**ESI** The Erwin Schrödinger International Institute for Mathematical Physics
Boltzmanngasse 9 A-1090 Wien, Austria

# Spin & Statistics, Localization Regions, and Modular Symmetries in Quantum Field Theory

**Bernd Kuckert**

Vienna, Preprint ESI 589 (1998) September 7, 1998

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at

# Spin & Statistics, Localization Regions, and Modular Symmetries in Quantum Field Theory

Dissertation
zur Erlangung des Doktorgrades
des Fachbereichs Physik
der Universität Hamburg

vorgelegt von
Bernd Kuckert
aus Krefeld

Hamburg 1998

## Abstract

Using a special version of the PCT-theorem which was found by Bisognano and Wichmann for finite-component Wightman fields, a proof of the spin-statistics theorem is given within the algebraic framework for quantum field theory. The proof covers massive bosons and fermions with ordinary as well as with parastatistics and, in contrast to earlier proofs, also works in 1+2 spacetime dimensions.

Two uniqueness theorems concerning the Bisognano-Wichmann symmetries whose $P_1$CT-part is used in the discussion of the spin-statistics theorem are presented for the algebraic setting. A derivation of the Bisognano-Wichmann result from standard assumptions of the algebraic setting is still lacking. The uniqueness theorems show that the operators which were found to implement the $P_1$CT-symmetry and the Lorentz boosts in the Bisognano-Wichmann setting cannot implement any other symmetry on a local net of observables than precisely the one found by Bisognano and Wichmann.

The analysis uses the notions of localization regions not only for algebras of observables, but also for single local observables. It is shown how these regions can be defined, and for the localization region of a single local observable it is investigated under what assumptions observables localized in spacelike separated region commute.

## Zusammenfassung

Es wird ein Beweis des Spin-Statistik-Theorems gegeben, der auf einer speziellen Form der PCT-Symmetrie beruht. Diese Symmetrie wurde von Bisognano und Wichmann für alle endlichkomponentigen Wightmanfelder nachgewiesen. Der Beweis des Spin-Statistik-Theorems deckt alle massiven Bosonen und Fermionen mit gewöhnlicher und mit Parastatistik ab und gilt — im Gegensatz zu früheren Beweisen des Satzes — auch in 1+2 Raumzeitdimensionen.

Es werden zei Eindeutigkeitssätze über die von Bisognano und Wichmann gefundenen Symmetrien für den algebraischen Rahmen der Quantenfeldtheorie bewiesen. Eine Herleitung des Bisognano-Wichmann-Resultates aus Standardannahmen dieses Zugangs gibt es bisher noch nicht. Die beiden Eindeutigkeitssätze besagen, daß die Operatoren, die im Bisognano-Wichmann-Fall die Lorentz-Boosts bzw. die $P_1$CT-Symmetrie darstellen, keine andere als eben diese Symmetriesn auf einem lokalen Observablennetz implementieren können.

Diese Untersuchung benutzt den Begriff des Lokalisationsgebietes sowohl für Algebren von Observablen als auch für einzelne lokale Observable. Es wird gezeigt, wie solche Lokalisationsgebiete definiert werden können, und für das lokalisationsgebiet einer lokalen Observablen wird untersucht, unter welchen Annahmen raumartig getrennt lokalisierte Observable miteinander vertauschen.

# Contents

# Chapter 1

# Introduction

The spin-statistics theorem due to Fierz and Pauli [43, 66] is one of the great successes of general quantum field theory. A proof for finite-component Wightman fields can be found in the monograph by Streater and Wightman [72]. The Wightman framework covers most of the known examples of quantum field theories. But one of the basic structures the spin-statistics theorem deals with, the Bose-Fermi alternative, has to be assumed as an axiom in the Wightman framework.

In the algebraic approach to quantum field theory, which is due to Haag and Kastler [52, 51], the Bose-Fermi alternative is a result, not an axiom. The input of the theory is a net $\mathfrak{A}$ of C$^*$-algebras $\mathfrak{A}(\mathcal{O})$ of bounded linear operators in a Hilbert space which are associated with every open, bounded space-time region $\mathcal{O} \subset \mathbb{R}^{1+s}$ in such a way that operators belonging to spacelike separated regions commute (**locality**). The basic structures of special classes of charged fields — including the possible particle statistics — can be recovered from the mere observable input [38, 39, 35].

A field system consisting of von Neumann algebras which generates, in particular, all massive parabosonic and parafermionic sectors from the vacuum and exhibits normal commutation relations has been constructed by Doplicher and Roberts [41] for every local net of observables satisfying the standard assumptions, and such a field system is unique up to unitary equivalence.

For the algebraic framework, the spin-statistics theorem in (1+3)-dimensional spacetime has been proven in [39] for charges which are localizable in bounded spacetime regions and in [33] for charges which are localizable in open convex cones in $\mathbb{R}^{1+s}$ extended to spacelike infinity (spacelike cones, see the definition in chapter 2); such charges appear in purely massive theories [35]. All these proofs use properties of the irreducible representations of the Poincaré group which heavily depend on the spacetime dimension. Therefore these arguments do not apply to lower dimensions.

In the most general case in lower dimensions, particles violating the familiar Bose-Fermi alternative can occur. The spin of these particles no longer needs to be integer or half-integer, it may be any real number. These particles are expected to play a role in the theory of the fractional quantum Hall effect [71]. In 1+2 dimensions, however, the rotation group is the circle, and the universal covering of the circle is of infinite order. For this reason, the familiar spinor structure does not describe the irreducible representations of the Poincaré group in 1+2 dimensions and of its universal covering. This makes it interesting to prove the spin-statistics theorem without making use of this structure.

In **Chapter 2**, such a proof will be given for massive (para-) bosonic and fermionic sectors. A first version of the proof already appeared in [57], a more detailed version has been published in [58]. The result includes parabosonic and parafermionic charges localizable in spacelike cones. The proof works for theories of local observables in at least 1+2 dimensions which — in addition to a couple of standard properties — exhibit a special form of PCT-symmetry.

