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INVARIANT CR STRUCTURES ON COMPACT HOMOGENEOUS MANIFOLDS

Dmitry V. Alekseevsky, Andrea F. Spiro

Abstract. An explicit classification of the simply connected homogeneous spaces G/L of a compact Lie group G, admitting a G-invariant CR structure of codimension one and Levi non degenerate, is given. For each such a homogeneous space, all admissible G-invariant CR structures are listed and classified up to CR equivalences.

It is also proved that if a compact homogeneous CR manifold G/L is not the covering space of a G-orbit in TS^n, $T\mathbb{H}P^n$ or $T\mathbb{O}P^2$, then there exists a holomorphic fibration $\pi\colon G/L \to G/K$, where G/K is a flag manifold endowed with an invariant complex structure and the typical fiber K/L is S^1 or it is equivalent to (the universal covering of) a K-orbit in TS^2 or in TS^{2n-1} with $2 \leq n \leq 7$.

1. Introduction.

An *almost CR structure* on a manifold M is a pair $(\mathcal{D}, J)$, where $\mathcal{D} \subset TM$ is a distribution and J is a complex structure on $\mathcal{D}$. The complexification $\mathcal{D}^{\mathbb{C}}$ can be decomposed as $\mathcal{D}^{\mathbb{C}} = \mathcal{D}^{10} + \mathcal{D}^{01}$ into sum of complex eigendistributions of J, with eigenvalues i and $-i$.

An almost CR structure is called *integrable* or *CR structure* if the distribution $\mathcal{D}^{01}$ (and hence also the $\mathcal{D}^{10}$) is involutive, that is the space of sections is closed under Lie brackets. This is equivalent to the following conditions:

$$J([JX, Y] + [X, JY]) \in \mathcal{D} \tag{1.1}$$

$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0 \ , \tag{1.2}$$

for any two fields X, Y in $\mathcal{D}$.

A map $\varphi\colon (M, \mathcal{D}, I) \to (M', \mathcal{D}', J')$ between two CR manifolds is called *holomorphic map* or *CR map* if $\varphi_*(\mathcal{D}) \subset \mathcal{D}'$ and $\varphi_*(JX) = J'\varphi_*(X)$.

Two CR structures $(\mathcal{D}, J)$ and $(\mathcal{D}', J')$ are called *equivalent* if there exists a diffeomorphism such that $\phi_*(\mathcal{D}) = \mathcal{D}'$ and $\phi_* J = J'$.

The codimension of $\mathcal{D}$ is called *the codimension of the CR structure*. Note that a CR structure of codimension zero is the same as a complex structure.

A codimension one CR structure $(\mathcal{D}, J)$ on a $2n+1$-dimensional manifold M is called Levi non degenerate if $\mathcal{D}$ is a contact distribution. This means that any local (contact) 1-form θ, which defines the distribution (i.e. such that $ker\theta = \mathcal{D}$) is maximally non degenerate, that is $(d\theta)^n \wedge \theta \neq 0$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Note that any real hypersurface M of a complex manifold N has a natural codimension one CR structure $(\mathcal{D}, J_{\mathcal{D}})$ induced by the complex structure J of N, where

$$\mathcal{D} = \{ X \in TM , JX \in TM \} , \qquad J_{\mathcal{D}} = J|_{\mathcal{D}} .$$

In the following, if the opposite is not stated, by *CR structure* we will mean *integrable codimension one Levi nondegenerate CR structure*. Sometimes, if the contact distribution $\mathcal{D}$ is given, we will identify a CR structure with the associated complex structure J.

A CR manifold, that is a manifold M with a CR structure $(\mathcal{D}, J)$, is called *homogeneous* if it admits a transitive Lie group of CR transformations.

The aim of this paper is to give a complete classification of simply connected homogeneous CR manifolds $M = G/L$ of a compact Lie group G. This gives a classification of all simply connected homogeneous CR manifolds, since any compact homogeneous CR manifold admits a compact transitive Lie group of CR transformations (see [Sp]).

The simplest example of compact homogeneous CR manifold is the standard sphere $S^{2n-1} \subset \mathbb{C}^n$ with the induced CR structure.

More elaborated examples are provided by the following construction of A. Morimoto and T. Nagano ([MN]). Let $N = G/H$ be a compact rank one symmetric space (shortly 'CROSS'). The tangent space TN can be identified with the homogeneous space $G^{\mathbb{C}}/H^{\mathbb{C}}$. Hence, it admits a natural $G^{\mathbb{C}}$-invariant complex structure J. Any regular orbit $G \cdot p \simeq G/L$ in $TN = G^{\mathbb{C}}/H^{\mathbb{C}}$ is a real hypersurface with (Levi non degenerate) G-invariant CR structure.

Moreover, these examples together with the standard sphere $S^{2n-1} \subset \mathbb{C}^n$ exhaust the class of CR structures induced on a codimension one orbit $M = G \cdot x \subset C$ of a compact Lie group G of holomorphic transformations of a Stein manifold C. We call the homogeneous CR manifolds which are equivalent to such orbits in tangent spaces of a CROSS *Morimoto-Nagano spaces*.

In the fundamental paper [AHR], H. Azad, A. Huckleberry and W. Richthofer showed that these manifolds play a basic role in the description of the compact homogeneous CR manifolds.

More precisely, for any compact homogeneous CR manifold $M = G/L$ they define a holomorphic map (called *anticanonical map*) $\phi\colon M = G/L \to \mathbb{C}P^N$. This map is G-equivariant with respect to some explicitly defined projective action of G on $\mathbb{C}P^N$. For any compact homogeneous CR manifold M only two possibilities may occur: the orbit $\phi(M) = G \cdot p$, $p \in \phi(M)$, is either a flag manifold with the complex structure induced by the complex structure J_P of $\mathbb{C}P^N$ and in this case $\phi\colon M \to \phi(M)$ is an S^1-fibering, or it is a CR manifold with CR structure induced by J_P and in this case $\phi\colon M \to \phi(M)$ is a finite covering.

This reduces the description of the CR homogeneous manifolds of the second type to the description of compact orbits $G \cdot p \subset \mathbb{C}P^N$ of a real subgroup $G \subset Aut(\mathbb{C}P^N)$ of projective transformations, on which J_P induces a CR structure.

A simple argument shows that an orbit $G \cdot p \subset \mathbb{C}P^N$ of a connected Lie subgroup $G \subset Aut(\mathbb{C}P^N)$ carries a (possibly Levi degenerate) CR structure induced by $\mathbb{C}P^N$

if and only if $G \cdot p$ is a real hypersurface of $G^{\mathbb{C}} \cdot p$. Moreover, if the orbit is compact, one may assume that G is a compact semisimple Lie group.

The following important result in [AHR] describes the structure of such orbits.

Theorem. *Let $G^{\mathbb{C}} \subset Aut(\mathbb{C}P^N)$ be a connected complex semisimple group of projective transformations and G its compact form. Assume that the orbit $M = G \cdot p = G/L$ carries a Levi non degenerate CR structure induced by J_P and hence it is a real hypersurface in $B = G^{\mathbb{C}} \cdot p = G^{\mathbb{C}}/H$. Denote by P a minimal parabolic subgroup of $G^{\mathbb{C}}$ which properly contains H. Then the fiber $C = P/H$ of the $G^{\mathbb{C}}$-equivariant fibration over the flag manifold $F = G^{\mathbb{C}}/P$*

$$\pi \colon B = G^{\mathbb{C}}/H \to F = G^{\mathbb{C}}/P$$

is a homogeneous Stein manifold biholomorphic to $\mathbb{C}^*$, $\mathbb{C}^n$ or to the tangent space of a CROSS.

This fibration is called *Stein-rational fibration*. Note that P not necessarily acts effectively on C.

The Stein-rational fibration induces a G-equivariant holomorphic fibration of the homogeneous CR manifold $M = G/L$ over the flag manifold F

$$\pi' \colon M = G/L \to F = G^{\mathbb{C}}/P$$

(it is a *CRF fibration* according to our definitions, see below). Moreover, in correspondence to a fiber of π, a fiber of π' is either S^1, S^{2n-1} or a Morimoto-Nagano spaces.

This Theorem gives necessary conditions for an orbit $M = G \cdot p \subset \mathbb{C}P^N$ in order to carry an induced CR structure. Our classification gives necessary and sufficient conditions. In particular, we show that not all Morimoto-Nagano spaces may occur as fibers of the fibration π'.

Now we describe the main results of this paper. Section §2 collects the basics facts on homogeneous CR manifolds.

Section §3 is devoted to the infinitesimal description of homogeneous contact manifolds $M = G/L$ of a compact Lie group.

We prove that the center of G is at most one dimensional and we establish a natural one to one correspondence between simply connected homogeneous manifolds $M = G/L$ with an invariant contact distribution $\mathcal{D}$ and an element $Z \in \mathfrak{g} = Lie(G)$ (defined up to scaling) such that:

a) the centralizer of Z has the following orthogonal decomposition w.r.t. the Cartan-Killing form $\mathcal{B}$

$$C_{\mathfrak{g}}(Z) = \mathfrak{l} \oplus \mathbb{R}Z\ , \qquad \mathfrak{l} = Lie(L)\ ;$$

b) the 1-parametric subgroup generated by Z is closed.

This element Z (called *contact element*) defines an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} \ .$$

The subspace $\mathfrak{m}$ is Ad_L-invariant and defines the contact distribution $\mathcal{D}$ on $M = G/L$, while the Ad_L-invariant 1-form $\theta = \mathcal{B} \circ Z \in \mathfrak{g}^*$ is extended to a G-invariant contact form θ on G/L.

We associate with Z a flag manifold F_Z, which is the adjoint orbit

$$F_Z = \mathrm{Ad}_G(Z) = G/K \ ,$$

where $K = C_G(Z)$ is the centralizer of Z. There is a natural principal S^1-fibration

$$\pi \colon M = G/L \to F_Z = G/K \ .$$

In general, a homogeneous manifold G/L admits no more then one invariant contact structure. If it admits more then one then it is called *special contact manifold*.

The main examples of such manifolds can be described as follows.

Let G be a simple compact Lie group without center and let $Q = G/Sp_1 \cdot H'$ be the associated Wolf space, that is the homogeneous quaternionic Kähler manifold, where $Sp_1 \cdot H'$ is the normalizer in G of the 3-dimensional subalgebra $\mathfrak{sp}_1(\mu)$ of $\mathfrak{g}$ associated with the maximal root μ. Then the associated 3-Sasakian homogeneous manifold $M = G/H'$ is a special contact manifold.

Any $0 \neq Z \in \mathfrak{sp}_1(\mu)$ is a contact element. Furthermore, any two invariant contact structures on M are equivalent under a transformation, which commutes with G, defined by the right action of an element from Sp_1.

We prove the following theorem.

Theorem 1.1. *Any special contact manifold $M = G/L$ is either the 3-Sasakian homogeneous manifold G/H' of a simple group G, as described above, or $M = G_2/Sp_1$, where Sp_1 is the 3-dimensional subgroup of the exceptional Lie group G_2, with Lie algebra $\mathfrak{sp}_1(\mu)$, where μ is the maximal root of G_2.*

In section §4 we establish some general properties of compact homogeneous CR manifolds. Let $(M = G/L, \mathcal{D})$ be a homogeneous contact manifold and

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$$

the associated decomposition of the Lie algebra $\mathfrak{g}$. Then any invariant (integrable) CR structure J is defined by the Ad_L-invariant decomposition

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01} \tag{1.1}$$

of the complexified tangent space $\mathfrak{m}^{\mathbb{C}} = T^{\mathbb{C}}_{eL}M$ into holomorphic and antiholomorphic subspaces, such that

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01} \quad \text{is a subalgebra of} \quad \mathfrak{g}^{\mathbb{C}} \ . \tag{1.2}$$

The subspace $\mathfrak{m}$ is naturally identified with the tangent space of the associated flag manifold $F_Z = G/K$, $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z = Lie(K)$. It is known that any invariant complex structure on F_Z is defined by an Ad_K-invariant decomposition (1.1), where $\mathfrak{m}^{01}$ is a subalgebra (in fact it is the nilradical of a parabolic subalgebra $\mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{01}$). Hence any invariant complex structure J_F on F_Z defines an invariant CR structure J_M on $M = G/L$. It is called *standard CR structure induced by* J_F.

The natural S^1-fibration $\pi\colon M = G/L \to F_Z = G/K$ is holomorphic with respect to the CR structure J_M and the complex structure J_F.

Since the description of all invariant complex structures on a flag manifold is known (see e.g. [Ni], [AP], [BFR], [Al1]), it is sufficient to classify the non standard CR structures.

The following notion is important for such classification.

A compact homogeneous CR manifold $(M = G/L, \mathcal{D}, J)$ is called *not primitive* if it admits a holomorphic G-equivariant fibration π (called *CRF-fibration*)

$$\pi\colon M = G/L \to F = G/Q$$

where $F = G/Q$ is a flag manifold of positive dimension, equipped with an invariant complex structure J_F. Note that a fiber of π will be a homogeneous compact CR manifold Q/L and that any standard CR manifold is not primitive.

The classification of primitive CR structures given in §5 and §6 is an important step for the description of all non standard CR structures.

A basic tool for studying the homogeneous CR manifolds is the anticanonical map ϕ defined in [AHR].

Let $(M = G/L, \mathcal{D}_Z, J)$ be a homogeneous CR manifold of a compact Lie group G and

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} + \mathbb{C}Z + \mathfrak{m}^{10} + \mathfrak{m}^{01}$$

the corresponding decomposition of $\mathfrak{g}^{\mathbb{C}}$. Then the anticanonical ϕ is the holomorphic map from M into the Grassmanian of k-planes, $k = \dim_{\mathbb{C}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$, given by

$$\phi\colon M = G/L \to Gr_k(\mathfrak{g}^{\mathbb{C}}) \subset \mathbb{C}P^N$$

$$\phi\colon gL \mapsto \mathrm{Ad}_g(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) \ .$$

Note that ϕ is a G-equivariant map onto the orbit $G \cdot p$ of $p = \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01} \in Gr_k(\mathfrak{g}^{\mathbb{C}})$ under the natural adjoint action of G on $Gr_k(\mathfrak{g}^{\mathbb{C}})$.

We obtain the following characterization of standard and non standard CR structures (see Theorems 4.10 and 4.12):

Theorem 1.2. *Let $(M = G/L, \mathcal{D}_Z, J)$ be a homogeneous CR manifold.*

(1) *If it is standard, then the image $\phi(M) = G \cdot p$ of the anticanonical map is the flag manifold $F_Z = G/K$, associated with the contact structure $\mathcal{D}_Z$. Hence $\phi\colon M \to \phi(M) = F_Z$ is the natural S^1-fibration.*

(2) *If it is non standard, then $\phi\colon M \to \phi(M) = G \cdot p$ is a finite holomorphic covering, with respect to the CR structure of $G \cdot p \subset Gr_k(\mathfrak{g}^{\mathbb{C}})$ induced by the complex structure of $Gr_k(\mathfrak{g}^{\mathbb{C}})$.*

In section §5, we classify all invariant CR structures on special contact manifolds G/L. The result is the following:

Theorem 1.3. *Let $M = G/L$ be a special contact manifold with an invariant contact structure $\mathcal{D}_Z$.*

(1) *if $G \neq SU_n$, then there exists (up to sign of J) only one invariant CR structure $(\mathcal{D}_Z, J)$, which is standard;*
(2) *if $G = SU_2$ and hence $M = SU_2$, then there exist (up to sign of J) one standard CR structure and one family of non standard CR structures; any non standard CR structure is primitive and all of them are equivalent to each other;*
(3) *if $G = SU_n$, $n > 2$, and hence $M = SU_n/U_{n-2}$, then there exist (up to sign of J) three standard CR structures, induced by the three invariant complex structures of the corresponding flag manifold $F_Z = SU_n/T^2 \cdot SU_{n-2}$ (which is the twistor space of the Wolf space $Q = SU_n/S(U_2 \cdot U_{n-2})$), and two families consisting of mutually equivalent non standard CR structures. Any non standard CR structure is not primitive and admits a CRF fibration*

$$\pi \colon M = SU_n/U_{n-2} \to SU_n/S(U_2 \cdot U_{n-2})$$

with a fiber SU_2 over the Wolf space $SU_n/S(U_2 \cdot U_{n-2})$ equipped with its (unique up to sign) complex structure.

The explicit description of all non standard CR structures on SU_2 and SU_n/U_{n-2} is given in §5.

In section §6, we obtain the classification of non standard invariant CR structures on non special homogeneous contact manifolds.

¿From the list of non standard CR structures and from the previous results on special contact manifolds, we obtain the following classification of primitive CR structures.

Theorem 1.4. *Let $(M = G/L, \mathcal{D}_Z, J)$ be a simply connected primitive, homogeneous CR manifold and $\theta = i\mathcal{B} \circ Z$ the dual form of the contact element Z restricted to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k} = C_{\mathfrak{g}}(Z) = \mathfrak{l} + \mathbb{R}Z$. Then G/L is the universal covering of a regular (codimension one) orbit of G in a homogeneous complex space $B = G^{\mathbb{C}}/H$ with the induced CR structure. G, $K = C_G(Z)$, θ and the complex homogeneous space B belong to the following table. In all cases, B is the tangent space of a CROSS.*

n^o	G	$K = C_G(Z)$	θ	$B = G^{\mathbb{C}}/H$
1	SU_2	T^1	ε_0	$TS^2 = \frac{SO(3,\mathbb{C})}{SO(2,\mathbb{C})}$
2	$SU_2 \times SU_2'$	$T^1 \times T^{1\prime}$	$\varepsilon_0 - \varepsilon_0'$	$TS^3 = \frac{SO(4,\mathbb{C})}{SO(3,\mathbb{C})}$
3	F_4	$T^1 \cdot SO(7)$	ε_1	$T\mathbb{O}P^2 = \frac{F_4(\mathbb{C})}{Spin_9(\mathbb{C})}$
4	SO_{2n+1}	$T^1 \cdot SO_{2n-1}$	ε_1	$TS^{2n} = \frac{SO_{2n+1}(\mathbb{C})}{SO_{2n}(\mathbb{C})}$
5	SO_{2n}	$T^1 \cdot SO_{2n-2}$	ε_1	$TS^{2n-1} = \frac{SO_{2n}(\mathbb{C})}{SO_{2n-1}(\mathbb{C})}$
6	Sp_n	$T^1 \cdot Sp_1 \cdot Sp_{n-2}$	$\varepsilon_1 + \varepsilon_2$	$T\mathbb{H}P^{n-1} = \frac{Sp_n(\mathbb{C})}{Sp_1(\mathbb{C}) \cdot Sp_{n-1}(\mathbb{C})}$

In each of these cases, the set of all CR structures (considered up to sign) on $M = G/L$ is parameterized by the points of the unit disc D in $\mathbb{R}^2$. The center of D corresponds to the (unique) standard CR structure of M and all other points correspond to mutually equivalent primitive CR structures.

For what concerns the non primitive and non standard CR structures, we have the following theorem.

Theorem 1.5. *Let $M = G/L$ be a simply connected homogeneous CR manifold with a non standard not primitive CR structure. Then G is either simple or a product of two simple Lie groups and there exists a unique CRF fibration*

$$\pi\colon M = G/L \to F = G/Q$$

over a flag manifold F with an invariant complex structure J_F, such that the fiber Q/L is a primitive CR manifold. Moreover, if G is not simple then $Q/L = SU_2 \times SU_2'/T^1$; if G is simple then Q/L is one of the following primitive homogeneous CR manifolds

$$SU_2\ , \qquad SO_{2n}/SO_{2n-2}\ \ ,\quad 3 \leq n \leq 7\ .$$

If the fiber is $Q/L = SU_2$, then G/L is a special contact manifold SU_n/SU_{n-2}; if the fiber is $Q/L = SO_{2n}/SO_{2n-2}$ with $n \geq 4$, then $G = E_6, E_7$ or E_8.

Corollary 1.6. *Let $\pi\colon M = G/L \to F = G/Q$ be the CRF fibration of not primitive non standard CR manifold $(G/L, \mathcal{D}, J_0)$ onto the flag manifold $F = G/Q$ with a fixed invariant complex structure J_F. Then the set of all invariant CR structures $(\mathcal{D}, J)$ on G/L (up to sign of J), such that the fibering $\pi\colon M = G/L \to F = G/Q$ is holomorphic, is parameterized by the points of the unit disc D in $\mathbb{R}^2$. The center of D corresponds to the unique standard CR structure J_s of this family and all other points correspond to mutually equivalent primitive CR structures.*

The unique standard CR structure J_s on $M = G/L$ such that the fibration $\pi\colon M = G/L \to F = G/Q$ is holomorphic w.r.t. J_s and J_F is called the *standard CR structure associated with the non-standard CR structure J_0.*

We conclude this introduction, by showing how the explicit description of all non primitive CR manifolds G/L of a given compact Lie group G can be done in terms of *painted Dynkin graphs of* $\mathfrak{g} = Lie(G)$, that is of Dynkin graphs of the Lie algebra $\mathfrak{g}$ with nodes painted in three colors: white, black and gray.

Recall that any flag manifold $F = G/Q$ with an invariant complex structure J_F is defined (up to equivalences) by a black-white Dynkin graph, where the subalgebra $\mathfrak{q} = Lie(Q)$ is generated by the Cartan subalgebra and the root vectors associated with the white nodes. The complex structure J_F is determined by the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{q}^{\mathbb{C}} + \mathfrak{m}^{10} + \mathfrak{m}^{01}$$

where $\mathfrak{m}^{10}$ is the nilpotent subalgebra generated by the root vectors associated to black nodes.

