^{\pm}_{w_j}(y)\cdot y)} = \Pi^k_{i=1} \frac{1}{\delta(g^{\pm}_{w_{j_i}}(y))} a(g^{\pm}_{w_{j_i}}(y))\, |y|^{-\frac{k}{2}(\frac{dim Y}{2} - 1)} |dy|^{\frac{1}{2}} \otimes (g^{\pm}_{w_{j_i}})_* \pi\,.$$

5.2 Sobolev smoothing properties of ray Markov operators: Proof of Theorem B(ii)

We begin by defining a scale of Sobolev spaces $W^s H^2(B)$ which are the Hardy space analogues of the usual Sobolev spaces $W^s(B)$ on a compact manifold B.

Recall that the W^s-norm of a distribution $f \in \mathcal{D}(B)$ is defined by

$$||f||_{W^s} := ||P^s f||_2$$

where P is a positive elliptic pseudodifferential operator of order 1 and $||\cdot||_2$ denotes the L^2-norm. In the Hardy setting, the role of P can be played by $D = \frac{1}{i}\frac{\partial}{\partial\theta}$. This operator is an elliptic Toeplitz operator on $H^2(B)$ in the sense that D has nowhere vanishing symbol on Y. Hence we make the following:

Definition 5.2.1 The Hardy-Sobolev space $W^s H^2(B)$ is the space of $f \in H^2(B)$ such that $||D^s f||_2 < \infty$.

We can describe the Hardy-Sobolev norms more concretely in terms of Fourier coefficients relative to the S^1-action on B. Recall that $H^2(B) = \bigoplus_{n=0}^{\infty} H^2_n(B)$ where $H^2_n(B)$ is the space of CR-functions of eigenvalue

n for D. Let $f \in H^2(B)$ and let f_n be its 'Fourier coefficient' of degree n, i.e., its component in $H^2_n(B)$. Then put

$$||f||^2_{W^s H^2} := \sum_{n=0}^{\infty} n^{2s} ||f_n||_2^2.$$

Obviously, $W^s H^s(B)$ for $s > 0$ is the space of $f \in H^2(B)$ for which $D^s f \in H^2(B)$.

We recall that in the case of Fourier integral operators associated to local canonical graphs, there is a general Sobolev smoothing result:

Theorem 5.2.2 ([Hö, Vol.IV Theorem 25.3.1 and Corollary 25.3.2]) *Let C be a homogeneous canonical relation that is locally the graph of a canonical transformation and let $A \in I^m(M \times M, C)$. Then A is a bounded operator from $W^s(M)$ to $W^{s-m}(M)$ for every real s.*

The Toeplitz analogue of this result is:

Theorem 5.2.3 *Let $C \subset Y \times Y$ be a homogeneous canonical relation that is locally the graph of a canonical transformation on Y and let $A \in I^m(B \times B, C)$. Then for every real s, A is a bounded operator from $W^s H^2(B)$ to $W^{s-m'} H^2(B)$ with $m' = m + \frac{\dim B - 1}{2}$. We call m' the* effective order *of A.*

Proof: Suppose first that $A \in I^{-\frac{\dim B-1}{2}}(B \times B, C)$, that is, A has effective order 0. We can write $A = \sum_{i=1}^n A_i$ as a finite sum of operators such that each A_i, $i = 1, ..., n$, is associated to a canonical graph. Hence $A_i^* A_i \in I^{-\frac{\dim B-1}{2}}(B \times B, \Delta(Y))$. We claim that Toeplitz operators of effective order 0 associated to the identity graph on Y are bounded on $H^2(B)$. Granted this statement, it follows that

$$(A_i u, A_i u) = (A_i^* A_i u, u) \leq const.(u, u), \qquad u \in H^2(B)\,.$$

Thus, for all $i = 1, ..., n$, A_i is bounded on $H^2(B)$ and hence so is A.

To prove that $A_i^* A_i \in I^{-\frac{\dim B-1}{2}}(B \times B, \Delta(Y))$ is bounded on $H^2(B)$ we use that there exists a pseudodifferential operator Q of order 0 such that $[\Pi, Q] = 0$ and such that $A_i^* A_i = \Pi Q \Pi$ (see [BdM.G, Proposition 2.13]). Then for $u \in H^2(B)$,

$$||Au||^2 = ||\Pi Q \Pi u||_2 = ||Q \Pi u||_2 \leq const.||\Pi u||_2 = const.||u||_2$$

by the L^2-boundedness of 0^{th} order pseudodifferential operators.