This symmetry is exhibited by every finite-component Wightman field $\phi$, as Bisognano and Wichmann already showed in the seventies [8, 9]. They considered the field operators $\phi(f)$ associated with test functions $f$ with support in the wedge region $W_1 := \{x \in \mathbb{R}^{1+s} : x_1 > |x_0|\}$. These operators generate a complex involutive algebra $\mathcal{R}(W_1)$. By the Reeh-Schlieder theorem, applying the operators of this algebra to the vacuum generates a dense subspace of the Hilbert space of the field system. On this space one defines the antilinear operator $R\Omega \mapsto R^*\Omega$, $R \in \mathcal{R}(W_1)$. This operator is closable, and its closure, the **Tomita operator** $S_{W_1}$, has a unique polar decomposition $S_{W_1} = J_{W_1}\Delta_{W_1}^{1/2}$ into an antiunitary operator $J_{W_1}$ called the **modular conjugation**, and a positive operator $\Delta_{W_1}^{1/2}$ which is the square root of the **modular operator** $\Delta_{W_1}$ (actually, $\Delta_{W_1}^{1/2}$ is the modulus of $S_{W_1}$). Bisognano and Wichmann showed that $J_{W_1}$ implements a $\mathrm{P}_1$CT-transformation on the field, i.e., a reflection in the 0- and the 1-direction together with a charge conjugation (**modular $\mathbf{P}_1$CT-symmetry**), whereas the unitary group $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, the **modular group** of the setting, implements the Lorentz-boosts in the 1-direction (**modular Lorentz symmetry**).

In chapter 2, modular $\mathrm{P}_1$CT-symmetry is assumed, whereas the modular group is free to behave or not to behave like in the Bisognano-Wichmann setting.

The modular objects considered bby Bisognano and Wichmann are also well-defined in algebraic quantum field theory. At present, however, a result like the Bisognano-Wichmann theorem is not known for this framework.

As an important step towards an appropriate generalization, Borchers recently showed that for every local net of observables which is covariant under a representation of the translation group satisfying the spectrum condition, the operators $J_{W_1}$ and $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, commute with the representation of the translation group like a $\mathrm{P}_1$CT-operator and Lorentz-boosts, respectively [16]. He concluded that in 1+1 dimensions, a local extension of the local net exists which exhibits the Bisognano-Wichmann symmetries.

For the higher-dimensional case, such a general result is not known. But using the commutation relations found by Borchers, one can show that as soon as $J_{W_1}$ or $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, acts — in a most general sense — geometrically on the net of observables, it implements Lorentz boost or the $\mathrm{P}_1$CT-symmetry, respectively. Two versions of this statement have been found so far; they are discussed in **Chapter 4**. In both cases it is assumed that the modular conjugation or group under consideration acts geometrically in a very general sense, and it is concluded that it implements, like in the Bisognano-Wichmann setting, a boost or a $\mathrm{P}_1$CT-operator, respectively.

In the first uniqueness theorem it will be assumed that under the adjoint action of $J_{W_1}$ or $\Delta_{W_1}^{it}$, all algebras associated with certain bounded regions in Minkowski space are mapped onto algebras associated with arbitrary open regions elsewhere in Minkowski space. It is due to the large impact of Borchers' commutation relations that it is, a priori, not even necessary to assume that the latter region is associated with the former by some point transformation from $\mathbb{R}^{1+s}$ to $\mathbb{R}^{1+s}$. The first uniqueness theorem for modular symmetries states that as soon as $J_{W_1}$ or $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, behaves this way, it behaves like in the Bisognano-Wichmann case. A preliminary version of this result occurred in [59, 57], but applying the double cone theorem due to Borchers and Vladimirov in a way proposed by Trebels [78], a last ambiguity left in the earlier version could be removed.

In the second uniqueness theorem, only the modular group $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, is considered. For a given local observable $A$, it is assumed that for small $t$, the operator $\Delta_{W_1}^{it} A \Delta_{W_1}^{-it} =: A_t$ is a local observable and that the localization region of this observable depends continuously on $t$. The second uniqueness theorem then states that for these small $t$ the localization region of $A_t$ develops like under the action of a Lorentz boost, as in the Bisognano-Wichmann setting.

The two uniqueness theorems play different roles in the analysis of modular symmetries: the first one investigates which net symmetries can be implemented by the Bisognano-Wichmann modular objects, whereas the second one rather may demonstrate that in order not to implement a symmetry, the modular group must act in an 'extremely non-geometric or discontinuous' way.

In both uniqueness theorems, the localization of algebras and observables plays a fundamental role not only for the proof, but even for the statement of the theorems. These notions are introduced in **Chapter 3**. The proof that the prescriptions considered there yield unique and nonempty regions is based on a classical result due to Landau which states that algebras associated with two double cones with disjoint closures are disjoint up to the multiples of the identity. A generalization of this theorem will be used; it will be proved as a main result of Chapter 3. It states that the algebras associated with a finite family of wedge regions whose closures have an empty common intersection do not have any local observables in common except for the multiples of the identity. This generalization implies that the intersection of all closed wedge-shaped regions in whose algebras a nontrivial local observable $A$ is contained is a well-defined and nonempty set.

This result, in turn, leads to the question which one of the localization prescriptions considered may be considered the most favourable one, and under which localzation prescriptions observables with spacelike separated localization regions commute. So far, no proof has been found that this holds under the standard assumptions which were sufficient to associate a nonempty localization region with every local observable.