With a painted Dynkin graph Γ, we associate two flag manifolds $F_1(\Gamma) = G/K$ and $F_2(\Gamma) = G/Q$ and two invariant complex structure $J_1(\Gamma)$ and $J_2(\Gamma)$ on $F_1(\Gamma)$ and $F_2(\Gamma_2)$, respectively, as follows. The pairs $(F_i(\Gamma) = G/Q, J_i(\Gamma))$, $i = 1, 2$, are the flag manifolds with invariant complex structures defined by the black-white graphs obtained from Γ by considering the gray nodes as white and, respectively, black.

Note that Q contains K and that the natural fibration

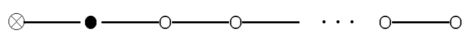
$$\varpi\colon F_1(\Gamma) = G/K \to F_2(\Gamma) = G/Q$$

is holomorphic and a fiber Q/K is a flag manifold with an induced invariant complex structure J'. Moreover, $J_1(\Gamma)$ is canonically defined by $J_2(\Gamma)$ and J'.

Conversely, if $F_1 = G/K$ and $F_2 = G/Q$ are two flag manifolds with invariant complex structures J_1 and J_2 such that $Q \supset K$ and the equivariant fibration $\varpi\colon F_1 \to F_2$ is holomorphic, then we may associate with F_1 and F_2 a painted Dynkin graph in an obvious way.

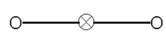
A painted Dynkin graph Γ of a semisimple Lie algebra $\mathfrak{g}$ is called *admissible* if

a) $\mathfrak{g} = A_\ell$ and Γ is


(1.3)

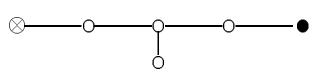
or

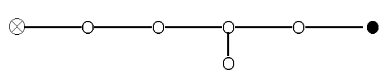
b) $\mathfrak{g} \neq A_\ell, E_6, E_7, E_8$ is simple, the black nodes are isolated and, after deleting the black nodes, Γ is of the following form, modulo connected components which consist of only white nodes,

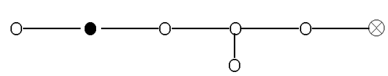
 + white (1.4)

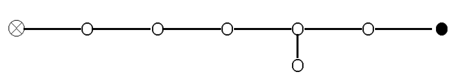
or

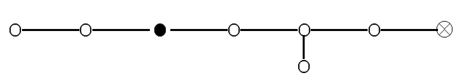
c) $\mathfrak{g} = E_6, E_7$ or E_8 and Γ is one of the following diagrams


(1.5)


(1.6)


(1.7)


(1.8)


(1.9)

d) $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ is sum of two simple Lie algebras, the black nodes are isolated and each connected component of Γ has exactly one grey node, which is not connected with a white node; in particular, after deleting the black nodes, the remaining graph is of the following form

$$\begin{array}{ccc} \otimes & + & \text{white} \\ \otimes & + & \text{white} \end{array} \tag{1.10}$$

Using the concept of admissible painted graph, the results of our classification may be stated as follows.

Theorem 1.7. *Let $M = G/L$ be a homogeneous CR manifold with a not primitive non standard CR structure $(\mathcal{D}, J)$. Denote by $\pi\colon G/L \to F_Z = G/K$ the natural (not holomorphic) fibration defined by the contact structure and $\pi'\colon G/L \to F = G/Q$ the unique CRF fibration with primitive fiber Q/L onto a flag manifold $F = G/Q$ with invariant complex structure J_F.*

Then $Q \supset K$ and the sequence of fibering

$$M = G/L \to F_Z = G/K \to F = G/Q$$

is holomorphic with respect to the standard CR structure $(\mathcal{D}, J_s)$ on M, associated to $(\mathcal{D}, J)$, the corresponding complex structure J_s on F_Z and the complex structure J_F on F.

Moreover, the painted Dynkin graph Γ associated to the flag manifolds $F_1 = F_Z$, $F_2 = F$ with the complex structures $J_1 = J_s$ and $J_2 = J_F$, respectively, is admissible.

Conversely, if Γ is an admissible painted Dynkin graph, then there exists a homogeneous contact manifold $(M = G/L, \mathcal{D}_Z)$ such that $F_Z = F_1(\Gamma) = G/K$ and the complex structure $J_1(\Gamma)$ defines the unique standard CR structure $(\mathcal{D}, J_1(\Gamma))$ on M such that the sequence of fibrations

$$M = G/L \to F_Z = F_1(\Gamma) = G/K \to F_2(\Gamma) = G/Q$$

is holomorphic w.r.t. $(\mathcal{D}, J_1(\Gamma))$, $J_1(\Gamma)$ and $J_2(\Gamma)$. The space of the invariant CR structures $(\mathcal{D}_Z, J)$ on M, such that the projection $\pi'\colon M \to F_2(\Gamma)$ is holomorphic, is parameterized by the points of a unit disc, with the center corresponding to the CR structure $J_1(\Gamma)$ and all other points corresponding to non standard CR structures which induce primitive CR structures on the fiber Q/L.

Part I

2. Basic facts about CR structures.

Definition 2.1.

(1) *A CR structure on a manifold M is a pair $(\mathcal{D}, J)$, where $\mathcal{D} \subset TM$ is a distribution on M and $J \in \operatorname{End}\mathcal{D}$, $J^2 = -1$, is a complex structure on $\mathcal{D}$.*

(2) *A CR structure $(\mathcal{D}, J)$ is called to be integrable if J satisfies the following integrability condition :*

$$J([JX, Y] + [X, JY]) \in \mathcal{D}$$

$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0 \tag{2.1}$$

for any pair of vector fields X, Y in $\mathcal{D}$.

In the sequel by CR manifold we will understand a manifold M with *integrable* CR structure.

If $(\mathcal{D}, J)$ is a CR structure then the complexification $\mathcal{D}^{\mathbb{C}} \subset T^{\mathbb{C}}M$ of the distribution $\mathcal{D}$ is decomposed into a sum $\mathcal{D}^{\mathbb{C}} = \mathcal{D}^{10} + \mathcal{D}^{01}$ of two mutually conjugated ($\mathcal{D}^{10} = \bar{\mathcal{D}}^{01}$) J-eigendistributions with eigenvalue i and $-i$. The integrability condition (2.1) means that these eigendistributions are involutive (i.e. closed under the Lie bracket).

The *codimension* of a CR structure $(\mathcal{D}, J)$ is defined as the codimension of the distribution $\mathcal{D}$. Remark that codimension zero CR structure is the same as a complex structure on a manifold. A codimension one CR structure $(\mathcal{D}, J)$ is called also a CR structure of *hypersurface type*, because such structure is induced on a real hypersurface of a complex manifold. In this case the distribution $\mathcal{D}$ can be described locally as the kernel of a 1-form θ. The form θ defines an Hermitian symmetric bilinear form

$$\mathcal{L}^\theta_q \colon \mathcal{D}_q \times \mathcal{D}_q \to \mathbb{R}$$

given by

$$\mathcal{L}^\theta(v, w) = (d\theta)(v, Jw)$$

for $v, w \in \mathcal{D}$. It is called the *Levi form*. Remark that the 1-form θ is defined up to the multiplication by a function f everywhere different from zero and $\mathcal{L}^{f\theta} = f\mathcal{L}^\theta$. In particular, the conformal class of a Levi form depends only on the CR structure.

A CR structure $(\mathcal{D}, J)$ of hypersurface type is called *non degenerate* if it has non degenerate Levi form or, in other words, if $\mathcal{D}$ is a contact distribution. In this case a 1-form θ with $\ker\theta = \mathcal{D}$ is called *contact form.*

A smooth map $\varphi \colon M \to M'$ of one CR manifold $(M, \mathcal{D}, J)$ into another one $(M', \mathcal{D}', J')$ is called *CR map or holomorphic map* if

a) $\varphi_*(\mathcal{D}) \subset \mathcal{D}'$;

b) $\varphi_*(Jv) = J'\varphi_*(v)$ for all $v \in \mathcal{D}$.

In particular, we may speak about CR *transformation* of a CR manifold $(M, \mathcal{D}, J)$ as a transformation φ such that φ and φ^{-1} are CR maps. In general, the group of all CR transformations is not a Lie group, but it is a Lie group when $(\mathcal{D}, J)$ is of hypersurface type and it is Levi non degenerate.

Definition 2.2. *A CR manifold $(M, \mathcal{D}, J)$ is called homogeneous if it admits a transitive Lie group G of CR transformations.*

Our aim is to classify compact homogeneous codimension one non degenerate CR manifolds. The following result shows that we may identify such manifold with a quotient space G/L of a compact Lie group G.

Theorem 2.3. *[Sp] Let $(M, \mathcal{D}, J)$ be a compact non degenerate CR manifold of hypersurface type. Assume that it is homogeneous, i.e. that there exists a transitive Lie group A of CR transformations. Then a maximal compact connected subgroup G of A acts on M transitively and one may identify M with the quotient space G/L where L is the stabilizer of a point $p \in M$.*

Now we fix some notations. If the opposite is not stated, we will assume that a CR structure is of hypersurface type, integrable and Levi non degenerate.

The Lie algebra of a Lie group is denoted by the corresponding gothic letter.

For any subset A of a Lie group G or of its Lie algebra $\mathfrak{g}$, we denote by $C_G(A)$ and $C_{\mathfrak{g}}(A)$ its centralizer in G and $\mathfrak{g}$, respectively. $Z(G)$ and $Z(\mathfrak{g})$ denote the center of a Lie group G and Lie algebra $\mathfrak{g}$. By homogeneous manifold $M = G/L$ we mean a homogeneous manifold of a compact connected Lie group G with *connected stability subgroup* L and such that the action of G on M is *effective*.

3. Compact Homogeneous Contact Manifold.

3.1 Homogeneous contact manifolds of a compact Lie group G.

Let $M = G/L$ be a homogeneous manifold of a compact Lie group G with connected stabilizer L.

An 1-form $\theta \in \mathfrak{g}^*$ on the Lie algebra $\mathfrak{g}$ of G is called *contact form* if it is $\mathrm{Ad}_{\mathfrak{l}}$-invariant and vanishes on $\mathfrak{l} = \mathrm{Lie}\, L$. Such form defines a global invariant 1-form θ on the manifold M which is a contact form of the contact distribution $\mathcal{D} = \ker \theta$. This establishes 1-1 correspondence between invariant contact structures $\mathcal{D}$ on M and contact 1-form $\theta \in \mathfrak{g}$ up to a scaling (see e.g.[Al]).

Fix now an Ad_G-invariant Euclidean metric $\mathcal{B}$ on $\mathfrak{g}$ and denote by $\mathfrak{l}^-$ the orthogonal complement to $\mathfrak{l}$ in $\mathfrak{g}$.

The vector $Z = \mathcal{B}^{-1} \circ \theta$ which corresponds to a contact form θ is called a *contact element* of the manifold $M = G/L$.

It is characterized by the properties that

(1) $Z \in \mathfrak{l}^-$ and
(2) the centralizer $C_{\mathfrak{g}}(Z) = \mathfrak{l} \oplus \mathbb{R}Z$.

Hence, we have the following

Proposition 3.1. *There exists a natural bijection between invariant contact structures on a homogeneous manifold $M = G/L$ and contact elements Z defined up to a scaling.*

We will denote by $\mathcal{D}_Z$ the contact structure on M defined by a contact element Z. A homogeneous manifold $M = G/L$ with an invariant contact structure $\mathcal{D}$ is called *homogeneous contact manifold*.

Proposition 3.1 implies the following

Corollary 3.2. *Let G/L be a homogeneous contact manifold of a compact Lie group G which acts effectively. Then the the center $Z(G)$ of G has dimension 0 or 1.*

Moreover, if $Z(G)$ is one dimensional, then any contact element Z has non zero orthogonal projections $Z_{Z(\mathfrak{g})}$, $Z_{\mathfrak{g}'}$ on $Z(\mathfrak{g})$ and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, and the stability subalgebra $\mathfrak{l}$ can be written as

$$\mathfrak{l} = [C_{\mathfrak{g}'}(Z_{\mathfrak{g}'})]_\varphi \overset{\text{def}}{=} \{X = Y + \varphi(Y)\ ,\ Y \in C_{\mathfrak{g}'}(Z_{\mathfrak{g}'})\}$$

where $\varphi\colon C_{\mathfrak{g}'}(Z_{\mathfrak{g}'}) \to Z(\mathfrak{g}) \approx \mathbb{R}$ is a non trivial Lie algebra homomorphism.

Proof. Clearly $C_{\mathfrak{g}}(Z) \supset Z(\mathfrak{g})$. If $\dim Z(\mathfrak{g}) \geq 2$ then $\mathfrak{l} \cap Z(\mathfrak{g}) \neq \{0\}$ and this contradicts the fact that G acts effectively. The other claims follow immediately. □

Remark that if Z is a contact element of a homogeneous manifold G/L and $Z_{\mathfrak{g}'}$ is its orthogonal projection of $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, then the adjoint orbit

$$F_Z \overset{\text{def}}{=} Ad_G Z = Ad_{G'}(Z_{\mathfrak{g}'})$$

is a flag manifold and the projection $\pi\colon M = G/L \to F_Z = G/K$ is a principal S^1-fibration over F_Z. We will call F_Z *the flag manifold associated to a contact element* Z. Note that the contact form $\theta = \mathcal{B} \circ Z$ is a connection (form) in the S^1 bundle $\pi : G/L \to F_Z$ and the corresponding contact structure $\mathcal{D} = \ker\theta$ is the horizontal distribution of this connection.

Let $F = G/K$ be a flag manifold of a semisimple compact group G. We describe now all homogeneous contact manifolds $(G/L, \mathcal{D}_Z)$ such that the associated flag manifold $F_Z = Ad_G Z$ is isomorphic to F.

Consider the orthogonal reductive decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$$

associated with the flag manifold $F = G/K$.

We say that an element Z of the center $Z(\mathfrak{k})$ is *regular* if it generates a closed 1-parametric subgroup of G and the centralizer $C_G(Z) = K$.

Note that if Z is regular, then the subalgebra

$$\mathfrak{l}_Z = \mathfrak{k} \cap (Z)^-$$

generates a closed subgroup, which we denote by L_Z. Indeed, this can be proved as follows. Consider the decomposition $\mathfrak{k} = \mathfrak{k}' + Z(\mathfrak{k})$, where $\mathfrak{k}'$ is the semisimple part of $\mathfrak{k}$. Then we have that $\mathfrak{l}_Z = \mathfrak{k}' + (Z(\mathfrak{k}) \cap (Z)^-)$ and it generates a closed subgroup if and only if the center $Z(\mathfrak{l}_Z) = (Z(\mathfrak{k}) \cap (Z)^-)$ generates a closed subgroup in the maximal torus corresponding to $Z(\mathfrak{k})$. Now, take an orthonormal basis $B = \{e_1, \dots, e_p\}$ for $Z(\mathfrak{k})$ and let us write $Z = \sum_i x^i e_i$. It is clear that Z generates a closed subgroup if and only if each x^i is rational. But this implies that $(Z(\mathfrak{k}) \cap (Z)^-)$ admits a basis $B' = \{f_1, \dots, f_p\}$, where each f_i has rational components in B. And from this, the claim follows. Therefore

Proposition 3.3. *Let $F = G/K$ be a flag manifold. There is a natural 1-1 correspondence between regular elements $Z \in Z(\mathfrak{k})_{reg} \subset Z(\mathfrak{k})$ up to a scaling and homogeneous manifolds G/L with an invariant contact structure $\mathcal{D}$ and the associated flag manifold $F = G/K$. The correspondence is*

$$Z(\mathfrak{k})_{reg} \mod \mathbb{R} \ni [Z] \Longleftrightarrow (G/L_Z, \mathcal{D}_Z).$$

Proof. Let $Z \in Z(\mathfrak{k})_{reg}$ and let $L_Z \subset G$ be the closed subgroup generated by $\mathfrak{l}_Z$. Then, clearly, Z is a contact element for G/L_Z and determines a contact structure $\mathcal{D}_Z$. Moreover, if $Z, Z' \in Z(\mathfrak{k})_{reg}$ are such that $L_Z = L_{Z'}$, then $Z' = \lambda Z$ for some λ. This shows that the map $Z \Rightarrow (G/L_Z, \mathcal{D}_Z)$ is injective on $Z(\mathfrak{k})_{reg} \mod \mathbb{R}$. The surjectivity is also clear. □

3.2 Invariant contact structures on a contact manifold $M = G/L$.

Now we describe all invariant contact structures on a given homogeneous manifold $M = G/L$. We will show that generically there is no more then one such structure.

Definition 3.4. A homogeneous manifold G/L is called homogeneous contact manifold of *generic type* (respectively, of *special type* or, shortly, *special*) if it admits a unique (respectively, more then one) invariant contact structure.

3.2.1 Main examples of special homogeneous contact manifolds.

Let $\mathfrak{g}$ be a compact semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and R the root system of the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$.

Recall that a root $\alpha \in R$ defines a 3-dimensional regular subalgebra $\mathfrak{g}^{\mathbb{C}}(\alpha) = \mathrm{span}_{\mathbb{C}} < E_\alpha, E_{-\alpha}, H_\alpha >$ and its intersection with $\mathfrak{g}$ is a 3-dimensional compact subalgebra $\mathfrak{g}(\alpha)$. We will call $\mathfrak{g}(\alpha)$ the *subalgebra associated with the root* α and denote by $G(\alpha)$ the 3-dimensional subgroup of the adjoint group $G = \mathrm{Int}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})^0$ generated by $\mathfrak{g}(\alpha)$.

Note that any two such subalgebras are conjugated by an inner automorphism of $\mathfrak{g}$ if and only if the corresponding roots have the same length.

Fix a system R^+ of positive roots of R and put $R^- = -R^+$. The highest root μ of R^+ defines the following gradation of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 \tag{3.1}$$

where

$$\mathfrak{g}_{-2} = \mathbb{C}E_{-\mu} \qquad \mathfrak{g}_2 = \mathbb{C}E_\mu \qquad \mathfrak{g}_0 = \mathbb{C}H_\mu + \mathfrak{g}_0' \qquad \mathfrak{g}_0' = C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}(\mu)) \tag{3.2}$$

$$\mathfrak{g}_{-1} = \sum_{\beta \in R^- \setminus (\{-\mu\} \cup R_0)} \mathbb{C}E_\beta \qquad \mathfrak{g}_1 = \sum_{\beta \in R^+ \setminus (\{\mu\} \cup R_0)} \mathbb{C}E_\beta$$

and $R_0 = \{\alpha \in R,\ \alpha - \mu\}$ is the root system of the subalgebra $\mathfrak{g}_0 = C_{\mathfrak{g}}(H_\mu)$.

(3.1) is called the *gradation associated with the highest root* .

The explicit decomposition (3.1) for any simple complex Lie algebra is given in Table 1 of the Appendix.

Denote by $\mathfrak{l} = C_{\mathfrak{g}}(\mathfrak{g}(\mu)) = \mathfrak{g}_0' \cap \mathfrak{g}$ the centralizer of $\mathfrak{g}(\mu)$ in $\mathfrak{g}$ and by L the corresponding connected subgroup of G. It is easy to check that $L = C_G(\mathfrak{g}(\mu))$.

Lemma 3.5. *Let G be a compact simple Lie group without center and let $L = C_G(\mathfrak{g}(\mu))$ be as defined above. Then any non zero vector $Z \in \mathfrak{g}(\mu)$ is a contact element of the manifold G/L. In particular, G/L is a homogeneous contact manifold of special type.*

Proof. Observe that $Z \in \mathfrak{g}(\mu)$ is a contact element if and only if $C_{\mathfrak{g}}(Z) = \mathfrak{l} + \mathbb{R}Z$. Moreover Z is a contact element if and only if $g \cdot Z$ is contact, for any $g \in G(\mu)$. Since $G(\mu)$ acts transitively on the unit sphere of $\mathfrak{g}(\mu)$, the Lemma follows from the fact that

$$C_{\mathfrak{g}}(iH_\mu) = \mathfrak{g}_0 \cap \mathfrak{g} = \mathfrak{l} + \mathbb{R}(iH_\mu)$$

and hence that iH_μ is a contact element. □

Remark that the manifolds $M = G/L = G/C_G(\mathfrak{g}(\mu))$ with G simple carry invariant 3-Sasakian structure and they exhaust all homogeneous 3-Sasakian manifolds (see [BGM]).

3.2.2 Classification of special homogeneous contact manifolds.

The previous examples almost exhaust the class of special homogeneous contact manifold. In fact, we have the following classification theorem.

Theorem 3.6. *Let $M = G/L$ be a special homogeneous contact manifold of a compact Lie group G. Then the group G is simple and either L is the centralizer of the subalgebra $\mathfrak{g}(\mu)$ associated with the highest root and M is a homogeneous 3-Sasakian manifold or $G = G_2$ and L is the centralizer of the subalgebra $\mathfrak{g}(\nu)$ associated with a short root ν.*

Proof. We prove now that if G is not semisimple and, hence, $\dim Z(\mathfrak{g}) = 1$, then a contact element Z is unique up to a scaling and M is generic. Indeed, we have the decomposition

$$\mathfrak{k} = C_{\mathfrak{g}}(Z) = \mathfrak{l} \oplus \mathbb{R}Z = \mathfrak{l} + Z(\mathfrak{g})$$

since $Z(\mathfrak{g}) \cap \mathfrak{l} = 0$, by effectivity. The line $\mathbb{R}Z$ is determined uniquely as the orthogonal complement to $\mathfrak{l}$ in $\mathfrak{k} = \mathfrak{l} + Z(\mathfrak{g})$.