Finally, let A have order m. To show that $A : W^s H^2(B) \to W^{s-m'} H^2(B)$ is bounded it suffices to show that $D^{s-m'} A : W^s H^2(B) \to H^2(B)$ is bounded. However, we can write

$$D^{s-m'} A = (D^{s-m'} A D^{-s}) D^s\,.$$

Since $D^{s-m'} A D^{-s}$ is bounded on $H^2(B)$, it follows that $D^{s-m'} A$ is bounded from $W^s H^2(B) \to H^2(B)$. ∎

In particular, an isotropic operator of degree $m < 0$ is smoothing of degree $-m' = -(m + \frac{\dim B - 1}{2})$ on these spaces. By Theorem B(i), the ray Markov operator $T_{\mu,L}$ for δ-functions on finite unions of regular conjugacy classes is of order $\dim T - \dim G$. Hence in this case $T_{\mu,L}$ is smoothing of degree $\frac{\dim B - 1}{2}$. This completes the proof of Theorem B(ii). ∎

5.3 The Hilbert-Schmidt property of ray Markov operators: Proof of Theorem B(iii)

Here we assume that G is a classical compact Lie group. We prove that the ray Markov operator $T_{\mu,L}$ for a δ-function on a finite union of regular conjugacy classes is a Hilbert-Schmidt operator.

Recall Theorem B(i) and Proposition 3.2.1. Since $\Gamma_{\mu,L}$ is a union of canonical graphs, the excess of the composition $T_{\mu,L}^{k-1} \circ T_{\mu,L}$ is $\dim B - 1$ and hence the order of $T_{\mu,L}^k$ is $k(\dim T - \dim G) - \frac{\dim B - 1}{2}$ for all $k > 1$. This implies that the effective order is $m'(k) = k(\dim T - \dim G)$. Thus, by Theorem 5.2.3,

$$T_{\mu,L}^k : H^2(B) \to W^{k(\dim G - \dim T)} H^2(B)\,.$$

Furthermore, we have

$$D^{k(\dim G - \dim T)} : W^{k(\dim G - \dim T)} H^2(B) \to H^2(B)\,.$$

It follows that

$$B := D^{k(\dim G - \dim T)} T_{\mu,L}^k : H^2(B) \to H^2(B)$$

is a bounded operator leaving the subspaces $H_n^2(B)$, $n = 1, 2, \ldots$ invariant. For $f = \sum_{n=1}^{\infty} f_n \in H^2(B)$ (according to $\bigoplus_{n=1}^{\infty} H_n^2(B)$) we then have

$$Bf = \sum_{n=1}^{\infty} B_n f_n \qquad \text{with } \|B_n\|_2 \le C \quad \forall n$$

for some constant C. Here $\|\cdot\|_2$ denotes the $L^2 - L^2$ mapping norm. Recall that the action of $T_{\mu,L}^k$ on $H^2(B)$ is given by

$$T_{\mu,L}^k(f) = \sum_{n=1}^{\infty} \hat{\mu}^k(n\rho) f_n\,.$$

Denoting the Hilbert Schmidt norm by $\|\cdot\|_{HS}$, we thus have

$$\|T_{\mu,L}^k\|_{HS}^2 = \sum_{n=1}^{\infty} \|\hat{\mu}^k(n\rho)\|_{HS}^2 = \sum_{n=1}^{\infty} n^{2k(\dim T - \dim G)} \|B_n\|_{HS}^2\,.$$

