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REDUCTION OF THE HERMITIAN-EINSTEIN EQUATION ON KÄHLERIAN FIBER BUNDLES

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Abstract. The technique of dimensional reduction of an integrable system usually requires symmetry arising from a group action. In this paper we study a situation in which a dimensional reduction can be achieved despite the absence of any such global symmetry. We consider certain holmorphic vector bundles over a Kahler manifold which is itself the total space of a fiber bundle over a Kahler manifold. We establish an equivalence between invariant solutions to the Hermitian–Einstein equations on such bundles, and general solutions to a coupled system of equations defined on holomorphic bundles over the base Kahler manifold. The latter equations are the Coupled Vortex Equations. Our results thus generalize the dimensional reduction results of García–Prada, which apply when the fiber bundle is a product and the fiber is the complex projective line.

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1. Introduction

Techniques involving dimensional reduction are important in many areas of mathematical physics when one is looking at solutions to partial differential equations which are invariant under a group of symmetries. The term 'dimensional reduction' then refers to the fact that the invariant solutions to the original equation can be interpreted as ordinary solutions to a related set of equations on the (lower dimensional) orbit space of the group action. The latter in their own right can be the equations of an equally important physical system and correspondences between these two systems involve exploiting a whole range of mathematical ideas (examples relating to monopole and soliton type equations can be found e.g. in **[37]**).

The *vortex equations* were first studied (over $\mathbb{R}^2$) by Ginsburg and Landau **[19]** in the study of superconductivity and their mathematical framework was later developed in the book of Jaffe and Taubes **[28]**. Taubes in **[44]** showed that a reduction of the anti–self–dual equations on $\mathbb{R}^2 \times S^2$ led to the vortex equations on $\mathbb{R}^2$ and by analogous means, Witten **[51]**, on taking $\mathbb{H}^2 \times S^2$, obtained the vortex equations on the hyperbolic plane $\mathbb{H}^2$. The holomorphic geometry of these equations over a compact Kähler manifold X along with their corresponding moduli spaces was studied by the first author in **[6]** **[7]** **[8]** **[10]**. García–Prada in **[17]** **[18]** showed that the coupled vortex equations over X could in fact be obtained as a dimensional reduction of the *Hermitian–Einstein equations* over $X \times \mathbb{CP}^1$, and in effect generalized the cases studied in **[44]** and **[51]**. When $\dim_{\mathbb{C}} X = 2$, the abelian vortex equations are known to be equivalent to the *Seiberg–Witten equations* **[43]** **[50]** **[10]**.

In **[11]** we generalized the García–Prada technique of dimensional reduction from the case $X \times \mathbb{CP}^1$ to a *projectively flat* $\mathbb{CP}^1$–bundle over X . In this paper, we consider a generalization from the fiber $\mathbb{CP}^1$ to the case of a fiber F which is a compact symmetric Kähler manifold. Thus we consider holomorphic Kählerian fiber bundles $F \to M \to X$, where M is taken to have a flat structure. The holomorphic vector bundles on M that we study are the analogues of the $SU(2)$–equivariant bundles considered in **[17]** **[18]** . We show that such bundles correspond to holomorphic objects on X. These objects are holomorphic triples consisting of a pair of holomorphic bundles together with a holomorphic bundle map between them. Our main result (Theorem 8.9) establishes an equivalence between special solutions to the Hermitian–Einstein equations on the bundles over M and general solutions to the Coupled Vortex Equations on the corresponding triples on X . It is in this sense that our main result can be viewed as a dimensional reduction result.

An outline of the paper is as follows. As apparent in **[11]** , an essential difference from **[18]** is that the $SU(2)$–orbits appearing there are generalized to the leaves of a foliation by the fibers F as above. So in § 2 , we establish some concepts from the theory of Riemannian foliations, in particular a general structure theorem for vector bundles over generalized flat bundles and the notion of extendability to M of bundles and forms defined over F ; here the flat structure on M plays a crucial role. Although foliation methods are motivationally important (cf. **[20]**), the eventual reduction procedure is achieved by making substantial modifications to the holomorphic–geometric approach of **[18]**.

In § 3 we present a short discussion of equivariant and homogeneous bundles. In § 4 we establish a generalization of the Borel–Leray spectral sequence (as given in **[27]**) for non–trivial coefficient bundles on the fiber, that in § 5 leads to a Kunneth formula in Hodge cohomology (with non–trivial coefficients). In § 6 we turn to an important class of examples namely, holomorphic projective bundles $\mathbb{CP}^l \hookrightarrow M \to X$. Quite naturally it is the most geometrically realizable case. It is necessary to look at the essential topological aspects and make note of the differences between the $GL(l+1)$ and $PGL(l+1)$ cases. Then we adapt the main results of § 5 to establish necessary vanishing theorems which are needed later.

Having recalled the notion of the Ext functor, we proceed in § 7 to apply the main results of § 5 and § 6 to obtain an essential parametrization of certain holomorphic extensions over M by basic sections. In § 8 we start by formulating some results which yield base and fiber degree invariants of the various bundles defined on X and F respectively. In the latter case, the extension of objects to M involves subtle technical work which is incumbent upon the flat structure and justifies certain calibration conditions on the fiber. In § 8 we establish the main result of the paper which proves that the classes of holomorphic vector bundles on M described in § 7 , when endowed with a Hermitian–Einstein metric connection, reduce to the Coupled Vortex equations on X, and conversely. We also establish a number of formulas relating the parameters of the vortex equations to the slopes of the various bundles featuring in the construction.

The important example of the $\mathbb{CP}^l$–fibration reappears in § 9 where the necessary hypotheses needed in the main result of § 8 are automatically satisfied. Following this we state explicit formulas for computing the fiberwise invariants. In § 10 we introduce the notion of holomorphic triples and the relationship with solutions of the Coupled Vortex equations. With regards to these, we establish a priori estimates for solutions and stability. We conclude with an appendix which accounts for the topological details needed in completing our description of the geometric structure of projectively flat bundles over X .

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2. Generalized flat fiber bundles and Riemannian foliations

Let M be a compact oriented Riemannian manifold and let $\mathcal{F}$ be an oriented foliation on M . Denoting by $T\mathcal{F}$ the tangent bundle along the leaves of $\mathcal{F}$ and by Q the normal bundle, we have the exact sequence

$$0 \to T\mathcal{F} \longrightarrow TM \longrightarrow Q \to 0 \ .$$

The metric g_M on M gives the identification $T\mathcal{F}^{\perp} \cong Q$ and $g_M = g_{T\mathcal{F}} + g_Q$. The foliation $(M, \mathcal{F})$ is said to be *Riemannian* if g_Q is *$\mathcal{F}$–holonomy invariant* ; specifically, for all $Z \in C^{\infty}(T\mathcal{F})$, the *Lie invariance* condition $L_Z g_Q = 0$ is satisfied. Henceforth we assume that $(M, \mathcal{F})$ is a Riemannian foliation (for further details see e.g. **[39]**). The type of Riemannian fibrations which we will consider are seen as particular cases. An important class of examples is provided by the following general construction which will be implemented in the following sections.

Example 2.1. Let X be a compact Riemannian manifold with fundamental group $\Gamma = \pi_1(X)$ and universal covering $\widetilde{X}$, F a compact Riemannian manifold and α a representation

$$\alpha \; : \; \Gamma \rightarrow \mathrm{Diff}_0(F) \; ,$$

into the *orientation–preserving diffeomorphisms* of F. We consider the action of $\gamma \in \Gamma$ on $\widetilde{X} \times F$ given by

$$\gamma(\tilde{x}, f) = \; (\tilde{x} \cdot \gamma^{-1}, \alpha(\gamma) f) \; .$$

Let M be the quotient of $\widetilde{X} \times F$ by this action. There are two foliations which can be considered on the *generalized flat bundle* **[29]**

$$F \hookrightarrow M = \widetilde{X} \times_\Gamma F \xrightarrow{\pi} X \; . \tag{2.1}$$

(1) the foliation $(M, \mathcal{F}_\pi)$ by the fibers of π ;

(2) the foliation $(M, \mathcal{F}_\alpha)$ by the holonomy covers $\widetilde{X}$.

For a bundle–like metric g_M on M of the form $\pi^* g_X + g_{T(\pi)}$, the fibration π is a Riemannian submersion and hence $(M, \mathcal{F}_\pi)$ is a Riemannian foliation such that the transverse metric $g_{Q(\mathcal{F}_\pi)} = \pi^* g_X$. It is well–known (e.g. **[30]**) that this is equivalent to $(M, \mathcal{F}_\alpha)$ being *totally geodesic*. If $\alpha : \Gamma \rightarrow \mathrm{Iso}(F)$ is a representation into the *isometries* of F, then a bundle–like metric g_M on M is induced by the *product metric* $\pi^* g_X + p^* g_F$ on $\widetilde{X} \times F$ via the natural projection $p \; : \; \widetilde{X} \times F \rightarrow F$. In this case, the foliation $(M, \mathcal{F}_\alpha)$ is also Riemannian and consequently $(M, \mathcal{F}_\pi)$ is totally geodesic.

Let G be a connected Lie group and $\psi\colon P \rightarrow M$ a principal G-bundle. We say that P is *foliated* if P has a foliation $\widetilde{\mathcal{F}}$ obtained by a choice of horizontal lifting of $\mathcal{F}$ (cf. **[29]** **[16]**). Specifically, for tangent vector fields $Z \in C^\infty(T\mathcal{F})$, there is a lift $\widetilde{Z} \in C^\infty(T\widetilde{\mathcal{F}})$ such that:

(1) $\widetilde{Z}$ is G–invariant and hence ψ–projectable, that is, $\psi_* \tilde{Z} = Z$ and $R_{g*} \tilde{Z} = \tilde{Z}$ for all $g \in G$;

(2) $[Z, Y]^\sim = [\tilde{Z}, \tilde{Y}]$ for all $Z, Y \in C^\infty(T\mathcal{F})$.

Thus for each $p \in P$, the differential $\psi_*\colon T_p P \rightarrow T_{\psi(p)} M$ maps the tangent space $T_p \widetilde{\mathcal{F}}$ isomorphically onto the tangent space $T_{\psi(p)} \mathcal{F}$ and the action of G on P permutes the leaves of $\widetilde{\mathcal{F}}$. The definition carries over in the usual way to any vector bundle $\mathcal{E} \rightarrow M$ associated with P.

Associated to $\mathcal{F}$ is its *holonomy groupoid* $\mathcal{G}_\mathcal{F}$ **[49]**, whereby a vector bundle $E \rightarrow M$ is said to be *$\mathcal{G}_\mathcal{F}$–equivariant* if there is an action of $\mathcal{G}_\mathcal{F}$ on the fibers of E via holonomy transport. Conversely, a foliated bundle $E \rightarrow M$ is naturally a $\mathcal{G}_\pi$–equivariant bundle with respect to the *fundamental groupoid* $\mathcal{G}_\pi \rightarrow \mathcal{G}_\mathcal{F}$ of homotopy classes of paths along the leaves of $\mathcal{F}$. For further details see **[13]** **[49]** .

Recall that $p : \widetilde{X} \times F \rightarrow F$ is the natural projection. Then if V is a Γ–equivariant vector bundle over F, we obtain an extension $\widetilde{V}$ of V to M by

$$\widetilde{V} = p^* V / \alpha \cong \widetilde{X} \times_\Gamma V \longrightarrow M = \widetilde{X} \times_\Gamma F \; , \tag{2.2}$$

with the restriction property

$$\widetilde{V}|_F = V \; . \tag{2.3}$$

Next, we state a structure theorem for $\mathcal{G}_\mathcal{F}$–equivariant bundles on a generalized flat bundle M.

Theorem 2.2. *For $M = \widetilde{X} \times_\Gamma F$ as in Example 2.1 , the equivariant vector bundles are described as follows.*

(1) *There is a one to one correspondence between vector bundles W over X and $\mathcal{G}_{\mathcal{F}_\pi}$–equivariant vector bundles on M, given by $W \to \pi^* W$.*

(2) *There is a one to one correspondence between Γ–equivariant vector bundles V over F and $\mathcal{G}_{\mathcal{F}_\alpha}$–equivariant vector bundles on M, given by $V \to \widetilde{V} = p^* V/\alpha$, where $\widetilde{V}$ is the extension of V to M in (2.2) .*

Proof. The foliation $(M, \mathcal{F}_\pi)$ is a fibration and the holonomy groupoid $\mathcal{G}_{\mathcal{F}_\pi}$ is given by the fiber product

$$\begin{array}{ccc} M \times_X M & \xrightarrow{pr_1} & M \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\pi} & X \ , \end{array} \tag{2.4}$$

which records the fact that $\mathcal{F}_\pi$ has trivial holonomy. The statement in (1) then follows immediately from the definition of $\mathcal{G}_\mathcal{F}$–equivariance.

Essentially the same argument applies in the case of the foliation $\mathcal{F}_\alpha$. Here, the global holonomy is given by the image of Γ under α in $\mathrm{Diff}_0(F)$ and one requires equivariance with respect to α . □

Remark 2.3. This result plays the role of the structure theorem in **[18]** (Proposition 3.1). One of our main observations is that the dimensional reduction procedure of **[17]** **[18]** can be generalized to the case where the (smooth) bundles over M are direct sums with summands of the form $\pi^* W \otimes_\mathbb{C} \widetilde{V}$. For instance, in going from bundles over $X \times \mathbb{CP}^1$ to bundles over a flat $\mathbb{CP}^1$–bundle over X, one replaces the $SU(2)$–action on $X \times \mathbb{CP}^1$ by the 'double foliation' of the $\mathbb{CP}^1$–bundle with respect to the foliations $\mathcal{F}_\pi$ and $\mathcal{F}_\alpha$. The analogues of the $SU(2)$–equivariant bundles on $X \times \mathbb{CP}^1$ are then the 'doubly $\mathcal{G}_\mathcal{F}$–equivariant' bundles on M. By Theorem 2.2 , these are of the indicated form.

We define the extension of a Γ–equivariant V–valued form φ on F to a $\widetilde{V}$–valued form $\tilde{\varphi}$ by the formula

$$\tilde{\varphi} = p^* \varphi / \alpha \ , \tag{2.5}$$

noting that $p^*\varphi$ is Γ–equivariant under the diagonal action of Γ on $\widetilde{X} \times_\Gamma V$. Let

$$q : \widetilde{X} \times F \longrightarrow M = \widetilde{X} \times_\Gamma F$$

be the quotient map under Γ . Then $\tilde{\varphi}$ and φ are related by

$$q^* \tilde{\varphi} = p^* \varphi \ . \tag{2.6}$$

For a Riemannian foliation $(M, \mathcal{F})$, the *basic forms* with coefficients in a foliated vector bundle E are defined by

$$\Omega_b^*(M, \mathcal{F}; E) = \{\alpha \in \Omega^*(M, E) \ \mid \ i_{\widetilde{Z}}\alpha = 0 \ , \ L_{\widetilde{Z}}\alpha = 0 \ ; \ Z \in C^\infty(M, T\mathcal{F})\} \ . \tag{2.7}$$

In degree 0, only the latter condition applies and the basic sections are also called *invariant.*

We remark that for a Riemannian fiber bundle

$$F \hookrightarrow M \xrightarrow{\pi} X \ ,$$

the basic forms relative to $\mathcal{F}_\pi$ are given by pull–backs from the base space X. In fact, there is a canonical isomorphism

$$\pi^* \ : \ \Omega^*(X, W) \xrightarrow{\cong} \Omega_b^*(M, \mathcal{F}_\pi; \pi^* W) \ , \tag{2.8}$$

where $T\mathcal{F}_\pi$ is given by the tangent bundle $T(\pi)$ along the fibers of π . This fact explains of course the origin of the terminology (cf. **[29]** **[39]**).

We also note that in the flat case (2.1) , the basic forms relative to the transverse foliation $\mathcal{F}_\alpha$ are exactly given by the extensions $\tilde{\varphi}$ of Γ–equivariant V–valued forms φ on F as described in (2.5) .

3. Equivariant and homogeneous bundles

In this and the following sections we consider a holomorphic fibration

$$F \hookrightarrow M \xrightarrow{\pi} X \ , \tag{3.1}$$

of compact complex analytic manifolds. We refer to **[27]** (Appendix Two by A. Borel) for the details of some of the following constructions. The structure group of this fibration is a complex Lie group G acting on F via a holomorphic map $\psi : G \times F \to F$. Let $g_{ij} : U_{ij} = U_i \cap U_j \to G$ be the transition functions defining the fibration where $\mathcal{U} = \{U_i\}$ is a suitable covering of X. If F is Kähler, the induced representation $\hat{\psi} : G \to GL\{H^{p,q}(F)\}$ is constant on the connected components of G, that is, $\hat{\psi}$ factors through $\pi_0(G)$. The composition $\hat{\psi} \circ g_{ij}$ defines then locally constant transition functions for the associated vector bundles

$$\mathcal{H}^{p,q}(F) = \bigcup_{x \in X} H^{p,q}(F_x) \ ,$$

and

$$\mathcal{H}_{\bar\partial}(F) = \bigoplus_{p,q} \mathcal{H}^{p,q}(F) \ ,$$

both of which are therefore *flat holomorphic* vector bundles on X .

If $P \to X$ is the holomorphic G–principal bundle determined by the cocycle $\{g_{ij}\}$, the fiber bundle (3.1) is associated to P by the formula

$$M \cong P \times_G F \xrightarrow{\pi} X \ . \tag{3.2}$$

To a G–equivariant holomorphic vector bundle $\mathcal{V} \xrightarrow{p} F$, we may associate a holomorphic vector bundle $\widetilde{\mathcal{V}} \to M$, called the *canonical extension* of $\mathcal{V}$ to M , by

$$\widetilde{\mathcal{V}} = P \times_G \mathcal{V} \xrightarrow{id \times p} M = P \times_G F \ . \tag{3.3}$$

The assignment $\mathcal{V} \to \widetilde{\mathcal{V}}$ is evidently an additive exact functor, compatible with tensor products, and $\widetilde{\mathcal{V}}$ satisfies the restriction property (2.3) .

Remark 3.1. On the topological level, the above construction determines a natural homomorphism of rings

$$\alpha_P \ : \ K_G(F) \longrightarrow K(P \times_G F) \ ,$$

which for $F = \mathrm{pt}$ specializes to the well–known homomorphism

$$\alpha_P \ : \ R(G) \longrightarrow K(X) \ .$$

The Hodge cohomology with coefficients in $\mathcal{V}$, defined by

$$H_{\bar\partial}(F, \mathcal{V}) = \bigoplus_{p,q} H^{p,q}(F, \mathcal{V}) \ ,$$
$$H^{p,q}(F, \mathcal{V}) = H^q(F, \Omega^p(F, \mathcal{V})) \ ,$$

is computed by the Dolbeault $\bar{\partial}$–complex (cf. **[27]**)

$$\mathcal{A}^{p,*}(F,\mathcal{V}) = \mathcal{V} \otimes_{\mathbb{C}} \Lambda^{p,*}_{\mathbb{C}}(T^*(F) \otimes_{\mathbb{R}} \mathbb{C}) \ .$$

Here $\Omega^p(F,\mathcal{V}) = \mathcal{V} \otimes_{\mathbb{C}} \Lambda^{p,0}_{\mathbb{C}}(T^*(F) \otimes_{\mathbb{R}} \mathbb{C})$ is the holomorphic bundle of $(p,0)$–forms with coefficients $\mathcal{V}$.