Now we may assume that $\mathfrak{g}$ is semisimple. We need the following

Lemma 3.7. *Let $\mathfrak{g}$ be compact semisimple and let $\mathfrak{l} \subset \mathfrak{g}$ be a closed subalgebra, which contains no ideal of $\mathfrak{g}$. If there exists two non proportional vectors Z, $Z' \in \mathfrak{l}^-$ such that*

$$C_{\mathfrak{g}}(Z) = \mathfrak{l} + \mathbb{R}Z, \quad \mathfrak{l} + \mathbb{R}Z' \subseteq C_{\mathfrak{g}}(Z')\ ,$$

then $\mathfrak{g}$ is simple and there exists a root $\alpha \in R$ such that:

1. *$\mathfrak{l} = C_{\mathfrak{g}}(\mathfrak{g}(\alpha))$;*
2. *$Z, Z' \in \mathfrak{g}(\alpha)$ and $C_{\mathfrak{g}}(Z') = C_{\mathfrak{g}}(\mathfrak{g}(\alpha)) + \mathbb{R}Z'$;*
3. *$C_{\mathfrak{g}}(\mathfrak{l}) = Z(\mathfrak{l}) + \mathfrak{g}(\alpha)$;*
4. *for any root β which is orthogonal to α, $\alpha \pm \beta$ is not a root.*

Proof. We put $\mathfrak{k} = C_{\mathfrak{g}}(Z)$ and consider the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = (\mathfrak{l} + \mathbb{R}Z) + \mathfrak{m}.$$

Denote by R the root system of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ with respect to a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ which is the complexification of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. Then the element Z' can be written as

$$Z' = cZ + \sum_{i=1}^{k} c_i E_{\alpha_i}$$

for some root vectors E_{α_i} and constants c, c_i. The condition $[\mathfrak{l}, Z'] = 0$ implies $\alpha_i(\mathfrak{h} \cap \mathfrak{l}) = 0$ if $c_i \neq 0$. Since $\mathfrak{h} \cap \mathfrak{l}$ is of codimension one in $\mathfrak{h}$, there exist exactly two (proportional) roots with this properties, say α and $-\alpha$. This shows that $\mathfrak{l} \subset C_{\mathfrak{g}}(\mathfrak{g}(\alpha))$. Moreover, since $Z \in \mathfrak{h} \cap \mathfrak{l}^{-}$, we obtain also that Z is proportional to $H_\alpha = [E_\alpha, E_{-\alpha}]$ and (1) follows. In particular, $\mathfrak{g}$ must be simple and now (2) is clear. (3) follows from (2).

To prove (4), assume that there is a root β which is orthogonal to α and such that $\alpha + \beta$ is a root. Then the vector $E_\beta + E_{-\beta} \in \mathfrak{g}^{\mathbb{C}}$ does not belong to $\mathfrak{l}^{\mathbb{C}} = C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}(\alpha))$, but it is orthogonal to Z (since Z is proportional to H_α) and belongs to the centralizer of Z: contradiction. □

Now we conclude the proof of Theorem 3.6. Let G be a compact semisimple Lie group and Z, Z' two non proportional contact elements for G/L. By Lemma 3.7, G is simple and $L = C_G(\mathfrak{g}(\alpha))$. By direct inspection of the root systems of simple Lie groups, a root α verifies the condition (4) of Lemma 3.6 if and only if it is a long root or it is a short root in the G_2 type system. This concludes the proof. □

3.3 Isotropy representation of a homogeneous contact manifold.

Let $M = G/L$ be a homogeneous contact manifold with invariant contact structure $\mathcal{D}$ associated to a contact element Z. Let $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ be the corresponding orthogonal decomposition. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which belongs to $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z = Z(\mathfrak{k}) + \mathfrak{k}'$. Then

$$\mathfrak{h} = Z(\mathfrak{k}) + \mathfrak{h}' = Z(\mathfrak{l}) + \mathbb{R}Z + \mathfrak{h}'$$

where we denote by $\mathfrak{h}'$ a Cartan subalgebra of $\mathfrak{k}'$ (=semisimple part of $\mathfrak{k}$). Remark that $\mathfrak{h}(\mathfrak{l}) = Z(\mathfrak{l}) + \mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{l}$.

Denote by R (resp. R_o) the root system of $\mathfrak{g}^{\mathbb{C}}$ (resp. $\mathfrak{k}^{\mathbb{C}}$) w.r.t. the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ and let $R_{\mathfrak{m}} = R \setminus R_o$. We will denote by $\mathfrak{h}(\mathbb{R})$ the standard real form of $\mathfrak{h}$, spanned by R, that is

$$\mathfrak{h}(\mathbb{R}) = \mathfrak{h} \cap \mathcal{B}^{-1}(< R >)$$

We put $\mathfrak{t} = \mathfrak{z}(\mathfrak{k}) \cap \mathfrak{h}(\mathbb{R})$. Then $Z \in i\mathfrak{t}$ and we may identify $\theta \overset{\text{def}}{=} i\mathcal{B}(Z, \cdot)$ with the corresponding element in $\mathfrak{t}^* \subset \mathfrak{h}(\mathbb{R})^* = \operatorname{span}_{\mathbb{R}} R$.

Consider the decomposition of the $\mathfrak{k}^{\mathbb{C}}$-module $\mathfrak{m}^{\mathbb{C}}$ into sum of irreducible $\mathfrak{k}^{\mathbb{C}}$-modules

$$\mathfrak{m}^{\mathbb{C}} = \sum \mathfrak{m}(\gamma) \tag{3.1}$$

Here, $\mathfrak{m}(\gamma)$ stands for the irreducible $\mathfrak{k}^{\mathbb{C}}$-module with highest weight $\gamma \in R_{\mathfrak{m}}$.

The following Lemma states a well known property of flag manifolds (see e.g. [AP]).

Lemma 3.8. *The $\mathfrak{k}^{\mathbb{C}}$-modules $\mathfrak{m}(\gamma)$ are pairwise not equivalent and, in particular, the decomposition (3.1) is unique. The moduli $\mathfrak{m}(\gamma)$ are irreducible also as $\mathfrak{l}^{\mathbb{C}}$-modules.*

Proof. We only need to check that a module $\mathfrak{m}(\gamma)$ is irreducible also as an $\mathfrak{l}^{\mathbb{C}}$-module. But it is sufficient to observe that the semisimple parts of $\mathfrak{l}^{\mathbb{C}}$ and of $\mathfrak{k}^{\mathbb{C}}$ coincide. In fact, whenever $\dim_{\mathbb{C}} \mathfrak{m}(\gamma) > 1$, the semisimple part of $\mathfrak{k}^{\mathbb{C}}$ acts not trivially and irreducibly on $\mathfrak{m}(\gamma)$. $\square$

From Lemma 3.8 we derive the following technical proposition, which will be useful in the following sections.

Proposition 3.9. *Let $M = G/L$ be a homogeneous contact manifold and let Z be a contact element for M. Assume that $G \neq G_2$ or that $G = G_2$ and $\theta = i\mathcal{B} \circ Z$ is not proportional to a short root of R.*

Then for any irreducible $\mathfrak{k}^{\mathbb{C}}$-module $\mathfrak{m}(\gamma)$ there exists at most one distinct $\mathfrak{k}^{\mathbb{C}}$-module $\mathfrak{m}(\gamma')$ which is isomorphic to $\mathfrak{m}(\gamma)$ as $\mathfrak{l}^{\mathbb{C}}$-module.

This is the case if and only if the highest weights γ and γ' are θ-congruent, i.e. $\gamma' = \gamma + \lambda\theta$ for some real number λ.

Corollary 3.10. *Let M and Z as in the Proposition 3.9. Then:*

a) *if the modules $\mathfrak{m}(\gamma)$, $\mathfrak{m}(\gamma')$ are equivalent as $\mathfrak{l}^{\mathbb{C}}$-modules, then for any weight $\alpha \in R_{\mathfrak{m}}$ of $\mathfrak{m}(\gamma)$, there exists exactly one weight $\alpha' \in R_{\mathfrak{m}}$ of $\mathfrak{m}(\gamma')$ which is θ-congruent to α;*
b) *for any root $\alpha \in R_{\mathfrak{m}}$ there exists at most one other root $\alpha' \in R_{\mathfrak{m}}$ which is θ-congruent to α, i.e. such that $\alpha' = \alpha + \lambda\theta$ for some real number λ.*

Proof of Proposition 3.9. Observe that two irreducible $\mathfrak{l}^{\mathbb{C}}$-modules $\mathfrak{m}(\gamma)$ and $\mathfrak{m}(\gamma')$ are isomorphic if and only if their highest weights $\gamma|_{\mathfrak{h}(\mathfrak{l})}$ and $\gamma'|_{\mathfrak{h}(\mathfrak{l})}$ coincide. This is if and only if $\gamma' = \gamma + \lambda\theta$ for some $\lambda \in \mathbb{R}$.

Assume now that there exist three distinct isomorphic $\mathfrak{l}^{\mathbb{C}}$-modules $\mathfrak{m}(\gamma)$, $\mathfrak{m}(\gamma')$ and $\mathfrak{m}(\gamma'')$. Then $\tilde{R} = \operatorname{span}_{\mathbb{R}}(\gamma, \gamma', \gamma'') \cap R$ is a 2-dimensional root system and γ, γ' and γ'' belong to the straight line $\gamma + \mathbb{R}\theta$. Checking all 2-dimensional root systems, $2A_1$, A_2, B_2, G_2, we conclude that this is possible only if $\tilde{R}$ is of type B_2 or G_2 and θ is proportional to a short root.

To conclude the proof, it is sufficient to observe that in case $\tilde{R} = B_2$, one of the roots γ, γ', γ'' should be orthogonal to θ and this is impossible because

$$\theta^- \cap R = R_o = R \setminus R_{\mathfrak{m}}$$

while γ, γ', $\gamma'' \in R_{\mathfrak{m}}$. $\square$

4. General Properties of Compact Homogeneous CR manifolds.

4.1 Basic properties and definitions.

Let $(M = G/L, \mathcal{D}_Z)$ be a homogeneous contact manifold of a connected compact Lie group G with connected stabilizer L and let $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ be the associated decomposition with the contact element Z orthogonal decomposition such that $\mathfrak{m} \simeq \mathcal{D}|_o, o = eL$ and $\mathfrak{k} = C_{\mathfrak{g}}(Z) = \mathfrak{l} + \mathbb{R}Z$.

Definition 4.1. An $\mathrm{ad}_{\mathfrak{l}}$-invariant complex structure J on $\mathfrak{m}$ is called *integrable* if

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01} \textit{ is a complex subalgebra of } \mathfrak{g}^{\mathbb{C}},$$

where

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01} \quad , \quad \mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}} \tag{4.1}$$

is the eigenspace decomposition of J.

Note that any $ad_{\mathfrak{l}^{\mathbb{C}}}$-invariant decomposition (4.1) defines an $ad_{\mathfrak{l}^{\mathbb{C}}}$-invariant complex structure J on $\mathfrak{m}$ which has this decomposition as the eigenspace decomposition.

The following Proposition can be checked directly.

Proposition 4.2. *Let $(M = G/L, \mathcal{D}_Z)$ be a homogeneous contact manifold and let $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ be the corresponding decomposition of the Lie algebra $\mathfrak{g}$ of G. Then there exists a natural one to one correspondence between the invariant CR structures $(\mathcal{D}, J)$ on M, with underlying contact distribution $\mathcal{D}$, and the integrable complex structures J on $\mathfrak{m}$.*

Consider the decomposition $\mathfrak{m}^{\mathbb{C}} = \sum \mathfrak{m}(\gamma)$ into irreducible $\mathfrak{k}$-submodules as in §3.3. Since any $\mathrm{ad}_{\mathfrak{l}}$-invariant complex structure J on $\mathfrak{m}$ preserves the $\mathfrak{l}^{\mathbb{C}}$-*isotopic components* (i.e. the sum of all mutually equivalent irreducible $\mathfrak{l}^{\mathbb{C}}$-modules) and since the multiplicity of any irreducible $\mathfrak{l}$-module $\mathfrak{m}(\gamma)$ is less or equal to 2 in the hypothesis of Proposition 3.10, we have the following corollary.

Corollary 4.3. *Let J be an $\mathfrak{l}$-invariant complex structure on $\mathfrak{m}$ and suppose that $G \neq G_2$ or that $G = G_2$ and that θ is not proportional to a short root.*

Then a minimal J-invariant $\mathfrak{k}^{\mathbb{C}}$-submodule of $\mathfrak{m}^{\mathbb{C}}$ is either $\mathfrak{k}^{\mathbb{C}}$-irreducible or is the sum $\mathfrak{m}(\gamma) + \mathfrak{m}(\gamma')$ of two such $\mathfrak{k}^{\mathbb{C}}$-modules, with γ' θ-congruent to γ (i.e. $\gamma' = \gamma + \lambda\theta$, for some λ).

4.2 Standard CR structures.

Many invariant CR structures $(\mathcal{D}, J)$ on a contact manifold $M = G/L$ may be constructed as follows. Let $(M = G/L, \mathcal{D}_Z)$ be a homogeneous contact manifold and let

$$\pi : M = G/L \longrightarrow F = G/K = \mathrm{Ad}_G(Z)$$

be the associated fibration over the flag manifold $F = G/K$.

Then the contact distribution $\mathcal{D}_Z$ is the horizontal distribution of π with respect to the invariant Riemannian metric on M defined by the invariant bilinear form $\mathcal{B}$ on $\mathfrak{g}$. Any invariant complex structure J_F on the flag manifold F defines an invariant CR structure $(\mathcal{D}, J)$ on M. The integrability of this CR structure follows from the integrability of J_F which is equivalent to the statement that $\mathfrak{p} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{01}$ is a (in fact parabolic) subalgebra of $\mathfrak{g}^{\mathbb{C}}$. (Here $\mathfrak{m}^{01}$ is the $(-i)$-eigenspace of the complex structure J_F on $\mathfrak{m}^{\mathbb{C}} = T_o^{\mathbb{C}}F, o = eK$).

Definition 4.4. An invariant CR structure $(\mathcal{D}, J)$ on a homogeneous contact manifold $(M = G/L, \mathcal{D})$, which is induced by an invariant complex structure J_F of the associated flag manifold $F = G/K$, is called a *standard CR structure.*

Remark 4.5. Since any flag manifold admits at least one invariant complex structure, we may conclude that *any homogeneous contact manifold* $(G/L, \mathcal{D})$*, with* G *compact, admits an invariant CR structure* $(\mathcal{D}, J)$.

The following Lemma gives an algebraic characterization of the standard CR structures.

Lemma 4.6. *An invariant CR structure* $(\mathcal{D}, J)$ *on a homogeneous contact manifold* $(M = G/L, \mathcal{D})$ *is standard if and only if the corresponding complex structure* J *on* $\mathfrak{m}$ *is* $\mathrm{Ad}(K)$*-invariant.*

Proof. The necessity is immediate from the definitions. In case G is semisimple, the sufficiency is also clear. Suppose now that $\dim Z(G) = 1$ and let (4.1) be the decomposition associated to an $\mathrm{Ad}(K)$-invariant complex structure J on $\mathfrak{m}$. Then let $\pi_o \colon \mathfrak{g} \to \mathfrak{g}'$ be the standard orthogonal projection onto the semisimple part and let $\mathfrak{m}'^{10} = \pi_o(\mathfrak{m}^{10})$, $\mathfrak{m}'^{01} = \pi_o(\mathfrak{m}^{01})$. Since $\mathfrak{m}^{10}$ and $\mathfrak{m}^{01}$ are $\mathrm{Ad}(K)$-invariant and $K = (K \cap G') \cdot Z(G)$, it is clear that $\mathfrak{m}'^{10}$ and $\mathfrak{m}'^{01}$ correspond to an invariant complex structure J_F on $G/K = G'/(K \cap G')$. $\square$

Since the description of all invariant complex structures on flag manifolds is well known (see [Na], [AP], [BFR], [Al1]), the problem of classification of the invariant CR structures on compact homogeneous spaces reduces to the description of *non-standard* invariant CR structure.

The following proposition reduces the problem to the case of G semisimple.

Proposition 4.7. *Let* $(M = G/L, \mathcal{D})$ *be a contact manifold of a compact Lie group* G *with* $\dim Z(G) = 1$*. Then any invariant CR structure with underlying distribution* $\mathcal{D}$ *is standard.*

Proof. It follows immediately from the fact that any $Ad(L)$-invariant decomposition (4.1) is clearly also $Ad(K)$-invariant, since $K = L \cdot Z(G)$. $\square$

4.3 Holomorphic fibering of homogeneous CR manifolds.

Let $(M = G/L, \mathcal{D}, J)$ be a homogeneous *standard* CR structure associated to a complex structure J_F on the associated flag manifold $F = G/K$. Then the natural projection

$$\pi \colon G/L \to F = G/K$$

is a G-equivariant S^1-fibration and it is a holomorphic map between the CR manifolds $(M = G/L, \mathcal{D}, J)$ and (F, TF, J_F).

More generally:

Definition 4.8. Let $M = G/L$ be a homogeneous manifold with invariant CR structure $(\mathcal{D}, J)$.

(1) Any G-equivariant holomorphic fibering

$$\pi\colon M = G/L \to F = G/Q$$

of $(M, \mathcal{D}, J)$ over a flag manifold $F = G/Q$ equipped with an invariant complex structure J_F is called *CRF fibration*;

(2) We say that a homogeneous CR manifold $(M = G/L, \mathcal{D}, J)$ is *primitive* if it doesn't admit a non trivial CRF fibration;

(3) a non primitive homogeneous CR manifold $(M = G/L, \mathcal{D}, J)$, which admits a CRF fibration with typical fiber S^1 is called *circular*.

Remark that *any standard CR structure is circular*.

The following Lemma give a characterization of primitive CR structures.

Lemma 4.9. *A homogeneous CR manifold $(G/L, \mathcal{D}, J)$ admits a non trivial CRF fibration if and only if there exists a proper parabolic subalgebra $\mathfrak{p} = \mathfrak{r} + \mathfrak{n} \subsetneq \mathfrak{g}^{\mathbb{C}}$ (here $\mathfrak{r}$ is the reductive part and $\mathfrak{n}$ the nilpotent part) such that*

$$a)\ \mathfrak{r} = (\mathfrak{p} \cap \mathfrak{g})^{\mathbb{C}}\ ; \qquad b)\ \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01} \subset \mathfrak{p}\ ; \qquad c)\ \mathfrak{l}^{\mathbb{C}} \subsetneq \mathfrak{r}\ .$$

In this case, G/L admits a CRF fibration with basis G/Q, where Q is the connected subgroup generated by $\mathfrak{q} = \mathfrak{r} \cap \mathfrak{g}$.

Proof. Suppose that $(M = G/L, \mathcal{D}, J)$ is not primitive and let $\pi\colon G/L \to G/Q$ be a CRF fibration over a flag manifold $F = G/Q$ with invariant complex structure J_F. Consider the decompositions associated to J and J_F

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} \qquad \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$$

$$\mathfrak{g} = \mathfrak{q} + \mathfrak{m}' \qquad \mathfrak{m}'^{\mathbb{C}} = \mathfrak{m}'^{10} + \mathfrak{m}'^{01}$$

Since π is holomorphic and not trivial, the subalgebra $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}$ is contained in the parabolic subalgebra $\mathfrak{p} = \mathfrak{q}^{\mathbb{C}} + \mathfrak{m}'^{01}$, with reductive part $\mathfrak{q}^{\mathbb{C}} = (\mathfrak{g} \cap \mathfrak{p})^{\mathbb{C}}$. Furthermore, since the fiber is at least 1 dimensional, $\mathfrak{l} \subsetneq \mathfrak{q}$.

Conversely, if $\mathfrak{p} = \mathfrak{r} + \mathfrak{n} \subset \mathfrak{g}^{\mathbb{C}}$ is a parabolic subalgebra with reductive subalgebra $\mathfrak{r} = \mathfrak{q}^{\mathbb{C}}$, where $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{g}$, then we may consider the orthogonal decompositions

$$\mathfrak{g} = \mathfrak{q} + \mathfrak{m}' \qquad \mathfrak{g}^{\mathbb{C}} = \mathfrak{r} + \mathfrak{m}'^{\mathbb{C}} = \mathfrak{r} + \mathfrak{n} + \mathfrak{n}'$$

where $\mathfrak{n}' = \mathfrak{n}^- \cap \mathfrak{m}'^{\mathbb{C}}$. It is well known that there exists a unique invariant complex structure J_F on $G^{\mathbb{C}}/P = G/Q$, such that $\mathfrak{n} = \mathfrak{m}'^{01}$ and $\mathfrak{n}' = \mathfrak{m}'^{10}$. Therefore if $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01} \subset \mathfrak{p}$, $\mathfrak{l} \subsetneq \mathfrak{q}$ and Q is the reductive subgroup generated by $\mathfrak{q}$, it is clear that $\pi\colon G/L \to G/Q$ is a non trivial CRF fibration. $\square$

4.4 The anticanonical map of a homogeneous CR manifold.

Let $(M = G/L, \mathcal{D}_Z, J)$ be a homogeneous CR manifolds of a compact Lie group G and

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} \quad , \quad \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$$

the associated decompositions of $\mathfrak{g}$ and of $\mathfrak{m}^{\mathbb{C}}$.