It follows, since $\|B_n\|_{HS}^2 \le d_n \|B_n\|_2^2$ with $d_n = \dim H_n^2(B)$, that

$$\|T_{\mu,L}^k\|_{HS}^2 \le C \sum_{n=1}^{\infty} n^{2k(\dim T - \dim G)} d_n\,,$$

and hence that $T_{\mu,L}^k$ is Hilbert-Schmidt if

$$\sum_{n=1}^{\infty} n^{2k(\dim T - \dim G)} d_n < \infty\,. \tag{25}$$

Since ρ is assumed to be an interior weight, $\dim H_1^2(B)$ is a polynomial of degree $|R^+| = (\dim G - \dim T)/2$ (the cardinality of the set of positive roots) in the components of ρ, and $d_n \le C' n^{(\dim G - \dim T)/2}$ for some constant C' and all n. It follows that the series (25) converges for $2k(\dim T - \dim G) + (\dim G - \dim T)/2 < -1$ and hence that $T_{\mu,L}^k$ is Hilbert-Schmidt for

$$k > \frac{1 + (\dim G - \dim T)/2}{2(\dim G - \dim T)}\,. \tag{26}$$

But for the right hand side in (26) we have:

$$\frac{1 + (\dim G - \dim T)/2}{2(\dim G - \dim T)} \le \frac{1}{2}$$

for all classical compact Lie groups. ∎

5.4 The limit spectral measure: Proof of Theorem B(iv)

Let us recall the statement of Theorem B(iv).

Theorem B(iv) *Let μ be a δ-function on the union of conjugacy classes $C_x \cup C_{x^{-1}}$ as defined above and let $L = \mathbb{N}\rho$ be an interior ray of representations of G. The asymptotics of the k^{th} moments of the spectral measures $m^{\mu}_{n\rho}$ of $T_{\mu,L}$ along L are given by*

$$M^{\mu}_{n\rho}(k) \sim n^{\frac{k}{2}(\dim T - \dim G)} \frac{1}{vol(B)} \sum_{\substack{(j_k, \ldots, j_1) \\ (\pm \cdots \pm)}} e(\rho_+(\pm w_{j_k}(X))) \cdots e(\rho_+(\pm w_{j_1}(X))))$$

$$\frac{(-1)^{w_{j_k}} \cdots (-1)^{w_{j_1}}}{\delta(\pm X) \ldots \delta(\pm X)} (\chi_\rho(w_{j_k} x^{\pm 1} w_{j_k}^{-1} \cdots w_{j_1} x^{\pm 1} w_{j_1}^{-1}))^n \left[\int_B \left(\prod_{i=1}^{k} a(g^{\pm}_{w_{j_i}}(b)) \right) dvol \right]$$

Proof: The proof is very similar to that of Proposition 4.3.2. As in the case of the spherical means operator, the $\Gamma^k_{\mu,L}$ here are also finite unions of graphs of translates by certain circle elements, namely the $\chi_\rho(w_{j_k} x^{\pm 1} w_{j_k}^{-1} \cdots w_{j_1} x^{\pm 1} w_{j_1}^{-1})$'s. This implies that the trace operation $Tr(e^{i\theta D} \circ T^k_{\mu,L})$ is clean and that the singular angles of the resulting distribution $\Upsilon_k(\theta)$ are precisely the $\chi_\rho(w_{j_k} x^{\pm 1} w_{j_k}^{-1} \cdots w_{j_1} x^{\pm 1} w_{j_1}^{-1})$'s, each of which corresponds to the fixed point set Y. Thus the excess of the trace operation is $e = 2 \dim B - 1$. Using Corollary 5.1.1(b) for the principal symbol and plugging into the general formula of Proposition 3.3.1 gives all the stated factors except for the ρ_+, i.e. symplectic spinor factor.

The latter is calculated as in Proposition 3.4.1 (iv): From (b) we know that $g^{\pm}_w(y)$ fixes $pr(y) = o$. Hence we get the induced map $g^{\pm}_w(y)_*$ on the space of symplectic spinors at o, that is, by applying $\mathcal{M}$, the metaplectic representation, to the normal part $dT^-_{g^{\pm}_w(y)}$ of the derivative. This operators on the symplectic orthogonal to $TY \cong T\mathcal{O} \times T(S^1 \times \mathbb{R}^+)$ in T^*B, of which the non-trivial part is the symplectic orthogonal to $T_o\mathcal{O}$ in $T^*\mathcal{O}$. Identifying $\mathcal{O} \cong G/T$, $T^*\mathcal{O} \cong G/T \times \mathbf{t}^{*-}$, $T_o\mathcal{O}^- \cong \mathbf{t}^{*-}$ and $dT^-_{g^{\pm}_w(y)} \cong Ad^*(g^{\pm}_w(y))$ on $k(y)\mathbf{t}^{*-}k^{-1}$. ¿From the fact that $g^{\pm}_w(y) = k(y) w x^{\pm 1} w^{-1} k(y)^{-1}$ it follows that the eigenvalues of $Ad^*(g^{\pm}_w(y))$ are the same as the eigenvalues of $Ad^*(w x^{\pm 1} w^{-1})$ on $\mathbf{t}^-$. These are given by the global roots $e(\alpha)$ evaluated at wxw^{-1}. As usual, we have (with $x = e^X$)