Even if F is Kähler, the induced representations

$$G_e \longrightarrow GL(H^{p,q}(F,\mathcal{V})) \ , \tag{3.4}$$

are no longer trivial in general. We will always assume that they are holomorphic and hence define a holomorphic associated bundle of fiber cohomology groups

$$\mathcal{H}_{\bar{\partial}}(F,\mathcal{V}) = P \times_G H_{\bar{\partial}}(F,\mathcal{V}) \longrightarrow X \ . \tag{3.5}$$

Suppose in particular that the above holomorphic fibration has the structure of a generalized flat bundle

$$F \hookrightarrow M = \widetilde{X} \times_\Gamma F \stackrel{\pi}{\longrightarrow} X \ , \tag{3.6}$$

where $\alpha : \Gamma \to G \subset \mathrm{Hol}(F)$ is taken to be a representation into the holomorphic diffeomorphisms of F. This is equivalent to saying that the transition functions $\{g_{ij}\}$ are locally constant. Thus the principal bundle P is flat as well, namely given by $P = \widetilde{X} \times_\Gamma G$. In this case, the bundle $\widetilde{\mathcal{V}}$ is given by formula (2.2) . The bundle of fiber cohomology groups (3.5) is a flat holomorphic vector bundle over X with respect to the representation $\tilde{\alpha} : \Gamma \to G \to GL\{H_{\bar{\partial}}(F,\mathcal{V})\}$. The bundle $\mathcal{H}_{\bar{\partial}}(F,\mathcal{V})$ may then be regarded as a *system of local coefficients*, whose associated sheaf of *locally constant sections* will be denoted by $\mathbf{H}_{\bar{\partial}}(F,\mathcal{V})$.

For a holomorphic vector bundle $\mathcal{W} \to X$, we denote by $\mathscr{W}$ the locally free holomorphic sheaf of $\mathscr{O}_X$–modules associated to $\mathcal{W}$ and mutatis mutandis for the other spaces involved.

Example 3.2. Homogeneous bundles :

Let $F = G/H \cong U/K$ be a compact symmetric Hermitian manifold, where

$$G = \mathrm{Hol}(F)_e \ , \ U = \mathrm{Hol}_{\mathrm{Iso}}(F)_e \ , \ K = U \cap H \ .$$

Much is known about these symmetric spaces and we refer to **[26]** Ch. VIII, **[34]** for details. In particular, F is simply connected, G and H are connected complex Lie groups with G semisimple and H parabolic. U and K are connected compact Lie groups; U is semisimple and K is the centralizer of a torus. Further, any K–invariant Hermitian metric on F is Kähler.

The equivariant holomorphic vector bundles on G/H are now homogeneous **[4]** **[47]** , that is, they are given by representations (ρ, V_ρ) of H :

$$\rho \mapsto \mathcal{V}_\rho = G \times_H V_\rho \ ,$$

inducing an isomorphism

$$R(H) \cong K_G(G/H) \ .$$

The canonical extension $\widetilde{\mathcal{V}}_\rho$ is then of the form

$$\widetilde{\mathcal{V}}_\rho \cong P \times_H V_\rho \to M = P/H \ .$$

For $(\rho', V_{\rho'}) \in R(G)$, the associated bundle

$$\widetilde{\mathcal{V}}_{\rho'} = P \times_G V_{\rho'} \to X \ ,$$

is a holomorphic vector bundle over X and $\widetilde{\mathcal{V}}_{i^*\rho'}$ and $\widetilde{\mathcal{V}}_{\rho'}$ are related by

$$\widetilde{\mathcal{V}}_{i^*\rho'} \cong \pi^* \widetilde{\mathcal{V}}_{\rho'} \ , \tag{3.7}$$

under the restriction map $i^* : R(G) \to R(H)$. In the flat case, $\widetilde{\mathcal{V}}_{\rho'} \to X$ is flat and so is therefore $\widetilde{\mathcal{V}}_{i^*\rho'} \to M$ by (3.7) .

We further have the Frobenius formula

$$H_{\bar\partial}(F, \mathcal{V}_{i^*\rho' \otimes \rho}) \cong V_{\rho'} \otimes_{\mathbb{C}} H_{\bar\partial}(F, \mathcal{V}_\rho) \ , \tag{3.8}$$

as G–modules and hence

$$\mathcal{H}_{\bar\partial}(F, \mathcal{V}_{i^*\rho' \otimes \rho}) \cong \widetilde{\mathcal{V}}_{\rho'} \otimes_{\mathbb{C}} \mathcal{H}_{\bar\partial}(F, \mathcal{V}_\rho) \ , \tag{3.9}$$

for $(\rho, V_\rho) \in R(H)$. The same is true if the pair of complex groups (G, H) is replaced by the corresponding compact pair (U, K) .

Relative to the Cartan sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{m} = \mathfrak{g}/\mathfrak{h} \to 0 \ ,$$

respectively the Cartan decomposition

$$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{m} \ , \tag{3.10}$$

and the complex adjoint representation $\rho_0 : H \to GL(\mathfrak{m}, \mathbb{C})$, the Dolbeault complex $\mathcal{A}^{p,*}(F, \mathcal{V}_\rho)$ is associated to the representation

$$V_\rho \otimes_{\mathbb{C}} \Lambda^{p,*}_{\mathbb{C}}(\mathfrak{m}^*_{\mathbb{C}}), \ \ \mathfrak{m}_{\mathbb{C}} = \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C} \ . \tag{3.11}$$

Since U is compact, semisimple and the differential $\bar\partial$ is U–invariant, the decomposition of the Dolbeault complex according to irreducible representations of U must preserve cohomology. The U–invariant forms are given by

$$A^{p,q}(F, \mathcal{V}_\rho)^U \cong (V_\rho \otimes_{\mathbb{C}} \Lambda^{p,q}_{\mathbb{C}}(\mathfrak{m}^*_{\mathbb{C}}))^K \ , \tag{3.12}$$

and we have therefore in particular

$$H^{p,q}(F, \mathcal{V}_\rho)^U \cong H^q(A^{p,*}(F, \mathcal{V}_\rho)^U) \cong H^q((V_\rho \otimes_{\mathbb{C}} \Lambda^{p,*}_{\mathbb{C}}(\mathfrak{m}^*_{\mathbb{C}}))^K) \ . \tag{3.13}$$

For $\rho = \mathbf{1}$, the Hodge cohomology $H_{\bar\partial}(F)$ is invariant under U and the total differential d on $\Lambda^{p,q}_{\mathbb{C}}(\mathfrak{m}^*_{\mathbb{C}})^K$ vanishes since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. Hence we have (cf. **[22]** IV)

$$H^{p,q}(F) \cong A^{p,q}(F)^U \cong \Lambda^{p,q}_{\mathbb{C}}(\mathfrak{m}^*_{\mathbb{C}})^K \ . \tag{3.14}$$

These formulas are very useful for explicit computations.

According to Bott's generalization of the Borel–Weil theorem **[4]** , $H^{0,*}(F, \mathcal{V}_\rho)$ is an irreducible U–module, if the induced highest weight of an irreducible representation $(\rho \ , V_\rho) \in R(K)$ is non–singular. The degree of the non-vanishing cohomology group is given by the index of the induced highest weight of ρ . If the induced highest weight of ρ is singular, then $H^{0,*}(F, \mathcal{V}_\rho) = 0$.

4. The Borel–Leray spectral sequence

Using the concept of extension of equivariant bundles, we give here a generalization of Borel's Theorem 2.1, p. 204 in **[27]** , to non-trivial coefficient bundles on the fiber.

Theorem 4.1. *Let $F \hookrightarrow M \to X$ be a holomorphic fiber bundle of compact complex analytic manifolds. Let $\mathcal{W}$ be a holomorphic vector bundle on X and $\mathcal{V}$ be a holomorphic G–equivariant vector bundle on F. Then there exists a spectral sequence (E_r, d_r), $(r \geq 0)$, with the following properties:*

1. *E_r is 4–graded by the base degree, the fiber degree and the complex type. Let ${}^{p,q}E_r^{s,t}$ be the subspace of elements of E_r of type (p,q), base degree s and fiber degree t . We have ${}^{p,q}E_r^{s,t} = 0$ if $p+q \neq s+t$, or if one of p,q,s,t is < 0 . The differential d_r maps ${}^{p,q}E_r^{s,t}$ into ${}^{p,q+1}E_r^{s+r,t\perp r+1}$.*
2. *The spectral sequence converges to $H_{\bar\partial}(M, \pi^*\mathcal{W} \otimes \widetilde{\mathcal{V}})$. For all $p,q \geq 0$, we have*

$${}^{p,q}E_\infty^{s,t} \cong \mathrm{Gr}^s \; H^{p,q}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}) \ ,$$

   for a suitable filtration of $H^{p,q}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}})$.

3. *For $p+q = s+t$, we have*

$${}^{p,q}E_2^{s,t} \cong \sum_{i \geq 0} H^{i,s\perp i}(X, \mathcal{W} \otimes_{\mathbb{C}} \mathcal{H}^{p\perp i, q\perp s+i}(F, \mathcal{V})) \ .$$

4. *If the fibration is a generalized flat bundle* (3.6)*, the bundle $\mathcal{H}_{\bar\partial}(F, \mathcal{V})$ is a flat holomorphic vector bundle and we have for $p+q = s+t$,*

$${}^{p,q}E_2^{s,t} \cong \sum_{i \geq 0} H^{i,s\perp i}(X, \mathscr{W} \otimes_{\mathbb{C}} \mathbf{H}^{p\perp i, q\perp s+i}(F, \mathcal{V})) \ .$$

   The conclusion also holds if $H_{\bar\partial}(F, \mathcal{V})$ is a trivial G_e–module, the flat structure being induced by the connecting homomorphism $\partial_* \ : \ \Gamma = \pi_1(X) \to \pi_0(G)$.

5. *If $\mathcal{W} = \mathbf{1}_X$ and $\mathcal{V} = \mathbf{1}_F$ are trivial of rank 1, then (E_r, d_r) is multiplicative and the isomorphisms of* (2) *and* (3) *are compatible with products.*
6. *If the fiber F is Kähler, $H_{\bar\partial}(F)$ is a trivial G_e–module.*

Proof. For our applications in § 5 to § 8 , it will be crucial to recognize the Borel spectral sequence as a special type of Leray spectral sequence. We therefore outline here an alternative proof to the one in **[27]** (Appendix 2) which allows non–trivial coefficients on the fiber. We will refer to this spectral sequence as the Borel–Leray spectral sequence.

First, we recall the Leray spectral sequence (cf. e.g. **[23]** **[24]**): For any (coherent) sheaf $\mathscr{F}$ of $\mathscr{O}_M$–modules, there exists a convergent spectral sequence

$$E_2^{s,t} = H^s(X, \mathscr{R}^t\pi_*(\mathscr{F})) \Rightarrow H^{s+t}(M, \mathscr{F}) \ . \tag{4.1}$$

We claim that for a suitable choice of $\mathscr{F}$, (4.1) coincides with the Borel spectral sequence of Theorem 4.1 .

Next, we need the projection formula for pull–backs **[25]** :

$$\mathscr{R}^t\pi_*(\pi^*\mathscr{W} \otimes_{\mathscr{O}_M} \mathscr{F}) = \mathscr{W} \otimes_{\mathscr{O}_X} \mathscr{R}^t\pi_*(\mathscr{F}) \ , \ t \geq 0 \ . \tag{4.2}$$

Third, the tangent bundle of the fiber F is obviously a G–bundle and so is the holomorphic bundle of forms $\Lambda^{p,0}_{\mathbb{C}}(T^*(F) \otimes_{\mathbb{R}} \mathbb{C})$ of type $(p,0)$. ¿From (3.3) it follows that we may extend the bundle $\mathcal{V} \otimes_{\mathbb{C}} \Lambda^{p,0}_{\mathbb{C}}(T^*(F) \otimes_{\mathbb{R}} \mathbb{C})$ to M and we will denote by $\Omega^p_{M/X}(\widetilde{\mathcal{V}}) = \widetilde{\mathscr{V}} \otimes_{\mathscr{O}_M} \widetilde{\Omega}^p_F$ the locally free $\mathscr{O}_M$–module of its holomorphic sections (this corresponds of course to relative forms on the tangent bundle along the fiber). Thus the graded sheaf of modules $\mathscr{F}^* = \Omega^*_M(\pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}})$ can be written as a bigraded sheaf

$$\begin{aligned}\mathscr{F}^* &= \Omega^*_M(\pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}) \\ &\cong \pi^*\Omega^*_X(\mathcal{W}) \otimes_{\mathscr{O}_M} \Omega^*_{M/X}(\widetilde{\mathcal{V}}) \ . \end{aligned} \tag{4.3}$$

The derived direct image $\mathscr{R}^t\pi_*(\widetilde{\mathcal{V}})$ of $\widetilde{\mathcal{V}}$ may be computed as the sheaf $\mathscr{H}^t(F,\mathcal{V})$ of holomorphic sections in the bundle $\mathcal{H}^{0,t}_{\bar\partial}(F,\mathcal{V})$ of fiber cohomologies **[4]**. Applied to the relative cotangent complex $\Omega^*_{M/X}(\widetilde{\mathcal{V}})$ this yields

$$\mathscr{R}^t\pi_*(\Omega^*_{M/X}(\widetilde{\mathcal{V}})) \cong \mathscr{H}^t(F,\Omega^*_F(\mathcal{V})) \cong \mathscr{H}^{*,t}_{\bar\partial}(F,\mathcal{V}) \ . \tag{4.4}$$

Thus we have, using (4.2) and (4.3)

$$\mathscr{R}^t\pi_*(\mathscr{F}^*) \cong \Omega^*_X(\mathcal{W}) \otimes_{\mathscr{O}_X} \mathscr{H}^t(F,\Omega^*_F(\mathcal{V})) \cong \Omega^*_X(\mathcal{W} \otimes_{\mathbb{C}} \mathcal{H}^{*,t}(F,\mathcal{V})) \ . \tag{4.5}$$

We observe that the total cohomology in the Borel–Leray spectral sequence in Theorem 4.1 (2) is given by the cohomology of M with coefficients $\mathscr{F}^*$ above, that is

$$H^{p,q}(M,\pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}) = H^p(M,\Omega^q_M(\pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}})) \ , \tag{4.6}$$

while the E_2–term in (4.1) , using (4.5) , is seen to coincide with the E_2–term of the Borel–Leray spectral sequence in Theorem 4.1 (3). In fact, we have

$$\begin{aligned} H^s(X,\mathscr{R}^t\pi_*(\mathscr{F}^*)) &\cong H^s(X,\Omega^*_X(\mathcal{W}) \otimes_{\mathscr{O}_X} \mathscr{H}^t(F,\Omega^*_F(\mathcal{V}))) \\ &\cong H^{*,s}(X,\mathcal{W} \otimes_{\mathbb{C}} \mathcal{H}^{*,t}(F,\mathcal{V})) \ . \end{aligned} \tag{4.7}$$

The equivalence of the two spectral sequences follows from a well–known argument in sheaf theory, using fine resolutions to compute derived functors **[21]** . The Leray spectral sequence is associated to the composition of left exact functors **[24]**

$$\Gamma(M,\mathscr{F}) = \Gamma(X,\pi_*\mathscr{F}) \ ,$$

while Borel's proof makes use of a Dolbeault–type resolution of the bigraded sheaf $\mathscr{F}^*$ of $\mathscr{O}_M$ modules in (4.3) , namely the 4–graded sheaf complex

$$\begin{aligned}\mathscr{M}^{a,b,c,d}(\mathcal{W},\mathcal{V}) &= \mathscr{A}(\pi^*\mathcal{W}) \otimes_{\mathscr{A}_M} \mathscr{A}(\widetilde{\mathcal{V}}) \otimes_{\mathscr{A}_M} \pi^*{\mathscr{A}_X}^{a,b} \otimes_{\mathscr{A}_M} \mathscr{A}^{c,d}_{M/X} \\ &\cong \pi^*\mathscr{W} \otimes_{\mathscr{O}_M} \widetilde{\mathscr{V}} \otimes_{\mathscr{O}_M} \pi^*{\mathscr{A}_X}^{a,b} \otimes_{\mathscr{A}_M} \mathscr{A}^{c,d}_{M/X} \\ &\cong \pi^*{\mathscr{A}_X}^{a,b}(\mathcal{W}) \otimes_{\mathscr{A}_M} \mathscr{A}^{c,d}_{M/X}(\widetilde{\mathcal{V}}) \ . \end{aligned} \tag{4.8}$$

Here $\mathscr{A}^{a,b}_{M/X} = \widetilde{\mathscr{A}_F}^{a,b}$ denotes the sheaf of smooth germs on M of fiberwise (a,b)–forms, that is (a,b)–forms along the tangent bundle of the fibers. Hence, applying π_* to (4.8) , we have

$$\pi_*\mathscr{M}^{a,b,c,d}(\mathcal{W},\mathcal{V}) = {\mathscr{A}_X}^{a,b}(\mathcal{W}) \otimes_{\mathscr{A}_X} \pi_*\mathscr{A}^{c,d}_{M/X}(\widetilde{\mathcal{V}}) \ . \tag{4.9}$$

It is then apparent that the $\bar\partial$–differentials defined in loc. cit. extend by linearity to the case with coefficients $\mathcal{V}$ and produce the required resolution.

The E_2–term (4.7) is now computed by

$$H^s(X,\mathscr{R}^t\pi_*(\mathscr{F}^*)) \cong H^s_{\bar\partial_X}\Gamma(X,\mathscr{A}^{*,*}_X(\mathcal{W} \otimes_{\mathbb{C}} \mathcal{H}^t_{\bar\partial_F}(\pi_*\mathscr{A}^{*,*}_{M/X}(\widetilde{\mathcal{V}})))) \ ,$$

while the total cohomology (4.6) is computed by

$$H^{p,q}(M,\pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}) = H^q_{\bar\partial}\,\Gamma(M,\mathscr{M}^{p,*}) \ .$$

Part (4) of the theorem follows by a standard argument about local systems of coefficients, since $\mathcal{H}_{\bar\partial}(F,\mathcal{V})$ is holomorphically flat. □

Let F be a compact complex manifold. The Hodge to DeRham spectral sequence

$$H^{p,q}(F) = H^p_\partial(H^q(F, \Omega^*(F))) \Rightarrow H^{p+q}_{DR}(F, \mathbb{C}) \ ,$$

associated to the Dolbeault double complex $A^{*,*}(F)$, has E_1–term given by the Hodge groups $H^q(F, \Omega^p(F))$ and the differential d_1 is given by the differential ∂ on the sheaf complex Ω^*_F of holomorphic forms. If (F, ω_F) is a Kähler manifold, this spectral sequence degenerates at the E_1–term.