To characterize the circular invariant CR structures we recall the definition of *anticanonical map* of a homogeneous CR manifold introduced for the first time in [AHR]. It is a G-equivariant holomorphic map

$$\phi : M = G/L \longrightarrow \mathrm{Gr}_k(\mathfrak{g}^{\mathbb{C}})$$

into the Grassmanian of complex k-planes, $k = \dim_{\mathbb{C}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$, of $\mathfrak{g}^{\mathbb{C}}$ given by

$$\phi \colon gL \mapsto Ad_g(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}) \ .$$

Due to the existence of standard holomorphic G-equivariant embedding

$$\imath \colon \mathrm{Gr}_k(\mathfrak{g}^{\mathbb{C}}) \longrightarrow \mathbb{C}P^N \ , \quad N = \binom{\dim \mathfrak{g}^{\mathbb{C}}}{k} - 1$$

$$V = \mathrm{span}(e_1, \dots, e_k) \overset{\imath}{\mapsto} [V] = \mathbb{C}(e_1 \wedge \dots \wedge e_k) \ ,$$

we may consider ϕ as a G-equivariant map into $\mathbb{C}P^N$. To prove that the map ϕ is holomorphic it is sufficient to check that the linear map

$$\phi_* \colon \mathcal{D}_0 = \ker \theta|_{T_0 M} = \mathfrak{m} \longrightarrow T_{[\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]}\mathrm{Gr}_k(\mathfrak{g}^{\mathbb{C}})$$

commutes with the complex structure.

Let $v = X + \bar{X} \in \mathfrak{m}$, where $X \in \mathfrak{m}^{10}$. Then

$$\phi_*(v) = ad_{(X+\bar{X})}([\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]) = ad_X([\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]).$$

Therefore

$$\phi_*(Jv) = \phi_*(iX - i\bar{X}) = ad_{iX}([\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]) = iad_X([\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]) = i\phi_*(v)$$

This shows that the map ϕ is holomorphic.

Remark that the stabilizer Q of the point $[\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]$ in $\phi(M) = G/Q$ is the normalizer $Q = N_G(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$.

Now, the following theorem gives some crucial properties of the anticanonical map.

Theorem 4.10. *Let*

$$\phi : M = G/L \longrightarrow Gr_k(\mathfrak{g}^{\mathbb{C}})$$

be the anticanonical map of a homogeneous CR manifold $(M = G/L, \mathcal{D}_Z, J)$*.*

(1) *If the CR structure is circular, then the image* $\phi(M) = G/Q$ *is a flag manifold and* ϕ *is a CRF fibration with fiber* S^1*.*
In this case the normalizer in $\mathfrak{g}$ *of* $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}$ *is*

$$\mathfrak{q} = N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) = \mathfrak{l} + \mathbb{R}Z'$$

where $Z' \neq 0$ *is an element from the centralizer of* $\mathfrak{l}$ *in* $\mathfrak{g}$*. Moreover,* $\mathfrak{q}$ *is the Lie algebra of the stabilizer of the point* $[\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}] \in \phi(M)$ *in* G*.*

(2) *If the CR structure is not circular then the image* $\phi(M) = G/Q$ *is a homogeneous CR manifold with CR structure induced by the complex structure of* $Gr_k(\mathfrak{g}^{\mathbb{C}})$ *and* $\phi \colon M \to \phi(M)$ *is a finite covering.*

Proof. We first need the following Lemma, which in fact was proved in [AHR].

Lemma 4.11. *Let* $G/Q = \phi(G/L)$ *be the image of the anticanonical map. Then* $\dim Q/L \leq 1$*.*

Proof. We need to prove that $\dim \mathfrak{q}/\mathfrak{l} \leq 1$, where $\mathfrak{q} = N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$ is the stability subalgebra of the flag manifold G/Q. Since $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$, it is sufficient to check that $\mathfrak{q} \cap \mathfrak{m} = 0$. Let $v \in \mathfrak{q} \cap \mathfrak{m}$. Then

$$\mathcal{B}(Z, [v, \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]) \subset \mathcal{B}(Z, \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) = \{0\}$$

and in particular

$$\{0\} = \mathcal{B}(Z, [v, \mathfrak{l} + \mathfrak{m}]) = -\mathcal{B}([v, Z], \mathfrak{l} + \mathfrak{m})$$

This means that $v \in N_{\mathfrak{g}}(Z) = \mathfrak{k} = \mathfrak{l} + \mathbb{R}Z$ and hence that $v \in \mathfrak{k} \cap \mathfrak{m} = 0$. □

Let us prove (1). In the case $\dim Z(G) = 1$, the invariant CR structure is standard and the normalizer $N_G(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$ coincides with $L \cdot Z(G)$. Therefore the image $\phi(G/L)$ of the anticanonical map coincides with the flag manifold $F = G/K = G/C_G(Z)$ associated to the contact structure. This proves (1) in this case.

Assume now that G is semisimple and consider a CRF a fibration with S^1 fiber, i.e. a G-equivariant holomorphic map $\pi \colon M = G/L \to F = G/Q$ onto a flag manifold with invariant complex structure J_F. As usual, consider the associated decompositions

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} \qquad \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$$

$$\mathfrak{g} = \mathfrak{q} + \mathfrak{m}' \qquad \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}'^{10} + \mathfrak{m}'^{01}$$

corresponding to the CR structure of M and to the complex structure J_F on $F = G/Q$. Clearly, the subalgebra $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}$ is a subalgebra of the parabolic subalgebra $\mathfrak{p} = \mathfrak{q}^{\mathbb{C}} + \mathfrak{m}'^{01}$.

Since the fiber is one dimensional, we may express $\mathfrak{q}$ as $\mathfrak{q} = \mathfrak{l} + \mathbb{R}Z$ for some element $Z \in \mathfrak{z}(\mathfrak{q})$ and, from the previous observations,

$$[Z, \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}] \subset [Z, \mathfrak{m}^{01}] \subset [\mathfrak{q}^{\mathbb{C}}, \mathfrak{q}^{\mathbb{C}}] + \mathfrak{m}'^{01} \subset [\mathfrak{q}^{\mathbb{C}}, \mathfrak{q}^{\mathbb{C}}] + \mathfrak{m}^{01}$$

where we used the fact that $\mathfrak{m}^{01} \subset \mathfrak{q}^{\mathbb{C}} + \mathfrak{m}'^{01}$ and that $\mathfrak{m}'^{01} \subset (\mathfrak{l}^{\mathbb{C}} + Z)^{-}$.

On the other hand, the semisimple parts of $\mathfrak{l}^{\mathbb{C}}$ and $\mathfrak{q}^{\mathbb{C}}$ coincide and therefore

$$[Z, \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}] \subset \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}$$

In particular $Z \in N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$. Lemma 4.11 implies that $\mathfrak{l} + \mathbb{R}Z = N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$ and that the anticanonical map is a CRF fibration onto the image $\phi(M) = G/N_G([\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}])$ with fiber S^1. The other part of the claim is clear.

To prove (2) it is sufficient to observe that if the CR structure is not circular, the fiber of the anticanonical map cannot be 1-dimensional, because otherwise it would give a CRF fibration with S^1 fiber. Lemma 4.11 shows that in this case $\varphi : G/L \to \phi(G/L)$ is a finite covering. The other part of the claim follows immediately by the holomorphicity and the G-equivariance of ϕ. □

4.5 Any circular CR structure is standard.

Now we will prove that any circular CR structure is standard. Let $(\mathcal{D}, J)$ be a circular CR structure on G/L and let $Z_{\mathcal{D}}$ be a contact element associated to $\mathcal{D}$. By $Z_J = Z'$ we denote the element given in Theorem 4.10 (1) such that the normalizer $\mathfrak{q} = N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$ is of the form

$$\mathfrak{q} = N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) = \mathfrak{l} + \mathbb{R}Z_J$$

¿From Theorem 4.10 and Lemma 4.6, the circular CR structure is standard if and only if $\mathbb{R}Z_{\mathcal{D}} = \mathbb{R}Z_J$. Since $Z_{\mathcal{D}}, Z_J \in C_{\mathfrak{g}}(\mathfrak{l}) \cap (\mathfrak{l})^{-}$, if G/L is a contact manifold of generic type, then $\dim C_{\mathfrak{g}}(\mathfrak{l}) \cap (\mathfrak{l})^{-} = 1$ and hence any circular CR structure is standard. But we will prove now that the same holds also for the special contact manifolds. In fact

Theorem 4.12. *Let G/L be a homogeneous contact manifold of a compact Lie group G. An invariant CR structure $(\mathcal{D}, J)$ on G/L is circular if and only if it is standard.*

Proof. By Proposition 4.7, we may clearly assume that G is semisimple. Furthermore, by the previous remarks, we may assume that G/L is a special contact manifold and we only need to prove that $\mathbb{R}Z_{\mathcal{D}} = \mathbb{R}Z_J$.

Since $C_{\mathfrak{g}}(Z_J) \supset \mathfrak{l} + \mathbb{R}Z_J$ and $C_{\mathfrak{g}}(Z_{\mathcal{D}}) = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}}$, by Lemma 3.7, we have that $C_{\mathfrak{g}}(Z_J) = \mathfrak{l} + \mathbb{R}Z_J$ and hence that Z_J is a contact element too.

It follows immediately from Theorem 3.6 that $\mathfrak{g}$ admits the following orthogonal decomposition

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{a} + \mathfrak{n} = \mathfrak{l} + \mathbb{R}Z_{\mathcal{D}} + \mathfrak{m}$$

where $\mathfrak{a} = \mathfrak{g}(\alpha)$, for some root α of $\mathfrak{g}^{\mathbb{C}}$, and $\mathfrak{l} = C_{\mathfrak{g}}(\mathfrak{a})$. Moreover, $Z_{\mathcal{D}}, Z_J \in \mathfrak{a}$. Since by Lemma 3.7 (3) $\mathfrak{n}^{\mathbb{C}}$ contains no trivial $\mathfrak{l}^{\mathbb{C}}$-module, we may decompose $\mathfrak{m}^{\mathbb{C}}$ as follows

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}\ , \qquad \mathfrak{m}^{10} = \mathfrak{a}^{10} + \mathfrak{n}^{10}\ , \qquad \mathfrak{m}^{01} = \mathfrak{a}^{01} + \mathfrak{n}^{01}\ ,$$

where $\mathfrak{a}^{10} = \mathfrak{a}^{\mathbb{C}} \cap \mathfrak{m}^{10}$, $\mathfrak{n}^{10} = \mathfrak{n}^{\mathbb{C}} \cap \mathfrak{m}^{10}$ and $\mathfrak{a}^{01} = \overline{\mathfrak{a}^{10}}$, $\mathfrak{n}^{01} = \overline{\mathfrak{n}^{10}}$. On the other hand, since $Z_J \in N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}) \cap \mathfrak{a}$, then

$$[Z_J, \mathfrak{a}^{10}] \subset \mathfrak{a}^{10} \qquad [Z_J, \mathfrak{a}^{01}] \subset \mathfrak{a}^{01}$$

and hence Z_J is orthogonal to $\mathfrak{a}^{10} + \mathfrak{a}^{01} = \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{a}^{\mathbb{C}}$, because $\mathfrak{a}^{\mathbb{C}} \simeq \mathfrak{sl}_2$. From this follows that Z_J and $Z_{\mathcal{D}}$ are proportional, because they are two elements of $\mathfrak{a}$, which are both orthogonal to the 2-plane $\mathfrak{a} \cap \mathfrak{m}$. $\square$

5. Classification of CR structures on special contact manifolds.

We describe here all the invariant CR structures $(\mathcal{D}, J)$ on a special contact manifold G/L. Recall that, by Theorem 3.6, G is simple and $L = C_G(\mathfrak{g}(\alpha))$, where either $\alpha = \mu$ is the highest root or $G = G_2$ and $\alpha = \nu$ is a short root. In all cases, $\mathfrak{g}$ admits the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{a} + \mathfrak{n} \tag{5.1}$$

where $\mathfrak{a} = \mathfrak{g}(\alpha)$ and $\mathfrak{l} = C_{\mathfrak{g}}(\mathfrak{a})$.

Let $(\mathcal{D}, J)$ be an invariant CR structure on G/L and let

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} \qquad \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$$

be the associated decompositions. As in the proof of Theorem 4.12, we may decompose $\mathfrak{m}^{10}$ and $\mathfrak{m}^{01}$ as

$$\mathfrak{m}^{10} = \mathfrak{a}^{10} + \mathfrak{n}^{10} \qquad \mathfrak{m}^{01} = \mathfrak{a}^{01} + \mathfrak{n}^{01} \tag{5.2}$$

with $\mathfrak{a}^{10} = \mathfrak{a}^{\mathbb{C}} \cap \mathfrak{m}^{10}$ and $\mathfrak{a}^{01} = \mathfrak{a}^{\mathbb{C}} \cap \mathfrak{m}^{01}$.

Since $\mathfrak{a} \simeq \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{a}^{10} + \mathfrak{a}^{01}$ is the orthogonal complement to $\mathbb{C}Z$ in $\mathfrak{a}^{\mathbb{C}}$, we have that $\dim_{\mathbb{C}}(\mathfrak{a}^{10}) = 1$ and we can write $\mathfrak{a}^{10} = \mathbb{C}Z'$, for some $Z' \in \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{a}^{\mathbb{C}}$.

Consider now a regular element X of $\mathfrak{a}$. There always exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with root system R associated to the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ so that $\mathfrak{a} = \mathfrak{g}(\alpha)$ and $\mathbb{C}X = \mathbb{C}H_\alpha$.

In the case in which $\alpha = \mu$, μ highest root of R^+, we may consider the associated gradation

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 \tag{5.3}$$

where $\mathfrak{g}_i$ are defined in (3.2). Recall that $\mathfrak{g}_0 = \mathbb{C}H_\mu + \mathfrak{g}_0'$, where $\mathfrak{g}_0' = C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}(\mu)) = \mathfrak{l}^{\mathbb{C}}$. The explicit decompositions of the moduli $\mathfrak{g}_{\pm 1}$ into irreducible $\mathfrak{g}_0$-moduli can be found in Table 1, for any simple Lie group. From Table 1 it appears that for $\mathfrak{g}^{\mathbb{C}} \neq A_\ell$, then $\mathfrak{g}_{\pm 1}$ is irreducible, $\dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 1/2 \dim_{\mathbb{C}} \mathfrak{n}^{\mathbb{C}}$ and

$$[\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] = \mathfrak{g}_{\pm 2} \tag{5.4}$$

In case $\mathfrak{g}^{\mathbb{C}} = A_\ell$, then each $\mathfrak{g}_0$-module $\mathfrak{g}_{\pm 1}$ decomposes into two not equivalent irreducible $\mathfrak{g}_0$-moduli: $\mathfrak{g}_{\pm 1} = \mathfrak{g}^{(1)}_{\pm 1} + \mathfrak{g}^{(2)}_{\pm 1}$. Moreover the following properties hold:

$$[\mathfrak{g}^{(i)}_1, \mathfrak{g}^{(i)}_1] = \{0\} = [\mathfrak{g}^{(i)}_{-1}, \mathfrak{g}^{(i)}_{-1}] \qquad [\mathfrak{g}^{(i)}_1, \mathfrak{g}^{(j)}_1] = \mathfrak{g}_2 \qquad [\mathfrak{g}^{(i)}_{-1}, \mathfrak{g}^{(j)}_{-1}] = \mathfrak{g}_{-2} \quad (i \neq j) \tag{5.5}$$

$$[\mathfrak{g}^{(i)}_1, \mathfrak{g}_{-2}] = \mathfrak{g}^{(j)}_{-1} \qquad [\mathfrak{g}^{(i)}_{-1}, \mathfrak{g}_2] = \mathfrak{g}^{(j)}_1 \qquad i \neq j \tag{5.6}$$

The moduli $\mathfrak{g}^{(i)}_1$ and $\mathfrak{g}^{(i)}_{-1}$ $(i \neq j)$ are isomorphic as $\mathfrak{g}'_0$-moduli and for both values of i, $\dim_{\mathbb{C}} \mathfrak{g}^i_{\pm 1} = 1/4 \dim_{\mathbb{C}} \mathfrak{n}^{\mathbb{C}}$.

In the case $\mathfrak{g}^{\mathbb{C}} = G_2$ and $\alpha = \nu$, ν short root, the vector H_ν determines on $\mathfrak{g}^{\mathbb{C}}$ a graded decomposition analogous to (5.3). In fact,

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-3} + \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3 \tag{5.7}$$

with (here $\nu = \varepsilon_1 - \varepsilon_2$)

$$\mathfrak{g}_0 = \mathfrak{g}'_0 + \mathbb{C}H_\nu = < E_{\pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2)}, H_{2\varepsilon_3 - \varepsilon_1 - \varepsilon_2} > + \mathbb{C}H_{\varepsilon_1 - \varepsilon_2}\ , \quad \mathfrak{g}'_0 = C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}(\nu))$$

$$\mathfrak{g}_2 = \mathbb{C}E_{\varepsilon_1 - \varepsilon_2} \quad , \quad \mathfrak{g}_{-2} = \mathbb{C}E_{-\varepsilon_1 + \varepsilon_2} \quad , \quad \mathfrak{g}^{\mathbb{C}}(\nu) = \mathfrak{g}_2 + \mathfrak{g}_{-2} + \mathbb{C}H_\nu$$

$$\mathfrak{g}_1 = < E_{-\varepsilon_2 + \varepsilon_3}, E_{\varepsilon_1 - \varepsilon_3} > \quad , \quad \mathfrak{g}_3 = < E_{-2\varepsilon_2 + \varepsilon_1 + \varepsilon_3}, E_{2\varepsilon_1 - \varepsilon_2 - \varepsilon_3} >$$

$$\mathfrak{g}_{-i} = \overline{\mathfrak{g}_i} \qquad \text{for} \quad i = 1, 3 \tag{5.8}$$

Note that all subspaces $\mathfrak{g}_i$ are irreducible $\mathfrak{g}'_0$ moduli and that the moduli $\mathfrak{g}_j$, $j = \pm 1, \pm 3$, are all equivalent $\mathfrak{g}'_0$-moduli. Furthermore, $[\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] = \mathfrak{g}_{\pm 2}$ and $[\mathfrak{g}_{\pm 3}, \mathfrak{g}_{\pm 3}] = \{0\}$.

For any regular element $X \in \mathfrak{a}^{\mathbb{C}}$, we will call (5.3) and (5.7) the *graded decompositions determined* X.

In this notation, any invariant CR structure on a special contact manifold is described by the following Theorem.

Theorem 5.1. *Let $(M = G/L, \mathcal{D}_Z)$ be a special contact manifold associated to a simple Lie group G. Then:*

a) *if $G \neq SU_{\ell+1}$, then there exists (up to sign) a unique invariant CR structure $(\mathcal{D}_Z, J_W)$ and it is the unique standard CR structure of G/L. It corresponds to the unique invariant complex structure J_F on the flag manifold $F_Z = G/L \cdot T$ (associated to the contact element Z), which is the twistor space of the Wolf space $G/L \cdot G(\mu)$.*

b) *if $G = SU_{\ell+1}$ and hence $M = SU_{\ell+1}/U_{\ell-1}$, then there exist (up to sign) three distinct standard CR structures $(\mathcal{D}_Z, J_W)$, $(\mathcal{D}_Z, J^{st}_o)$, $(\mathcal{D}_Z, J^{st\prime}_o)$ and two families of* *non standard invariant structures* $(\mathcal{D}_Z, J_o)$ *and* $(\mathcal{D}_Z, J_o')$, *which correspond to the following holomorphic subspaces* $\mathfrak{a}^{10}$ *and* $\mathfrak{m}^{10}$ *of* $\mathfrak{m}^{\mathbb{C}}$*:*

(1) $\mathfrak{a}^{10}_{J_W} = \mathbb{C}Z'$, *where* Z' *is a non regular element in* $\mathfrak{a}^{\mathbb{C}}$ *and considering the graded decomposition determined by the (regular) contact element* Z,

$$\mathfrak{a}^{10}_{J_W} = \mathfrak{g}_2 \ , \qquad \mathfrak{m}^{10}_{J_W} = \mathfrak{g}_1 + \mathfrak{g}_2 \ , \qquad \mathfrak{m}^{01}_{J_W} = \mathfrak{g}_{-1} + \mathfrak{g}_{-2} \ ;$$

(2) $\mathfrak{a}^{10}_{J_o^{st}} = \mathbb{C}Z'$, *where* Z' *is a non regular element in* $\mathfrak{a}^{\mathbb{C}}$*; in the graded decomposition determined by the contact element* Z,

$$\mathfrak{a}^{10}_{J_o^{st}} = \mathfrak{g}_2 \ , \ \ \mathfrak{m}^{10}_{J_o^{st}} = \mathfrak{g}_1^{(1)} + \mathfrak{g}_{-1}^{(2)} + \mathfrak{g}_2 \ , \ \ \mathfrak{m}^{01}_{J_o^{st}} = \mathfrak{g}_{-1}^{(1)} + \mathfrak{g}_1^{(2)} + \mathfrak{g}_{-2} \ ;$$

(3) $\mathfrak{a}^{10}_{J_o^{st\prime}} = \mathbb{C}Z'$, *where* Z' *is a non regular element in* $\mathfrak{a}^{\mathbb{C}}$*; in the graded decomposition determined by the contact element* Z,

$$\mathfrak{a}^{10}_{J_o^{st\prime}} = \mathfrak{g}_2 \ , \ \ \mathfrak{m}^{10}_{J_o^{st}} = \mathfrak{g}_1^{(2)} + \mathfrak{g}_{-1}^{(1)} + \mathfrak{g}_2 \ , \ \ \mathfrak{m}^{01}_{J_o^{st}} = \mathfrak{g}_{-1}^{(2)} + \mathfrak{g}_1^{(1)} + \mathfrak{g}_{-2} \ ;$$

(4) $\mathfrak{a}^{10}_{J_o} = \mathbb{C}Z'$, *where* Z' *is a regular element in* $\mathfrak{a}^{\mathbb{C}}$*; in the graded decomposition determined by* Z',

$$\mathfrak{a}^{10}_{J_o} = \mathbb{C}H_\mu \ , \ \ \mathfrak{m}^{10}_{J_o} = \mathbb{C}H_\mu + \mathfrak{g}_1^{(1)} + \mathfrak{g}_{-1}^{(2)} \ , \ \ \mathfrak{m}^{01}_{J_o} = \mathbb{C}Z'' + \mathfrak{g}_{-1}^{(1)} + \mathfrak{g}_1^{(2)} \ ,$$

where Z'' *is some element in* $\mathfrak{g}_2 + \mathfrak{g}_{-2}$ *which is conjugate to* H_μ *w.r.t. the compact form* $\mathfrak{g}$ *of* $\mathfrak{g}^{\mathbb{C}}$*;*

(5) $\mathfrak{a}^{10}_{J_o'} = \mathbb{C}Z'$, *where* Z' *is a regular element in* $\mathfrak{a}^{\mathbb{C}}$*; in the graded decomposition determined by* Z'

$$\mathfrak{a}^{10}_{J_o'} = \mathbb{C}H_\mu \ , \ \ \mathfrak{m}^{10}_{J_o} = \mathbb{C}H_\mu + \mathfrak{g}_1^{(2)} + \mathfrak{g}_{-1}^{(1)} \ , \ \ \mathfrak{m}^{01}_{J_o} = \mathbb{C}Z'' + \mathfrak{g}_{-1}^{(2)} + \mathfrak{g}_1^{(1)} \ ,$$

where Z'' *is some element in* $\mathfrak{g}_2 + \mathfrak{g}_{-2}$ *which is conjugate to* H_μ *w.r.t. the compact form* $\mathfrak{g}$ *of* $\mathfrak{g}^{\mathbb{C}}$.