It turns out that if the net of observables satisfies wedge duality (which, by the Bisognano-Wichmann theorem, is a property of all finite-component Wightman fields) those of the considered localization prescriptions which yield the smallest localization regions do coincide. For this case, a necessary and sufficient condition for locality of this localization prescription is that for every finite family of wedges, the intersection of the algebras associated with these wedges contains the same local observables as the algebra that is associated with the intersection of the wedge regions by the dual net, which is the maximal extension of the net $\mathfrak{A}$ which satisfies locality.

A sufficient condition for locality of the localization prescription is strong additivity for wedge regions, a technical assumption which is typically fulfilled by nets arising from Wightman fields.

# Chapter 2

# A new approach to Spin & Statistics

In this chapter the spin-statistics theorem for massive particles will be proved within the setting of algebraic quantum field theory. The line of argument is as follows: from modular $P_1CT$-symmetry and a compactness assumption discussed below, it will be derived that any rotation is represented by a product of two $P_1CT$-operators (i.e. the $P_1CT$-operators associated with respect to two (in general) distinct Lorentz frames). This result and modular $P_1CT$-symmetry will then be extended from the net of observables to the $\widetilde{\mathcal{P}_+^\uparrow}$-covariant Bose-Fermi field constructed by Doplicher and Roberts. The straightforward computation of any rotation by $2\pi$ in the corresponding representation, finally, yields the Bose-Fermi operator of the field; this implies the familiar spin-statistics connection.

The chapter is structured as follows: in Section 2.1 basic assumptions, the notation and preliminaries are introduced, these include the foundations of Tomita-Takesaki theory, the Buchholz-Fredenhagen sector analysis and the Doplicher-Roberts field system based on this analysis. In Section 2.2 the results, which are the steps leading to the spin-statistics theorem, are stated and proved. In Section 2.3 a couple of related results and open problems are discussed.

## 2.1 Notation, preliminaries, and assumptions

For some integer $s \geq 2$, denote by $\mathbb{R}^{1+s}$ the (1+s)-dimensional Minkowski space, and let $V_+$ be the (open) forward light cone. The class $\mathcal{K}$ of all **double cones**, i.e., all open sets $\mathcal{O}$ of the form

$$\mathcal{O} := (a + V_+) \cap (b - V_+), \qquad a, b \in \mathbb{R}^{1+s},$$

is a convenient topological base of $\mathbb{R}^{1+s}$. Each nonempty double cone is fixed by two points, its upper and its lower apex, and the class $\mathcal{K}$ is invariant under the action of the Poincaré group.

In the sequel, $\mathfrak{A}$ will denote a **local net of observables** in an infinite-dimensional Hilbert space $\mathcal{H}$: it associates with every double cone $\mathcal{O} \in \mathcal{K}$ a C$^*$-algebra $\mathfrak{A}(\mathcal{O})$ which consists of bounded operators in $\mathcal{H}$ and contains the identity operator; this mapping is assumed to be **isotonous**, i.e. if $\mathcal{O} \subset \mathcal{P}$, $\mathcal{O}, P \in \mathcal{K}$, then $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(P)$, and to satisfy **locality**, i.e., if $\mathcal{O}$ and $P$ are spacelike separated double cones and if $A \in \mathfrak{A}(\mathcal{O})$, $B \in \mathfrak{A}(P)$, then $AB = BA$.

For an arbitrary region[1] $M \subset \mathbb{R}^{1+s}$, define $\mathfrak{A}(M)$ to be the C$^*$-algebra generated by the C$^*$-algebras $\mathfrak{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$, $\mathcal{O} \subset M$. $\widetilde{\mathfrak{A}} := \mathfrak{A}(\mathbb{R}^{1+s})$ will be the C$^*$-algebra of **quasilocal observables**. Note that every state of the normed, involutive algebra $\mathfrak{A}_{\mathrm{loc}} = \bigcup_{\mathcal{O} \in \mathcal{K}} \mathfrak{A}(\mathcal{O})$ of all **local observables** has a unique continuous extension to a state of the C$^*$-algebra $\widetilde{\mathfrak{A}}$.

For every region $M$ in $\mathbb{R}^{1+s}$, $M'$ will denote the **spacelike complement** of $M$, i.e., the set of all points in $\mathbb{R}^{1+s}$ which are spacelike with respect to all points of $M$, and for every algebra $\mathfrak{M}$ of bounded operators in some Hilbert space $\mathfrak{H}$, $\mathfrak{M}'$ will denote the algebra of all bounded operators which commute with all elements of $\mathfrak{M}$. Using this notation, the above locality assumption reads $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')'$ for all $\mathcal{O} \in \mathcal{K}$.

Another kind of regions in Minkowski space that will be used are the **spacelike cones**: for every open, salient, convex circular cone $\vec{C}$ in $\mathbb{R}^s$, i.e., every cone in $\mathbb{R}^s$ which is generated by some open $\varepsilon$-ball around a vector $\vec{x} \in \mathbb{R}^s$ with euclidean length $\|\vec{x}\|_2 > \varepsilon$, the causal completion $\vec{C}'' =: C$ of $\vec{C}$ and its Poincaré transforms will be called spacelike cones; their set will be denoted by $\mathcal{S}$. Note that this definition, which is based on remarks in [33], singles out the **causally complete** spacelike cones in the sense of [35], i.e., cones with $C'' = C$.

[1] Any subset of a topological space may happen to be called a region in the sequel. Typically, the topological space will be some spacetime or a finite-dimensional complex vector space. In the mathematical literature, the word 'region' is sometimes used in a more restrictve way, but the use of the word is far from uniform.

The last class of regions in Minkowski space which will be important in this chapter is the class $\mathcal{W}$ of **wedges**, which contains all Poincaré transforms of the region

$$W_1 := \{x \in \mathbb{R}^{1+s} : x_1 > |x_0|\}$$

and the causal complements of these regions (which are closed wedges).