Theorem 5.1. [48] *For a Kähler manifold F, there is a multiplicative isomorphism*

$$H^*_{\bar\partial}(F) \cong H^*_{DR}(F, \mathbb{C}) \ . \tag{5.1}$$

Thus the homotopy invariance of DeRham cohomology implies that G_e acts trivially on $H_{\bar\partial}(F)$. We also recall Deligne's theorem on the degenerescence of the Borel–Leray spectral sequence.

Theorem 5.2. [14] *If the total space M and the base space X in the fiber bundle* (3.1) *are compact Kähler manifolds (hence the fiber F is Kähler), $\mathcal{W} = \mathbf{1}_X$ and $\mathcal{V} = \mathbf{1}_F$, then the Borel–Leray spectral sequence degenerates at the E_2–term and we have*

$$E_2 \cong E_\infty \ ,$$

that is, there is a multiplicative isomorphism

$$\mathrm{Gr}^* \ H^{*,*}_{\bar\partial}(M) \cong H^{*,*}_{\bar\partial}(X, \mathbf{H}^{*,*}_{\bar\partial}(F)) \ . \tag{5.2}$$

For connected G there are therefore module isomorphisms over X, that is, additive Kunneth formulas:

$$H^{*,*}_{\bar\partial}(M) \cong H^{*,*}_{\bar\partial}(X) \otimes_{\mathbb{C}} H^{*,*}_{\bar\partial}(F) \ ,$$

and

$$H^*_{DR}(M, \mathbb{C}) \cong H^*_{DR}(X, \mathbb{C}) \otimes_{\mathbb{C}} H^*_{DR}(F, \mathbb{C}) \ .$$

Via the Chern character, this gives a Kunneth formula for rational K–theory as well

$$K^*(M)_{\mathbb{Q}} \cong K^*(X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} K^*(F)_{\mathbb{Q}} \ .$$

We state now the main result of this section, using Theorem 4.1 to obtain a Kunneth type formula for Hodge cohomology with non–trivial coefficients. This formula is crucial in our discussion of holomorphic extensions in § 7 .

Theorem 5.3. *Fix $0 \le p_0 \le l$ and assume there exists an integer m , $p_0 < m \le 2l$ so that*

$$H^{u,v}(F, \mathcal{V}) = H^v(F, \Omega^u(F, \mathcal{V})) = 0 \ , \ \textit{for } 0 \le u+v < m \ , \ 0 \le u \le p_0 \ . \tag{5.3}$$

For $0 \le p \le p_0$, the Borel–Leray spectral sequence has the following properties:

1. ${}^{p,q}E_2^{s,t} = 0$, *for $p+q < s+m$.*
2. *The total cohomology $H^{p,q}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}})$ vanishes for $p+q < m$.*
3. *For $p+q = m$, there is a canonical isomorphism*
$$H^{p,q}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}) \cong {}^{p,q}E_2^{0,m} \cong H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathcal{H}^{p,q}(F, \mathcal{V})) \ .$$
4. *For $(\rho', V_{\rho'}) \in R(G)$, we have in the homogeneous case*
$$\begin{aligned} H^{p,q}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_{i^*\rho'}) &\cong H^{p,q}(M, \pi^*(\mathcal{W} \otimes_{\mathbb{C}} \mathcal{V}_{\rho'})) \\ &\cong H^0(X, (\mathcal{W} \otimes_{\mathbb{C}} \mathcal{V}_{\rho'}) \otimes_{\mathbb{C}} \mathbf{H}^{p,q}(F)) \ . \end{aligned}$$
5. *If $H^{p,q}(F, \mathcal{V})$ is a trivial G–module, or if $F \to M \to X$ is flat and the Γ–action trivial, the holomorphic vector bundle $\mathcal{H}^{p,q}(F, \mathcal{V})$ is trivial :*
$$H^{p,q}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}) \cong H^0(X, \mathcal{W}) \otimes_{\mathbb{C}} H^{p,q}(F, \mathcal{V}) \ .$$

Proof. First, we observe that the condition $p+q<s+m$ is equivalent to $t<m$, since non–zero terms occur only for $p+q=s+t$. Part (1) follows from the assumption (5.3) and Theorem 4.1 (3). In fact, the total fiber degrees in the formula for the E_2–terms satisfy $u+v=(p-i)+(q-s+i)=p+q-s=t<m$ and $u=p-i\leq p\leq p_0$. Thus we have ${}^{p,q}E_2^{s,t}=0$, for $p+q<m,\ s\geq 0$ or $p+q=m,\ s>0$. Part (2) follows immediately from this and the expression for the total cohomology in Theorem 4.1 (2). For $p+q=m$, we have further ${}^{p,q}E_2^{0,m}=H^0(X,\mathcal{W}\otimes_{\mathbb{C}}\mathcal{H}^{p,q}(F,\mathcal{V}))$. Since the spectral sequence has no non-zero terms for $t<m, p<p_0$, the assertion in part (3) follows from a standard argument, e.g. the 5–term exact sequence for $p+q=m$,

$$0\to {}^{p,q}E_2^{m,0}\to H^{p,q}(M,\pi^*\mathcal{W}\otimes_{\mathbb{C}}\widetilde{\mathcal{V}})\to {}^{p,q}E_2^{0,m}\xrightarrow{d_{m+1}} {}^{p,q+1}E_2^{m+1,0}\to H^{p,q+1}(M,\pi^*\mathcal{W}\otimes_{\mathbb{C}}\widetilde{\mathcal{V}})\ .$$

Observing that the base terms are zero by part (1), we conclude that the edge homomorphism

$$H^{p,q}(M,\pi^*\mathcal{W}\otimes_{\mathbb{C}}\widetilde{\mathcal{V}})\to {}^{p,q}E_2^{0,m}$$

is an isomorphism. Part (4) follows from (3.7) , the Frobenius formula (3.9) and Theorem 5.1 . Part (5) follows from Theorem 4.1 (4). In fact, under our assumption, the coefficient bundle $\mathcal{H}^{p,q}(F,\mathcal{V})$ is holomorphically trivial. □

Corollary 5.4. *Suppose G is connected, semisimple and*

$$H^0(F,\mathcal{V})=0\ ,\ \dim_{\mathbb{C}}H^{0,1}(F,\mathcal{V})=1\ .$$

Then $H^0(M,\pi^*\mathcal{W}\otimes_{\mathbb{C}}\widetilde{\mathcal{V}})=0$, the bundle of fiber cohomologies $\mathcal{H}^{0,1}(F,\mathcal{V})$ is a holomorphically trivial line bundle and we have

$$H^{0,1}(M,\pi^*\mathcal{W}\otimes_{\mathbb{C}}\widetilde{\mathcal{V}})\cong H^0(X,\mathcal{W}\otimes_{\mathbb{C}}\mathcal{H}^{0,1}(F,\mathcal{V}))\cong H^0(X,\mathcal{W})\otimes_{\mathbb{C}}H^{0,1}(F,\mathcal{V})\cong H^0(X,\mathcal{W})\ ,$$

for any holomorphic vector bundle $\mathcal{W}$ on X .

Proof. We only need to observe that G, being connected and semisimple, has no non–trivial 1–dimensional representation. □

6. Projective fiber bundles

We now consider the case where M is the total space of a (holomorphic) projective bundle over the compact Kähler manifold X with structure group $G=PGL(l+1,\mathbb{C})$:

$$\mathbb{CP}^l\hookrightarrow M=P\times_{PGL}\mathbb{CP}^l\xrightarrow{\pi}X\ . \tag{6.1}$$

The linear and projective groups are related by the commutative diagram

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}_{l+1} & \xrightarrow{\Delta} & SL(l+1) & \xrightarrow{j_0} & PSL(l+1) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle\cong} & & \\ 1 & \longrightarrow & \mathbb{C}^\times & \xrightarrow{\Delta} & GL(l+1) & \xrightarrow{j} & PGL(l+1) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle(\cdot)^{l+1}} & & \downarrow{\scriptstyle\det} & & & & \\ & & \mathbb{C}^\times & \xrightarrow{=} & \mathbb{C}^\times & & & & \end{array} \tag{6.2}$$

Let $E \to X$ be a holomorphic vector bundle of rank $l+1$. The projectivisation $\mathbb{P}(E)$ gives rise to a holomorphic projective bundle

$$\mathbb{CP}^l \hookrightarrow \mathbb{P}(E) \xrightarrow{\pi} X \ . \tag{6.3}$$

If $P_E = F(E)$ denotes the holomorphic frame bundle of E, then the $PGL(l+1,\mathbb{C})$–principal bundle is given by $P = j_*(P_E)$ and

$$\mathbb{P}(E) \cong P_E \times_{GL} \mathbb{CP}^l \cong P \times_{PGL} \mathbb{CP}^l \ .$$

The topology of projective bundles is essentially derived from the commutative diagram of groups (6.2) at the level of classifying spaces. There are obstructions for the linearization (respectively the unimodular linearization) of a projective fiber bundle. For convenience, we discuss these topics in an Appendix (§ 11).

Here we mention only the following facts. If E is projectively flat, that is

$$\mathbb{CP}^l \hookrightarrow M \cong \widetilde{X} \times_{\hat{\alpha}} \mathbb{CP}^l \xrightarrow{\pi} X \ , \tag{6.4}$$

with holonomy $\hat{\alpha} : \Gamma \to PGL(l+1,\mathbb{C})$, it follows from Chern–Weil theory (cf. **[34]**) that the Chern classes of E are determined by the first Chern class $c_1(E)$, namely

$$\begin{aligned} c_k(E) &= \binom{l+1}{k} \{\frac{c_1(E)}{l+1}\}^k \ , \\ c_t(E) &= \{1 + t\ \frac{c_1(E)}{l+1}\}^{l+1} \ . \end{aligned} \tag{6.5}$$

For $k=2$, we obtain therefore the strong Bogomolov relation

$$c_2(E) = \frac{l}{2(l+1)}\ c_1(E)^2 \ . \tag{6.6}$$

At this point, we recall the Hitchin–Kobayashi correspondence **[34]** for a holomorphic vector bundle $E \to X$ over a compact Kähler manifold (X, ω_X) . We define the normalized degree of $E \to X$ relative to ω_X by the formula

$$\begin{aligned} \deg_X(E) &= \frac{1}{(n-1)!\ \mathrm{Vol}(X)} \int_X c_1(E) \wedge \omega_X^{n-1} \\ &= \frac{1}{n!\ \mathrm{Vol}(X)} \int_X \Lambda_X\ c_1(E)\ \omega_X^n \ , \\ &= \frac{1}{\mathrm{Vol}(X)} \int_X \Lambda_X\ c_1(E)\ \mathrm{dvol}_X \ , \end{aligned} \tag{6.7}$$

where Λ_X is contraction against the Kähler form ω_X and the volume element dvol_X is taken to be

$$\mathrm{dvol}_X = \frac{\omega_X^n}{n!} \ .$$

The *slope* of E is defined to be

$$\mu_E = \frac{\deg_X(E)}{\mathrm{rank}(E)} \ .$$

$(E,h) \to X$ is a *Hermitian–Einstein* bundle if

$$\iota\ \Lambda_X F_h = 2\pi\ \lambda\ \mathbf{I}_E \ , \tag{6.8}$$

where F_h is the curvature of type $(1,1)$ of the unitary, integrable connection (Chern connection) determined by the Hermitian metric h .

The above definition of the degree has the advantage that the Hermitian–Einstein constant λ is given by the slope μ_E. In fact, we have the Chern–Weil formula $c_1(E) = \frac{\iota}{2\pi}[\mathrm{Tr}\, F_h]$ and hence

$$\int_X \Lambda_X c_1(E) \cdot \omega_X^n = \frac{\iota}{2\pi} \int_X \Lambda_X\ \mathrm{Tr}\, F_h \cdot \omega_X^n = \lambda\ \mathrm{rank}(E)\ n!\ \mathrm{Vol}(X) \ ,$$

which implies $\lambda = \mu_E$.

The Hitchin–Kobayashi correspondence (cf. **[34]**) may then be stated as follows.

Let $E \to X$ be a holomorphic vector bundle over the compact Kähler manifold (X, ω_X) . Then the following conditions are equivalent:

(1) E is polystable, that is

$$E = \bigoplus_j E_j \ ,$$

where the E_j are stable (relative to ω_X) holomorphic bundles of equal slope $\mu_{E_j} = \mu_E$;

(2) E admits a solution of the Hermitian–Einstein equation

$$\iota \, \Lambda_X F_h = 2\pi \ \mu_E \ \mathbf{I}_E \ .$$

Together with the Bogomolov relation below (for $n = \dim_{\mathbb{C}}(X) \geq 2$), (2) is further equivalent to :

(3) E is projectively flat, that is $\mathbb{P}(E)$ is flat with holonomy $\hat{\alpha} \ : \ \Gamma \to PU(l+1)$.

The existence theorem (1) $\Rightarrow$ (2) is due to Donaldson **[15]** and Uhlenbeck–Yau **[45]** . If E is projectively flat, E is obviously Hermitian–Einstein and the equation (6.6) implies the Bogomolov relation

$$\int_X (l \ c_1(E)^2 - 2(l+1) \ c_2(E)) \wedge \omega_X^{n-2} = 0 \ . \tag{6.9}$$

This latter relation is sufficient to prove (2) $\Rightarrow$ (3). In fact, if (2) holds, then (3) is equivalent to (6.9) (cf. **[36]**).

Example 6.1. The case of $\mathrm{rank}(E) = 2$:

$\mathbb{P}(E)$ has a flat $PU(2)$–structure exactly in one of the following two cases.

(1) E is stable and $\int_X (c_1(E)^2 - 4c_2(E)) \wedge \omega_X^{n-2} = 0$;

(2) $E = L_1 \oplus L_2$, with $c_1(L_1) = c_1(L_2)$ and hence $\int_X (c_1(L_1) - c_1(L_2))^2 \wedge \omega_X^{n-2} = 0$. For a proof of (2), see **[11]** .

On the projectivized bundle $\mathbb{P}(E)$, we have the tautological line bundle $\mathcal{H}_E^* \to M$ which is defined by

$$\mathcal{H}_E^* = \{(l, v) \in M \times_X E \ \mid \ v \in l\} \subset \pi^* E \ . \tag{6.10}$$

We follow the common notation and denote the powers $\mathcal{H}_E^k$ by $\mathcal{O}_M(k)$, for $k \in \mathbb{Z}$. Similarly, we denote $\mathcal{H}^k = \mathcal{H}_E^k|_{\mathbb{CP}^l}$ on $\mathbb{CP}^l$ by $\mathcal{O}(k)$.

¿From the definition (6.10) and the multiplicativity of the extension (3.3) , we obtain directly the following Lemma.

Lemma 6.2. *The tautological bundle $\mathcal{H}_E^*$ on $M = \mathbb{P}(E)$ is the canonical extension of the $GL(l+1, \mathbb{C})$–equivariant tautological bundle $\mathcal{H}^*$ on $\mathbb{CP}^l$ and we have for any $k \in \mathbb{Z}$:*

$$\mathcal{O}_M(k) \cong \widetilde{\mathcal{O}}(k) \ . \tag{6.11}$$

The exact Euler sequence

$$0 \to \Omega^1_{M/X} \longrightarrow (\pi^* E^*)(-1) \longrightarrow \mathcal{O}_M \to 0 \ , \tag{6.12}$$

derived from (6.10) , is the canonical extension of the $GL(l+1, \mathbb{C})$–equivariant exact sequence

$$0 \to \Omega^1(\mathbb{CP}^l) \longrightarrow \mathcal{V}_0^*(-1) \longrightarrow \mathcal{O} \to 0 \ , \tag{6.13}$$

where $\mathcal{V}_0 = \mathbb{CP}^l \times \mathbb{C}^{l+1}$ is the $GL(l+1, \mathbb{C})$–equivariant bundle with the standard action of $GL(l+1, \mathbb{C})$.

The relative canonical bundle $\mathcal{K}_{M/X}$ of $\mathbb{P}(E)$ is computed from the determinant bundle of (6.12) . Setting $\mathcal{L} = \det E = \Lambda^{l+1}E$, we have (cf. **[25]**)

$$\mathcal{K}_{M/X} = \Omega^l_{M/X} \cong \pi^*(\Lambda^{l+1}E^*) \otimes_{\mathbb{C}} \mathcal{O}_M(-l-1) \cong \pi^*\mathcal{L}^* \otimes_{\mathbb{C}} \mathcal{O}_M(-l-1) \; . \tag{6.14}$$

Equivalently, $\mathcal{K}_{M/X}$ may be computed as the extension of the $PGL(l+1,\mathbb{C})$–equivariant canonical bundle

$$\mathcal{K}_{\mathbb{CP}^l} = \Omega^l(\mathbb{CP}^l) \cong (\det \mathcal{V}_0)^* \otimes_{\mathbb{C}} \mathcal{O}(-l-1) \; , \tag{6.15}$$

obtained from the Euler sequence (6.13) on $\mathbb{CP}^l$.

It follows from the projection formula and (6.14) that

$$\mathcal{L} = \pi_*(\mathcal{K}^*_{M/X}(-l-1)) \; . \tag{6.16}$$

The *multiplicative* structure of the cohomology ring of M is determined by the Leray–Hirsch theorem (cf. **[5]** **[23]** **[31]**) :

$$H^*(\mathbb{P}(E),\mathbb{Z}) \cong H^*(X,\mathbb{Z}) \; [t] \; / \; \{\sum_{j=0}^{l+1} c_j(E) t^{l+1-j}\} \; , \tag{6.17}$$

where t corresponds to the first Chern class $c_1(\mathcal{H}_E)$ of the tautological bundle $\mathcal{H}_E$. In other words, $H^*(\mathbb{P}(E),\mathbb{Z})$ is generated as an $H^*(X,\mathbb{Z})$–algebra by $c_1(\mathcal{H}_E)$ subject to the defining equation

$$\sum_{j=0}^{l+1} \pi^* c_j(E) \; c_1(\mathcal{H}_E)^{l+1-j} = 0 \; .$$

This shows that the Chern classes $c_j(E)$ measure how the ring structure of $H^*(\mathbb{P}(E))$ deviates from that of the product $H^*(X \times \mathbb{CP}^l)$.