The CR structures J_o *and* J_o' *admit a CRF fibration with* SU_2 *fiber from* $M = SU_\ell/U_{\ell-1}$ *onto the Wolf space* $G_2(\mathbb{C}^{\ell+1}) = SU_{\ell+1}/S(U_2 \cdot U_{\ell-1})$, *endowed with the complex structure* $\tilde{J}_o$ *or* $-\tilde{J}_o$, *respectively; here* $\tilde{J}_o$ *is the unique complex structure commuting with the quaternionic structure of* $G_2(\mathbb{C}^{\ell+1})$.

The CR structures J_W, J_o^{st} *and* $J_o^{st\prime}$ *are induced by three distinct invariant complex structures* J_F, J_F' *and* J_F'' *on the flag manifold* $F_Z = SU_{\ell+1}/SU_{\ell-1} \cdot T^2$ *which is associated to the contact element* Z *(note:* J_F' *and* J_F'' *are biholomorphic;* J_F *and* J_F' *are not biholomorphic).*

The complex structure J_F *is the canonical complex structure of* F_Z, *considered as twistor space of the Wolf space* $G_2(\mathbb{C}^{\ell+1})$. *The complex structures* J_F' *and* J_F'' *admit a holomorphic fibration on* $(G_2(\mathbb{C}^{\ell+1}), \tilde{J}_o)$ *and* $(G_2(\mathbb{C}^{\ell+1}), -\tilde{J}_o)$, *respectively, with typical fiber* $SU_2/U_1 = S^2$.

Note. In case $G = SU_2$ and hence $M = SU_2$, the cases (1), (2) and (3) of the previous theorem coincide and they correspond to the unique (up to sign) standard

CR structure on $(M, \mathcal{D}_Z)$; cases (4) and (5) coincide up to sign and they correspond to a family on non standard CR structure on $(M, \mathcal{D}_Z)$.

The proof of Theorem 5.1 is done considering two cases. If $\mathfrak{a}^{10} = \mathbb{C}Z'$, there are only two possibilities: *Case 1*: Z' is a regular element of $\mathfrak{a}^{\mathbb{C}}$; *Case 2*: Z' is a not regular (hence nilpotent) element of $\mathfrak{a}^{\mathbb{C}} \simeq \mathfrak{sl}_2(\mathbb{C})$.

In the following two subsections, we are going to determine all invariant CR structures in Case 1 and in Case 2.

5.1 Proof of Theorem 5.1: case in which there exists a regular holomorphic element $Z' \in \mathfrak{a}^{10}$.

Assume first that the special manifold is *associated to a long root μ of $\mathfrak{g}^{\mathbb{C}}$*, i.e. that $\mathfrak{a} = \mathfrak{g}(\mu)$. We may assume that $Z' = H_\mu$ and that it defines a gradation of the form (5.3) for $\mathfrak{g}^{\mathbb{C}}$. Recall that $\mathfrak{l}^{\mathbb{C}} = C_{\mathfrak{g}}(\mathfrak{g}(\mu)) = \mathfrak{g}_0'$.

Hence, using the decomposition (5.2), we have that

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} = \mathfrak{g}^0 + \mathfrak{n}^{10} \subset \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_{-1}$$

since $\mathfrak{n}^{\mathbb{C}}$ is orthogonal to $\mathfrak{a}^{\mathbb{C}} = \mathbb{C}H_\mu + \mathfrak{g}_2 + \mathfrak{g}_{-2}$. Recall that by integrability condition, $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ is a subalgebra.

In case $\mathfrak{g}^{\mathbb{C}} \neq A_\ell$, $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are irreducible $\mathfrak{g}_0$-modules (see Table 1) and hence either $\mathfrak{g}_1$ or $\mathfrak{g}_{-1}$ is included in $\mathfrak{n}^{10}$. Since $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ and $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$, there is no subalgebra included in $\mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_{-1}$ and this contradiction shows that *if the manifold is associated to a long root μ and $\mathfrak{g}^{\mathbb{C}} \neq A_\ell$, this case cannot occur.*

Consider now the case $\mathfrak{g}^{\mathbb{C}} = A_\ell$ and take the decomposition (5.3) determined by $Z' = H_\mu$. Recall that each $\mathfrak{g}_{\pm 1}$ decomposes into two inequivalent irreducible $\mathfrak{g}_0$-moduli $\mathfrak{g}^{(i)}_{\pm 1}$, $i = 1, 2$, which verify (5.4) - (5.6). Since all $\mathfrak{g}_0$-moduli $\mathfrak{g}^{(i)}_{\pm 1}$ have dimension equal to $1/4 \dim_{\mathbb{C}} \mathfrak{n}^{\mathbb{C}}$, the subalgebra $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ is of the form $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} = \mathfrak{g}^0 + \mathfrak{n}^{10}$ where $\mathfrak{n}^{10}$ can be written as

$$\mathfrak{n}^{10} = \mathfrak{g}^{(i)}_1 + \mathfrak{g}^{(j)}_{-1}$$

for some choice of i and j.

If $i = j = 1$, then $\mathfrak{n}^{01} = \mathfrak{g}^{(2)}_1 + \mathfrak{g}^{(2)}_{-1}$, because $\mathfrak{n}^{10} \cap \mathfrak{n}^{01} = \{0\}$. Then $\mathbb{C}H_\mu \in [\mathfrak{n}^{01}, \mathfrak{n}^{01}] \subset \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}$ and this is a contradiction because $H_\mu \in \mathfrak{a}^{10}$. A similar contradiction arises when $i = j = 2$.

Hence only two cases are admissible:

$$\mathfrak{n}^{10} = \mathfrak{g}^{(1)}_1 + \mathfrak{g}^{(2)}_{-1}\,, \qquad \mathfrak{n}^{10} = \mathfrak{g}^{(2)}_1 + \mathfrak{g}^{(1)}_{-1}$$

It is immediate to check that they both define two invariant CR structures $(\mathcal{D}_Z, J_o)$ and $(\mathcal{D}_Z, J_o')$ on $G/L = SU_{\ell+1}/U_{\ell-1}$ associated to the following decompositions

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} \qquad \mathfrak{l}^{\mathbb{C}} = \mathfrak{g}_0' \qquad Z \in \mathfrak{g} \cap (\mathfrak{g}_2 + \mathfrak{g}_{-2}) \tag{5.9}$$

$$\mathfrak{m}^{10}_{J_o} = \mathbb{C}H_\mu + \mathfrak{g}^{(1)}_1 + \mathfrak{g}^{(2)}_{-1} \qquad \mathfrak{m}^{10}_{J_o'} = \mathbb{C}H_\mu + \mathfrak{g}^{(2)}_1 + \mathfrak{g}^{(1)}_{-1} \tag{5.10}$$

The subalgebras $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}_{J_o}$ and $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}_{J'_o}$ are not circular (and hence not standard), because in both cases

$$N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}_{J_o}) = \mathfrak{g} \cap N_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}_0 + \mathfrak{g}^{(1)}_1 + \mathfrak{g}^{(2)}_{-1}) = \mathfrak{g} \cap (\mathfrak{g}_0 + \mathfrak{g}^{(1)}_1 + \mathfrak{g}^{(2)}_{-1}) = \mathfrak{l}$$

$$N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}_{J'_o}) = \mathfrak{g} \cap N_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}_0 + \mathfrak{g}^{(2)}_1 + \mathfrak{g}^{(1)}_{-1}) = \mathfrak{g} \cap (\mathfrak{g}_0 + \mathfrak{g}^{(2)}_1 + \mathfrak{g}^{(1)}_{-1}) = \mathfrak{l}$$

and the claim follows from Theorem 4.10.

On the other hand, the subalgebras $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}_{J_o}$ and $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}_{J'_o}$ are contained in the parabolic subalgebras

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}_{J_o} \subset \mathfrak{p}_{J_o} = \mathfrak{g}_0 + \mathfrak{g}^{(1)}_1 + \mathfrak{g}^{(2)}_{-1} + \mathfrak{g}_2 + \mathfrak{g}_{-2} \tag{5.11}$$

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}_{J'_o} \subset \mathfrak{p}_{J'_o} = \mathfrak{g}_0 + \mathfrak{g}^{(2)}_1 + \mathfrak{g}^{(1)}_{-1} + \mathfrak{g}_2 + \mathfrak{g}_{-2} \tag{5.11'}$$

respectively. For both such parabolic subalgebras the reductive parts are equal to $\mathfrak{r}_{J_o} = \mathfrak{r}_{J'_o} = \mathfrak{q}^{\mathbb{C}}$ where $\mathfrak{q} = \mathfrak{l} + \mathfrak{a}$. Therefore, by Lemma 4.9, the CR structures $(\mathcal{D}, J_o)$ and $(\mathcal{D}, J'_o)$ are not primitive and they admit a CRF fibration on the Wolf space $SU_{\ell+1}/S(U_2 \cdot U_{\ell-1})$ with typical fiber $S(U_2 \cdot U_{\ell-1})/U_{\ell-1} = SU_2$.

It remains to consider the case in which $G = G_2$ and the special manifold is *associated to a short root* ν *of* $\mathfrak{g}^{\mathbb{C}}$. We may assume that $Z' = H_\nu$ and that it defines a gradation of the form (5.7) for $\mathfrak{g}^{\mathbb{C}} = G_2$.

Since $\mathfrak{l}^{\mathbb{C}} = \mathfrak{g}'_0 = C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}(\nu))$, we have that

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} = \mathfrak{g}_0 + \mathfrak{n}^{10} \subset \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_{-1} + \mathfrak{g}_{-3} + \mathfrak{g}_3$$

because $\mathfrak{n}^{\mathbb{C}}$ is orthogonal to $\mathfrak{a}^{\mathbb{C}} = \mathbb{C}H_\mu + \mathfrak{g}_2 + \mathfrak{g}_{-2}$. Since $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} = \mathfrak{g}^0 + \mathfrak{n}^{10}$ is a subalgebra and

$$2 = \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = \dim_{\mathbb{C}} \mathfrak{g}_{\pm 3} = \frac{1}{4} \dim_{\mathbb{C}} \mathfrak{n}^{10}$$

$\mathfrak{n}^{10}$ contains at least two of the four irreducible $\mathfrak{g}_0$-moduli $\mathfrak{g}_{\pm 1}$ and $\mathfrak{g}_{\pm 3}$. The only subalgebra $\mathfrak{g}_0 + \mathfrak{n}^{10}$ with $\mathfrak{n}^{10}$ of this kind is $\mathfrak{g}_0 + \mathfrak{g}_{-3} + \mathfrak{g}_3$ and hence $\mathfrak{n}^{10} = \mathfrak{g}_{-3} + \mathfrak{g}_3$. The same argument would imply that $\mathfrak{n}^{01} = \mathfrak{g}_{-3} + \mathfrak{g}_3 = \mathfrak{n}^{10}$ and this contradicts the hypothesis that $\mathfrak{m}^{10} \cap \overline{\mathfrak{m}^{10}} = \{0\}$.

5.2 Proof of Theorem 5.1: case in which there exists a non regular holomorphic element $Z' \in \mathfrak{a}^{10}$.

Since Z' is non regular, it is a nilpotent element of $\mathfrak{a} = \mathfrak{sl}_2(\mathbb{C})$. Then we may always choose a Cartan subalgebra $\mathbb{C}H_\mu$ of $\mathfrak{a}$ so that $Z' \in \mathbb{C}E_\mu$.

Consider first the case in which the special manifold is *associated to a long root* μ *of* $\mathfrak{g}^{\mathbb{C}}$ and take the gradation (5.3) of $\mathfrak{g}^{\mathbb{C}}$ determined with H_μ. Then $\mathfrak{g}_2 = \mathbb{C}Z' = \mathfrak{a}^{10}$ and hence we have that

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} = \mathfrak{g}'_0 + \mathfrak{g}_2 + \mathfrak{n}^{10} \subset \mathfrak{g}'_0 + \mathfrak{g}_2 + \mathfrak{g}_1 + \mathfrak{g}_{-1}$$

Assume that $\mathfrak{g}^{\mathbb{C}} \neq A_\ell$. Then the $\mathfrak{g}_0'$-moduli $\mathfrak{g}_{\pm 1}$ are irreducible and $[\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] = \mathfrak{g}_{\pm 2}$. This implies that the only subalgebra of $\mathfrak{g}_0' + \mathfrak{g}_2 + \mathfrak{g}_1 + \mathfrak{g}_{-1}$ which properly contains $\mathfrak{g}_0' + \mathfrak{g}_2$ is $\mathfrak{g}_0' + \mathfrak{g}_1 + \mathfrak{g}_2$. Hence

$$\mathfrak{m}^{10} = \mathfrak{g}_1 + \mathfrak{g}_2$$

and it defines the unique CR structure on G/L.

Since $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01} = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_{-1} + \mathfrak{g}_{-2}$, we have that $\mathfrak{l}^{\mathbb{C}} + \mathbb{C}Z = \mathfrak{g}^0$ and that $\mathbb{C}Z = \mathbb{C}H_\mu$. Therefore

$$N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) = N_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}_0' + \mathfrak{g}_{-1} + \mathfrak{g}_{-2})) \cap \mathfrak{g} = \mathfrak{g}_0 \cap \mathfrak{g} = \mathfrak{l} + \mathbb{R}Z$$

and the CR structure is standard because the contact element Z is in the normalizer of $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}$.

Assume now that $\mathfrak{g}^{\mathbb{C}} = A_\ell$ and again consider the decomposition (5.3) determined by H_μ. Note that, when $\mathfrak{g}^{\mathbb{C}} = A_\ell$, the $\mathfrak{g}_0$-moduli $\mathfrak{g}^{(i)}_{\pm 1}$ and $\mathfrak{g}^{(j)}_{\pm 1}$ are equivalent as $\mathfrak{g}_0'$-moduli. In fact, $\mathfrak{g}_1^{(1)} \simeq \mathfrak{g}_{-1}^{(2)}$ and $\mathfrak{g}_1^{(2)} \simeq \mathfrak{g}_{-1}^{(1)}$. Since $\dim_{\mathbb{C}} \mathfrak{g}^{(i)}_{\pm 1} = 1/4 \dim_{\mathbb{C}} \mathfrak{n}^{10}$, the $\mathfrak{g}_0'$-module $\mathfrak{n}^{10}$ must have one of the following five structures:

$$1)\ \mathfrak{n}^{10} = (\mathfrak{g}_1^{(1)})_\varphi + (\mathfrak{g}_{-1}^{(1)})_\psi \qquad 2)\ \mathfrak{n}^{10} = \mathfrak{g}_1^{(1)} + \mathfrak{g}_{-1}^{(2)} \qquad 3)\ \mathfrak{n}^{10} = \mathfrak{g}_1^{(2)} + \mathfrak{g}_{-1}^{(1)}$$

$$4)\ \mathfrak{n}^{10} = \mathfrak{g}_1 \qquad 5)\ \mathfrak{n}^{10} = \mathfrak{g}_{-1}$$

where $\varphi : \mathfrak{g}_1^{(1)} \to \mathfrak{g}_{-1}^{(2)}$ and $\psi : \mathfrak{g}_{-1}^{(1)} \to \mathfrak{g}_1^{(2)}$ are two $\mathfrak{g}_0'$-equivariant homomorphisms and by $(\mathfrak{g}_1^{(1)})_\varphi$ and $(\mathfrak{g}_{-1}^{(1)})_\psi$ we denote the subspaces of the form

$$(\mathfrak{g}_1^{(1)})_\varphi = \{X + \varphi(X)\ : X \in \mathfrak{g}_1^{(1)}\} \qquad (\mathfrak{g}_{-1}^{(1)})_\psi = \{X + \psi(X)\ : X \in \mathfrak{g}_{-1}^{(1)}\}$$

5) cannot occur because in that case $[\mathfrak{n}^{10}, \mathfrak{n}^{10}] = \mathfrak{g}_{-2}$ and this contradicts the fact that $\mathfrak{g}_0' + \mathfrak{n}^{10} + \mathfrak{g}_2$ is a subalgebra.

We claim that also case 1) may not occur. In fact, φ is either trivial or an isomorphism. In case φ is an isomorphism, the subspace $[\mathfrak{n}^{10}, \mathfrak{n}^{10}] \mod \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$ contains non trivial elements of the form

$$[X + \varphi(X), Y + \psi(Y)] \mod \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 = [\varphi(X), Y] \in \mathfrak{g}_{-2}$$

and this is a contradiction with the fact that $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ is a subalgebra included in $\mathfrak{g}_0' + \mathfrak{g}_1 + \mathfrak{g}_2$. Therefore, if case 1) occurred, $\mathfrak{n}^{10} = \mathfrak{g}_1^{(1)} + (\mathfrak{g}_{-1}^{(1)})_\psi$. Now, for any $X \in \mathfrak{g}_1^{(1)}$ we may consider an element $Y \in (\mathfrak{g}_{-1}^{(1)})_\psi$ so that

$$[X, Y] = \lambda H_\mu \mod \mathfrak{g}_0' + \mathfrak{g}_2$$

for some $\lambda \neq 0$. This gives a contradiction with the fact that $\mathfrak{g}_0' + \mathfrak{n}^{10} + \mathfrak{g}_2$ is a subalgebra and the claim is proved.

It is immediate to check that, for the cases 2), 3) and 4), we obtain three subalgebras

$$\mathfrak{g}_0' + \mathfrak{g}_1^{(1)} + \mathfrak{g}_{-1}^{(2)} + \mathfrak{g}_2 \tag{5.12}$$

$$\mathfrak{g}_0' + \mathfrak{g}_1^{(2)} + \mathfrak{g}_{-1}^{(1)} + \mathfrak{g}_2 \tag{5.12'}$$

$$\mathfrak{g}' + \mathfrak{g}_1 + \mathfrak{g}_2 \tag{5.13}$$

They determine three distinct CR structures $(\mathcal{D}, J_o^{st})$, $(\mathcal{D}, J_o^{st\prime})$ and $(\mathcal{D}, J_W)$, respectively. For any of the three subalgebras (5.12), (5.12') and (5.13), the normalizer $N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10})$ contains $\mathfrak{g}_0 \cap \mathfrak{g} = \mathfrak{l} + \mathbb{R}Z$ and hence it is strictly larger then $\mathfrak{l}$. By Theorem 4.10, this implies that all those CR structures are circular and hence standard.

Observe also that if $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ equals either (5.12) or (5.12'), then $\mathfrak{p} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} + \mathfrak{g}^{\mathbb{C}}(\mu)$ is a parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$. The parabolic subgroup $P \subset G^{\mathbb{C}}$, which is generated by $\mathfrak{p}$ is the parabolic subgroup associated either to a complex structure $\tilde{J}_o$ or to its opposite $-\tilde{J}_o$ on $G_2(\mathbb{C}^{\ell+1})$, which commutes with the quaternionic structure. Therefore, $(\mathcal{D}, J_o^{st})$, $(\mathcal{D}, J_o^{st\prime})$ admit a CRF fibration on $(G_2(\mathbb{C}^{\ell+1}), \tilde{J}_o)$ and $(G_2(\mathbb{C}^{\ell+1}), -\tilde{J}_o)$, respectively.