## Assumptions, I:

The above local net $\mathfrak{A}$ of local observables in the Hilbert space $\mathcal{H}$ will be assumed to be **covariant** under a strongly continuous representation $U$ in $\mathcal{H}$ of the universal covering $\widetilde{\mathcal{P}_+^\uparrow}$ of the restricted Poincaré group $\mathcal{P}_+^\uparrow$, i.e., $U(g)\mathfrak{A}(\mathcal{O})U(g)^* = \mathfrak{A}(\Lambda(g)\mathcal{O})$ for all $g \in \widetilde{\mathcal{P}_+^\uparrow}$, where $\Lambda : \widetilde{\mathcal{P}_+^\uparrow} \to \mathcal{P}_+^\uparrow$ denotes the covering map. $U$ will be assumed to have the following properties:

**Spectrum condition:** The joint spectrum of the four-momentum operator generating the translations in $U$ is contained in the closure of the forward light cone.

**Existence and uniqueness of a cyclic vacuum vector:** There is an up to a phase unique unit vector $\Omega$ in $\mathcal{H}$ which is invariant under $U$ and **cyclic** with respect to $(\mathcal{H}, \widetilde{\mathfrak{A}})$, i.e. $\overline{\widetilde{\mathfrak{A}}\Omega} = \mathcal{H}$; $\Omega$ will be called the **vacuum vector**.

The Hilbert space $\mathcal{H}$ is **separable**.

That $U$ is not only a representation of $\widetilde{\mathcal{P}_+^\uparrow}$, but even of $\mathcal{P}_+^\uparrow$, will be obtained as a result in Proposition 2.2.1.

The second and the third assumption imply that the identical representation $(\mathcal{H}, \mathrm{id}_{\widetilde{\mathfrak{A}}})$ of $\widetilde{\mathfrak{A}}$ is irreducible, i.e., $\widetilde{\mathfrak{A}}'' = \mathfrak{B}(\mathcal{H})$, which means that the vacuum state is pure (cf., e.g., Prop. 7.3.30 in [7]).

To see that the assumption of a unique vacuum vector does not mean a loss of generality for the discussion of the spin-statistics theorem, some remarks about the analysis of massive particle representations are at hand. The Gelfand-Naimark-Segal construction (GNS-construction) associates with every state $\omega$ of $\widetilde{\mathfrak{A}}$ in a unique way a Hilbert space $\mathfrak{H}_\omega$, a vector $\Omega_\omega$ and a representation $\pi_\omega$ of $\widetilde{\mathfrak{A}}$ in $\mathfrak{H}_\omega$ which is cyclic with respect to the vector $\Omega_\omega$, i.e., $\pi_\omega(\widetilde{\mathfrak{A}})\Omega_\omega$ is dense in $\mathfrak{H}_\omega$. Since conversely, the linear functional $\widetilde{\mathfrak{A}} \ni A \mapsto \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$ is a state of the algebra $\widetilde{\mathfrak{A}}$, this establishes a one-to-one correspondence between the states and the (unitary-equivalence classes of) cyclic representations of $\widetilde{\mathfrak{A}}$.

The spin-statistics theorem concerns states/representations associated with massive particles. Following Buchholz and Fredenhagen [35], a representation of $\widetilde{\mathfrak{A}}$ will be called a **massive single-particle representation** if it may be associated with a single massive particle species. In mathematical terms, such a representation is assumed to be a factor representation in order to fix all charges, it is assumed to be translation covariant in order to define energy and momentum, and the energy-momentum spectrum is assumed to consist of the whole positive energy branch of a mass shell with some mass $m > 0$ and a subset of the region 'above' (in the energy direction) the positive energy mass shell of some higher mass $M > m$.

Given such a representation $(\mathcal{H}_\pi, \pi)$, there is a state $\omega_0$ on $\widetilde{\mathfrak{A}}$ with the property that $\pi(U(x)AU(-x))$ tends to $\omega_0(A)\mathrm{id}_{\mathfrak{H}_\pi}$ in the weak operator topology as $x$ tends to spacelike infinity ([35], Theorem 3.4). By construction, this state is unique and invariant under translations. The GNS-representation of this state is covariant under a representation $U_\pi$ of the translation group which satisfies the spectrum condition, and its cyclic vector is invariant under translations. Such a representation is called a **vacuum representation**, the translation invariant state associated with it is called a **vacuum state**.

If the massive single-particle representation $(\mathfrak{H}_\pi, \pi)$ is irreducible, it is unitarily equivalent to its associated vacuum representation $(\mathfrak{H}_\pi^{\mathrm{vac}}, \pi^{\mathrm{vac}})$ when restricted to $\mathfrak{A}(C')$ for any spacelike cone $C$ ([35], Theorem 3.5), so any irreducible massive single-particle representation may be regarded as an excitation of a vacuum. In the sequel, only excitations of one given vacuum will be considered; therefore, it means no loss of generality to assume that $(\mathfrak{H}_\pi^{\mathrm{vac}}, \pi^{\mathrm{vac}}) = (\mathcal{H}, \mathrm{id}_{\widetilde{\mathfrak{A}}})$. The set of all parabosonic and parafermionic spacelike-cone excitations (in the above sense) of the vacuum will be called $\Pi_{\mathcal{S}}$. This justifies the assumption of a unique vacuum vector[2]. Now consider the algebra associated with the wedge $W_1$ defined above. As a weakened form of the Reeh-Schlieder theorem it can be shown that $\Omega$ is cyclic with respect to the von Neumann algebra $(\mathcal{H}, \mathfrak{A}(W_1)'')$ ([27], p. 279), and using a standard argument (see, e.g., Prop. 2.5.3 in [22]), one obtains from locality that $\Omega$ is also **separating** with respect to $(\mathcal{H}, \mathfrak{A}(W_1)'')$, i.e. if $A \in \mathfrak{A}(W_1)''$ and $A\Omega = 0$, then $A = 0$.