If X is a compact Kähler manifold and E holomorphic, there is an analogous result for Hodge cohomology. The Chern class $t = c_1(\mathcal{H}_E)$ is of type $(1,1)$ and the classes $c_j(E)$ of type (j,j). We obtain then from (5.1) and (6.5) corresponding *multiplicative* Kunneth formulas for Hodge cohomology.

Theorem 6.3. *Let $E \to X$ be a holomorphic vector bundle over a compact Kähler manifold X.*

(1) *There is a multiplicative isomorphism*

$$H^{*,*}_{\bar\partial}(\mathbb{P}(E)) \cong H^{*,*}_{\bar\partial}(X) \; [t] \; / \; \{\sum_{j=0}^{l+1} c_j(E) \; t^{l+1-j}\} \; ; \tag{6.18}$$

(2) *If E is projectively flat, there is a multiplicative isomorphism*

$$H^{*,*}_{\bar\partial}(\mathbb{P}(E)) \cong H^{*,*}_{\bar\partial}(X) \; [t] \; / \; \{(t + \frac{c_1(E)}{l+1})^{l+1}\} \; ; \tag{6.19}$$

(3) *If E is topologically flat, there is a multiplicative isomorphism*

$$H^{*,*}_{\bar\partial}(\mathbb{P}(E)) \cong H^{*,*}_{\bar\partial}(X) \otimes_{\mathbb{C}} H^{*,*}_{\bar\partial}(\mathbb{CP}^l) \; .$$

Thus for $M = \mathbb{P}(E)$, we have a more precise form of Deligne's Theorem 5.2 on the degenerescence of the Borel–Leray spectral sequence. If E is flat, we have indeed a *multiplicative* Kunneth formula for the Hodge cohomology of such a twisted product.

We now apply the degeneracy results of the previous section to the case of a projective fiber bunde (6.1) . The *PGL*–equivariant holomorphic line bundles on $\mathbb{CP}^l$ are the powers of the canonical bundle $\mathcal{K}_{\mathbb{CP}^l}$ and the relative canonical bundle $\mathcal{K}_{M/X}$ is given by the *PSL*–extension $\mathcal{K}_{M/X} = \widetilde{\mathcal{K}}_{\mathbb{CP}^l}$. The line bundle $\mathcal{O}(-l-1)$ and its powers carry a canonical *PSL*–structure isomorphic to that on $\mathcal{K}_{\mathbb{CP}^l}$ and we have

$$\mathcal{K}_{M/X} \cong \mathcal{O}(-l-1)^{\sim}_{PSL} \ . \tag{6.20}$$

Proposition 6.4. *The vanishing conditions* (5.3) *are satisfied and* $H^{0,m}(\mathbb{CP}^l, \mathcal{V})$ *is a trivial* $PGL(l+1,\mathbb{C})$*–module of rank* 1*, for* $\mathcal{V} = \Omega^m(\mathbb{CP}^l)$, $p_0 = 0 < m \leq l$:

$$H^{0,m}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \Omega^m_{M/X}) \cong H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathbf{H}^{0,m}(\mathbb{CP}^l, \Omega^m(\mathbb{CP}^l))) \cong H^0(X, \mathcal{W}) \ .$$

Proof. This follows from Corollary 5.4 , since $H^{p,q}(\mathbb{CP}^l) = H^q(\mathbb{CP}^l, \Omega^p(\mathbb{CP}^l)) \cong \mathbb{C}$ for $p = q$ and zero otherwise. □

In the linear case (6.3) , Theorem 5.3 and the Bott formulas for $H^{p,q}(\mathbb{CP}^l, \mathcal{O}(k))$ (cf. **[41]**, **[25]**III, §8) yield the following result.

Proposition 6.5. *The vanishing conditions* (5.3) *are satisfied for* $\mathcal{V} = \mathcal{O}(k)$, $0 \leq p_0 \leq l$, $m = p_0 + l$ *and* $p_0 - l - 1 \leq k < 0$ *. Hence we have for* $0 \leq p \leq p_0$, $p + q \leq p_0 + l$:

$$H^{p,q}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \mathcal{O}_M(k)) \cong H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathcal{H}^{p,q}(\mathbb{CP}^l, \mathcal{O}(k))) \ .$$

In addition, we have:

(1) $\mathscr{H}^{0,l}(\mathbb{CP}^l, \mathscr{O}(k)) = \mathscr{R}^l\pi_*(\mathscr{O}_M(k)) \cong \mathscr{L} \otimes_{\mathscr{O}_X} \pi_*(\mathscr{O}_M(-k-l-1))^*$ *, and hence*

$$H^{0,l}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \mathcal{O}_M(k)) \cong H^0(X, (\mathcal{W} \otimes_{\mathbb{C}} \mathcal{L}) \otimes_{\mathbb{C}} \pi_*(\mathcal{O}_M(-k-l-1))^*) \ .$$

(2) *In particular, for* $\mathcal{V} = \mathcal{O}(-l-1)$, $\mathcal{H}^{0,l}(\mathbb{CP}^l, \mathcal{O}(-l-1)) \cong \mathcal{L}$ *is the determinant bundle* $\mathcal{L} = \det E$ *and*

$$H^{0,l}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \mathcal{O}_M(-l-1)) \cong H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathcal{L}) \ .$$

(3) *If E is unimodular, we have*

$$H^{0,l}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \mathcal{O}_M(-l-1)) \cong H^0(X, \mathcal{W}) \otimes_{\mathbb{C}} \mathbb{C} \cong H^0(X, \mathcal{W}) \ .$$

Proof. We need to check the vanishing conditions (5.3) , that is

$$H^{u,v}(\mathbb{CP}^l, \mathcal{O}(k)) = 0 \ , \ \text{for } 0 \leq u + v < m = p_0 + l \ , \ 0 \leq u \leq p_0 \ .$$

It follows from the Bott formulas that $H^{u,v}(\mathcal{O}(k)) = 0$ for $v < l$ and $k < 0$. For $v = l$, we have $u \leq p_0 - 1 < l$ and the Bott formulas imply that $H^{u,l}(\mathcal{O}(k)) = 0$ for $k \geq p_0 - l - 1 \geq u - l$. □

We also note that we have the formula

$$\mathscr{R}^l\pi_*(\mathscr{K}_{M/X}) = \mathscr{H}^{0,l}(\mathbb{CP}^l, \mathscr{K}_{\mathbb{CP}^l}) = \mathscr{H}^{l,l}(\mathbb{CP}^l) \cong \mathscr{O}_X \ , \tag{6.21}$$

with the last isomorphism being induced by the volume form $\eta = \omega^l$, where ω is the Kähler form of $\mathbb{CP}^l$. Using (6.14) and (6.21) we may reformulate Proposition 6.5 (2) as follows:

$$\begin{aligned} H^{0,l}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \mathcal{O}_M(-l-1)) &\cong H^{0,l}(M, \pi^*(\mathcal{W} \otimes_{\mathbb{C}} \mathcal{L}) \otimes_{\mathbb{C}} (\pi^*\mathcal{L}^* \otimes_{\mathbb{C}} \mathcal{O}_M(-l-1))) \\ &\cong H^{0,l}(M, \pi^*(\mathcal{W} \otimes_{\mathbb{C}} \mathcal{L}) \otimes_{\mathbb{C}} \mathcal{K}_{M/X}) \\ &\cong H^0(X, (\mathcal{W} \otimes_{\mathbb{C}} \mathcal{L}) \otimes_{\mathbb{C}} \mathcal{H}^{0,l}(\mathbb{CP}^l, \mathcal{K}_{\mathbb{CP}^l})) \\ &\cong H^0(X, (\mathcal{W} \otimes_{\mathbb{C}} \mathcal{L}) \otimes_{\mathbb{C}} \mathcal{H}^{l,l}(\mathbb{CP}^l)) \\ &\cong H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathcal{L}) \ , \end{aligned} \tag{6.22}$$

with the last isomorphism again being induced by the fiber volume η .

The above observation allows us to construct an explicit inverse to the edge isomorphism in Proposition 6.5 (2) .

Proposition 6.6.

(1) *The fiber volume form η is $PSL(l+1)$–invariant and extends to a closed form $\tilde{\eta}$ of (fiber) type (l,l) on M.*

(2) *The inverse of the edge isomorphism in Corollary* 6.5 (2) *is induced by the assigment $\phi \mapsto \beta_\phi$, where*

$$\beta_\phi = \pi^*\phi \otimes \tilde{\eta} \ , \tag{6.23}$$

for $\phi \in H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathcal{L})$.

Proof. The Kähler form ω is harmonic and hence $PU(l+1)$–invariant. Its canonical extension $\tilde{\omega}$ is of fiber type $(1,1)$ and further extends to a closed form on M, which we also denote by $\tilde{\omega}$. In fact, we may take for $\tilde{\omega}$ the first Chern form of the holomorphic line bundle $\mathcal{H}_E = \mathcal{O}_M(1)$, using a suitable connection (compare (6.17)). The form $\tilde{\eta} = \tilde{\omega}^l$, evidently restricts to a generator of $H^{l,l}(\mathbb{CP}^l) \cong \mathbb{C}$ on each fiber and one checks easily that (6.23) defines an inverse to the above edge isomorphism, provided the volume form η is normalized, that is $\int_{\mathbb{CP}^l} \eta = 1$. □

7. Holomorphic extensions and their parametrization

We begin with a flat holomorphic bundle (3.6)

$$F \hookrightarrow M = \widetilde{X} \times_\Gamma F \xrightarrow{\pi} X \ ,$$

and consider a type of holomorphic bundle over M on which a dimensional reduction of the Hermitian–Einstein equations will be possible. Recall that in the case where $M = X \times \mathbb{CP}^1$, the appropriate class of bundles consists of those with specially chosen $SU(2)$–equivariance properties (cf. **[18]**). In the present, more general setting, we generalize the $SU(2)$–equivariance by the requirement of compatibility with the two foliations of M. In particular, we consider holomorphic bundles $\mathcal{E} \to M$, having the following properties :

(1) $\mathcal{E}$ is a holomorphic extension of $\mathcal{E}_2$ by $\mathcal{E}_1$;

(2) their smooth structure is that of a direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$, where $\mathcal{E}_i$ is a tensor product of two bundles with one factor being foliated with respect to $\mathcal{F}_\pi$, and the other being foliated with respect to $\mathcal{F}_\alpha$.

Let $\mathcal{W}_i \to X$ and $\mathcal{V}_i \to F$, $i = 1, 2$, be holomorphic vector bundles, where the $\mathcal{V}_i$ are G–equivariant and thus extend to the bundles $\widetilde{\mathcal{V}}_i$ on M. Then we set $\mathcal{E}_i = \pi^*\mathcal{W}_i \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_i$, and observe that by Theorem 2.2, the holomorphic bundles $\mathcal{E}_i$ satisfy the second of the above requirements.

For the remainder of this section we will be concerned with the first requirement, namely the nature of the holomorphic extension. Except for one Lemma, the flatness condition is not needed in this section and it will be sufficient to have a complex fiber bundle as in (3.1) , respectively a projective bundle as in (6.1) .

Recall from **[25]** III, §6, that in terms of the corresponding locally free sheaves of $\mathscr{O}_M$–modules on M, an extension of $\mathscr{E}_2$ by $\mathscr{E}_1$ is a short exact sequence

$$0 \longrightarrow \mathscr{E}_1 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}_2 \longrightarrow 0 \ , \tag{7.1}$$

over M. In view of the properties of the functor 'Ext' **[25]** III, §6, such holomorphic extensions are parametrized by classes in

$$\begin{aligned} \operatorname{Ext}^1_{\mathscr{O}_M}(\mathscr{E}_2, \mathscr{E}_1) &\cong \operatorname{Ext}^1_{\mathscr{O}_M}(\mathscr{O}_M, \mathscr{E}_1 \otimes_{\mathscr{O}_M} \mathscr{E}_2{}^*) \\ &\cong \operatorname{Ext}^1_{\mathscr{O}_M}(\mathscr{O}_M, \mathscr{H}om_{\mathscr{O}_M}(\mathscr{E}_2, \mathscr{E}_1)) \\ &\cong H^{0,1}(M, \mathcal{H}om_{\mathbb{C}}(\mathcal{E}_2, \mathcal{E}_1)) \ . \end{aligned} \tag{7.2}$$

For the bundles $\mathcal{E}_i = \pi^*\mathcal{W}_i \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_i$ as above, we have then

$$\operatorname{Ext}^1_{\mathscr{O}_M}(\mathscr{E}_2, \mathscr{E}_1) \cong H^{0,1}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}) \ , \tag{7.3}$$

where $\mathcal{W} = \mathcal{H}om_{\mathbb{C}}(\mathcal{W}_2, \mathcal{W}_1)$ and $\mathcal{V} = \mathcal{H}om_{\mathbb{C}}(\mathcal{V}_2, \mathcal{V}_1)$.

Proposition 7.1. *Suppose that G is connected, semisimple and*

$$H^0(F,\mathcal{V}) = 0 \ , \ \dim_{\mathbb{C}} H^{0,1}(F,\mathcal{V}) = 1 \ .$$

Then $\mathcal{H}^{0,1}(F,\mathcal{V})$ is holomorphically trivial and

$$\mathrm{Ext}^1_{\mathscr{O}_M}(\mathscr{E}_2,\mathscr{E}_1) \cong H^{0,1}(M,\pi^*\mathcal{W}\otimes_{\mathbb{C}}\widetilde{\mathcal{V}}) \cong H^0(X,\mathcal{W})\otimes_{\mathbb{C}} H^{0,1}(F,\mathcal{V}) \cong H^0(X,\mathcal{W}) \ , \tag{7.4}$$

for any holomorphic vector bundle $\mathcal{W}$ on X.

This follows directly from Corollary 5.4 .

In § 8 it will be important to have an explicit realization of the isomorphism in Proposition 7.1 .

Lemma 7.2. *Suppose that the fiber bundle (3.1) is flat with holonomy $\alpha : \Gamma \to U$, with $U \subseteq \mathrm{Hol}_{\mathrm{Iso}}(F)_e$ connected, compact semisimple, and that the U–equivariant holomorphic vector bundle $\mathcal{V}$ satisfies the conditions in Proposition 7.1 .*

Then the following hold:

1. *$H^{0,1}(F,\mathcal{V}) \cong \mathbb{C}$ is generated by an invariant, $\bar\partial$–closed $(0,1)$–form $\eta \in A^{0,1}(F,\mathcal{V})^U$.*
2. *η extends to a $\bar\partial$–closed $(0,1)$–form $\tilde\eta \in A^{0,1}(M,\widetilde{\mathcal{V}})$.*
3. *$[\tilde\eta]$ is a generator of $H^{0,1}(M,\widetilde{\mathcal{V}})$, that is $H^{0,1}(M,\widetilde{\mathcal{V}}) \cong \mathbb{C}$.*
4. *There is a one–one–correspondence between holomorphic sections $\phi \in H^0(X,\mathcal{W})$ and classes $[\beta_\phi] \in H^{0,1}(M,\pi^*\mathcal{W}\otimes_{\mathbb{C}}\widetilde{\mathcal{V}})$, given by*

$$\beta_\phi = \pi^*\phi \otimes \tilde\eta \ .$$

Proof. Since $H^{0,1}(F,\mathcal{V})$ is a trivial U–module, we have from (3.13)

$$H^{0,1}(F,\mathcal{V}) \cong H^1_{\bar\partial}(A^{0,*}(F,\mathcal{V})^U)) \ .$$

This implies (1). If $p : \widetilde{X}\times F \to F$ denotes the projection, the form $p^*\eta$ is $\bar\partial$–closed and Γ–invariant by construction. Thus it defines $\tilde\eta = (p^*\eta)/\alpha$ on M, satisfying $\bar\partial\tilde\eta = \bar\partial(p^*\eta)/\alpha = (\bar\partial p^*\eta)/\alpha = 0$. ¿From Proposition 7.1 it follows that $\tilde\eta$ has the remaining required properties (2) to (4). □

We remark that Proposition 7.1 and Lemma 7.2 apply in particular to the situation in Proposition 6.4 for $m = l = 1$.

In the linear case

$$\mathbb{CP}^l \hookrightarrow \mathbb{P}(E) \stackrel{\pi}{\longrightarrow} X \ ,$$

we consider line bundles $\mathcal{V}_i = \mathcal{O}(k_i)$ with $k_1 - k_2 = -\ l - 1$, satisfying $\mathcal{V} = \mathcal{H}om_{\mathbb{C}}(\mathcal{V}_2,\mathcal{V}_1) = \mathcal{O}(-l-1)$. By Corollary 6.5 (2), the coefficient bundle $\mathcal{H}^{0,l}(\mathbb{CP}^1,\mathcal{O}(-l-1))$ is given by

$$\mathcal{H}^{0,l}(\mathbb{CP}^l,\mathcal{O}(-l-1)) = \mathcal{R}^l\pi_*(\mathcal{O}_M(-l-1)) \cong \det E = \mathcal{L} \ . \tag{7.5}$$

The induced representation

$$GL(l+1,\mathbb{C}) \longrightarrow GL\{H^{0,l}(\mathbb{CP}^l,\mathcal{O}(-l-1))\} \ , \tag{7.6}$$

is therefore the determinant representation. Thus for $l = 1$, we have the following characterisation of extensions by basic sections.

Proposition 7.3. *For $l=1$ and $\mathcal{E}_i = \pi^*\mathcal{W}_i \otimes_{\mathbb{C}} \mathcal{O}_M(k_i)$, $k_1 - k_2 = -2$, we have*

$$\begin{aligned}\mathrm{Ext}^1_{\mathscr{O}_M}(\mathscr{E}_2, \mathscr{E}_1) &\cong H^{0,1}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \mathcal{O}_M(-2)) \\ &\cong H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathcal{H}^{0,1}(\mathbb{CP}^1, \mathcal{O}(-2)) \\ &\cong H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathcal{L}) \ . \end{aligned} \tag{7.7}$$

Proof. This follows from Proposition 6.5 (2), compare also **[35]** §2 . □

Recalling (6.14) , we may rephrase Proposition 6.6 in the case $l=1$, in order to obtain an explicit parametrization of the extension classes in (7.7) .

Lemma 7.4. *Let $\eta \in A^{0,1}(\mathbb{CP}^1, \mathcal{K}_{\mathbb{CP}^1}) \cong A^{1,1}(\mathbb{CP}^1)$ be a PSL–invariant, closed form generating*

$$H^{0,1}(\mathbb{CP}^1, \mathcal{K}_{\mathbb{CP}^1}) \cong H^{1,1}(\mathbb{CP}^1) \cong \mathbb{C} \ ,$$

e.g. we may take η corresponding to the Kähler form ω of $\mathbb{CP}^1$.