It remains to consider the case in which $G = G_2$ and the special manifold is *associated to a short root* ν *of* $\mathfrak{g}^{\mathbb{C}}$. Consider the decomposition (5.7) determined by H_ν so that $\mathbb{C}Z' = \mathbb{C}E_\nu = \mathfrak{g}_2$.

In analogy with the previous discussions, we have that

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} = \mathfrak{g}_0' + \mathfrak{a}^{10} + \mathfrak{n}^{10} \subset \mathfrak{g}_0' + \mathfrak{g}_1 + \mathfrak{g}_{-1} + \mathfrak{g}_{-3} + \mathfrak{g}_3 + \mathfrak{g}_2$$

because $\mathfrak{n}^{\mathbb{C}}$ is orthogonal to $\mathfrak{a}^{\mathbb{C}} = \mathbb{C}H_\mu + \mathfrak{g}_2 + \mathfrak{g}_{-2}$. ¿From the fact that $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ is a subalgebra, we claim that $\mathfrak{g}_3 \subset \mathfrak{n}_1$. In fact, for any element $X \in \mathfrak{n}^{10}$ consider the decomposition:

$$X = X_{-3} + X_{-1} + X_1 + X_3 \qquad X_i \in \mathfrak{g}_i$$

Then, one of the four vectors X, $X' = [E_\mu, X]$, $X'' = [E_\mu, [E_\mu, X]]$, $X''' = [E_\mu, [E_\mu, [E_\mu, X]]]$ is a non trivial element of $\mathfrak{g}_3$ and it belongs to $\mathfrak{n}^{10}$. Since $\mathfrak{g}_3$ is $\mathfrak{g}_0'$-irreducible, the claim follows.

Similarly, we claim that $\mathfrak{g}_1 \subset \mathfrak{n}^{10}$. In fact, take any element $X \in \mathfrak{n}^{10}$ which has a decomposition of the form

$$X = X_{-3} + X_{-1} + X_1 \qquad X_i \in \mathfrak{g}_i$$

Then, either X or $X' = [E_\mu, X]$ or $X'' = [E_\mu, [E_\mu, X]]$ is a non trivial element of $\mathfrak{g}_1 + \mathfrak{g}_3$, with non vanishing projection on $\mathfrak{g}_1$. This implies that $\mathfrak{g}_1 \cap \mathfrak{n}^{10} \neq \{0\}$ and hence that $\mathfrak{g}_1 \subset \mathfrak{n}^{10}$. Since $\dim_{\mathbb{C}}(\mathfrak{g}_1 + \mathfrak{g}_3) = \dim_{\mathbb{C}} \mathfrak{n}^{10}$, we conclude that $\mathfrak{n}_1 = \mathfrak{g}_1 + \mathfrak{g}_3$ and hence that $\mathfrak{m}^{10} = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3$. This defines an integrable CR structure and it is simple to check that $N_{\mathfrak{g}}([\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]) = \mathfrak{g}_0 \cap \mathfrak{g} = \mathfrak{l} + \mathbb{R}iH_\mu$. Since this normalizer contains properly $\mathfrak{l}$, by Theorem 4.10, this CR structure is circular and hence standard.

6. Classification of non circular CR structures.

6.1 Case of non simple Lie group.

¿From §4, the classification of the invariant CR structures can be now reduced to the analysis of non circular CR structures.

Lemma 6.1. *Let $(G/L, \mathcal{D}, J)$ be a homogeneous CR manifold with non circular CR structure. Then G is either simple or of the form $G = G_1 \times G_2$, with each G_i simple.*

Moreover, if $G = G_1 \times G_2$ and $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ is the decomposition associated to the contact structure $\mathcal{D}$, then $\mathfrak{m}^{\mathbb{C}}$ decomposes into $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}_1 + \mathfrak{m}_2$, with $\mathfrak{m}_i \in \mathfrak{g}_i^{\mathbb{C}}$ and each $\mathfrak{m}_i$ contains at least a 1-dimensional irreducible $\mathfrak{l}^{\mathbb{C}}$-modules.

Proof. Consider a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ and let R be the corresponding root system of G. If $\theta|_{\mathfrak{h}} \overset{\text{def}}{=} \mathcal{B}(Z, *)|_{\mathfrak{h}}$ is parallel to some root α, then this root belongs to some summand $\mathfrak{g}_1$ of $\mathfrak{g}$. Hence, $\mathfrak{k} = C_{\mathfrak{g}}(Z)$ contains all other simple summands of $\mathfrak{g}$ and the same holds for $\mathfrak{l}$. By effectivity, this implies that $\mathfrak{g} = \mathfrak{g}_1$.

If $\theta|_{\mathfrak{h}} \overset{\text{def}}{=} \mathcal{B}(Z, *)|_{\mathfrak{h}}$ is not parallel to any root α, it can be assumed to be difference of two (but no more) roots β and γ. If they both belong to the same summand $\mathfrak{g}_1$, then $\mathfrak{g} = \mathfrak{g}_1$ as before. Assume that they belong to two different summands $\mathfrak{g}_1$ and $\mathfrak{g}_2$. The same arguments of before show that this time $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Moreover it is clear that $\pm(\alpha, \beta)$ are the only pairs of roots which are congruent modulo $\theta|_{\mathfrak{h}}$. This also implies that the only pair of $\mathfrak{k}$-modules $\mathfrak{m}'_1$ and $\mathfrak{m}'_2$ which are $\mathfrak{l}$-equivalent consists in those spanned by $E_\alpha, E_{-\alpha}$ and $E_\beta, E_{-\beta}$, respectively. Therefore $\mathfrak{m}'_1 \subset \mathfrak{m}_1$ and $\mathfrak{m}'_2 \subset \mathfrak{m}_2$ are 1-dimensional and $\mathfrak{l}$-irreducible. $\square$

Proposition 6.2. *Let G/L a contact manifold with $G = G_1 \times G_2$, where each G_i is simple. Let also $G/K = G_1/K_1 \times G_2/K_2$ be the flag manifold associated to the contact structure. Then:*

1. *G/L admits a non-standard CR structure if and only if there exists a painted Dynkin diagram of a complex structure on each G_i/K_i, which contains one black node not connected to any white node and such that, if deleted, all other black nodes are isolated;*
2. *if $G \neq SU_2 \times SU_2$ and $(G/L, \mathcal{D}_Z)$ admits a non-standard CR structure, then G/L admits a CRF fibration with fiber $SU_2 \times SU_2/T^1$;*
3. *let $(\mathcal{D}_Z, J)$ be an invariant CR structure on $G/L = SU_2 \times SU_2/T^1$ and let us denote by μ and μ' the roots of the first and the second copy of $\mathfrak{su}_2$ in $\mathfrak{g}$; then there exists a Cartan subalgebra $\mathfrak{h} = \mathbb{C}H_\mu + \mathbb{C}H_{\mu'}$ so that*

$$Z = iH_\mu - iH_{\mu'} \qquad \mathfrak{l}^{\mathbb{C}} = \mathbb{C}(H_\mu + H_{\mu'})$$

$$\mathfrak{m}^{10} = \mathbb{C}(aE_\mu + bE_{\mu'}) + \mathbb{C}(aE_{-\mu} + bE_{-\mu'})$$

for some $[a : b] \in \mathbb{C}P^1$ so that $|a|^2 - |b|^2 \neq 0$. J is standard if and only if $a \cdot b = 0$.

Proof. Consider the usual decomposition of the Lie algebra

$$\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$$

Note that if we denote by $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}]$, $\mathfrak{k}'_i = [\mathfrak{k}_i, \mathfrak{k}_i]$, $i = 1, 2$, then

$$\mathfrak{g} = \mathfrak{l}' + \mathfrak{z}(\mathfrak{l}) + \mathbb{R}Z + \mathfrak{m} = \mathfrak{k}'_1 + \mathfrak{k}'_2 + \mathfrak{z}(\mathfrak{l}) + \mathbb{R}Z + \mathfrak{m}_1 + \mathfrak{m}_2$$

¿From the proof of Lemma 6.1, we have that if G/L admits a non standard CR structure, then there exists exactly two $\mathfrak{l}$-equivalent irreducible moduli in $\mathfrak{m}^{\mathbb{C}}$ and they are of the form

$$\mathfrak{n}_1 =< E_{\alpha_1}, E_{-\alpha_1} >\subset \mathfrak{m}_1^{\mathbb{C}} \qquad \mathfrak{n}_2 =< E_{\alpha_2}, E_{-\alpha_2} >\subset \mathfrak{m}_2^{\mathbb{C}}$$

for two suitable roots α and β of G_1 and G_2, respectively. This means that for any integrable complex structure J on $\mathfrak{m}$ the associated eigenspaces $\mathfrak{m}^{10}$ and $\mathfrak{m}^{01}$ are of the form

$$\mathfrak{m}^{01} = \mathbb{C}(\lambda E_{\alpha_1} + \mu E_{\alpha_2}) + \mathfrak{m}_1'^{01} + \mathfrak{m}_2'^{01}$$

where $a, b \in \mathbb{C}$ are such that $aE_{\alpha_1} + bE_{\alpha_2}$ is linearly independent on $\overline{aE_{\alpha_1} + bE_{\alpha_2}}$ and each $\mathfrak{m}_i'^{01}$ is in $\mathfrak{m}_i$.

Consider now the parabolic subalgebras $\mathfrak{p}_i = \mathfrak{k}^{\mathbb{C}} + \mathfrak{n}_i + \mathfrak{m}_i'^{01}$, for $i = 1, 2$ and let $\tilde{\mathfrak{k}}_i^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{n}_i$ the corresponding reductive parts. From Lemma 4.9, they determine two flag manifolds $G/\tilde{K}_i = G^{\mathbb{C}}/P_i$, with invariant complex structures so that the projection

$$\pi \colon G/L \to G_1/\tilde{K}_1 \times G_2/\tilde{K}_2$$

is a CRF fibration.

The typical fiber of this fibration is, up to covering,

$$\tilde{K}_1 \times \tilde{K}_2/L \simeq SU(2) \times SU(2)/T^1$$

as it can be checked by looking at the Lie algebras. ¿From this, (2) follows immediately.

To conclude the proof of (1), observe that $\mathbb{C}E_{\alpha_i} + \mathfrak{m}'^{01}$, $i = 1, 2$, is an eigenspace for an integrable complex structure on G_i/K_i. The corresponding painted Dynkin diagram of this complex structure must contain a black node (associated with the root α_i) which is not connected to any white root, because any complex line $\mathbb{C}E_{\alpha_i}$ must be a 1-dimensional irreducible $\mathfrak{k}_i$-module in $\mathfrak{m}_i$. Moreover all nodes which are not connected to the node associated to the root α_i correspond to roots which belong to the centralizer $C_{\mathfrak{g}}((Z)$, where $Z = iH_{\alpha_1} - iH_{\alpha_2} = \mathcal{B} \circ \theta$ and hence they must belong to the subdiagram of the white nodes.

Vice versa, if the flag manifolds G_i/K_i admit painted Dynkin diagrams which verifies the conditions given in (1) exists, the associated complex structure can be used to construct a non standard CR structure as described above.

It remains to prove (3). Since Z is a contact element for $SU_2 \times SU_2/T^1$, it is a regular element a Cartan subalgebra of $\mathfrak{sl}_2 + \mathfrak{sl}_2$ and hence we may assume that $Z = iH_\mu - iH_{\mu'}$; in this case, $\mathfrak{l}^{\mathbb{C}} = C_{\mathfrak{g}^{\mathbb{C}}}(Z) \cap Z^- = \mathbb{C}(H_\mu + H_{\mu'})$ and any $\mathfrak{l}^{\mathbb{C}}$-module in $\mathfrak{m}^{\mathbb{C}}$ is of the form $\mathbb{C}(aE_\mu + bE_{\mu'})$ or $\mathbb{C}(cE_{-\mu} + dE_{-\mu'})$. Since $\mathfrak{m}^{10}$ is 2-dimensional and it is so that $[\mathfrak{m}^{10}, \mathfrak{m}^{10}] \subset \mathfrak{m}^{10} + \mathfrak{l}^{\mathbb{C}}$, we obtain that it must be of the form

$$\mathfrak{m}^{10} = \mathbb{C}(aE_\mu + bE_{\mu'}) + \mathbb{C}(aE_{-\mu} + bE_{-\mu'})$$

¿From $\overline{\mathfrak{m}^{10}} = \mathfrak{m}^{01}$, we have that $\mathfrak{m}^{01} = \mathbb{C}(\bar{a}E_\mu + \bar{b}E_{\mu'}) + \mathbb{C}(\bar{a}E_{-\mu} + \bar{c}E_{-\mu'})$. Since $\mathfrak{m}^{10} \cap \mathfrak{m}^{01} = \{0\}$ we conclude that

$$\det \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} = |a|^2 - |b|^2 \neq 0$$

For each such pair (a,b) (defined up to multiple) the subalgebra $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ defines an integrable CR structure on G/L. $N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) \supsetneq \mathfrak{l}$ if and only if $a \cdot b = 0$ and hence these are the only cases corresponding to a standard CR structure. In all other cases, it can be checked that there exists no proper parabolic subalgebra $\mathfrak{p} \supset \mathfrak{l}^{\mathbb{C}}$ which verifies conditions a), b) and c) of Lemma 4.9. This concludes the proof. □

Lemma 6.1 and Proposition 6.2 reduces the classification of non-standard CR structures to the analysis of homogeneous spaces of simple compact Lie groups.

For this purpose, we are going to consider two mutually exclusive cases.

Case 1: G/L is of generic type and the contact form $\theta = \mathcal{B} \circ Z$, is proportional to a root, when restricted to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$;

Case 2: G/L is of generic type and the contact form $\theta = \mathcal{B} \circ Z$ is proportional to no root, when restricted to the Cartan subalgebra.

6.2 Case when the contact form is proportional to a root.

This first case is quite easily solved, by considering the list of all compact simple Lie groups and checking for each of them the contact elements with the desired property. Note that if $\theta = \mathcal{B} \circ Z$ is proportional to a long root of the compact simple group G or if $G = G_2$ and θ is proportional to a short root, then G/L is a special contact manifold. Therefore it is sufficient to consider only those groups which have roots of different length and which are not G_2. Therefore we get that:

Proposition 6.3. *Let $(G/L, \mathcal{D})$, G simple, be a contact manifold with associated contact element Z such that $\theta = \mathcal{B} \circ Z|_{\mathfrak{h}}$ is parallel to a root. If G/L is not a special contact manifold, then:*

(1) *G/L is $SO(2n+1)/SO(2n-1)$, $Sp(n)/Sp(1) \times Sp(n-2)$ or $F_4/SO(7)$ and θ is proportional to a short root of G;*

(2) *for any invariant CR structure $(\mathcal{D}_Z, J)$, the associated decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$ is one of the following table:*

G/L	θ	$\mathfrak{m}^{01}$	*Space of parameters*
$\frac{SO(2n+1)}{SO(2n-1)}$	ε_1	$a\mathfrak{m}(\varepsilon_1+\varepsilon_2) + b\mathfrak{m}(-\varepsilon_1+\varepsilon_2)$	$[a:b] \in \mathbb{C}P^1$ $\lvert a\rvert^2 - \lvert b\rvert^2 \neq 0$
$\frac{Sp(n)}{Sp(1)\times Sp(n-2)}$	$\varepsilon_1+\varepsilon_2$	$[a^2\mathfrak{m}(2\varepsilon_1) + b^2\mathfrak{m}(-2\varepsilon_2)] \oplus$ $[a\mathfrak{m}(\varepsilon_1+\varepsilon_3) + b\mathfrak{m}(-(\varepsilon_2+\varepsilon_3))]$	$[a:b] \in \mathbb{C}P^1$ $\lvert a\rvert^2 - \lvert b\rvert^2 \neq 0$
$\frac{F_4}{SO(7)}$	ε_1	$a^2\mathfrak{m}(\varepsilon_1+\varepsilon_2) + b^2\mathfrak{m}(-\varepsilon_1+\varepsilon_2) \oplus$ $[a\mathfrak{m}(1/2(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)) +$ $b\mathfrak{m}(-1/2(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)]$	$[a:b] \in \mathbb{C}P^1$ $\lvert a\rvert^2 - \lvert b\rvert^2 \neq 0$

where $\mathfrak{m}(\alpha)$ denotes the $\mathfrak{k}^{\mathbb{C}}$-irreducible module with $\mathfrak{h}$-weight $\alpha \in R_{\mathfrak{m}}$ and $[a\mathfrak{m}(\alpha) + b\mathfrak{m}(\alpha')]$ denotes the $\mathfrak{l}^{\mathbb{C}}$-module generated by the highest weight vector $aE_\alpha + bE_{\alpha'}$;

(3) *the standard CR structures in (2) are exactly those corresponding to pairs of parameters with $a \cdot b = 0$;*

(4) *if $a \cdot b \neq 0$, any CR structure of point (2) is primitive.*

Proof. For any choice of the group G, there is only one possibility for the contact form θ. Once θ is given, the decomposition $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ is deducible from Table 2 in the Appendix. It remains to find all the decompositions $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$ into two $\mathfrak{l}^{\mathbb{C}}$-modules such that: a) $\mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}}$; b) $\mathfrak{m}^{10} \cap \mathfrak{m}^{01}$; c) $[\mathfrak{m}^{01}, \mathfrak{m}^{01}] \subset \mathfrak{m}^{01} + \mathfrak{l}^{\mathbb{C}}$. ¿From Table 2 in the Appendix, one may find all irreducible $\mathfrak{l}^{\mathbb{C}}$-moduli in $\mathfrak{m}^{\mathbb{C}}$ and hence to determine that the only $\mathfrak{l}^{\mathbb{C}}$-moduli which have half the dimension of $\mathfrak{m}^{\mathbb{C}}$ and which verify conditions a) and c) are just those given in the third colomun of the table in (2). Condition b) implies that the admissible cases are exactly those such that $\det \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \neq 0$ and this justifies the fourth column of the table.

(3) follows from the fact that, in all cases listed in the table of (2), $N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) \subsetneq \mathfrak{l}$ only if $a \cdot b = 0$.

(4) is proved by checking that in no case, when $a \cdot b \neq 0$, there exists a proper parabolic subalgebra $\mathfrak{p} \supset \mathfrak{l}^{\mathbb{C}}$ which verifies the conditions of Lemma 4.9. □

6.3 Case when the contact form is not proportional to a root.

In all this section we will suppose G simple, that $(G/L, \mathcal{D})$ has an associated contact element Z such that $\theta \overset{\text{def}}{=} \mathcal{B} \circ Z|_{\mathfrak{h}}$ is not parallel to any root and that $(\mathcal{D}, J)$ is a non standard CR structure on G/L.

We also need to introduce the following notation. Let R be the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} + \mathbb{C}Z)$, with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ contained in $\mathfrak{k}^{\mathbb{C}}$. Then let us denote by R_o the roots corresponding to the root vectors in $\mathfrak{k}^{\mathbb{C}}$ and let $R' = R \setminus R_o$: it is known that the root vectors E_α with $\alpha \in R'$ generate $\mathfrak{m}^{\mathbb{C}} = \mathfrak{k}^{-}$. Then let us define

$$R_J = \{\alpha \in R' \colon J(E_\alpha) = \pm i E_\alpha\}$$

Note that J is standard if and only if $R_J = R'$. Then let

$$R_{\mathfrak{e}} \overset{\text{def}}{=} R' \setminus R_J$$

$$\tilde{R}_{\mathfrak{e}} \overset{\text{def}}{=} R \cap span_{\mathbb{R}} < R_{\mathfrak{e}} >$$

$$\mathfrak{e} \overset{\text{def}}{=} \sum_{\beta \in R_{\mathfrak{e}}} \mathbb{C} E_\beta$$

Lemma 6.4.

(1) *$R_J = -R_J$ and $R_{\mathfrak{e}} = -R_{\mathfrak{e}}$;*

(2) *for any $\alpha \in R_{\mathfrak{e}}$ there exists exactly one root $\beta \in R_{\mathfrak{e}}$ which is θ-congruent to α;*

(3) *for $\alpha \in R_{\mathfrak{e}}$ there exist exactly one $\lambda \neq 0$ and $\mu \neq 0$ such that, for the $\beta \in R_{\mathfrak{e}}$ which is θ-congruent to α (see Cor. 3.10),*

$$e_{\alpha,\beta} \overset{\text{def}}{=} E_\alpha + \lambda E_\beta \in \mathfrak{m}^{10} \qquad f_{\alpha,\beta} \overset{\text{def}}{=} E_\alpha + \mu E_\beta \in \mathfrak{m}^{01} \ . \tag{6.2}$$

(4) *$(R_J^{\pm} + R_o) \cap R \subset R_J^{\pm}$ and $(R_{\mathfrak{e}} + R_o) \cap R \subset R_{\mathfrak{e}}$;*

(5) *$(R_J^{\pm} + R_{\mathfrak{e}}) \cap R \subset R_J^{\pm} \cup R_{\mathfrak{e}} \cup R_o$.*

Proof. (1) is clear. To see (2), (3) and (4), observe that $\alpha \in R_J$ if and only if E_α belongs to an irreducible $\mathfrak{k}^{\mathbb{C}}$-module which is also J-invariant; hence (2), (3) and (4) follow from Corollary 4.3 and Corollary 3.11.