[2]In a theory where spontaneous symmetry breaking occurs, the vacuum state may be degenerate. But since the spin-statistics theorem deals with a single particle rather than a full gauge theory it occurs in, the discussion of the spin-statistics theorem can, without any loss of generality, be confined to a minimal setting which contains precisely the massive particle under investigation.

A triple $(\mathfrak{H}, \mathfrak{M}, \xi)$ consisting of a von Neumann algebra $(\mathfrak{H}, \mathfrak{M})$ and a cyclic and separating vector $\xi$ is called a **standard von Neumann algebra**; such a triple is the setting of Tomita-Takesaki theory ([74], see also [22, 51]): the antilinear operator

$$S_0 : \mathfrak{M}\xi \ni A\xi \mapsto A^*\xi \in \mathfrak{M}\xi$$

is closable, and its closure $S$, which is called the **Tomita operator** admits a (unique) polar decomposition into an antiunitary operator $J$ (its 'phase') and a positive operator $\Delta^{1/2}$ (its 'modulus') defined on the domain of $S$ such that

$$S = J\Delta^{1/2}.$$

$J$ satisfies $J^2 = \mathrm{id}_{\mathfrak{H}}$ and is called the **modular conjugation**, whereas $\Delta = (\Delta^{1/2})^2$ is called the **modular operator** of the standard von Neumann algebra $(\mathfrak{H}, \mathfrak{M}, \xi)$. The unitary group $(\Delta^{it})_{t\in\mathbb{R}}$ is called the **modular group** of $(\mathfrak{H}, \mathfrak{M}, \xi)$. $\Delta$, $\Delta^{it}$, $t \in \mathbb{R}$, and $J$, which we refer to as the **modular objects** of the standard von Neumann algebra $(\mathfrak{H}, \mathfrak{M}, \xi)$, leave $\xi$ invariant.

The first main result of modular theory is the following theorem due to Tomita and Takesaki [74]:

### 2.1.1 Theorem (Tomita, Takesaki)

*Let $(\mathfrak{H}, \mathfrak{M}, \xi)$ be a standard von Neumann algebra, let $\Delta$ be its modular operator, and let $J$ be its modular conjugation. Then*

$$\Delta^{it}\mathfrak{M}\Delta^{-it} = \mathfrak{M};$$
$$J\mathfrak{M}J = \mathfrak{M}'.$$

Two further basic facts from Tomita-Takesaki theory will be used below; they are recalled here for the reader's convenience:

### 2.1.2 Lemma

*Let $(\mathfrak{H}_1, \mathfrak{M}_1, \xi_1)$ and $(\mathfrak{H}_2, \mathfrak{M}_2, \xi_2)$ be two standard von Neumann algebras with modular objects $\Delta_1$, $J_1$ and $\Delta_2$, $J_2$, respectively, and let $V : \mathfrak{H}_1 \to \mathfrak{H}_2$ be a unitary operator with $V\mathfrak{M}_1V^* = \mathfrak{M}_2$ and $V\xi_1 = \xi_2$. Then we have $V\Delta_1^{1/2}V^* = \Delta_2^{1/2}$ and $VJ_1V^* = J_2$.*

**Proof.** If $S_1$ and $S_2$ are the respective Tomita operators of $(\mathfrak{H}_1, \mathcal{M}_1, \Omega_1)$ and $(\mathfrak{H}_2, \mathcal{M}_2, \Omega_2)$, one has

$$S_2 = VS_1V^* = VJ_1V^*V\Delta_1^{\frac{1}{2}}V^*,$$

and since the polar composition of a closed operator is unique, the statement follows. □

### 2.1.3 Theorem (Takesaki, Winnink)

*For every standard von Neumann algebra $(\mathfrak{H}, \mathfrak{M}, \xi)$, the modular automorphism group $(\mathrm{Ad}(\Delta^{it}))_{t\in\mathbb{R}}$ is the unique one-parameter group $(\sigma_t)_{t\in\mathbb{R}}$ of automorphisms of the von Neumann algebra $(\mathfrak{H}, \mathfrak{M})$ which satisfies the following conditions:*

*(i) for every $A \in \mathfrak{M}$, the function $\mathbb{R} \ni t \mapsto \sigma_t(A) \in \mathfrak{M}$ is a continuous function from $\mathbb{R}$ into the von Neumann algebra $(\mathfrak{H}, \mathfrak{M})$ endowed with the strong operator topology;*

*(ii) $(\sigma_t)_{t\in\mathbb{R}}$ satisfies the* **KMS-condition (at the inverse temperature $\beta = 1$) with respect to** $(\mathfrak{H}, \mathfrak{M}, \xi)$*: for any $A, B \in \mathfrak{M}$, the function $\mathbb{R} \ni t \mapsto \langle \xi, A\sigma_t(B)\xi\rangle$ may be extended to a continuous function $f$ on the complex strip $-1 \leq \operatorname{Im} z \leq 0$ which is analytic in the interior of this strip and satisfies*

$$f(t-i) = \langle \xi, \sigma_t(B)A\xi\rangle \qquad \text{for all } t \in \mathbb{R}.$$

**Proof.** See [74], Theorems 13.1 and 13.2. □

For every wedge $W \in \mathcal{W}$, the modular conjugation $J_{\mathfrak{A}(W)''}$ and the modular operator $\Delta_{\mathfrak{A}(W)''}$ of the standard von Neumann algebra $(\mathcal{H}, \mathfrak{A}(W)'', \Omega)$ will be shorthanded by $J_W$ and $\Delta_W$.

As mentioned above, Bisognano and Wichmann have shown that if $\mathfrak{A}$ arises from a finite-component Wightman field, $J_{W_1}$ and $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, implement the $\mathrm{P_1CT}$-symmetry and the Lorentz boosts in the 1-direction, respectively. That $J_{W_1}$ implements a $\mathrm{P_1CT}$-symmetry, will be assumed throughout this chapter.