There is a one–one–correspondence between holomorphic sections $\phi \in H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathcal{L})$ and classes $[\beta] \in H^{0,1}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \mathcal{O}_M(-2))$, given by

$$\beta_\phi = \pi^*\phi \otimes \tilde{\eta} \ .$$

If E is holomorphically flat and Γ acts via $SL(2,\mathbb{C})$, or more generally, if $\mathcal{L} = \det E$ is holomorphically trivial, the formulas in Proposition 7.3 and Lemma 7.4 simplify accordingly. In particular, the generator η defines then a class in $H^{0,1}(\mathbb{CP}^1, \mathcal{O}(-2))$.

8. Reduction to the coupled vortex equations

In this section we will establish the main result on the reduction of the Hermitian–Einstein equation on the total space M to the *Coupled Vortex equations* on the base manifold X.

First we construct a family of Kähler metrics on the total space M of the flat fiber bundle (3.6) . We wish to combine Kähler metrics on X and F to define a 1–parameterfamily of Kähler metrics on M. To this end, we now assume that the base manifold (X, ω_X) and the fiber (F, ω_F) come equipped with Kähler structures.

Proposition 8.1. *Let X and F be Kähler manifolds and let*

$$F \hookrightarrow M = \widetilde{X} \times_\Gamma F \stackrel{\pi}{\longrightarrow} X$$

be a generalized flat bundle with holonomy $\alpha \ : \Gamma \to U \subseteq \mathrm{Hol}_{\mathrm{Iso}}(F)$, where U is a connected compact subgroup of the group of holomorphic isometries of F. Let ω_X and ω_F denote the respective Kähler $(1,1)$–forms on X and on F. Then for a (constant) parameter $\sigma > 0$, there exists a family of Kähler metrics defined on M with corresponding weighted Kähler forms

$$\omega_\sigma = \pi^*\omega_X + \sigma\ \tilde{\omega}_F \ , \tag{8.1}$$

where $\tilde{\omega}_F = (p^*\omega_F)/\alpha$ is the extension of the invariant Kähler form ω_F to M.

Proof. Let g_X and g_F denote the Kähler metrics on X and the fiber F. The product metric $\pi^*g_X + p^*g_F$ on $\widetilde{X} \times F$ via the natural projection $p \ : \ \widetilde{X} \times F \to F$, defines a Kähler metric on $\widetilde{X} \times F$. The same applies on introducing the constant parameter σ fiberwise. Since α is a representation into the holomorphic isometries of F, the metric g_F is Γ–invariant. Hence $\pi^*g_X + \sigma p^*g_F$ descends to $M = \widetilde{X} \times_\Gamma F$, thus defining a Kähler metric on M having the required Kähler form. □

Remark 8.2. In the last result, the flat structure of M was crucial. For more general fiber bundles, such a combination of g_X and g_F may not define a Kähler structure on M (see e.g. **[12]**).

With respect to (8.1) , the definition of the normalized degree in (6.7) for a holomorphic vector bundle $\mathcal{E}$ on M , takes the form

$$\begin{aligned} \deg_\sigma(\mathcal{E}) = \deg_\sigma(\det \mathcal{E}) &= \frac{1}{(m-1)!\ \mathrm{Vol}_\sigma(M)} \int_M c_1(\mathcal{E}) \wedge \omega_\sigma^{m-1} \\ &= \frac{1}{m!\ \mathrm{Vol}_\sigma(M)} \int_M \Lambda_\sigma\ c_1(\mathcal{E})\ \omega_\sigma^m \\ &= \frac{1}{\mathrm{Vol}_\sigma(M)} \int_M \Lambda_\sigma\ c_1(\mathcal{E})\ \mathrm{dvol}_\sigma\ , \end{aligned} \tag{8.2}$$

where $\dim_{\mathbb{C}} X = n$, $\dim_{\mathbb{C}} F = l$, $m = n + l$ and Λ_σ denotes contraction against the Kähler form ω_σ. The binomial expansion of ω_σ in (8.1) gives

$$\mathrm{dvol}_\sigma = \frac{\omega_\sigma^{n+l}}{(n+l)!} = \frac{\pi^*\omega_X^n}{n!} \wedge \frac{\tilde{\omega}_F^l}{l!}\ \sigma^l\ , \tag{8.3}$$

and

$$\frac{\omega_\sigma^{n+l-1}}{(n+l-1)!} = \frac{\pi^*\omega_X^n}{n!} \wedge \frac{\tilde{\omega}_F^{l-1}}{(l-1)!}\ \sigma^{l-1} + \frac{\pi^*\omega_X^{n-1}}{(n-1)!} \wedge \frac{\tilde{\omega}_F^l}{l!}\ \sigma^l\ . \tag{8.4}$$

Using formula (8.3) , the proof of the following Lemma is essentially the same as in **[11]** , Lemma 4.9.

Lemma 8.3. *For a complex smooth function f on X, we have*

$$\int_M \pi^*(f)\,\mathrm{dvol}_\sigma = \int_X f\,\mathrm{dvol}_X\ \cdot\ \mathrm{Vol}(F)\ \sigma^l\ .$$

In particular, for $f = 1$:

$$\mathrm{Vol}_\sigma(M) = \mathrm{Vol}(X)\ \ \mathrm{Vol}(F)\ \sigma^l\ .$$

The following formulas for pull-backs and extensions will be needed in the proof of the main theorem. We will make use of the defining equation for the 'Lambda'–operator on any Kähler manifold (X, ω_X) (compare (8.2))

$$\Lambda_X \varphi \otimes \omega_X^n = n\ \varphi \wedge \omega_X^{n-1}\ . \tag{8.5}$$

Lemma 8.4. *Let $\mathcal{W} \to X$ be a complex vector bundle and φ a $(1,1)$–form with values in $\mathcal{W}$. Then*

$$\Lambda_\sigma \pi^*(\varphi) = \pi^*(\Lambda_X \varphi)\ .$$

Proof. Note that the flatness of the fiber bundle is not required here. The Lemma is proved by direct calculation, using (8.4) and (8.5) :

$$\begin{aligned} \Lambda_\sigma \pi^* \varphi \otimes \mathrm{dvol}_\sigma &= \pi^*\varphi \wedge \frac{\omega_\sigma^{n+l-1}}{(n+l-1)!} \\ &= \frac{1}{n!\ (l-1)!}\ \pi^*(\varphi \wedge \omega_X^n) \wedge \tilde{\omega}_F^{l-1}\ \sigma^{l-1} + \frac{1}{(n-1)!\ l!}\ \pi^*(\varphi \wedge \omega_X^{n-1}) \wedge \tilde{\omega}_F^l\ \sigma^l \\ &= \frac{1}{n!\ l!}\ \pi^*(\Lambda_X \varphi \otimes \omega_X^n) \wedge \tilde{\omega}_F^l\ \sigma^l \\ &= \frac{1}{n!\ l!}\ \pi^*\Lambda_X \varphi \otimes (\pi^*\omega_X^n \wedge \tilde{\omega}_F^l)\ \sigma^l \\ &= \pi^*\Lambda_X \varphi \otimes \mathrm{dvol}_\sigma\ . \end{aligned}$$

□

Proposition 8.5. *Let $\mathcal{W} \to X$ be a holomorphic vector bundle. Then*

$$\deg_\sigma(\pi^*\mathcal{W}) = \deg_X(\mathcal{W}) \ ,$$

and so $\deg_\sigma(\pi^*\mathcal{W})$ is a base invariant.

Proof. This follows from (8.2) , Lemma 8.3 and Lemma 8.4 for $\varphi = c_1(\mathcal{W})$:

$$\begin{aligned} \deg_\sigma(\pi^*\mathcal{W}) &= \frac{1}{\mathrm{Vol}_\sigma(M)} \int_M \Lambda_\sigma \pi^* c_1(\mathcal{W}) \ \mathrm{dvol}_\sigma \\ &= \frac{1}{\mathrm{Vol}_\sigma(M)} \int_M \pi^* \Lambda_X c_1(\mathcal{W}) \ \mathrm{dvol}_\sigma \\ &= \frac{1}{\mathrm{Vol}(X)} \int_X \Lambda_X c_1(\mathcal{W}) \ \mathrm{dvol}_X \\ &= \deg_X(\mathcal{W}) \ . \end{aligned}$$

□

Lemma 8.6. *Let the flat fiber bundle (3.6) be given as in Proposition 8.1 and let $\mathcal{V} \to F$ be an U–equivariant complex vector bundle. For an equivariant $(1,1)$–form φ with values in $\mathcal{V}$ with extension $\tilde{\varphi} = (p^*\varphi)/\alpha$, we have*

$$\Lambda_\sigma \tilde{\varphi} = \frac{1}{\sigma} \ (\Lambda_F \ \varphi)^\sim \ .$$

Proof. The flatness of the the fiber bundle is essential here, as we make use of the extension $\tilde{\varphi}$ of φ described in (2.5) . The Lemma is proved by direct calculation, using (8.4) and (8.5) :

$$\begin{aligned} \Lambda_\sigma \tilde{\varphi} \otimes \mathrm{dvol}_\sigma &= \tilde{\varphi} \wedge \ \frac{\omega_\sigma^{n+l-1}}{(n+l-1)!} \\ &= \frac{1}{n! \ (l-1)!} \ \pi^*\omega_X^n \wedge (\tilde{\varphi} \wedge \tilde{\omega}_F^{l-1}) \ \sigma^{l-1} + \frac{1}{(n-1)! \ l!} \ \pi^*\omega_X^{n-1} \wedge (\tilde{\varphi} \wedge \tilde{\omega}_F^l) \ \sigma^l \\ &= \frac{1}{n! \ (l-1)!} \ \pi^*\omega_X^n \wedge (\varphi \wedge \omega_F^{l-1})^\sim \ \sigma^{l-1} + \frac{1}{(n-1)! \ l!} \ \pi^*\omega_X^{n-1} \wedge (\varphi \wedge \omega_F^l)^\sim \ \sigma^l \\ &= \frac{1}{\sigma \ n! \ l!} \ \pi^*\omega_X^n \wedge (\Lambda_F \varphi \otimes \omega_F^l)^\sim \ \sigma^l \\ &= \frac{1}{\sigma \ n! \ l!} \ (\Lambda_F \ \varphi)^\sim \otimes (\pi^*\omega_X^n \wedge \tilde{\omega}_F^l) \ \sigma^l \\ &= \frac{(\Lambda_F \ \varphi)^\sim}{\sigma} \otimes \mathrm{dvol}_\sigma \ . \end{aligned}$$

□

Proposition 8.7. *Suppose that $U \subset \mathrm{Hol}_{\mathrm{Iso}}(F)$ acts transitively on F and let $\mathcal{V} \to F$ be an U–equivariant holomorphic vector bundle. Then*

$$\deg_\sigma(\widetilde{\mathcal{V}}) = \frac{1}{\sigma} \ \deg_F(\mathcal{V}) \ ,$$

and so $\deg_\sigma(\widetilde{\mathcal{V}})$ is a fiber invariant for the flat structure on $M \to X$.

Proof. We represent $c_1(\mathcal{V})$ by a closed U–invariant $(1,1)$–form α on F . Then using the flat structure on M, we have $c_1(\widetilde{\mathcal{V}}) = [\tilde{\alpha}]$, where $\tilde{\alpha}$ is the extension of α and $\tilde{\alpha} \wedge \tilde{\omega}_F^l = (\alpha \wedge {\omega_F}^l)^\sim = 0$.

By U–invariance, $\Lambda_F\alpha$ is constant. The result follows from (8.2) and Lemma 8.6 by setting $\varphi = \alpha$:

$$\begin{aligned}
\deg_\sigma(\widetilde{\mathcal{V}}) &= \frac{1}{\mathrm{Vol}_\sigma(M)} \int_M \Lambda_\sigma c_1(\widetilde{\mathcal{V}}) \ \mathrm{dvol}_\sigma \\
&= \frac{1}{\mathrm{Vol}_\sigma(M)} \int_M \Lambda_\sigma \tilde{\alpha} \ \mathrm{dvol}_\sigma \\
&= \frac{1}{\sigma \ \mathrm{Vol}_\sigma(M)} \int_M (\Lambda_F\alpha)^\sim \ \mathrm{dvol}_\sigma \\
&= \frac{\Lambda_F\alpha}{\sigma} \ .
\end{aligned}$$

Finally, we observe that

$$\begin{aligned}
\deg_F(\mathcal{V}) &= \frac{1}{(l-1)! \ \mathrm{Vol}(F)} \int_F \alpha \wedge \omega_F^{l-1} \\
&= \frac{1}{\mathrm{Vol}(F)} \int_F \Lambda_F\alpha \ \mathrm{dvol}_F \\
&= \Lambda_F\alpha \ .
\end{aligned}$$

□

8.8 Calibration conditions on the fiber.

We assume now that $F = U/K$ is a compact irreducible symmetric Hermitian space, equipped with its unique (up to homothety) invariant Kähler structure. Further, we are given homogeneous holomorphic bundles $\mathcal{V}_{\rho_i} = U \times_K V_{\rho_i} \to F = U/K$ associated to complex representations $(\rho_i, V_{\rho_i}) \in R(K)$ as in Example 3.2 . In order to establish the main result below, we need to impose a number of conditions for the data on the fiber.

(1) The representations $(\rho_i, V_{\rho_i}) \in R(K)$ are irreducible.
By **[34]** VI, Prop. 6.2, the $\mathcal{V}_{\rho_i} \to F$ are irreducible U–equivariant Hermitian–Einstein bundles and therefore stable (cf. **[33]** **[42]** **[46]**).

(2) $\mu_\rho = \mu_{\rho_1} - \mu_{\rho_2} < 0$.
It follows that and $V_\rho^K = \mathrm{Hom}_K(V_{\rho_2}, V_{\rho_1}) = 0$ and

$$H^0(F, \mathcal{V}_\rho) = H^0(F, \mathcal{H}om_{\mathbb{C}}(\mathcal{V}_{\rho_2}, \mathcal{V}_{\rho_1})) = 0 \ .$$

(3) $\dim_{\mathbb{C}} H^{0,1}(F, \mathcal{V}_\rho) = 1$.
Since U is simple, $H^{0,1}(F, \mathcal{V}_\rho)$ is a trivial U–module and we have by (3.13)

$$H^{0,1}(F, \mathcal{V}_\rho) \cong H^1_{\bar\partial}(A^{0,*}(F, \mathcal{V}_\rho)^U) \cong \mathbb{C} \ .$$

Hence Proposition 7.1 and Lemma 7.2 apply.

Observe that $\deg_F(\mathcal{V}_{\rho_i})$ and hence $\mu_{\rho_i} = \mu_{\mathcal{V}_{\rho_i}}$ are computable in terms of the weights of the representations $(\rho_i, V_{\rho_i}) \in R(K)$ by the methods of **[3]**.

The following theorem is the main result of this paper.

Theorem 8.9. *Let $F \hookrightarrow M = \widetilde{X} \times_\Gamma F \xrightarrow{\pi} X$ be a flat holomorphic fiber bundle of compact Kähler manifolds where the fiber $F = U/K$ is a compact irreducible symmetric Kähler manifold as above.*

Suppose that the homogeneous holomorphic bundles $\mathcal{V}_{\rho_i}$ on F satisfy the conditions in 8.8 *and let k_i be the U–equivariant solution of the Hermitian–Einstein equation on $\mathcal{V}_{\rho_i}$.*

Consider the proper holomorphic extension

$$\mathbb{E}_\phi \ : \ 0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0 \ ,$$

as in (7.4) *, where $\mathcal{E}_i = \pi^* \mathcal{W}_i \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_{\rho_i}$, and $\mathbb{E}_\phi$ corresponds to $\phi \in H^0(X, \mathcal{W}) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{W}_2, \mathcal{W}_1)$ for holomorphic vector bundles $\mathcal{W}_i$ on X .*

Let $\mathbf{h} = \mathbf{h}_1 \oplus \mathbf{h}_2$, where the Hermitian metric

$$\mathbf{h}_i = h_i' \otimes \tilde{k}_i$$

on $\mathcal{E}_i$ is defined by an invariant (basic) Hermitian metric h_i' on $\pi^*\mathcal{W}_i$ and the extension $\tilde{k}_i$ of the U–equivariant Hermitian–Einstein metric k_i on $\mathcal{V}_{\rho_i}$. Let $F_{\mathbf{h}}$ be the curvature of the Chern connection determined by $\mathbf{h}$ and Λ_σ the contraction against the Kähler form $\omega_\sigma = \pi^*\omega_X + \sigma\, \tilde{\omega}_F$.

For $\sigma > 0$, let

$$\lambda = \mu_{\mathcal{E}}(\sigma) = \frac{\deg_\sigma(\mathcal{E})}{\mathrm{rank}(\mathcal{E})} \ ,$$

and define the vortex parameters τ_i by

$$\tau_i = \tau_i(\sigma) = \mu_{\mathcal{E}}(\sigma) - \frac{\mu_{\rho_i}}{\sigma} \ .$$

Then the following statements are equivalent :

(1) *There exist Hermitian metrics of the form $\mathbf{h}$ on the extension bundle $\mathcal{E}$ which satisfy the Hermitian–Einstein equation*

$$\iota\, \Lambda_\sigma F_{\mathbf{h}} = 2\pi\ \lambda\ \mathbf{I}_{\mathcal{E}} \ ,$$

relative to (M, ω_σ) .

(2) *There exist Hermitian metrics h_i on $\mathcal{W}_i$ which satisfy the coupled σ–Vortex equations :*

$$\begin{aligned} \iota\, \Lambda_X F_{h_1} + \frac{1}{\sigma}\ \phi \circ \phi^* &= 2\pi\ \tau_1\ \mathbf{I}_{\mathcal{W}_1} \ , \\ \iota\, \Lambda_X F_{h_2} - \frac{1}{\sigma}\ \phi^* \circ \phi &= 2\pi\ \tau_2\ \mathbf{I}_{\mathcal{W}_2} \ , \end{aligned} \tag{8.6}$$

where the adjoint in ϕ^* is taken with respect to the metrics h_1 and h_2 .

There is a one–to–one correspondence between solutions in (1) *and* (2)*, given by the assignment $h_i \mapsto h_i' = \pi^* h_i$.*

Proof. First, we observe that the assignment $h_i \mapsto h_i' = \pi^* h_i$ realizes the isomorphism between Hermitian metrics h_i on $\mathcal{W}_i$ and invariant Hermitian metrics h_i' on $\pi^*\mathcal{W}_i$. This follows from (2.8) . Without loss of generality, we may therefore assume that h_i' is of the form $h_i' = \pi^* h_i$.

We continue by analyzing the Hermitian–Einstein condition on the holomorphic vector bundle $\mathcal{E}$ on M as in **[18]** §3; compare also **[11]** . The main part of the proof relies substantially on the technical results established in this section and the previous section.