The proof of (5) is the following. Let $\gamma \in R_J^+$ and $\alpha, \beta \in R_{\mathfrak{e}}$ a pair of two θ-congruent roots. If $\gamma + \alpha \in R_J^-$, consider the element $f_{-\alpha,-\beta} \in \mathfrak{m}^{01}$ as defined in (6.2). Then

$$[E_{\gamma+\alpha}, f_{-\alpha,-\beta}] = CE_\gamma + X \in \mathfrak{m}^{01}$$

for some $C \neq 0$ and $X \notin \mathbb{C}E_\gamma$. This implies that $\gamma \in R_J^-$: contradiction. $\square$

For any $\alpha \in R_{\mathfrak{e}}$ we will call $\mathfrak{e}$*-dual of* α the unique root $\beta \in R_{\mathfrak{e}}$ which is θ-congruent to α.

Lemma 6.5. *Let α and α' be an $\mathfrak{e}$-dual pair. Then:*

(1) *$\tilde{R} = R \cap span_{\mathbb{R}}(\alpha, \alpha')$ is $A_1 \cup A_1$*
(2) *$\alpha - \alpha'$ and $\alpha \pm \alpha' \notin R$.*

Proof. For (1) we have to show that if $\tilde{R} \neq A_1 \cup A_1$. (2) is an immediate corollary of (1).

Suppose that $\tilde{R} = A_2, B_2$ or G_2. Since $\alpha - \alpha'$ is proportional to no root, the only possibilities for α and α' are as in the picture (i.e. α short, α' long and forming an obtuse angle; or vice versa).

It follows that in all these cases $\alpha + \alpha' = \beta \in R$.

Let us first discuss the case $\tilde{R} = A_2$. In this case β is orthogonal to $\theta = \alpha - \alpha'$ and hence $\beta \in R_o$. Using the convention for representing the roots of a system of type A_2 as described in the Appendix, there is no lost of generality if we denote α and α' as $\alpha = \varepsilon_0 - \varepsilon_2$ and $\alpha' = \varepsilon_2 - \varepsilon_1$. Therefore we may assume that

$$\theta = (\varepsilon_0 - \varepsilon_2) - (\varepsilon_2 - \varepsilon_1) = \varepsilon_0 + \varepsilon_1 - 2\varepsilon_2$$

Then $\mathfrak{l} = C_{\mathfrak{g}}(Z)$, $Z = \mathcal{B}^{-1} \circ \theta$ contains the subalgebra

$$\mathfrak{l}' = \mathbb{C}H_{\varepsilon_0 - \varepsilon_1} + \mathbb{C}E_{\varepsilon_0 - \varepsilon_1} + \mathbb{C}E_{\varepsilon_1 - \varepsilon_0}$$

At the same time, by Lemma 6.4 (3), $\mathfrak{m}^{01}$ contains the element

$$f_{\varepsilon_0 - \varepsilon_2, \varepsilon_2 - \varepsilon_1} = E_{\varepsilon_0 - \varepsilon_2} + \mu E_{\varepsilon_2 - \varepsilon_1}$$

for some $\mu \neq 0$. Since $\mathfrak{m}^{01}$ is $\mathfrak{l}^{\mathbb{C}}$-invariant, $\mathfrak{m}^{01}$ has to contain the subspace

$$\mathbb{C}(E_{\varepsilon_0 - \varepsilon_2} + \mu E_{\varepsilon_2 - \varepsilon_1}) + \mathbb{C}(E_{\varepsilon_1 - \varepsilon_2} - \mu E_{\varepsilon_2 - \varepsilon_0})$$

This implies that

$$[E_{\varepsilon_0 - \varepsilon_2} + \mu E_{\varepsilon_2 - \varepsilon_1}, E_{\varepsilon_1 - \varepsilon_2} - \mu E_{\varepsilon_2 - \varepsilon_0}] = \mu(-H_{\varepsilon_0 - \varepsilon_2} + H_{\varepsilon_2 - \varepsilon_1}) \in [\mathfrak{m}^{01}, \mathfrak{m}^{01}]$$

By integrability, $[\mathfrak{m}^{01}, \mathfrak{m}^{01}] \subset \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}$ and hence we should have

$$-H_{\varepsilon_0 - \varepsilon_2} + H_{\varepsilon_2 - \varepsilon_1} \in l^{\mathbb{C}}$$

But this cannot be because $-H_{\varepsilon_0-\varepsilon_2} + H_{\varepsilon_2-\varepsilon_1}$ is not orthogonal to Z: this contradiction shows that the case $\tilde{R} = A_2$ may not occur.

Let us now suppose that $\tilde{R} = B_2$ or $\tilde{R} = C_2$. In this case, $\beta = \alpha + \alpha'$ is not orthogonal to $\theta = \alpha - \alpha'$ and, moreover,

$$(\beta + \mathbb{R}\theta) \cap R = \emptyset \ .$$

¿From this we conclude that $\beta \in R_J = R \setminus R_{\mathfrak{e}}$. Changing the sign of α, α' if necessary, we may assume that $E_\beta \in \mathfrak{m}^{10}$, that is $JE_\beta = iE_\beta$.

Let us consider the vector $f_{\alpha,\alpha'} = E_\alpha + \mu E'_\alpha \in \mathfrak{m}^{01}$ (see Lemma 6.4 (3)). Then $\overline{E_\alpha + \mu E'_\alpha} = E_{-\alpha} + \bar{\mu} E_{-\alpha'} \in \mathfrak{m}^{10}$ and by integrability of J its commutator with E_β is also in $\mathfrak{m}^{10}$. Therefore

$$[E_{-\alpha} + \bar{\mu} E_{-\alpha'}, E_{\alpha+\alpha'}] = N_{-\alpha,\alpha+\alpha'} E_{\alpha'} + \bar{\mu} N_{-\alpha',\alpha+\alpha'} E_\alpha \in \mathfrak{m}^{10}$$

By Lemma 6.4 (3), we get that the coefficient λ in $e_{\alpha,\alpha'}$ is

$$\lambda = \frac{N_{-\alpha,\alpha+\alpha'}}{\bar{\mu} N_{-\alpha',\alpha+\alpha'}} \tag{6.3}$$

We recall that for any two roots α, β, the integer $N_{-\alpha,\beta}$ equals

$$N_{-\alpha,\beta} = \pm(p+1)$$

where $p \geq 0$ is the maximal integer such that $\beta + p\alpha \in \tilde{R}$. In our case, we obtain from (6.3) that if $\tilde{R} = G_2$, $\lambda\bar{\mu} = \pm 3$, while if $\tilde{R} = B_2$, $\lambda\bar{\mu} = \pm 2$.

On the other hand, by integrability, we also have that

$$[e_{\alpha,\alpha'}, \overline{f_{\alpha,\alpha'}}] = [E_\alpha + \lambda E'_\alpha, E_{-\alpha} + \bar{\mu} E_{-\alpha'}] = H_\alpha + \lambda\bar{\mu} H_{\alpha'} \in \mathfrak{l}^{\mathbb{C}}$$

This means that $(H_\alpha + \lambda\bar{\mu} H_{\alpha'}, \theta) = 0$, i.e. that

$$2\frac{(\theta,\alpha)}{(\alpha,\alpha)} + 2\lambda\bar{\mu}\frac{(\theta,\alpha')}{(\alpha',\alpha')} = 0$$

Using $\theta = \alpha - \alpha'$, we obtain

$$2 - <\alpha'|\alpha> + \lambda\bar{\mu}[-2 + <\alpha|\alpha'>] = 0$$

In case $\tilde{R} = B_2$, $<\alpha'|\alpha> = -2$ and $<\alpha|\alpha'> = -1$ so that $\lambda\bar{\mu} = 3/4$; in case $\tilde{R} = G_2$, $<\alpha'|\alpha> = -3$ and $<\alpha|\alpha'> = -1$ so that $\lambda\bar{\mu} = -3/5$. In both cases we get a contradiction with the previously determined values for $\lambda\bar{\mu}$. $\square$

Consider now the root subsystem $\tilde{R}_e \stackrel{\text{def}}{=} R \cap span_{\mathbb{R}} < R_{\mathfrak{e}} >$.

Lemma 6.6. *If $\tilde{R}_{\mathfrak{e}}$ is not of the form $A_1 \cup A_1$, then $\tilde{R}_{\mathfrak{e}}$ is an indecomposable root subsystem.*

Proof. Note that if rank$\tilde{R}_{\mathfrak{e}} = 2$, by Lemma 6.5, $\tilde{R}_{\mathfrak{e}} = A_1 \cup A_1$. Therefore we may suppose that rank $\tilde{R}_{\mathfrak{e}} > 2$.

Suppose that $\tilde{R}_{\mathfrak{e}} = R_1 \cup R_2$ with R_1 orthogonal to R_2. Let $\alpha \in R_1 \cap R_{\mathfrak{e}}$, $\alpha' \in R_2 \cap R_{\mathfrak{e}}$ and let β, β' the corresponding $\mathfrak{e}$-dual roots; we may also suppose that they are not contained in a rank two subsystem. Since θ cannot be contained in the span of R_1, it is clear that $\beta \in R_2$ and that $\beta' \in R_1$; in this case we have that

$$\mathbb{R}(\alpha - \beta) = \mathbb{R}(\alpha' - \beta')$$

only if $\alpha + \rho\beta' = \rho\alpha' + \beta = 0$ for some $\rho \neq 0$. From this follows that $\beta' = -\alpha$ and $\beta = -\alpha'$: contradiction. $\square$

Lemma 6.7.

a) *if rank$\tilde{R}_{\mathfrak{e}} = 2$ then $\tilde{R}_{\mathfrak{e}}$ is of type $A_1 + A_1$;*

b) *if rank$\tilde{R}_{\mathfrak{e}} = 3$, then $\tilde{R}_{\mathfrak{e}}$ is of type $A_3 (\simeq D_3)$ and θ is a multiple of $\varepsilon_0 - \varepsilon_1 - \varepsilon_2 + \varepsilon_3$;*

c) *if rank$\tilde{R}_{\mathfrak{e}} = \ell \geq 4$, $\tilde{R}_{\mathfrak{e}}$ is of type D_ℓ and θ is a multiple of ε_1.*

Proof of a). See Lemma 6.5.

Proof of b). Assume that rank$\tilde{R}_{\mathfrak{e}} = 3$. By Lemma 6.6 all $\mathfrak{e}$-pairs are made of orthogonal roots and no other root in R is linear combination of any two of them.

Let α, α' be an $\mathfrak{e}$-pair of orthogonal roots and let us assume $\theta = \alpha - \alpha'$. Since θ is not proportional to any root, if we consider the list of all simple root systems of rank 3, up to renaming and change of orientation of the unit vectors ε_i, we have only the following possibilities for α, α' and θ:

$$\tilde{R}_{\mathfrak{e}} = A_3 \ : \qquad \alpha = \varepsilon_0 - \varepsilon_1 \ , \quad \alpha' = \varepsilon_1 - \varepsilon_3 \ , \quad \theta = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 + \varepsilon_3 \tag{6.4'}$$

$$\tilde{R}_{\mathfrak{e}} = B_3 \ : \qquad \alpha = \varepsilon_1 + \varepsilon_2 \ , \quad \alpha' = -\varepsilon_3 \ , \quad \theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{6.4"}$$

$$\tilde{R}_{\mathfrak{e}} = C_3 \ : \qquad \alpha = \varepsilon_1 + \varepsilon_2 \ , \quad \alpha' = -2\varepsilon_3 \ , \quad \theta = \varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 \tag{6.4"'}$$

We claim that the case $\tilde{R}_{\mathfrak{e}} = C_3$ cannot occur. In fact, $\varepsilon_2 + \varepsilon_3$ cannot be in $R_{\mathfrak{e}}$, because in that case its $\mathfrak{e}$-dual root is $-\varepsilon_1 - \varepsilon_3$ and it is not orthogonal to $\varepsilon_2 + \varepsilon_3$, contradicting our hypothesis. Therefore we may assume that $\varepsilon_2 + \varepsilon_3 \in R^+_J$ and hence, by Lemma 6.4(5),

$$\varepsilon_2 - \varepsilon_3 = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_3 \in R^+_J \cup R_{\mathfrak{e}} \cup R_o$$

Indeed $\varepsilon_2 - \varepsilon_3 \in R^+_J$ because it is not orthogonal to θ nor admits an $\mathfrak{e}$-dual root. Since the roots $\pm(\varepsilon_1 - \varepsilon_2)$ are orthogonal to θ and hence are in R_o, we also obtain that $\varepsilon_1 - \varepsilon_3 \in R^+_J$ and $\varepsilon_1 + \varepsilon_3 \in R^+_J$. This implies that the only admissible $\mathfrak{e}$-pair is the one given by α and α' and this contradicts the fact that $\tilde{R}_{\mathfrak{e}}$ is of rank 3.

We claim that also the case $\tilde{R}_{\mathfrak{e}} = B_3$ is not admissible. Suppose not. Then R_o contains $\varepsilon_i - \varepsilon_j$, $i, j = 1,2,3$ and $R_{\mathfrak{e}}$ contains the pairs of θ-congruent roots $\{\varepsilon_1 + \varepsilon_3, -\varepsilon_2\}$, $\{\varepsilon_1 + \varepsilon_2, -\varepsilon_3\}$ and $\{\varepsilon_2 + \varepsilon_3, -\varepsilon_1\}$. Since $rank \tilde{R}_{\mathfrak{e}} = 3$, we may assume that $\alpha = \varepsilon_1 + \varepsilon_3$, $\beta = -\varepsilon_2$, $\alpha' = \varepsilon_1 + \varepsilon_2$ and $\beta' = -\varepsilon_3$ are all roots in $R_{\mathfrak{e}}$.

Consider the corresponding two vectors $e_{\alpha,\beta}$ and $e_{\alpha',\beta'}$ as defined in (6.2). Note that $e_{\alpha,\beta}$ and $e_{\alpha',\beta'}$ are in the same $\mathfrak{l}^{\mathbb{C}}$-module; therefore we may assume that they are of the form

$$e_{\alpha,\beta} = E_{\varepsilon_1+\varepsilon_3} + \lambda E_{-\varepsilon_2} \qquad e_{\alpha',\beta'} = E_{\varepsilon_1+\varepsilon_2} + \lambda E_{-\varepsilon_3}$$

Then

$$[e_{\alpha,\beta}, e_{\alpha',\beta'}] = \lambda([E_{\varepsilon_1+\varepsilon_3}, E_{-\varepsilon_3}] + [E_{-\varepsilon_2}, E_{\varepsilon_1+\varepsilon_2}]) + \lambda^2 C E_{-(\varepsilon_2+\varepsilon_3)}$$

for some $C \neq 0$.

Note that

$$[E_{\varepsilon_1+\varepsilon_3}, E_{-\varepsilon_3}] = N_{\varepsilon_1+\varepsilon_3,-\varepsilon_3} E_{\varepsilon_1} = [E_{\varepsilon_1+\varepsilon_2}, E_{-\varepsilon_2}]$$

so that

$$[e_{\alpha,\beta}, e_{\alpha',\beta'}] = \lambda^2 C E_{-(\varepsilon_2+\varepsilon_3)} \in \mathfrak{m}^+$$

But this implies that $\varepsilon_2 + \varepsilon_3 \in R_J$. Since $\varepsilon_i - \varepsilon_j \in R_o$ for $i, j = 1,2,3$, by Lemma 6.4(4) we get that also α and α' are in R_J: contradiction.

So we remain only with the case $\tilde{R}_{\mathfrak{e}} = A_3$. Note that $A_3 = D_3$ and that if we write the roots of A_3 using the same notation used for the root systems of type D_ℓ, (6.4') can be rewritten as

$$\alpha = \varepsilon_1 + \varepsilon_2 \qquad \alpha' = -\varepsilon_1 + \varepsilon_2 \qquad \theta = 2\varepsilon_1 \tag{6.5}$$

Proof of c). Suppose that rank $\tilde{R}_{\mathfrak{e}} = 4$. It is then easy to see that there is only one possibility for θ, in order to be the difference of two orthogonal roots and such that all admissible $\mathfrak{e}$-pairs are not contained in a 3-dimensional root subsystem, that is

$$\theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \tag{6.7}$$

This case may not occir if $\tilde{R}_{\mathfrak{e}} = A_4$ and, furthermore, $R_o \cap \tilde{R}_{\mathfrak{e}} = A_3$. However, we claim that this situation is not possible if $\tilde{R}_{\mathfrak{e}}$ is of type B_4 or F_4, because all roots of the form $\pm\varepsilon_i$ must be in R_J (in fact, they do not admit any $\mathfrak{e}$-dual root) and hence also all root vectors $E_{\varepsilon_i+\varepsilon_j} = C[E_{\varepsilon_i}, E_{\varepsilon_j}]$ are in $\mathfrak{m}_J$. This would imply that $\tilde{R}_{\mathfrak{e}} \cap R_{\mathfrak{e}} = \emptyset$, which is impossible.

θ cannot be as in (6.7) also if $\tilde{R}_{\mathfrak{e}}$ is of type C_3: in fact any root vector $E_{\varepsilon_i+\varepsilon_j}$, $1 \leq i, j \leq 4$ should be in the $\mathfrak{l}^{\mathbb{C}}$-module of $E_{2\varepsilon_i}$ and this root vector should be in $\mathfrak{m}_J$.

So the only possible case is $\tilde{R}_{\mathfrak{e}} = D_4$. In this case, we may consider a different representation of the root vectors so that any pair of $\mathfrak{e}$-dual roots is of the form $\{\varepsilon_1 - \varepsilon_i, -\varepsilon_1 - \varepsilon_j\}$ and θ is proportional to ε_1.

Suppose now that rank $\tilde{R}_{\mathfrak{e}} \geq 4$ and that some $\mathfrak{e}$-pair consist of not orthogonal root. It is simple to verify that there is only possibility, i.e. that $\theta \in \mathbb{R}\varepsilon_1$ and that $\tilde{R}_{\mathfrak{e}} = D_\ell$, with $\ell \geq 4$. $\square$

Lemma 6.8.

a) *If $\tilde{R}_{\mathfrak{e}} \neq A_1 \cup A_1$, then $\tilde{R}_{\mathfrak{e}} \cap R_J = \emptyset$ and $\tilde{R}_{\mathfrak{e}} = R_{\mathfrak{e}} \cup (R_o \cap \tilde{R}_{\mathfrak{e}})$;*

b) *$R_{\mathfrak{e}} \cup R_J^- \cup R_o$ is a closed subsystem of roots.*

Proof. a) In all cases of Lemma 6.7, the contact form θ is explicitly given, so that also $R_o \cap \tilde{R}_{\mathfrak{e}} = \theta^- \cap R_{\mathfrak{e}}$ can be explicitly determined. In all cases, but when $\tilde{R}_{\mathfrak{e}} = A_1 \cup A_1$, it turns out that if $\alpha \in R_J \cap \tilde{R}_{\mathfrak{e}}$, then $\tilde{R}_{\mathfrak{e}} = R_o \cup R_J$ (we use the fact $\mathfrak{m}_J^{\mathbb{C}}$ is $\mathfrak{l}^{\mathbb{C}}$-invariant and invariant by conjugation): contradiction. This proves that $R_J \cap \tilde{R}_{\mathfrak{e}} = \emptyset$.

b) First observe that $R_{\mathfrak{e}} + R_{\mathfrak{e}} \subset R_{\mathfrak{e}} + R_o$: it follows immediately from the fact that $(R_{\mathfrak{e}} + R_{\mathfrak{e}}) \cap R \subset \tilde{R}_{\mathfrak{e}}$ and from a). Then recall that, by Lemma 6.4, $(R_{\mathfrak{e}} + R_o) \cap R \subset R_{\mathfrak{e}}$, $(R_J^- + R_o) \cap R \subset R_J^-$ and that $(R_J^- + R_{\mathfrak{e}}) \cap R \subset R_o + R_{\mathfrak{e}} + R_J^-$. □

Lemma 6.9. *If $\tilde{R}_{\mathfrak{e}} \neq R$ then:*

a) *The subspace $\mathfrak{p} = \mathfrak{e} + \mathbb{C}Z + \mathfrak{l}^{\mathbb{C}} + \sum_{\alpha \in R_J^-} \mathbb{C}E_\alpha$ is a parabolic subalgebra of $\mathfrak{g}$, with reductive part $\mathfrak{r} = \mathfrak{q}^{\mathbb{C}}$, where $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{g}$ and $\mathfrak{l} \subsetneq \mathfrak{q}$;*

b) *if $Q \subset G$ is the subgroup of maximal rank generated by $\mathfrak{q}$, and G/Q is not trivial, the fibering $\pi\colon G/L \to F = G/Q$ is a CRF fibration, where on G/Q is considered the complex structure J_F of the flag manifold $F = G/Q$ associated to the parabolic subalgebra $\mathfrak{p}$;*

c) *Q/L is SO_{2n}/SO_{2n-2} for some $n \geq 3$;*

d) *if $(\mathcal{D}_Z, J)$ is an invariant CR structure on SO_{2n}/SO_{2n-2} with $\theta = \mathcal{B} \circ Z$ not parallel to any root, then θ is a multiple to ε_1 and in the decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$ the subspace $\mathfrak{m}^{01}$ is of the form*

$$a\mathfrak{m}(\varepsilon_1 - \varepsilon_2) + b\mathfrak{m}(-\varepsilon_1 - \varepsilon_2)$$

for some $[a : b] \in \mathbb{C}P^1$, $|a|^2 - |b|^2 \neq 0$ (we use the same notation as the one of table of Prop. 6.3). The CR structures which are not standard are exactly those such that $a \cdot b \neq 0$ and they are all primitive;

e) *if $(G/L, \mathcal{D}_Z, J)$, with $G/L \neq SO_{2n}/SO_{2n-2}$, is an invariant non standard CR structure with $\theta = \mathcal{B} \circ Z$ not parallel to any root, then the associated flag manifold $F_Z = G/K$ admits a complex structure, whose associated painted diagram contains a subdiagram of type D_n, with the first node black and all other white, it is connected only to black nodes and such that, if it is deleted, the black nodes of the remaining diagram are isolated; using the same convention used in Appendix for the roots of D_n, θ is parallel to the vector $2\varepsilon_1 = (\varepsilon_1 - \varepsilon_2) + (\varepsilon_1 + \varepsilon_2)$, where $\varepsilon_1 - \varepsilon_2$ and $\varepsilon_1 + \varepsilon_2$ are two roots of the subgroup associated to the subdiagram D_n.*

Proof of a). It follows immediately from Lemma 6.8 b) and the fact that $\mathfrak{q} = \mathfrak{g} \cap \mathfrak{p} = \mathbb{R}Z + \mathfrak{g}(\tilde{R}_{\mathfrak{e}} \cup R_o) \cap \mathfrak{g} \supsetneq \mathfrak{l}$.