## Assumption II:

The net $\mathfrak{A}$ will be assumed to satisfy **modular $\mathrm{P_1}$CT-symmetry**:

$$J_{W_1}\mathfrak{A}(\mathcal{O})J_{W_1} = \mathfrak{A}(j\mathcal{O}) \qquad \text{for all } \mathcal{O} \in \mathcal{K},$$

where $j$ denotes the $\mathrm{P_1T}$-reflection given by

$$j(x_0, x_1, x_2, \ldots x_s) := (-x_0, -x_1, x_2, \ldots, x_s).$$

Note that due to $\widetilde{\mathcal{P}_+^\uparrow}$-covariance, this condition automatically holds in all Lorentz frames as soon as it holds in one. What is assumed appears to be $\mathrm{P}_1$T-symmetry rather than $\mathrm{P}_1$CT-symmetry. But it has been shown by Guido and Longo that under this assumption, $J_{W_1}$ indeed implements the correct charge conjugation [48]. The same authors have derived modular $\mathrm{P}_1$CT-symmetry from modular Lorentz covariance [50]. This assumption, which, evidently, is stronger than modular $\mathrm{P}_1$CT-symmetry, is not made here. In 1+3 dimensions, a full PCT-operator may be constructed as a product of such modular conjugations; in 1+2 dimensions, the corresponding product yields a rotation and leaves charge and time direction invariant (cf. also [65] for a discussion in the Wightman framework).

It follows from the Tomita-Takesaki theorem, $\widetilde{\mathcal{P}_+^\uparrow}$-covariance, and Lemma 2.1.2 that modular $\mathrm{P}_1$CT-symmetry implies **wedge duality**:

$$\mathfrak{A}(W)'' = \mathfrak{A}(W')' \qquad \text{for all } W \in \mathcal{W}.$$

This duality property is sufficient for the Doplicher-Roberts field construction discussed below.

### Assumption III:

> The group of **internal symmetries** of $(\mathcal{H}, \mathfrak{A}, \Omega)$, i.e., the group of all unitaries $\gamma$ in $\mathcal{H}$ such that $\gamma\Omega = \Omega$ and $\gamma\mathfrak{A}(\mathcal{O})\gamma^* = \mathfrak{A}(\mathcal{O})$ for all $\mathcal{O} \in \mathcal{K}$, will be assumed to be compact in the strong operator topology.

This property has been derived in [37] from assumptions concerning the scattering theory of the system. Another sufficient condition is the distal split property [40]. The distal split property, in turn, has been derived by Buchholz and Wichmann from their so-called **nuclearity condition**, for which they have given a thermodynamical justification [36].

On the other hand, the compactness of the internal symmetries implies that all internal symmetries commute with all $U(g)$, $g \in \widetilde{\mathcal{P}_+^\uparrow}$, and that $U$ is the unique strongly continuous unitary representation of $\widetilde{\mathcal{P}_+^\uparrow}$ in $\mathcal{H}$ with respect to which $\mathfrak{A}$ is covariant and $\Omega$ is invariant [40, 24]. Streater has given an example of a relativistic field theory [73] which violates the familiar spin-statistics connection. The model is an infinite-component Wightman field which is covariant under several unitary representations of $\widetilde{\mathcal{P}_+^\uparrow}$, and for which, a fortiori, the above compactness assumption is violated.

Finally, the definitions and results of the Doplicher-Roberts field construction performed in [41], which are used in the sequel, will be recalled for the reader's convenience. We recall the general version of their result which is based on the Buchholz-Fredenhagen analysis of massive single-particle representations and gives an algebraic field-theoretic structure to the analysis of the massive particle sectors.

### 2.1.4 Definition

Let $\mathcal{H}$, $\mathfrak{A}$, $U$ and $\Omega$ be as above, let $\mathfrak{H}$ be a (not necessarily separable) Hilbert space, and let $(\mathfrak{F}(C))_{C\in\mathcal{S}}$ be an isotonous family of von Neumann algebras. Let $\pi$ be a faithful representation of $\widetilde{\mathfrak{A}}$ in $\mathfrak{H}$, and let $G$ be a strongly compact group of unitaries in $\mathfrak{H}$. The quadruple $(\mathfrak{H}, \mathfrak{F}, \pi, G)$ is called an **extended field system with gauge symmetry** — we shall simply say: a **field** — **over** $(\mathcal{H}, \mathfrak{A}, U, \Omega)$ if the following conditions are satisfied:

(i) $(\mathfrak{H}, \pi)$ contains $(\mathcal{H}, \mathrm{id}_{\widetilde{\mathfrak{A}}})$ as a subrepresentation;
(ii) $\mathcal{H}$ is the subspace of all $G$-invariant vectors in $\mathfrak{H}$;
(iii) for every $C \in \mathcal{S}$, the maps $\mathrm{Ad}(\gamma)$, $\gamma \in G$, act as automorphisms on $\mathfrak{F}(C)$, and $\pi(\mathfrak{A}(C))''$ is the algebra of those elements of $\mathfrak{F}(C)$ which are invariant under all $\mathrm{Ad}(\gamma)$, $\gamma \in G$, i.e.:

$$\pi(\mathfrak{A}(C))'' = \mathfrak{F}(C) \cap G' \qquad \text{for all } C \in \mathcal{S};$$

(iv) $\mathfrak{F}$ is **irreducible** and **weakly additive**:

$$\left( \bigcup_{a\in\mathbb{R}^{1+s}} \mathfrak{F}(C+a) \right)'' = \mathfrak{B}(\mathfrak{H}) \qquad \text{for all } C \in \mathcal{S};$$

(v) $\mathfrak{F}$ has the **Reeh-Schlieder property for space-like cones**:

$$\overline{\mathfrak{F}(C)\Omega} = \mathfrak{H} \qquad \text{for all } C \in \mathcal{S};$$