Relative to a smooth decomposition $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$, the unitary integrable connection $\widehat{A}$ on $(\mathcal{E}, \mathbf{h})$ can be expressed in the form

$$\widehat{A} = \begin{pmatrix} \mathbf{A}_1 & \beta \\ -\beta^* & \mathbf{A}_2 \end{pmatrix} \ , \tag{8.7}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ are the metric connections of $(\mathcal{E}_1, \mathbf{h}_1)$ and $(\mathcal{E}_2, \mathbf{h}_2)$ respectively, and

$$\beta \in A^{0,1}(M, \mathcal{H}om_{\mathbb{C}}(\mathcal{E}_2, \mathcal{E}_1))$$

is the representative of the extension class in $\mathrm{Ext}^1_{\mathscr{O}_M}(\mathscr{E}_2, \mathscr{E}_1)$ as in (7.2) . A routine calculation (cf. e.g. **[34]**) shows that the curvature of $\widehat{A}$ has the form

$$F_{\mathbf{h}} = F_{\widehat{A}} = \begin{pmatrix} F_{\mathbf{h}_1} - \beta \wedge \beta^* & D'\beta \\ -D''\beta^* & F_{\mathbf{h}_2} - \beta^* \wedge \beta \end{pmatrix} , \tag{8.8}$$

where $D : A^1(M, \mathcal{H}om_{\mathbb{C}}(\mathcal{E}_1, \mathcal{E}_2)) \to A^2(M, \mathcal{H}om_{\mathbb{C}}(\mathcal{E}_1, \mathcal{E}_2))$ is constructed from $\mathbf{A}_1$ and $\mathbf{A}_2$ in the standard way.

Now if we take

(1) A_i to be the integrable unitary connection on $(\mathcal{W}_i, h_i)$, and
(2) $\tilde{A}_i$ to be the Hermitian–Einstein metric connection on $(\tilde{\mathcal{V}}_{\rho_i}, \tilde{k}_i)$,

then

$$\mathbf{A}_i = \pi^* A_i \otimes \mathbf{1} + \mathbf{1} \otimes \tilde{A}_i . \tag{8.9}$$

The corresponding curvature form of type (1, 1) can be expressed as

$$F_{\mathbf{h}_i} = \pi^* F_{h_i} \otimes \widetilde{\mathbf{I}}_{\rho_i} + \widetilde{\mathbf{I}}_i \otimes F_{\tilde{k}_i} , \tag{8.10}$$

where $\widetilde{\mathbf{I}}_i = \pi^* \mathbf{I}_{\mathcal{W}_i}$ and $\widetilde{\mathbf{I}}_{\rho_i} = \mathbf{I}_{\widetilde{\mathcal{V}}_{\rho_i}}$.

Under the assumptions 8.8 (2) and (3), Proposition 7.1 gives the required parametrization of

$$\mathrm{Ext}^1_{\mathscr{O}_M}(\mathscr{E}_2, \mathscr{E}_1) \cong H^{0,1}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_\rho) \cong H^0(X, \mathcal{W}) \otimes_{\mathbb{C}} H^{0,1}(F, \mathcal{V}_\rho) \cong H^0(X, \mathcal{W}) .$$

The one–to–one correspondence in Lemma 7.2 states that β is of the form $\beta = \beta_\phi = \pi^*\phi \otimes \tilde{\eta}$, where $\tilde{\eta} \in A^{0,1}(M, \tilde{\mathcal{V}}_\rho)$ is the extension of the invariant, $\bar{\partial}$–closed $(0,1)$–form $\eta \in A^{0,1}(F, \mathcal{V}_\rho)^U$ generating $H^{0,1}(F, \mathcal{V}_\rho) \cong \mathbb{C}$.

The following Lemma implements the results which are necessary to carry out the reduction process to the coupled vortex equations.

Lemma 8.10. *With $\tilde{\eta}$ and β as above, we have*

(1) $\Lambda_\sigma D'\beta = 0$;
(2) $\Lambda_\sigma D''\beta^* = 0$;
(3) $\Lambda_\sigma F_{\pi^* h_i} = \pi^* \Lambda_X F_{h_i}$;
(4) $\Lambda_\sigma F_{\tilde{k}_i} = \frac{1}{\sigma} \, (\Lambda_F F_{k_i})^\sim$;
(5) $\Lambda_\sigma(\tilde{\eta} \wedge \tilde{\eta}^*) = \frac{1}{\sigma} \, \Lambda_F(\eta \wedge \eta^*)^\sim = \frac{\iota}{\sigma} \, \widetilde{\mathbf{I}}_{\rho_1}$ *and* $\Lambda_\sigma(\tilde{\eta}^* \wedge \tilde{\eta}) = \frac{1}{\sigma} \, \Lambda_F(\eta^* \wedge \eta)^\sim = - \, \frac{\iota}{\sigma} \, \widetilde{\mathbf{I}}_{\rho_2}$, *for a suitable calibration of η, independent of σ.*

Proof. (1) and (2) follow essentially by the arguments in **[18]** and **[11]**. We observe that $\Lambda_F D'\eta \in A^0(F, \mathcal{V}_\rho)^U \cong {V_\rho}^K$ must vanish as a consequence of assumption 8.8 (2). (3) follows from Lemma 8.4 by taking $\varphi = F_{h_i}$ and noting that $F_{\pi^* h_i} = \pi^* F_{h_i}$. (4) follows from Lemma 8.6 by taking $\varphi = F_{k_i}$ and noting that $F_{\tilde{k}_i} = \widetilde{F}_{k_i}$.

To prove (5), we first apply Lemma 8.6 to the $(1,1)$–form $\varphi = \eta \wedge \eta^*$, to obtain

$$\Lambda_\sigma(\tilde{\eta} \wedge \tilde{\eta}^*) = \frac{1}{\sigma} \, \Lambda_F(\eta \wedge \eta^*)^\sim .$$

In terms of the Cartan decomposition (3.10) and formula (3.12) , we have

$$A^{0,1}(F, \mathcal{V}_\rho)^U \cong \mathrm{Hom}_K(\mathfrak{m}_{\mathbb{C}}^{\perp}, V_\rho) ,$$

where $\mathfrak{m}_{\mathbb{C}} = \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$. Thus we see that, as an U–equivariant form, Tr $\Lambda_F(\eta \wedge \eta^*)$ is realized by the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_K(\mathfrak{m}_{\mathbb{C}}^{\perp}, V_\rho) \otimes_{\mathbb{C}} \mathrm{Hom}_K(\mathfrak{m}_{\mathbb{C}}^{+}, V_\rho^*) & \xrightarrow{\wedge} & (\Lambda_{\mathbb{C}}^{1,1}(\mathfrak{m}_{\mathbb{C}}^*) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(V_{\rho_1}))^K & \xrightarrow{\Lambda} & \mathrm{End}_K(V_{\rho_1}) \\ \downarrow \cong & & \downarrow \mathrm{Tr} & & \cong \downarrow \mathrm{Tr} \\ \mathrm{Hom}_K(\mathfrak{m}_{\mathbb{C}}^{+*}, V_\rho) \otimes_{\mathbb{C}} \mathrm{Hom}_K(\mathfrak{m}_{\mathbb{C}}^{+}, V_\rho^*) & \xrightarrow{\mathrm{Tr} \circ \wedge} & \Lambda_{\mathbb{C}}^{1,1}(\mathfrak{m}_{\mathbb{C}}^*)^K & \xrightarrow{\Lambda} & \mathbb{C} \end{array} \tag{8.11}$$

As $(\rho_1, V_{\rho_1}) \in R(K)$ is irreducible by assumption 8.8 (1), it is simple by Schur's lemma. Thus we have $\mathrm{End}_K(V_{\rho_1}) \cong \mathbb{C}$ and the (normalized) trace Tr is an isomorphism. Consequently, the invariant endomorphism

$$\Lambda_F(\eta \wedge \eta^*) \in \mathrm{End}_U(\mathcal{V}_{\rho_1}) \cong \mathrm{End}_K(V_{\rho_1}) \ ,$$

must be a constant multiple of the identity.

In terms of the Kähler structure, the pointwise norm of η is given by $\frac{1}{\iota}$ Tr $\Lambda_F(\eta \wedge \eta^*)$. Hence we have

$$\Lambda_\sigma(\tilde{\eta} \wedge \tilde{\eta}^*) = \frac{1}{\sigma}\ (\Lambda_F(\eta \wedge \eta^*))^{\sim} = \frac{c\ \iota}{\sigma}\ \widetilde{\mathbf{I}}_{\rho_1}\ ,\ c > 0\ .$$

Since the generator η is determined up to a complex constant $\xi \in \mathbb{C}^\times$ and

$$\Lambda_F(\xi\eta \wedge (\xi\eta)^*) = |\ \xi\ |^2\ \Lambda_F(\eta \wedge \eta^*)\ ,$$

we may calibrate η by ξ and ϕ by $\xi^{\perp 1}$, with $|\ \xi\ |^2 = c^{\perp 1}$ to have the desired property. The second equation is proved likewise.

□

By 8.8 (1), the bundles $\mathcal{V}_{\rho_i} \to F = U/K$ have U–equivariant Hermitian–Einstein metrics k_i, unique up to a positive constant. For the extension $\tilde{k}_i$ of k_i to $\widetilde{\mathcal{V}}_{\rho_i} \to M$, we obtain the following result from Proposition 8.7 and Lemma 8.10 (4).

Proposition 8.11. *The holomorphic bundles $\widetilde{\mathcal{V}}_{\rho_i} \to M$ have Hermitian–Einstein structures*

$$\iota\ \Lambda_\sigma F_{\tilde{k}_i} = 2\pi\ \tilde{\mu}_{\rho_i} \widetilde{\mathbf{I}}_{\rho_i}\ , \tag{8.12}$$

with constant given by

$$\tilde{\mu}_{\rho_i} = \mu_{\widetilde{\mathcal{V}}_{\rho_i}} = \frac{\mu_{\rho_i}}{\sigma}\ . \tag{8.13}$$

We may now complete the proof of the main theorem as follows.

We use Lemma 8.10 to compute Λ_σ of the following forms :

$$\beta \wedge \beta^* = \pi^*(\phi \circ \phi^*) \otimes (\tilde{\eta} \wedge \tilde{\eta}^*) \in A^{1,1}(M, \mathcal{E}nd_{\mathbb{C}}(\pi^*\mathcal{W}_1) \otimes_{\mathbb{C}} \mathcal{E}nd_{\mathbb{C}}(\widetilde{\mathcal{V}}_{\rho_1}))\ ,$$

and

$$\beta^* \wedge \beta = \pi^*(\phi^* \circ \phi) \otimes (\tilde{\eta}^* \wedge \tilde{\eta}) \in A^{1,1}(M, \mathcal{E}nd_{\mathbb{C}}(\pi^*\mathcal{W}_2) \otimes_{\mathbb{C}} \mathcal{E}nd_{\mathbb{C}}(\widetilde{\mathcal{V}}_{\rho_2}))\ .$$

Thus we get

$$-\iota\ \Lambda_\sigma(\beta \wedge \beta^*) = -\iota\ \pi^*(\phi \circ \phi^*) \otimes \Lambda_\sigma(\tilde{\eta} \wedge \tilde{\eta}^*) = \frac{1}{\sigma}\ \pi^*(\phi \circ \phi^*) \otimes \widetilde{\mathbf{I}}_{\rho_1}\ .$$

Similarly, using $\mathrm{Tr}(\eta^* \wedge \eta') = -\ \mathrm{Tr}(\eta' \wedge \eta^*)$ and the irreducibility of (ρ_2, V_{ρ_2}), we get

$$\iota\ \Lambda_\sigma(\beta^* \wedge \beta) = \frac{1}{\sigma}\ \pi^*(\phi^* \circ \phi) \otimes \widetilde{\mathbf{I}}_{\rho_2}\ .$$

Substituting the previous formulas into (8.8) , and using (8.10) , (8.12) and Lemma 8.10 , we see that the Hermitian–Einstein condition on $\mathbf{h}$ on (M, ω_σ) is now equivalent to

$$\begin{pmatrix} \pi^*(\iota\, \Lambda_X F_{h_1} + \frac{1}{\sigma}\, \phi\phi^* + 2\pi\tilde{\mu}_{\rho_1}\mathbf{I}_{\mathcal{W}_1}) \otimes \widetilde{\mathbf{I}}_{\rho_1} & 0 \\ 0 & \pi^*(\iota\, \Lambda_X F_{h_2} - \frac{1}{\sigma}\, \phi^*\phi + 2\pi\tilde{\mu}_{\rho_2}\mathbf{I}_{\mathcal{W}_2}) \otimes \widetilde{\mathbf{I}}_{\rho_2} \end{pmatrix}$$
$$= 2\pi\lambda \begin{pmatrix} \mathbf{I}_{\mathcal{E}_1} & 0 \\ 0 & \mathbf{I}_{\mathcal{E}_2} \end{pmatrix}$$
$$= 2\pi\lambda \begin{pmatrix} \widetilde{\mathbf{I}}_1 \otimes \widetilde{\mathbf{I}}_{\rho_1} & 0 \\ 0 & \widetilde{\mathbf{I}}_2 \otimes \widetilde{\mathbf{I}}_{\rho_2} \end{pmatrix}$$

and hence equivalent to the system of equations on X :

$$\iota\, \Lambda_X F_{h_1} + \frac{1}{\sigma}\, \phi \circ \phi^* + 2\pi(\tilde{\mu}_{\rho_1} - \lambda)\, \mathbf{I}_{\mathcal{W}_1} = 0 \ ,$$
$$\iota\, \Lambda_X F_{h_2} - \frac{1}{\sigma}\, \phi^* \circ \phi + 2\pi(\tilde{\mu}_{\rho_2} - \lambda)\, \mathbf{I}_{\mathcal{W}_2} = 0 \ .$$

□

The vortex parameters $\tau_i(\sigma)$ are not independent. Recall that they are given by

$$\tau_i = \tau_i(\sigma) = \mu_{\mathcal{E}}(\sigma) - \frac{1}{\sigma}\, \mu_{\rho_i} \ .$$

The following Lemma is immediate from the definition.

Lemma 8.12.

$$\tau_1(\sigma) - \tau_2(\sigma) = -\frac{1}{\sigma}\, (\mu_{\rho_1} - \mu_{\rho_2}) = -\frac{\mu_\rho}{\sigma} \ . \tag{8.14}$$

In the following, we use the additivity of the degree under holomorphic extensions

$$\deg_\sigma(\mathcal{E}) = \deg_\sigma(\mathcal{E}_1) + \deg_\sigma(\mathcal{E}_2) \ ,$$

and the additivity of the slope under tensor products

$$\mu_{\mathcal{E}\otimes_{\mathbb{C}}\mathcal{F}} = \mu_{\mathcal{E}} + \mu_{\mathcal{F}} \ .$$

Lemma 8.13. *Set* $s_i = \mathrm{rank}(\mathcal{V}_{\rho_i})$ *and* $r_i = \mathrm{rank}(\mathcal{W}_i)$ *. Then*

$$\begin{aligned} r_1 s_1 \tau_1(\sigma) \ + \ r_2 s_2 \tau_2(\sigma) &= s_1 \deg_X(\mathcal{W}_1) + s_2 \deg_X(\mathcal{W}_2) \\ &= r_1 s_1\ \mu_{\mathcal{W}_1} + r_2 s_2\ \mu_{\mathcal{W}_2} \\ &= (r_1 s_1 + r_2 s_2)\ \mu_{s_1\mathcal{W}_1 \oplus s_2\mathcal{W}_2} \ . \end{aligned} \tag{8.15}$$

Proof. Observe that

$$\deg_\sigma(\mathcal{E}_i) = s_i \deg_\sigma(\pi^*\mathcal{W}_i) + r_i \deg_\sigma(\tilde{\mathcal{V}}_{\rho_i}) \ .$$

Since $\mathrm{rank}(\mathcal{E}) = s_1 r_1 + s_2 r_2$, it follows that

$$\begin{aligned} r_1 s_1 \tau_1 + r_2 s_2 \tau_2 &= \deg_\sigma(\mathcal{E}) - (r_1 \deg_\sigma(\tilde{\mathcal{V}}_{\rho_1}) + r_2 \deg_\sigma(\tilde{\mathcal{V}}_{\rho_2})) \\ &= (\deg_\sigma(\mathcal{E}_1) - r_1 \deg_\sigma(\tilde{\mathcal{V}}_{\rho_1})) + (\deg_\sigma(\mathcal{E}_2) - r_2 \deg_\sigma(\tilde{\mathcal{V}}_{\rho_2})) \\ &= s_1 \deg_\sigma(\pi^*\mathcal{W}_1) + s_2 \deg_\sigma(\pi^*\mathcal{W}_2) \\ &= s_1 \deg_X(\mathcal{W}_1) + s_2 \deg_X(\mathcal{W}_2) \ . \end{aligned}$$

□

As an immediate consequence of (8.14) and (8.15) , we obtain explicit formulas for $\tau_i(\sigma)$ and $\mu_{\mathcal{E}}(\sigma)$:

$$\tau_1(\sigma) = \mu_{s_1\mathcal{W}_1 \oplus s_2\mathcal{W}_2} - \frac{r_2 s_2}{r_1 s_1 + r_2 s_2} \cdot \frac{\mu_\rho}{\sigma} \ , \tag{8.16}$$

$$\tau_2(\sigma) = \mu_{s_1\mathcal{W}_1 \oplus s_2\mathcal{W}_2} + \frac{r_1 s_1}{r_1 s_1 + r_2 s_2} \cdot \frac{\mu_\rho}{\sigma} \ , \tag{8.17}$$

$$\mu_{\mathcal{E}}(\sigma) = \mu_{s_1\mathcal{W}_1 \oplus s_2\mathcal{W}_2} + \frac{1}{\sigma}\ \mu_{r_1\mathcal{V}_{\rho_1} \oplus r_2\mathcal{V}_{\rho_2}} \ . \tag{8.18}$$

9. Reduction to the coupled vortex equations: The projective case

In this section we take $F = \mathbb{CP}^l$ and recall from (6.4) the flat projective bundle

$$\mathbb{CP}^l \hookrightarrow M \cong \widetilde{X} \times_{\hat{\alpha}} \mathbb{CP}^l \xrightarrow{\pi} X \ ,$$

with structure group $U = \mathrm{Hol}_{\mathrm{Iso}}(\mathbb{CP}^l) \cong PU(l+1) \cong U(l+1)/\,U(1)$ and $K = U(l)$.