$$\mathcal{M}(Ad^*(w x^{\pm 1} w^{-1})) = \Pi_{\alpha \in R_+} exp(\langle \alpha, X \rangle \hat{I}_\alpha \tag{27}$$

where $\hat{I}_\alpha$ is the harmonic oscillator. It follows that the diagonal matrix element

$$\langle \mathcal{M}(Ad^*(w x^{\pm 1} w^{-1})) e_\Lambda, e_\Lambda \rangle = e(\frac{1}{2}\rho_+)|_{w x^{\pm 1} w^{-1}}. \tag{28}$$

This factor is constant as we integrand over the fiber of the trace, and thus it persists to the moment asymptotics as stated above. ∎

5.5 Example: uniform measure on conjugacy classes

As a check on our caluculations, let us consider the case of the symmetric uniform (conjugacy-invariant) probability measure

$$d\mu = \frac{1}{2} d\dot{x}_{C_g} + \frac{1}{2} d\dot{x}_{C_{g^{-1}}},$$

on $C_x \cup C_{x^{-1}}$ with $x = e^X$ a regular element of T. Random walks of this kind, but with highly singular x, were considered in [Ro][Po].

Since μ is conjugacy invariant, it follows from Schur's Lemma that for each $\rho \in \hat{G}$, the Fourier transform $\hat{\mu}(\rho)$ is a scalar. Namely

$$\hat{\mu}(\rho) = \frac{1}{2}\int_G (\rho(gxg^{-1}) + \rho(gx^{-1}g^{-1}))dg = \frac{1}{d_\rho}\int_G ReCh_\rho(gxg^{-1})dg I_{d_\rho} = \frac{ReCh_\rho(x)}{d_\rho} I_{d_\rho}$$

where the integral is with repect to Haar measure on G, Ch_ρ denotes the character with $ReCh_\rho$ its real part, and d_ρ denotes the dimension of the representation ρ. Hence the spectral measures m^μ_ρ are the delta functions

$$m^\mu_\rho = \delta(x - \lambda^x(\rho))$$

where $\lambda^x(\rho) = \frac{ReCh_\rho(x)}{d_\rho}$ is the eigenvalue of $\hat{\mu}(\rho)$. It is obvious that $M^\mu_{n\rho}(k) = \lambda^x(n\rho)^k$ so the asymptotics of any moment is determined by the asymptotics of the first moment $M^\mu_{n\rho}(1) = \lambda^x(n\rho)$. However, the first moment is simply the trace

$$M^\mu_{n\rho}(1) = \frac{1}{d_{n\rho}} Tr\hat{\mu}(n\rho) == \frac{1}{d_{n\rho}} Re\int_G Ch_{n\rho}(gxg^{-1})dg = \frac{1}{d_{n\rho}} ReCh_{n\rho}(x).$$

So the calculation comes down to the asympotics of $Ch_{n\rho}(x)$ as $n \to \infty$.

The result must of course duplicate the Weyl character formula, for which we are about to give a Toeplitz operator proof. The calculation follows the pattern of [P.Z, §3], except that here we are assuming x is a regular element and there we assumed it was central. Hence the dimensions of fixed point sets are entirely different and so therefore is the order of the asymptotics.