(vi) $\mathfrak{F}$ is **local with respect to the net** $(\pi(\mathfrak{A}(\mathcal{O})))_{\mathcal{O}\in\mathcal{K}}$:

$$\mathfrak{F}(C) \subset \pi(\mathfrak{A}(C'))' \qquad \text{for all } C \in \mathcal{S}.$$

In this case, $\mathfrak{F}$ is called the **field system**, and $G$ is called the **(global) gauge group**[3]. A field system $(\mathfrak{H}, \mathfrak{F}, \pi, G)$ is called **normal** if it satisfies the **normal commutation relations**, i.e., if

[3]Under standard assumptions, this gauge group can only include gauge transformations of the first kind, since its elements commute with the unitaries representing the translations (see the discussion at the end of [31]). This does not mean any loss of generality, as far as the discussion of the spin-statistics theorem is concerned, since, as remarked before, it is sufficient to consider a framework which precisely contains a given massive particle.

the gauge group contains an involution $k$ such that with the notations

$$F^{\pm} := \tfrac{1}{2}(F \pm kFk^*), \qquad F \in \mathfrak{F}(C), C \in \mathcal{S},$$

we have for any two spacelike separated cones $C_1$ and $C_2$:

$$F_1^+ F_2^+ = F_2^+ F_1^+, \qquad F_1^+ F_2^- = F_2^- F_1^+, \qquad F_1^- F_1^- = -F_2^- F_1^-$$

for all $F_{1,2} \in \mathfrak{F}(C_{1,2})$. $k$ is called a **Bose-Fermi operator**.

Using the separability of $\mathcal{H}$, Doplicher and Roberts have shown that, given any field $(\mathfrak{H}, \mathfrak{F}, \pi, G)$ over $(\mathcal{H}, \mathfrak{A}, U, \Omega)$, every irreducible subrepresentation of $(\mathfrak{H}, \pi)$ is contained in the class $\Pi_{\mathcal{S}}$ of spacelike-cone excitations of the vacuum, as introduced above (Theorem 5.4 in [41], cf. also the remarks on p. 19 in [35]), i.e., for some index set $I$, there is a family $(\pi_\iota)_{\iota \in I}$ of irreducible representations in $\Pi_{\mathcal{S}}$ such that $\pi = \bigoplus_{\iota \in I} \pi_\iota$. If the field is normal and $k$ is a Bose-Fermi operator of the field, then for every $\iota \in I$, the restriction of the Bose-Fermi operator $k$ to $\mathfrak{H}_\iota$ coincides with the identity operator $\mathrm{id}_{\mathcal{H}_\iota}$ for every (para-)bosonic $\pi_\iota$, while it coincides with $-\mathrm{id}_{\mathcal{H}_\iota}$ for every (para-) fermionic $\pi_\iota$ (Theorem 5.4 in [41]).

If $(\mathfrak{H}, \mathfrak{F}, \pi, G)$ is a normal field over $(\mathcal{H}, \mathfrak{A}, U, \Omega)$, the unitary operator defined by

$$V := \frac{1}{1+i}(\mathrm{id}_{\mathfrak{H}} + \mathrm{ik})$$

implements a **twist** of the field: the field system $(\mathfrak{H}, \mathfrak{F}, \pi, G)$ over $(\mathcal{H}, \mathfrak{A}, U, \Omega)$ given by

$$\mathfrak{F}^t(C) := V\mathfrak{F}(C)V^*, \qquad C \in \mathcal{S},$$

is local with respect to $\mathfrak{F}$, i.e. $\mathfrak{F}(C) \subset \mathfrak{F}^t(C')'$ for all $C \in \mathcal{S}$. Doplicher and Roberts even established **twisted duality**, i.e.,

$$\mathfrak{F}^t(C) = \mathfrak{F}(C')' \qquad \text{for all } C \in \mathcal{S};$$

see Theorem 5.4. in [41]. The same arguments show that wedge duality of the net of observables implies **twisted wedge duality** of the field:

$$\mathfrak{F}^t(W) = \mathfrak{F}(W')' \qquad \text{for all } W \in \mathcal{W},$$

where

$$\mathfrak{F}(W) := \left( \bigcup_{\substack{C \in \mathcal{S} \\ C \subset W}} \mathfrak{F}(C) \right)'' .$$

Note that the phase $\frac{1}{1+i}$ of $V$ has been chosen such that $V$ leaves $\Omega$ invariant; with this choice, $V^2 = k$.

Given, conversely, $(\mathcal{H}, \mathfrak{A}, U, \Omega)$ as assumed above, Doplicher and Roberts have shown that there is an up to unitary equivalence unique normal field $(\mathfrak{H}, \mathfrak{F}, \pi, G)$ over $(\mathcal{H}, \mathfrak{A}, U, \Omega)$ such that each irreducible representation in $\Pi_{\mathcal{S}}$ is unitarily equivalent to a subrepresentation of $(\mathfrak{H}, \pi)$ (Theorem 5.3 in [41][4]). There also is, up to unitary equivalence, a unique normal field $(\mathfrak{H}, \mathfrak{F}, \pi, G)$ over $(\mathcal{H}, \mathfrak{A}, U, \Omega)$ such that $(\mathfrak{H}, \pi)$ contains all irreducible $\widetilde{\mathcal{P}_+^\uparrow}$-**covariant** representations contained in the set $\Pi_{\mathcal{S}}$ and is, conversely, a direct sum of such representations ([41], top of p. 98). There is a unique strongly continuous unitary representation $U_\pi$ of $\widetilde{\mathcal{P}_+^\uparrow}$ in $\mathfrak{H}$ with

$$U_\pi(g)\pi(A)U_\pi(g)^* = \pi(U(g)AU(g)^*) \qquad \text{for all } g \in \widetilde{\mathcal{P}_+^\uparrow},\ A \in \widetilde{\mathfrak{A}}$$

([41], pp. 98-101, cf. also Lemma 2.2. in [39]). The vacuum vector is invariant under $U_\pi$, and the field net $\mathfrak{F}$ is covariant with respect to $U_\pi$. Such a field will be called a $\widetilde{\mathcal{P}_+^\uparrow}$**-covariant** (normal) field over $(\mathcal{H}, \mathfrak{A}, U, \Omega)$.