Theorem 9.1. *Theorem* 8.9 *holds for* $\rho_1 = \mathbf{1}$ *and* $\mathcal{V}_\rho = \Omega^1(\mathbb{CP}^l)$ *.*

Remark 9.2. The case $l = 1$, $\rho_1 = \mathbf{1}$ and $\mathcal{V}_\rho \cong \mathcal{K}_{\mathbb{CP}^1}$ was established in **[11]**, Theorem 5.1. In this case, we could normalize η so that $\eta \wedge \eta^* = \iota\, \omega_F$ and $\Lambda_F(\eta \wedge \eta^*) = \iota$.

Theorem 9.1 is actually a special case of a more general result.

Theorem 9.3. *Let* $F \hookrightarrow M = \widetilde{X} \times_\Gamma F \xrightarrow{\pi} X$ *be a flat holomorphic fiber bundle of compact Kähler manifolds where the fiber* $F = U/K$ *is a compact, irreducible symmetric Kähler manifold. Then Theorem* 8.9 *holds for* $\rho_1 = \mathbf{1}$, $\mathcal{V}_{\rho_2} = T^{1,0}(F)$ *and* $\mathcal{V}_\rho = \mathcal{V}^*_{\rho_2} = \Omega^1(F)$ *.*

Proof. We have to verify the conditions 8.8 (1) to (3). Referring to (8.11) , we note that $T^{1,0}(F)$ is associated to the irreducible K–representation $\rho_0 \ : \ K \to U(\mathfrak{m}^+_{\mathbb{C}})$ and therefore (1) is satisfied. By a result of Matsushima **[38]**, the invariant Hermitian metric k on $T(F)$ is Kähler–Einstein, that is (cf. **[2]** Ch. XI)

$$\rho = \iota\ \mathrm{Tr}\, F_k = \frac{s}{2l}\, \omega_F \ , \ \text{or}\ c_1(T^{1,0}(F)) = \frac{\iota}{2\pi}\, [\mathrm{Tr}\, F_k] = \frac{s}{4\pi l}\, [\omega_F] \ ,$$

where $s > 0$ is the (constant) scalar curvature of F. Therefore $\deg(T^{1,0}(F)) = \frac{s}{4\pi} > 0$, $\deg(\Omega^1_F) < 0$ and (2) is satisfied. (3) follows from (3.14) and $H^{0,1}(F, \Omega^1(F)) \cong H^{1,1}(F) \cong \Lambda^{1,1}_{\mathbb{C}}(\mathfrak{m}^*_{\mathbb{C}})^K \cong \mathbb{C}$. In this case, we may use diagram (8.11) and the fact that $V_\rho = \mathfrak{m}^{+*}_{\mathbb{C}}$, to choose the generator η as the canonical isomorphism $\eta \ : \ \mathfrak{m}^{\perp}_{\mathbb{C}} \cong \mathfrak{m}^{+*}_{\mathbb{C}}$ determined by the invariant Hermitian metric k on $\mathfrak{m}$. The identities

$$-2\iota\, \omega_F(\xi_1^+, \xi_2^\perp) = k(\xi_1, \xi_2) = \langle \xi_1^+, \eta(\xi_2^\perp) \rangle \ ,$$

for $\xi_i \in \mathfrak{m}$ and $\xi^\pm = \frac{1}{2}\, (\xi \mp \iota\, J\xi)$, may be reformulated as

$$\mathrm{Tr}\ (\eta \wedge \eta^*) = \frac{2\iota}{l}\, \omega_F \ ,$$

and therefore $\mathrm{Tr}\ \Lambda(\eta \wedge \eta^*) = \Lambda\ \ \mathrm{Tr}(\eta \wedge \eta^*) = 2\iota$. This gives an explicit verification of part (5) of Lemma 8.10 . □

Corollary 9.4.

$$\deg_\sigma(\Omega^1_{M/X}) = \deg_\sigma(\Omega^l_{M/X}) = \deg_\sigma(\mathcal{K}_{M/X}) = -\ \frac{s}{4\pi\ \sigma} < 0 \ .$$

Proof. This follows from the above formula for $c_1(T^{1,0}(F))$ and Proposition 8.7 . □

In the following Lemmas we compute the corresponding degree invariants for $F = \mathbb{CP}^l$.

Proposition 9.5. *On* $M = \widetilde{X} \times_{\hat{\alpha}} \mathbb{CP}^l$ *we have the projective invariant on* M

$$\deg_\sigma(\Omega^1_{M/X}) = \deg_\sigma(\Omega^l_{M/X}) = \deg_\sigma(\mathcal{K}_{M/X}) = -\ \frac{l+1}{\sigma} \ .$$

Proof. Recall from (6.20) that

$$\mathcal{K}_{M/X} \cong \widetilde{\mathcal{K}}_{\mathbb{CP}^l} \cong \mathcal{O}(-l-1)^{\sim}_{PSL} \ .$$

The result follows from Proposition 8.7 , noting that, relative to the Fubini–Study metric on $\mathbb{CP}^l$, we have $s = 4\pi(l+1)$ (cf. **[2]** loc. cit.) and therefore deg $(\mathcal{K}_{\mathbb{CP}^l}) = -(l+1)$. □

To consider linear invariants of $M = \mathbb{P}(E)$ for E projectively flat, we set

$$e = \deg_\sigma(\pi^* E) = \deg_X(E) = \deg_X(\mathcal{L}), \ \mathcal{L} = \det(E) \ .$$

Proposition 9.6. *On $M = \mathbb{P}(E)$ we have linear invariants, for $k \in \mathbb{Z}$, given by*

$$\deg_\sigma(\mathcal{O}_M(k)) = k \ (\ \frac{1}{\sigma} - \frac{e}{l+1} \ ) \ .$$

Proof. Recall from (6.14) that

$$\mathcal{O}_M(-l-1) \cong \pi^*\mathcal{L} \otimes_{\mathbb{C}} \mathcal{K}_{M/X} \ ,$$

from which we obtain

$$\deg_\sigma(\mathcal{O}_M(-l-1)) = \deg_\sigma(\pi^*\mathcal{L}) + \deg_\sigma(\mathcal{K}_{M/X}) = e - \ \frac{l+1}{\sigma} = -(l+1) \ (\ \frac{1}{\sigma} - \frac{e}{l+1} \ ) \ .$$

Using $\deg_\sigma(\mathcal{O}_M(k)) = k \ \deg_\sigma(\mathcal{O}_M(1))$, for $k \in \mathbb{Z}$, the result follows. □

Remark 9.7. Note that for $e \neq 0$, the $GL(l+1,\mathbb{C})$–equivariant tautological bundle $\mathcal{O}_M(1)$ is not associated to the projective flat structure of $M = \mathbb{P}(E)$ and Proposition 8.7 does not apply.

Finally we determine the condition under which the Kähler form ω_σ on M satisfies the Hodge condition, that is $[\omega_\sigma] = c_1(\mathcal{L})$, for some holomorphic line bundle $\mathcal{L} \to M$, provided the same is true for (X, ω_X), that is $[\omega_X] = c_1(\mathcal{L}_X)$.

The problem here is that the extension $\tilde{\omega}_F$ does not necessarily represent an integral cohomology class, even if ω_F does. Using (3.14) , we represent all cohomology classes on $\mathbb{CP}^l$ by $PU(l+1)$–invariant forms. If ω_0 is the Kähler form representing the canonical generator in $H^{1,1}(\mathbb{CP}^l, \mathbb{Z}) \cong \mathbb{Z}$, the Fubini–Study metric is determined by $\omega_F = l \ \omega_0$ (cf. **[2]** loc. cit.). Then the integral class $c_1(\mathcal{K}^*_{\mathbb{CP}^l})$ of type $(1,1)$ is represented by $\alpha = (l+1) \ \omega_0 = \frac{l+1}{l} \ \omega_F$. As $\mathcal{K}_{\mathbb{CP}^l}$ is PU–equivariant, we may use the argument in the proof of Proposition 8.7 to see that the Chern class $c_1(\mathcal{K}^*_{M/X})$ of the dual relative canonical bundle $\mathcal{K}^*_{M/X} = \widetilde{\mathcal{K}}^*_{\mathbb{CP}^l}$ is represented by the form $\tilde{\alpha} = \frac{l+1}{l} \ \tilde{\omega}_F$. This shows in particular that $\tilde{\omega}_F$ represents a rational class. For any positive integer k we have then

$$c_1(\pi^*\mathcal{L}_X \otimes_{\mathbb{C}} (\mathcal{K}^*_{M/X})^k) = [\pi^*\omega_X] + k \ [\tilde{\alpha}] = [\pi^*\omega_X] + k \ \frac{l+1}{l} \ [\tilde{\omega}_F] \ .$$

Comparison with (8.1) shows that we have proved the following result.

Theorem 9.8. *On $M = \widetilde{X} \times_{\hat{\alpha}} \mathbb{CP}^l$, the Kähler form $\omega_{\sigma_k} = \pi^*\omega_X + \sigma_k \ \tilde{\omega}_F$ satisfies the Hodge condition $[\omega_{\sigma_k}] = c_1(\mathcal{L}_k)$ for $\mathcal{L}_k = \pi^*\mathcal{L}_X \otimes_{\mathbb{C}} (\mathcal{K}^*_{M/X})^k$ and $\sigma_k = k \ \frac{l+1}{l}$, for any $k \in \mathbb{Z}^+$. In particular, the flat projective fiber space (M, ω_{σ_k}) is algebraic.*

By the same argument it follows that for any rational $\sigma \in \mathbb{Q}^+$, there exists a smallest positive integer k_σ such that $k_\sigma \ \omega_\sigma$ satisfies the Hodge condition. We also note that the above argument holds in particular if $M = \mathbb{P}(E) \to X$, for E projectively flat.

10. A priori estimates for solutions and stability

So far, the parameter σ appears as a fiberwise scaling parameter for the Kähler form ω_σ on the total space M and solutions of the Hermitian–Einstein equation are with respect to this Kähler structure. On the other hand, σ appears in the coupled vortex equations via the parameters $\tau_i(\sigma)$. We will see that the existence of solutions implies strong a priori restrictions on the range of σ, respectively τ_i .

We return now to the general case where $F = U/K$ is a compact symmetric Kähler manifold and to the context of § 8 . There we considered holomorphic extensions of the type

$$\mathbb{E}_\phi \ : \ 0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0 \ ,$$

with $\mathcal{E}_i = \pi^*\mathcal{W}_i \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_{\rho_i}$, $\rho_i \in R(K)$. Define the *deficiency* of the extension $\mathbb{E}_\phi$ by

$$\Delta\mu_i(\sigma) = \mu_{\mathcal{E}}(\sigma) - \mu_{\mathcal{E}_i}(\sigma) \ .$$

The vortex parameters are then given by

$$\tau_i(\sigma) = \mu_{\mathcal{W}_i} + \Delta\mu_i(\sigma) \ . \tag{10.1}$$

In terms of the deficiencies, the coupled vortex equations (8.6) can now be written as

$$\begin{aligned} \iota \, \Lambda_X F_{h_1} - 2\pi \ \mu_{\mathcal{W}_1} \ \mathbf{I}_{\mathcal{W}_1} &= 2\pi \ \Delta\mu_1(\sigma) \ \mathbf{I}_{\mathcal{W}_1} - \frac{1}{\sigma} \ \phi\circ\phi^* \ , \\ \iota \, \Lambda_X F_{h_2} - 2\pi \ \mu_{\mathcal{W}_2} \ \mathbf{I}_{\mathcal{W}_2} &= 2\pi \ \Delta\mu_2(\sigma) \ \mathbf{I}_{\mathcal{W}_2} + \frac{1}{\sigma} \ \phi^*\circ\phi \ . \end{aligned} \tag{10.2}$$

A straightforward calculation, using (8.14) and (8.15) , shows that

$$\Delta\mu_1(\sigma) = - \ \frac{r_2 s_2}{r_1 s_1 + r_2 s_2} \cdot (\mu_{\mathcal{W}} + \frac{\mu_\rho}{\sigma}) \ , \tag{10.3}$$

and

$$\Delta\mu_2(\sigma) = + \ \frac{r_1 s_1}{r_1 s_1 + r_2 s_2} \cdot (\mu_{\mathcal{W}} + \frac{\mu_\rho}{\sigma}) \ . \tag{10.4}$$

The data for the vortex equation are encoded in the fundamental notion from **[9]** :

Definition 10.1. A *holomorphic triple* $T = (\mathcal{W}_1, \mathcal{W}_2, \phi)$ is given by two holomorphic vector bundles $\mathcal{W}_i \to X$ and a holomorphic homomorphism $\phi : \mathcal{W}_2 \to \mathcal{W}_1$, that is $\phi \in H^0(X, \mathcal{W})$, where $\mathcal{W} = \mathcal{H}om_{\mathbb{C}}(\mathcal{W}_2, \mathcal{W}_1)$.

Now if the holomorphic triple T admits a solution of the vortex equations for a given value of σ, the corresponding Hermitian–Einstein equation on $\mathcal{E} \to M$ relative to ω_σ, admits a solution as described in Theorem 8.9 . The Hitchin–Kobayashi correspondence implies that $\mathcal{E}$ must be (poly–) stable, that is $\Delta\mu_1(\sigma) > 0$ and $\Delta\mu_2(\sigma) < 0$ in the stable case. In the degenerate, polystable case, where the extension $\mathbb{E}_\phi$ is split and hence $\phi = 0$, we have $\Delta\mu_1(\sigma) = \Delta\mu_2(\sigma) = 0$. Therefore we must have

$$\mu_{\mathcal{W}} + \frac{\mu_\rho}{\sigma} \leq 0 \ ,$$

with a strict inequality in the stable case. We will see that $\mu_{\mathcal{W}} \geq 0$ in the presence of solutions for a non–degenerate triple T and therefore

$$0 \leq \sigma \ \mu_{\mathcal{W}} < -\mu_\rho \ . \tag{10.5}$$

Thus the bundle $\mathcal{V}_\rho$ must have negative degree, which is consistent with our assumption (2) in 8.8 .

Theorem 10.2. *If the Hermitian–Einstein equation on $\mathcal{E} \to M$ relative to ω_σ admits an invariant solution as in Theorem 8.9 , the corresponding solution of the coupled σ–vortex equations for the holomorphic triple $T = (\mathcal{W}_1, \mathcal{W}_2, \phi)$ satisfies the L^2–formulas for ϕ :*

(1)

$$\frac{1}{2\pi\ \sigma}\ \|\ \phi\ \|^2 = \tau_1(\sigma) - \mu_{\mathcal{W}_1} = \Delta\mu_1(\sigma)\ ;$$

(2)

$$\frac{1}{2\pi\ \sigma}\ \|\ \phi^*\ \|^2 = \mu_{\mathcal{W}_2} - \tau_2(\sigma) = -\Delta\mu_2(\sigma)\ .$$

Proof. Taking $\int_X$ Tr of the first vortex equation and dividing by $2\pi r_1 \operatorname{Vol}(X)$ gives

$$\frac{1}{2\pi\ \sigma}\ \|\ \phi\ \|^2 = \tau_1(\sigma) - \mu_{\mathcal{W}_1}\ .$$

Here the normalized L^2–norm $\|\ \phi\ \|^2$ is defined by

$$\|\ \phi\ \|^2 = \frac{1}{r_1 \operatorname{Vol}(X)} \int_X \operatorname{Tr}(\phi \circ \phi^*)\, \mathrm{dvol}_X\ ,$$

where ϕ^* is taken with respect to the metrics h_i on $\mathcal{W}_i$. This establishes (1). The other equality is proved in the same way, using the second vortex equation. □

In order to simplify formulas, we assume now that $s_1 = s_2$, where $s_i = \operatorname{rank}(\mathcal{V}_{\rho_i})$.

Theorem 10.3. *Stable case* $(\Delta\mu_1 > 0)$:
If the Hermitian–Einstein equation on $\mathcal{E} \to M$ relative to ω_σ admits an invariant solution as in Theorem 8.9 , the corresponding solution of the coupled σ–vortex equations for the holomorphic triple $T = (\mathcal{W}_1, \mathcal{W}_2, \phi)$ satisfies the following a priori estimates.

(1)

$$\mu_{\mathcal{W}_1} < \tau_1(\sigma)\ ,\ \tau_2(\sigma) < \mu_{\mathcal{W}_2}\ ;$$

(2)

$$(\tau_1 - \tau_2)(\sigma) = -\ \frac{\mu_\rho}{\sigma} > \mu_{\mathcal{W}} = \mu_{\mathcal{W}_1} - \mu_{\mathcal{W}_2} \geq 0\ .$$

In the case of unequal rank $(r_1 \neq r_2)$:

(3)

$$\tau_1(\sigma) \leq \mu_{\mathcal{W}_1} + \frac{r_2}{|\ r_1 - r_2\ |}\ \mu_{\mathcal{W}}\ ,$$

$$\mu_{\mathcal{W}_2} - \frac{r_1}{|\ r_1 - r_2\ |}\ \mu_{\mathcal{W}} \leq \tau_2(\sigma)\ ,$$

and hence

$$\mu_{\mathcal{W}} = \mu_{\mathcal{W}_1} - \mu_{\mathcal{W}_2} > 0\ ;$$

(4)

$$0 < \sigma_1 \leq \sigma < \sigma_0 = -\ \frac{\mu_\rho}{\mu_{\mathcal{W}}}\ ,$$

where

$$\sigma_1 = -\ \frac{\mu_\rho}{\mu_{\mathcal{W}}} \cdot \{\ 1 + \frac{r_1 + r_2}{|\ r_1 - r_2\ |}\ \}^{\perp 1}\ ;$$

In the case of equal rank $(r = r_1 = r_2)$:

(5) $\deg_X(\mathcal{W}_1) \geq \deg_X(\mathcal{W}_2)$, *hence*

$$\mu_{\mathcal{W}} = \mu_{\mathcal{W}_1} - \mu_{\mathcal{W}_2} \geq 0 \ .$$

If ϕ *is an isomorphism, then the* $\mathcal{W}_i$ *are polystable bundles.*

(6)
$$0 < \sigma < \sigma_0 = - \ \frac{\mu_\rho}{\mu_{\mathcal{W}}} \ ; \ \sigma_0 = \infty \ , \ \textit{for} \ \mu_{\mathcal{W}} = 0 \ ;$$

(7)
$$\Delta\mu_1(\sigma) = - \ \Delta\mu_2(\sigma) = - \ \frac{1}{2} \ (\mu_{\mathcal{W}} + \frac{\mu_\rho}{\sigma}) > 0 \ .$$

Thus there is a gap of length $\mu_{\mathcal{W}} \geq 0$ between τ_1 and τ_2 :

$$\tau_2(\sigma) < \mu_{\mathcal{W}_2} \leq \mu_{\mathcal{W}_1} < \tau_1(\sigma) \ .$$

Proof. Statement (1) follows from Theorem 10.2 . (2) follows from (8.14) , (1) and (3), (5) below. (3) follows from **[9]** , Proposition 3.18. (4) follows from (1) and (3). (5) follows from **[9]** , Corollary 3.20 and Lemma 4.6 . (6) and (7) follow from (10.3) and (10.4) . □

Theorem 10.4. *For the degenerate, polystable case* $(\Delta\mu_1 = 0)$, *we have* :

(1)
$$\phi = 0 \ , \ \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \ , \ \mu_{\mathcal{E}} = \mu_{\mathcal{E}_1} = \mu_{\mathcal{E}_2} \ .$$

(2) *We have* $\tau_i = \mu_{\mathcal{W}_i}$ *and the vortex equations degenerate to the uncoupled Hermitian–Einstein equations on each* $\mathcal{W}_i$.