As usual, we form the generating function

$$\Upsilon(\theta) = Tr\Pi e^{i\theta D} T_x$$

and determine the principal singularities. They occur at the fixed points of x acting on $\mathcal{O}_\rho$. We may assume $x \in T$ and then the fixed points form the Weyl orbit $W \cdot \rho$ of ρ. By a well-known argument [A.B]B.G.V, T_x (conjugation by x) defines a Lefschetz map of $\mathcal{O}_\rho$ with fixed points at the Weyl orbit $W\rho$. The eigenvalues of $dT_x|_{T_{w\cdot\rho}}$ are the values of the global roots $e(\alpha)$ at $w^{-1}xw$. Hence the 1/2-density part of the symplectic spinor trace of T_x at $w \cdot \rho$ is given by $\frac{1}{\sqrt{det(I - dT_x|_{w^{-1}\rho w})}}$. Recall from the Weyl character and integration formulae that

$$det(I - dT_x|_{w^{-1}\rho w}) = det(I - Ad_{G/T}(wxw^{-1}) = \delta\bar{\delta}.$$

The formula makes it clear that $det(I - dT_x|_{w^{-1}\rho w}) > 0$ so we may take its square root. We also see that $\bar{\delta} = (-1)^{|R+|}\delta$ so that up to sign the square root equals

$$det(I - dT_x|_{w^{-1}\rho w}) = \pm i^{|R+|}\delta(X).$$

To determine the sign one would have to analyse the Maslov factors, or else work with 1/2-forms rather than with 1/2-densities. For the sake of brevity, we have ignored the analysis of Maslov factors in this article. A careful discussion of the correct signs, in a closely related context, is given in [B.G.V, Theorem 8.7] and it shows that the correct square root gives $(-1)^w\delta$.

The remaining ingredient in the trace is the symplectic spinor part. Thus, we must consider dT_x acting on the symplectic normal space T_yY^- at a fixed point y. We may identify T_yY^- with $\mathbf{g}^*/\mathbf{t}^*$ and since the action of T_x on T^*B is the lift of the base action, dT_x acts on $\mathbf{g}^*/\mathbf{t}^*$ by the usual linear (conjugation) action. In particular, it has the same eigenvalues as dT_g does on $T_o\mathcal{O}$ where o is the projection of y. Thus, $dT_x|_{T_yY^-}$ is a sum of rotations with eigenvalues $e(\alpha), e(-\alpha)|_{wxw^{-1}}$. Let us write $e(\alpha)|_{wxw^{-1}}$ by $e^{2\pi i\langle w(X),\alpha\rangle}$. Under the metaplectic representation $\mathcal{M}$ it therefore becomes

$$\mathcal{M}(dT_x|_{T_yY^-}) = \Pi_{\alpha\in R^+} e^{i\langle w(X),\alpha\rangle \hat{I}_\alpha}$$

where $\hat{I}_\alpha$ denotes the Harmonic oscillator Hamiltonian $D^2 + u^2$ with $D = \frac{d}{i\,du}$ and with u the αth coordinate of $\mathbf{g}^*/\mathbf{t}^*$. Since the symplectic spinor part of the symbol is the projection $\pi = e_\Lambda \otimes e_\Lambda^*$ onto the ground state e_Λ, the contribution to the trace of the spinor factor is

$$Tr\mathcal{M}(dT_x|_{T_yY^-})\pi = \langle \Pi_{\alpha \in R_+} e^{i\langle w(X),\alpha\rangle \hat{I}_\alpha} e_\Lambda, e_\Lambda \rangle.$$

But the positive definite Lagrangean Λ is precisely defined by the complex structure on $\mathcal{O}_\rho$ specified by the choice of positive roots. Hence $e_\Lambda = \bigotimes_{\alpha \in R_+} e_\alpha$ where e_α is the ground state of $\hat{I}_\alpha$. Since the lowest eigenvalue of the harmonic oscillator is $\frac{1}{2}$ we get

$$\langle \Pi_{\alpha \in R_+} e^{i\langle w(X),\alpha\rangle \hat{I}_\alpha} e_\Lambda, e_\Lambda \rangle = \Pi_{\alpha \in R_+} \langle e^{i\langle w(X),\alpha\rangle \hat{I}_\alpha} e_\alpha, e_\alpha \rangle = \Pi_{\alpha \in R_+} e(\frac{1}{2}\alpha)|_{wxw^{-1}}.$$

Putting together the 1/2-density, Maslov and spinor factors we get the Weyl character formula

$$Ch_{n\rho}(e^X) = \frac{1}{\delta(X)} \sum_{w \in W} (-1)^w \chi_\rho(we^Xw^{-1})^n e(\rho_+(wX)).$$

Dividing by the dimension we get

$$\lambda^x(n\rho) \sim n^{-\frac{1}{2}dim\mathcal{O}_\rho} Ch_{n\rho}(x)$$

corroborating the general formula. ■

References

[A.B] M.F.Atiyah and R.Bott, The Lefshetz fixed point formula for elliptic complexes II, *Ann. Math.* **86** (1967), 374–407.