Proof of b) It follows immediately from Lemma 4.9 and claim a).

Proof of c) Consider the largest ideal $\mathfrak{i}^{\mathbb{C}} \subset \mathfrak{q}^{\mathbb{C}}$ such that $\mathfrak{i}^{\mathbb{C}} \subset \mathfrak{l}^{\mathbb{C}}$. We claim that

$$\mathfrak{i}^{\mathbb{C}} = \mathfrak{g}(R_o \setminus span_{\mathbb{R}} < R_{\mathfrak{e}} >)$$

so that $\mathfrak{q}^{\mathbb{C}}/\mathfrak{i}^{\mathbb{C}} \simeq \mathfrak{g}(\tilde{R}_{\mathfrak{e}})$.

For this it is enough to show that if $\gamma \in R_o \setminus span_{\mathbb{R}} < R_{\mathfrak{e}} >$, then E_γ and H_γ are in $\mathfrak{i}^{\mathbb{C}}$. But this is clear because if $\gamma \in R_o \setminus span_{\mathbb{R}} < R_{\mathfrak{e}} >$ and $\alpha \in span_{\mathbb{R}} < R_{\mathfrak{e}} >$, then $\alpha+\gamma \in R$ only if it belongs to $R_o \setminus span_{\mathbb{R}} < R_{\mathfrak{e}} >$. From this the claim follows.

Now, set $\mathfrak{q}' = \mathfrak{q} \mod \mathfrak{i}$, $Z' = Z \mod \mathfrak{i}$ and $\mathfrak{l}' \mod \mathfrak{i}$. ¿From the previous observations, we have that $\mathfrak{q}'$ is a compact form of $\mathfrak{g}(\tilde{R}_{\mathfrak{e}})$ and hence, by Lemma 6.7, $\mathfrak{q}'$ is $A_1 \cup A_1$, $A_3 (\simeq D_3)$ or D_ℓ, $\ell \geq 4$. These cases correspond to CRF fibrations defined at the point b) with fiber Q/L equal to $SU_2 \times SU_2/T_1$ or to SO_{2n}/SO_{2n-2}, $n \geq 3$.

However we claim that the case $Q/L = SU_2 \times SU_2/T_1$ cannot occur. To see this, observe that, in case $Q/L = SU_2 \times SU_2/T_1$, any painted diagram associated to the complex structure J_F on G/Q has to contain a subdiagram of white nodes (corresponding to the roots of the isotropy $\mathfrak{q}$) which contain two isolated white nodes (corresponding to the roots α and α' of $SU_2 \times SU_2$). The contact form θ would be proportional to $\alpha - \alpha'$. But in this case, it can be checked that if G is simple and α, α' are two roots associated to two isolated white nodes in a black-white diagram for G, then the centralizer $C_{\mathfrak{g}}(Z)$, $Z = \mathcal{B}^{-1} \circ \theta$, with $\theta = \alpha - \alpha'$ has a semisimple part which is strictly larger then subalgebra associated to the white root subdiagram obtained by deleting the nodes α and α'. This gives a contradiction with our hypothesis, because the semisimple part of $C_{\mathfrak{g}}(Z)$ must coincide with the semisimple part of $\mathfrak{l}$, which is associated to the white nodes (minus the nodes α and α') of the black-white diagram of the complex structure J_F on G/Q.

Proof of d) It is proved with the same line of arguments used for Proposition 6.3 (2) and it is consequence of c).

Proof of e) It follows directly from b), c) and d). $\square$

Theorem 6.10. *Let $(G/L, \mathcal{D})$ with G simple, L connected and with contact form $\theta = \mathcal{B} \circ Z$ not parallel to any root. Then:*

(1) *if $(G/L, \mathcal{D})$ admits a primitive CR structure $(\mathcal{D}, J)$ then G/L is of the form SO_{2n}/SO_{2n-2} with $n \geq 3$; in this case θ is proportional to ε_1 and all invariant primitive CR structures are the non standard CR structures described in Lemma 6.9 d);*

(2) *if $(G/L, \mathcal{D})$, with $G \neq E_6, E_7, E_8$, admits a non primitive, non standard CR structure $(\mathcal{D}, J)$, then it admits a CRF fibration $\pi \colon G/L \to F = G/Q$, where $F = G/Q$ is a flag manifold with invariant complex structure J_F and the fiber Q/L is equal to SO_6/SO_4;*

(3) *if $(G/L, \mathcal{D})$, with $G = E_6$, E_7 or E_8, admits a non primitive, non standard CR structure $(\mathcal{D}, J)$, then it admits a CRF fibration $\pi \colon G/L \to F = G/Q$, where $F = G/Q$ is a flag manifold with invariant complex structure J_F and the fiber Q/L is a manifold SO_{2n}/SO_{2n-2}, with $5 \leq n \leq 7$; the fibrations which may occur are exactly those described by the admissible painted digrams (1.5) - (1.9) of the Introduction.*

Proof. (1) follows from Lemma 6.9.

(2) is proved with the following argument. By Lemma 6.9 e), if G/L admits a non standard, non primitive CR structure, then the Dynkin diagram of the root

system of G contains a subdiagram of type D_n. If we suppose that it admits a subdiagram of type D_n, with $n > 3$, since $\mathfrak{g}^{\mathbb{C}} \neq E_i$, we conclude that and the $\mathfrak{g}^{\mathbb{C}}$ is of type D_m, with $m > n$. However, this case is not possible, because in this case, using Lemma 6.9 e), we may compute θ and find out that the centralizer of $Z = \mathcal{B}^{-1} \circ \theta$ has a semisimple part which is strictly larger then the semisimple part of $\mathfrak{l}$, as defined by the black-white diagram: contradiction.

(3) can be obtained by a direct application of Lemma 6.9 e) and checking that the painted Dynkin graphs (1.5) - (1.9) do correspond to non primitive non standard CR structures. $\square$

APPENDIX

The notation used in the following Tables for the roots of the simple Lie groups A_ℓ, B_ℓ, C_ℓ, D_ℓ, F_4 and G_2 are as in [Hu]. For the roots of E_6, E_7 and E_8 the following conventions of [OV] has been used: the weights of the groups E_ℓ, $\ell = 7, 8$ are expressed using vectors $\varepsilon_1, \dots, \varepsilon_{\ell+1}$ such that

$$\sum \varepsilon_i = 0 \qquad (\varepsilon_i, \varepsilon_j) = \begin{cases} \frac{\ell}{\ell+1} & i = j \\ -\frac{1}{\ell+1} & i \neq j \end{cases} \tag{T.1}$$

It is useful to rember that, in this last case, if $\sum a_i = 0$, then $(\sum a_i \varepsilon_i, \sum b_j \varepsilon_j) = \sum a_i b_i$. For E_6, the weights are expressed by vectors $\varepsilon_1, \dots, \varepsilon_6$, which verify (1), and by an auxiliary vector ε which is orthogonal to all ε_i and verifies $(\varepsilon, \varepsilon) = 1/2$.

In Table 1, for any simple complex Lie group $\mathfrak{g}^{\mathbb{C}}$, we give the corresponing root system R, the longest root μ (unique up to inner automorphisms), the subalgebra $\mathfrak{g}_0' = C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}(\mu))$, the subsystem of roots R_0 corresponding to $\mathfrak{g}_0'$, the decomposion into irreducible submodules of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ which appear in the decomposition (3.1) and and the set of roots $R_1 = R^+ \setminus (\mu \cup R_0)$.

For a set of simple roots of $\mathfrak{g}_0'$, we denote by $\{\pi_1, \dots, \pi_\ell\}$ the corresponding system of fundamental weights and, for any weight $\lambda = \sum a_i \pi_i$, we denote by $V(\lambda)$ the irreducible $\mathfrak{g}_0'$-module with highest weight λ.

Table 1

$\mathfrak{g}$	R	μ	$\mathfrak{g}_0'$	R_0	$\mathfrak{g}_1$	R_1
A_ℓ	$\varepsilon_i-\varepsilon_j$ $0\leq i,j\leq \ell$	$\varepsilon_0-\varepsilon_\ell$	$A_{\ell-2}+\mathbb{R}$	$\varepsilon_a-\varepsilon_b$ $1\leq a,b\leq \ell-1$	$V(\pi_1)+$ $V(\pi_{\ell-2})$	$\varepsilon_0-\varepsilon_a,\ \varepsilon_a-\varepsilon_\ell$ $1\leq a\leq \ell-1$
B_ℓ	$\pm\varepsilon_i\pm\varepsilon_j,\ \pm\varepsilon_i$ $1\leq i,j\leq \ell$	$\varepsilon_1+\varepsilon_2$	$A_1+B_{\ell-2}$	$\pm(\varepsilon_1-\varepsilon_2),\ \pm\varepsilon_a\pm\varepsilon_b$ $\pm\varepsilon_a$ $3\leq a,b\leq \ell$	$V(\pi_1)\otimes$ $V(\pi_1')$	$\varepsilon_1,\ \varepsilon_2$ $\varepsilon_1\pm\varepsilon_a,\ \varepsilon_2\pm\varepsilon_a$ $3\leq a\leq \ell$
C_ℓ	$\pm\varepsilon_i\pm\varepsilon_j\ ,\ \pm 2\varepsilon_i$ $1\leq i,j\leq \ell$	$2\varepsilon_1$	$C_{\ell-1}$	$\pm\varepsilon_a\pm\varepsilon_b,\ \pm 2\varepsilon_a$ $2\leq a,b\leq \ell$	$V(\pi_1)$	$\varepsilon_1\pm\varepsilon_a$ $2\leq a\leq \ell$
D_ℓ	$\pm\varepsilon_i\pm\varepsilon_j$ $1\leq i,j\leq \ell$	$\varepsilon_1+\varepsilon_2$	$A_1+D_{\ell-2}$	$\pm(\varepsilon_1-\varepsilon_2),\ \pm\varepsilon_a\pm\varepsilon_b$ $3\leq a,b\leq \ell$	$V(\pi_1)\otimes$ $V(\pi_1')$	$\varepsilon_1\pm\varepsilon_a,\ \varepsilon_2\pm\varepsilon_a$ $3\leq a\leq \ell$
E_6	$\varepsilon_i-\varepsilon_j,\ \pm 2\varepsilon$ $\varepsilon_i+\varepsilon_j+\varepsilon_k\pm\varepsilon$ $1\leq i,j,k\leq 6$	2ε	A_5	$\varepsilon_i-\varepsilon_j$	$V(\pi_1)$	$\varepsilon_i+\varepsilon_j+\varepsilon_k\pm\varepsilon$
E_7	$\varepsilon_i-\varepsilon_j$ $\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_\ell$ $1\leq i,j,k,\ell\leq 8$	$-\varepsilon_7+\varepsilon_8$	D_6	$\varepsilon_a-\varepsilon_b$ $\varepsilon_7+\varepsilon_8+\varepsilon_a+\varepsilon_b$ $\varepsilon_a+\varepsilon_b+\varepsilon_c+\varepsilon_d$ $1\leq a,b,c,d\leq 6$	$V(\pi_1)$	$-\varepsilon_7+\varepsilon_a,\ \varepsilon_8-\varepsilon_a$ $\varepsilon_8+\varepsilon_a+\varepsilon_b+\varepsilon_c$ $1\leq a,b,c\leq 6$
E_8	$\varepsilon_i-\varepsilon_j$ $\pm(\varepsilon_i+\varepsilon_j+\varepsilon_k)$ $1\leq i,j,k\leq 9$	$\varepsilon_1-\varepsilon_9$	E_7	$\varepsilon_a-\varepsilon_b$ $\pm(\varepsilon_1+\varepsilon_9+\varepsilon_a)$ $\pm(\varepsilon_a+\varepsilon_b+\varepsilon_c)$ $2\leq a,b,c\leq 8$	$V(\pi_1)$	$\varepsilon_1-\varepsilon_a,\ -\varepsilon_9+\varepsilon_a$ $\varepsilon_1+\varepsilon_a+\varepsilon_b$ $2\leq a,b\leq 8$
F_4	$\frac{\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4}{2}$ $\pm\varepsilon_i\pm\varepsilon_j,\ \pm\varepsilon_i$ $1\leq i,j\leq 4$	$\varepsilon_1+\varepsilon_2$	C_3	$\pm(\varepsilon_1-\varepsilon_2)$ $\pm\frac{\varepsilon_1-\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4}{2}$ $\pm\varepsilon_a,\ \pm\varepsilon_a\pm\varepsilon_b$ $3\leq a,b\leq 4$	$V(\pi_1)$	$\varepsilon_1,\ \varepsilon_2$ $\frac{\varepsilon_1+\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4}{2}$
G_2	$\pm(\varepsilon_i-\varepsilon_j)$ $\pm(\varepsilon_i-\varepsilon_j-\varepsilon_k)$ $1\leq i,j,k\leq 3$	$2\varepsilon_1-\varepsilon_2-\varepsilon_3$	A_1	$\pm(\varepsilon_2-\varepsilon_3)$	$V(\pi_1)$	$2\varepsilon_2-\varepsilon_1-\varepsilon_3$ $\varepsilon_2-\varepsilon_1$

In the next Table 2, we give all information needed to determine the admissible decompositions $\mathfrak{g}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} + \mathbb{C}Z + \mathfrak{m}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$ associated to an invariant CR structure, when $\theta = \mathcal{B}\circ Z$ is parallel to a root and $\mathfrak{g}^{\mathbb{C}}$ is simple. Recall that when $\theta = \mu$, the associated contact manifold G/L is special and hence all needed informations can be recovered from Table 1.

In Table 2 we consider only the case of $\theta = \mathcal{B}\circ Z = \nu$ is equal to a short root. For each simple Lie algebra with roots of different length, we give the root system R, the short root ν (unique up to inner automorphisms), the cetralizer $C_{\mathfrak{g}^{\mathbb{C}}}(H_\nu)$, the root subsystem R_0 of $C_{\mathfrak{g}^{\mathbb{C}}}(H_\nu)$, the list of the highest weights for the irreducible $\mathfrak{k}^{\mathbb{C}}$-moduli in $\mathfrak{m}^{\mathbb{C}}$, $\mathfrak{k}^{\mathbb{C}} = C_{\mathfrak{g}^{\mathbb{C}}}(H_\nu) + \mathbb{C}H_\nu$, and the sets of the $\mathfrak{k}^{\mathbb{C}}$-moduli which are equivalent as $C_{\mathfrak{g}^{\mathbb{C}}}(H_\nu)$-moduli.

Table 2

$\mathfrak{g}$	R	ν	$C^{\mathbb{C}}_{\mathfrak{g}}(H_\nu)$	R_0	highest weights for $\mathfrak{m}^{\mathbb{C}}$	sets of equivalent $C^{\mathbb{C}}_{\mathfrak{g}}(H_\nu)$−moduli (denoted by their highest weights)
B_ℓ	$\pm\varepsilon_i\pm\varepsilon_j,\ \pm\varepsilon_i$ $1\leq i,j\leq\ell$	ε_1	$B_{\ell-1}$	$\pm\varepsilon_a\pm\varepsilon_b,\ \pm\varepsilon_a$ $2\leq a,b\leq\ell$	$\varepsilon_1+\varepsilon_2,\ -\varepsilon_1+\varepsilon_2$	$\{\varepsilon_1+\varepsilon_2,-\varepsilon_1+\varepsilon_2\}$
C_ℓ	$\pm\varepsilon_i\pm\varepsilon_j,\ \pm2\varepsilon_i$ $1\leq i,j\leq\ell$	$\varepsilon_1+\varepsilon_2$	$A_1+C_{\ell-2}$	$\pm(\varepsilon_1-\varepsilon_2),\ \pm2\varepsilon_a$ $\pm\varepsilon_a\pm\varepsilon_b$ $3\leq a,b\leq\ell$	$2\varepsilon_1,\ \varepsilon_1+\varepsilon_3$ $-2\varepsilon_2,\ -\varepsilon_2-\varepsilon_3$	$\{2\varepsilon_1,-2\varepsilon_2\}$ $\{\varepsilon_1+\varepsilon_3,\ -\varepsilon_2-\varepsilon_3\}$
F_4	$\frac{\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4}{2}$ $\pm\varepsilon_i\pm\varepsilon_j,\ \pm\varepsilon_i$ $1\leq i,j\leq4$	ε_1	B_3	$\pm\varepsilon_a\pm\varepsilon_b,\ \pm\varepsilon_a$ $2\leq a,b\leq4$	$\varepsilon_1+\varepsilon_2,\ -\varepsilon_1+\varepsilon_2$ $\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2},$ $-\frac{\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4}{2}$	$\{\varepsilon_1+\varepsilon_2,-\varepsilon_1+\varepsilon_2\}$ $\{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2},$ $-\frac{\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4}{2}\}$
G_2	$\pm(\varepsilon_i-\varepsilon_j)$ $\pm(\varepsilon_i-\varepsilon_j-\varepsilon_k)$ $1\leq i,j,k\leq3$	$\varepsilon_1-\varepsilon_2$	A_1	$\pm(2\varepsilon_3-\varepsilon_1-\varepsilon_2)$	$\varepsilon_1-\varepsilon_2,\ \varepsilon_2-\varepsilon_1$ $\varepsilon_2-\varepsilon_3,\ \varepsilon_1-\varepsilon_3$ $2\varepsilon_2-\varepsilon_1-\varepsilon_3$ $2\varepsilon_1-\varepsilon_2-\varepsilon_3$	$\{\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_1\}$ $\{2\varepsilon_2-\varepsilon_1-\varepsilon_3,$ $\varepsilon_1-\varepsilon_3,\varepsilon_2-\varepsilon_3,$ $2\varepsilon_1-\varepsilon_2-\varepsilon_3\}$

In Table 3, we give the same list of Table 2, when $\theta = \mathcal{B}\circ Z$ is parallel to no root and $\mathfrak{g}^{\mathbb{C}} = A_2$ or $\mathfrak{g}^{\mathbb{C}} = D_\ell$.

Table 3

$\mathfrak{g}$	R	$\theta=\mathcal{B}\circ Z$	$C^{\mathbb{C}}_{\mathfrak{g}}(Z)$	R_0	highest weights for $\mathfrak{m}^{\mathbb{C}}$	sets of equivalent $C^{\mathbb{C}}_{\mathfrak{g}}(Z)$−moduli (denoted by their highest weights)
A_2	$\varepsilon_i-\varepsilon_j$ $0\leq i,j\leq2$	$\varepsilon_0+\varepsilon_1-2\varepsilon_2$	A_1	$\pm(\varepsilon_0-\varepsilon_1)$	$\varepsilon_0-\varepsilon_2,\ \varepsilon_1-\varepsilon_2$	$\{\varepsilon_0-\varepsilon_1,\varepsilon_1-\varepsilon_2\}$
D_ℓ	$\pm\varepsilon_i\pm\varepsilon_j$ $1\leq i,j\leq\ell$	$2\varepsilon_1$	$D_{\ell-1}$	$\pm\varepsilon_i\pm\varepsilon_j$ $2\leq i,j\leq\ell$	$\varepsilon_1+\varepsilon_2,\ -\varepsilon_1+\varepsilon_2$	$\{\varepsilon_1-\varepsilon_2,-\varepsilon_1-\varepsilon_2\}$

In Table 4, we give the same list of of Onishchik ([On]) of the only three cases, where the Lie algebra $\mathfrak{g}$ of a compact simple Lie group G, which acts transitively on a flag manifold $F = G/K$ with an invariant complex structure J_F, is not the compact real form of the Lie algebra $\tilde{\mathfrak{g}}^{\mathbb{C}}$ of the Lie group of all holomorphic transformations of (F, J_F).

Table 4

$Case$	$\mathfrak{g}^{\mathbb{C}}$	$\mathfrak{k}^{\mathbb{C}}$	$\tilde{\mathfrak{g}}^{\mathbb{C}}$
1	$C_\ell\ (\ell>1)$	$C_{\ell-1}+\mathbb{C}$	$A_{2\ell-1}$
2	G_2	$A_1+\mathbb{C}$	B_3
3	$B_\ell\ (\ell>2)$	$A_{\ell-1}+\mathbb{C}$	$D_{\ell+1}$

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