(3) *If the bundles* $\mathcal{W}_i$ *are polystable, the Hermitian–Einstein equation on* $\mathcal{E} \to M$ *relative to* ω_σ *admits an invariant solution as in Theorem* 8.9 *exactly for*

$$\mu_{\mathcal{W}} > 0 \ , \ \sigma = \sigma_0 = -\frac{\mu_\rho}{\mu_{\mathcal{W}}} \ .$$

Proof. This follows immediately from Theorem 10.2 and formulas (10.2) , (10.3) .

□

11. Appendix : Obstructions for projectively flat holomorphic bundles

The topology of projective bundles is derived from the commutative diagram of groups (6.2) . At the level of classifying spaces this gives rise to the following diagram.

$$\begin{array}{ccc}
H^*(BSU(l+1),\mathbb{Q}) & \xleftarrow[\cong]{Bj_0^*} & H^*(BPSU(l+1),\mathbb{Q}) \\
\uparrow{\scriptstyle Bi_0^*} & & \uparrow{\scriptstyle\cong} \\
H^*(BU(l+1),\mathbb{Q}) & \xleftarrow{Bj^*} & H^*(BPU(l+1),\mathbb{Q}) \\
\uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\
\mathbb{Q}[c_1,\dots,c_{l+1}] & \xleftarrow{Bj^*} & \mathbb{Q}[\bar{c}_2,\dots,\bar{c}_{l+1}]
\end{array} \tag{11.1}$$

Here we choose the rational generators $\bar{c}_k, \ k \geq 2$ so that they correspond to the ordinary Chern classes of $SU(l+1)$, that is $Bj_0^*(\bar{c}_k) = c_k$ and of course we have $Bi_0^*(c_k) = c_k, \ Bi_0^*(c_1) = 0$.

The image of Bj^* consists of those rational characteristic classes of complex vector bundles E which are *projective invariants*. Computing on the maximal tori, we obtain the formulas

$$Bj^*(\bar{c}_k) = c_k + \binom{l+1}{k} (1-k) \{\frac{-c_1}{l+1}\}^k + \sum_{\alpha=2}^{k\perp 1} \binom{l+1-\alpha}{k-\alpha} c_\alpha \{\frac{-c_1}{l+1}\}^{k\perp\alpha} , \tag{11.2}$$

and

$$c_k = \binom{l+1}{k} \{\frac{c_1}{l+1}\}^k + Bj^*(\bar{c}_k) + \sum_{\alpha=2}^{k\perp 1} \binom{l+1-\alpha}{k-\alpha} Bj^*(\bar{c}_\alpha) \{\frac{c_1}{l+1}\}^{k\perp\alpha} . \tag{11.3}$$

If E is projectively flat, that is

$$\mathbb{CP}^l \hookrightarrow \mathbb{P}(E) \cong \widetilde{X} \times_{\hat{\alpha}} \mathbb{CP}^l \xrightarrow{\pi} X ,$$

with holonomy $\hat{\alpha} : \Gamma \to PGL(l+1, \mathbb{C})$, it follows from Chern–Weil theory (cf. [**34**]) that the projective classes $\bar{c}_k(P) = 0$ and therefore from (11.3)

$$c_k(E) = \binom{l+1}{k} \{\frac{c_1(E)}{l+1}\}^k ,$$

$$c_t(E) = \{1 + t \ \frac{c_1(E)}{l+1}\}^{l+1} .$$

Consider a (holomorphic) projective bundle over the compact Kähler manifold X with structure group $G = PGL(l+1, \mathbb{C})$:

$$\mathbb{CP}^l \hookrightarrow M = P \times_{PGL} \mathbb{CP}^l \xrightarrow{\pi} X .$$

In order to simplify the exposition, we assume now that $H^2(X, \mathbb{Z})$ is torsionfree, and that $X \simeq B_\Gamma$; in which case $H^*(\Gamma, A) \cong H^*(X, A)$ for any coefficient group A .

The obstruction for the linearization of a projective fiber bundle is given by an element $\mathfrak{o}_{GL}(P) \in H^2(X, \mathscr{O}_X^\times)$. For $\mathfrak{o}_{GL}(P) = 0$, that is $M = \mathbb{P}(E)$, the vector bundle E is determined up to $E \otimes_{\mathbb{C}} \mathcal{M}$, where $\mathcal{M}$ is a holomorphic line bundle on X. The relative obstruction for $M = \mathbb{P}(E)$ with E *unimodular*, that is $\mathcal{L} = \det(E)$ holomorphically trivial, is given by $\mathfrak{o}_{SL}(P) = r \ c_1(\mathcal{L}) \in H^2(X, \mathbb{Z}_{l+1})$. In fact, $\mathfrak{o}_{SL}(P) = 0$ means that the determinant bundle is of the form $\mathcal{L} = \mathcal{M}^{l+1}$ and thus $\mathbb{P}(E) \cong \mathbb{P}(E \otimes_{\mathbb{C}} \mathcal{M}^*)$, with $E \otimes_{\mathbb{C}} \mathcal{M}^*$ unimodular.

A similar discussion applies if (6.1) is flat, that is

$$\mathbb{CP}^l \hookrightarrow M \cong \widetilde{X} \times_{\hat{\alpha}} \mathbb{CP}^l \xrightarrow{\pi} X ,$$

with holonomy $\hat{\alpha} : \Gamma \to PGL(l+1, \mathbb{C})$. In this case the relative obstruction for $M = \mathbb{P}(E)$ with E *holomorphically flat* is given by

$$\mathfrak{o}_{GL}(\hat{\alpha}) \in \ker(H^2(\Gamma, \mathbb{C}^\times) \to H^2(X, \mathscr{O}_X^\times)) ,$$

and the relative obstruction for *unimodular flatness* of E is given by

$$\mathfrak{o}_{SL}(\hat{\alpha}) \in \ker(H^2(\Gamma, \mathbb{Z}_{l+1}) \to H^2(X, \mathbb{Z}_{l+1})) .$$

The obstruction $\mathfrak{o}_{GL}(P) = 0$, that is $M = \mathbb{P}(E)$, if and only if

$$\mathfrak{o}_{SL}(P) = r \ c_1(\mathcal{L}) \ , \ \mathcal{L} = \det(E) ,$$

where E is determined up to $E \otimes_{\mathbb{C}} \mathcal{M}$, for holomorphic line bundles $\mathcal{M}$ on X . If $\mathfrak{o}_{GL}(P) = 0$, then

$$\mathfrak{o}_{SL}(P) = 0 \text{ if and only if } c_1(\mathcal{L}) \text{ is divisible by } l+1 \ .$$

The obstruction $\mathfrak{o}_{GL}(\hat{\alpha}) = 0$, that is $M = \mathbb{P}(E)$ with E flat, if and only if $E \otimes_{\mathbb{C}} \mathcal{M}$ is unimodular (flat), for some $\mathcal{M} \in \mathrm{Pic}_0(X)$.

These four obstructions are related by the commutative diagram

$$\begin{array}{ccc} H^2(\Gamma, \mathbb{C}^\times) & \longrightarrow & H^2(X, \mathscr{O}_X^\times) \\ \uparrow i_* & & \uparrow i_* \\ H^2(\Gamma, \mathbb{Z}_{l+1}) & \xrightarrow{\cong} & H^2(X, \mathbb{Z}_{l+1}) \ . \end{array} \tag{11.4}$$

Thus we have the following relations :

$$\mathfrak{o}_{GL}(P) = i_* \, \mathfrak{o}_{SL}(P) \ ,$$
$$\mathfrak{o}_{GL}(P)^{l+1} = 0 \ ;$$

$$\mathfrak{o}_{GL}(\hat{\alpha}) = i_* \, \mathfrak{o}_{SL}(\hat{\alpha}) \ , \ i_* \text{ injective} \ ,$$
$$\mathfrak{o}_{GL}(\hat{\alpha})^{l+1} = 0 \ ,$$
$$\mathfrak{o}_{GL}(\hat{\alpha}) = \mathfrak{o}_{SL}(\hat{\alpha}) = \mathfrak{o}_{SL}(P) \ .$$

The above obstructions are defined by the 'exact' sequences associated to diagram (6.2), using the classification of (flat) holomorphic principal bundles

$$\begin{array}{ccccccc} H^1(X, \mathscr{O}_X^\times) & \xrightarrow{\Delta_*} & \check{H}^1(X, \mathscr{O}_X(GL)) & \xrightarrow{j_*} & \check{H}^1(X, \mathscr{O}_X(PGL)) & \xrightarrow{\mathfrak{o}_{GL}} & H^2(X, \mathscr{O}_X^\times) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H^1(\Gamma, \mathbb{C}^\times) & \xrightarrow{\Delta_*} & H^1(\Gamma, GL) & \xrightarrow{j_*} & H^1(\Gamma, PGL) & \xrightarrow{\mathfrak{o}_{GL}} & H^2(\Gamma, \mathbb{C}^\times) \end{array} \tag{11.5}$$

$$\begin{array}{ccccccc} H^1(X, \mathbb{Z}_{l+1}) & \xrightarrow{\Delta_*} & \check{H}^1(X, \mathscr{O}_X(SL)) & \xrightarrow{j_{0*}} & \check{H}^1(X, \mathscr{O}_X(PSL)) & \xrightarrow{\mathfrak{o}_{SL}} & H^2(X, \mathbb{Z}_{l+1}) \\ \uparrow \cong & & \uparrow & & \uparrow & & \uparrow \cong \\ H^1(\Gamma, \mathbb{Z}_{l+1}) & \xrightarrow{\Delta_*} & H^1(\Gamma, SL) & \xrightarrow{j_{0*}} & H^1(\Gamma, PSL) & \xrightarrow{\mathfrak{o}_{SL}} & H^2(\Gamma, \mathbb{Z}_{l+1}) \end{array} \tag{11.6}$$

as well as the classification of holomorphic line bundles

$$0 \to \mathrm{Pic}_0(X) = H^1(X, \mathscr{O}_X)/\iota_* H^1(X, \mathbb{Z}) \to H^1(X, \mathscr{O}_X^\times) \xrightarrow{c_1} H^{1,1}(X, \mathbb{Z}) \to 0 \ . \tag{11.7}$$

Combining the exact diagrams

$$\begin{array}{ccccccccc} & & \mathbb{Z}_{l+1} & & & & & & \\ & & \uparrow & & & & & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota} & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times & \longrightarrow & 0 \\ & & \uparrow l+1 & & \uparrow l+1 & & \uparrow (\cdot)^{l+1} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota} & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times & \longrightarrow & 0 \\ & & & & & & \uparrow i & & \\ & & & & & & \mathbb{Z}_{l+1} & & \end{array}$$

$$
\begin{array}{ccccccccc}
 & & \mathbb{Z}_{l+1} & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota} & \mathscr{O}_X & \xrightarrow{\exp} & \mathscr{O}_X^\times & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle l+1} & & \uparrow{\scriptstyle l+1} & & \uparrow{\scriptstyle (\cdot)^{l+1}} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota} & \mathscr{O}_X & \xrightarrow{\exp} & \mathscr{O}_X^\times & \longrightarrow & 0 \\
 & & & & & & \uparrow{\scriptstyle i} & & \\
 & & & & & & \mathbb{Z}_{l+1} & &
\end{array}
$$

via their corresponding exact cohomology sequences, one obtains the following commutative diagram with exact columns :

$$
\begin{array}{ccccc}
\uparrow & & \uparrow & & \uparrow \\
H^2(\Gamma,\mathbb{C}^\times) & \longrightarrow & H^2(X,\mathscr{O}_X^\times) & \longrightarrow & H^3(X,\mathbb{Z}) \\
\uparrow{\scriptstyle (\cdot)_*^{l+1}} & & \uparrow{\scriptstyle (\cdot)_*^{l+1}} & & \uparrow{\scriptstyle l+1} \\
H^2(\Gamma,\mathbb{C}^\times) & \longrightarrow & H^2(X,\mathscr{O}_X^\times) & \longrightarrow & H^3(X,\mathbb{Z}) \\
\uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle \beta} \\
H^2(\Gamma,\mathbb{Z}_{l+1}) & \xrightarrow{\cong} & H^2(X,\mathbb{Z}_{l+1}) & \xrightarrow{=} & H^2(X,\mathbb{Z}_{l+1}) \\
\uparrow{\scriptstyle \beta=0} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle r} \\
H^1(\Gamma,\mathbb{C}^\times) & \longrightarrow & H^1(X,\mathscr{O}_X^\times) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) \\
\uparrow{\scriptstyle (\cdot)_*^{l+1}} & & \uparrow{\scriptstyle (\cdot)_*^{l+1}} & & \uparrow{\scriptstyle l+1} \\
H^1(\Gamma,\mathbb{C}^\times) & \longrightarrow & H^1(X,\mathscr{O}_X^\times) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) \\
\uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle \beta=0} \\
H^1(\Gamma,\mathbb{Z}_{l+1}) & \xrightarrow{\cong} & H^1(X,\mathbb{Z}_{l+1}) & \xrightarrow{=} & H^1(X,\mathbb{Z}_{l+1}) \\
\uparrow & & \uparrow & & \uparrow
\end{array}
\tag{11.8}
$$

The listed properties of the obstructions are obtained by diagram chasing in (11.5) to (11.8) .

Example 11.1. Ruled surfaces :

Consider a *ruled surface*, that is a flat projective fiber bundle

$$\mathbb{CP}^1 \hookrightarrow M = \widetilde{X} \times_{\hat{\alpha}} \mathbb{CP}^1 \xrightarrow{\pi} X \ ,$$

with holonomy $\hat{\alpha} : \Gamma \to PGL(2,\mathbb{C})$, where X is a Riemann surface of genus $g > 1$. Observe that $X \simeq B_\Gamma$ and $\Gamma/[\Gamma,\Gamma] = \mathbb{Z}^{2g}$. In this case, diagram (11.8) has the form

$$\begin{array}{ccccc}
\mathbb{Z}_{l+1} & \longrightarrow & 0 & & \\
\Big\uparrow{\scriptstyle =} & & \Big\uparrow & & \\
\mathbb{Z}_{l+1} & \xrightarrow{=} & \mathbb{Z}_{l+1} & \xrightarrow{=} & \mathbb{Z}_{l+1} \\
\Big\uparrow{\scriptstyle 0} & & \Big\uparrow{\scriptstyle \beta} & & \Big\uparrow{\scriptstyle r} \\
(\mathbb{C}^\times)^{2g} & \longrightarrow & H^1(X, \mathscr{O}_X^\times) & \xrightarrow{c_1} & \mathbb{Z} \\
\Big\uparrow{\scriptstyle (\cdot)^{l+1}_*} & & \Big\uparrow{\scriptstyle (\cdot)^{l+1}_*} & & \Big\uparrow{\scriptstyle l+1} \\
(\mathbb{C}^\times)^{2g} & \longrightarrow & H^1(X, \mathscr{O}_X^\times) & \xrightarrow{c_1} & \mathbb{Z} \\
\Big\uparrow{\scriptstyle i_*} & & \Big\uparrow{\scriptstyle i_*} & & \Big\uparrow{\scriptstyle \beta=0} \\
(\mathbb{Z}_{l+1})^{2g} & \xrightarrow{=} & (\mathbb{Z}_{l+1})^{2g} & \xrightarrow{=} & (\mathbb{Z}_{l+1})^{2g}
\end{array} \tag{11.9}$$

The obstruction $\mathfrak{o}_{GL}(P)$ for linearity vanishes, since $H^2(X, \mathscr{O}_X^\times) \cong H^3(X, \mathbb{Z}) = 0$. Thus $M = \mathbb{P}(E)$ is given by a projectively flat holomorphic vector bundle E of rank 2 . The remaining obstructions are all identical. They are given by

$$\mathfrak{o}_{SL}(P) = \mathfrak{o}_{SL}(\hat{\alpha}) = r\ c_1(\det\ E) \in H^2(X, \mathbb{Z}_2) \cong \mathbb{Z}_2 ,$$

and can be identified with the Stiefel–Whitney class $w_2(P)$. Thus we know that the ruled surface $M = \widetilde{X} \times_{\hat{\alpha}} \mathbb{CP}^1$ is always linear, that is $M = \mathbb{P}(E)$, for a (not necessarily unique) projectively flat holomorphic vector bundle E of rank 2 .

The following properties are then equivalent:

(1) E is of degree 0, that is $c_1(E) = 0$ or $\det(E) \in \mathrm{Pic}_0(X)$;
(2) $\det(E)$ is trivial as a smooth complex line bundle ;
(3) $E \otimes_{\mathbb{C}} \mathcal{M}$ is unimodular, that is $\det(E \otimes_{\mathbb{C}} \mathcal{M}) = \det(E) \otimes_{\mathbb{C}} \mathcal{M}^2$ is holomorphically trivial, for some line bundle $\mathcal{M} \in \mathrm{Pic}_0(X)$;
(4) $E \otimes_{\mathbb{C}} \mathcal{M}$ is a flat holomorphic bundle with holonomy $\alpha : \Gamma \to SL(2, \mathbb{C})$, for some line bundle $\mathcal{M} \in \mathrm{Pic}_0(X)$;
(5) E is a flat holomorphic bundle with holonomy $\alpha : \Gamma \to GL(2, \mathbb{C})$;
(6) E admits a holomorphic connection.

In fact, the equivalence of (1) to (5) follows from § 9 and the diagrams above. The equivalence of (5) and (6) follows from the vanishing condition for the Atiyah obstruction **[1]** :

$$a(E) = [\Omega^{1,1}_\nabla] \in H^{1,1}(X, \mathcal{E}nd_{\mathbb{C}}(E)) ,$$

since the curvature of a connection ∇ of type $(1,0)$ is given by $\Omega^{1,1}_\nabla$, $\Omega^{2,0}_\nabla$ being automatically zero.

We note that this remains valid for flat projective bundles over X with fiber $\mathbb{CP}^l$, $l > 1$. Moreover, the flat holomorphic bundles with (irreducible) isometric holonomy $\alpha : \Gamma \to SU(2)$, correspond by the theorem of Narasimhan–Seshadri **[40]** to the semi–stable (stable) holomorphic bundles of degree 0 and rank 2 .

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