[B.G.V] N.Berline, E.Getzler and M.Vergne, *Heat Kernels and Dirac Operators*, Grundl.Math.Wiss. 298, Springer Verlag, New York (1992).

[BdM.G] L. Boutet de Monvel and V. Guillemin, *The Spectral Theory of Toeplitz Operators,* Princeton University Press, Princeton, 1981.

[BtD] T.Brocker and T. tom Dieck, *Representations of Compact Lie Groups*, Grad. Texts in Math., Springer Verlag, New York (1985).

[D.G] H. Duistermaat, V. Guillemin: The spectrum of elliptic operators and periodic bicharacteristics, *Inv. Math.* **29** (1975), 39–79.

[Gr.U] A. Greenleaf and G. Uhlmann, Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms, I, II, *Ann. Inst. Fourier 40* (1990), 443–466; Duke Math. J. 64 (1991), 415- 444.

[G] V.Guillemin, Symplectic spinors and partial differential equations, *Geomtrie Symplectique et Physique Mathematique*, Coll.Inst. CNRS no. 237.

[G.S.1] V. Guillemin and S. Sternberg, Homogeneous quantization and multiplicities of group representations, *J. Fun. Anal.* **47** No.3 (1982), 344–380.

[G.S.2] V. Guillemin and S. Sternberg, *Geometric Asymptotics,* American Mathematical Society, Providence, Rhode Island, 1977.

[G.U] V.Guillemin and A.Uribe, Monodromy in the quantum spherical pendulum, *Comm. Math. Phys.* **122** (1989), 563-574.

[H.R] E.Hewitt and K.Ross, *Abstract Harmonic Analysis, Volumes I–II*, Die Grundlehren der Math.Wiss.in Einz. Band 115, 152, Springer- Verlag (N.Y.) (1963, 1970).

[Hö] L. Hörmander, *The Analysis of Linear Partial Differential Operators, Volumes I–IV*, Springer-Verlag Berlin Heidelberg, 1983.

[L.P.S] A. Lubotzky, R. Phillips, and P. Sarnak, Hecke operators and distributing points on the sphere I; *Comm. Pure Appl. Math.* **39** No. S, suppl., S149–S186 (1986).

[Ph] D. H. Phong, Regularity of Fourier integral operators, *Proc. of the ICM, Zurich 1994*, 862–874, Birkhauser Basel (1995).

[Po] U. Porod, The cut-off phenomenon for random reflections, *Ann. Prob.* **24** No.1 (1996), 74–96.

[Po.2] U.Porod, L_2-lower bounds for a special class of random walks, Probab.Theory Relat.Fields 101 (1995), 277-289.

[P.Z] U.Porod and S.Zelditch, Semi-classical limit for random walks I (1996 preprint).

[Ra] D. L. Ragozin, Central measures on compact simple Lie groups, *J. Fun. Anal.* **10** (1972), 212–229.

[Ro] J. Rosenthal, Random rotations: characters and random walks on $SO(N)$, *Ann. Prob.* **22**, 398–423.

[R.S.II] F. Ricci and E. M. Stein, Harmonic analysis on nilpotent groups and singular integrals II: singular kernels supported on submanifolds, *J. Fun. Anal.* **78** (1988), 56-84.

[R.S.III] F. Ricci and E. M. Stein, Harmonic analysis on nilpotent groups and singular integrals III: fractional integration along manifolds, *J. Fun. Anal.* **86** (1989), 360–389.

[R.T] F. Ricci and G. Travaglini, $L^p - L^q$ estimates for orbital measures and Radon transforms on compact Lie groups and Lie algebras, *J. Fun. Anal.* **129** (1995), 132-147.

[S] E. M. Stein, *Harmonic Analysis,* Princeton Mathematical Series, 43, Princeton University Press, Princeton, 1993.

[T.U] M.E.Taylor and A.Uribe, Semiclassical spectra of gauge fields, J.Funct.Anal. 110 (1992), 1-46.