It follows from property (v) in Definition 2.1.4 that $\Omega$ is cyclic and separating with respect to the von Neumann algebras $(\mathfrak{H}, \mathfrak{F}(W))$, $W \in \mathcal{W}$,

---

[4]Note that the full $\widetilde{\mathcal{P}_+^\uparrow}$-covariance is not needed at this stage; it would suffice to assume translation covariance.

Furthermore, Doplicher and Roberts make an additional assumption they call **property B'**. However, the following is sufficient:

**Borchers property for spacelike cones:**

> $(\mathcal{H}, \mathfrak{A})$ is said to have the **Borchers property for spacelike cones** if, given any two spacelike cones $C_1$ and $C_2$ with $\overline{C_1} \subset C_2$ which are chosen in such a way that there is a third spacelike cone $C^\times$ with $C^\times \subset C_1' \cap C_2$, we can find for each nonzero projection $E \in \mathfrak{A}(C_1)''$ an isometry $W \in \mathfrak{A}(C_2)''$ such that $WW^* = E$ (and, trivially, $W^*W = \mathrm{id}_{\mathcal{H}}$, i.e., $E$ and $\mathrm{id}_{\mathcal{H}}$ are equivalent in $\mathfrak{A}(C_2)''$).

Noting that for any spacelike cone $C$, we have additivity:

$$\left( \bigcup_{a \in \mathbb{R}^{1+s}} \mathfrak{A}(C+a) \right)'' = \widetilde{\mathfrak{A}}'' = \mathfrak{B}(\mathcal{H}),$$

and using the spectrum condition and irreducibility, the Borchers property for spacelike cones can be proven applying the arguments from [14]. It is emphasized that Borchers proves in [14] the corresponding result for double cones and therefore has to *assume* for double cones the above additivity property.

Doplicher's and Roberts' property B' is stronger: the same assumption as in the Borchers property for spacelike cones is made for any two spacelike cones $C_1$ and $C_2$ with $\overline{C_1} \subset C_2$ even if there is no spacelike cone $C^\times \subset C_1' \cap C_2$ (this is, e.g., the case if $C_1$ is a translate of $C_2$). However, in order to prove this stronger form of the Borchers property for spacelike cones by means of the arguments taken from [14], one has to assume weak additivity for double cones.

whence the existence of a corresponding modular conjugation $J_{\mathfrak{F}(W_1)}$ and a modular group $\Delta_{\mathfrak{F}(W_1)}$ follows.

## 2.2 Results

The proof of the spin-statistics theorem (Corollary 2.2.5) will be based on a couple of results which are interesting on their own. Proposition 2.2.1 and Corollary 2.2.2 prove that rotations are represented by products of modular conjugations associated with the algebras of two wedges, which is in the spirit of the ancient result that every rotation in a two-dimensional plane is a product of two reflections.

The adjoint action $\mathrm{Ad}(j)$ of $j$ on $\mathcal{P}_+^\uparrow$ has a unique lift to a group homomorphism of $\widetilde{\mathcal{P}_+^\uparrow}$ (cf. Section III.4 in [23]) which will be denoted by $\widetilde{\mathrm{Ad}}(j)$.

A group homomorphism $r : \mathbb{R} \to \widetilde{\mathcal{P}_+^\uparrow}$ which plays a role in the sequel is constructed as follows: denote by $\exp(i\cdot)$ the covering map $\phi \mapsto \exp(i\phi)$ from $\mathbb{R}$ onto $S^1$, and let $\iota : S^1 \to \mathcal{P}_+^\uparrow$ be the group homomorphism embedding $S^1$ into $\mathcal{P}_+^\uparrow$ as the group of rotations in the 1-2-plane. $\exp(i\cdot)$ and $\iota$ are continuous, so $\iota \circ \exp(i\cdot)$ is a continuous curve in $\mathcal{P}_+^\uparrow$. There is a unique lift of this curve to a continuous curve $r$ in $\widetilde{\mathcal{P}_+^\uparrow}$ with $r(0) = 1_{\widetilde{\mathcal{P}_+^\uparrow}}$ (see, e.g., Theorem III.3.3. in [23]); $r$ is a group homomorphism of $\mathbb{R}$ into $\widetilde{\mathcal{P}_+^\uparrow}$.

### 2.2.1 Proposition

$J_{W_1} U(g) J_{W_1} = U(\widetilde{\mathrm{Ad}}(j)g)$ *for all* $g \in \widetilde{\mathcal{P}_+^\uparrow}$.

**Proof:** The representation $U$ and the strongly continuous unitary representation $U^J$ of $\widetilde{\mathcal{P}_+^\uparrow}$ defined by

$$U^J(g) := J_{W_1} U(\widetilde{Ad}(j)g) J_{W_1}, \qquad g \in \widetilde{\mathcal{P}_+^\uparrow},$$

implement the same spacetime transformations on the net $\mathfrak{A}$ and leave $\Omega$ invariant. As stated in the previous section, it follows from the strong compactness of the group of internal symmetries that there can be at most one such representation; this implies $U = U^J$. $\square$

### 2.2.2 Corollary (modular rotation symmetry)

*For every $\phi \in [0, 2\pi]$, denote by $W_1^\phi$ the image of the wedge $W_1$ under the rotation of angle $\phi$ in the 1-2-plane. With $r$ as above, define $R(\phi) := U(r(\phi))$, $\phi \in \mathbb{R}$. Then*

$$R(\phi) = J_{W_1^{\phi/2}} J_{W_1} \qquad \textit{for all } \phi \in \mathbb{R}.$$

*In particular, the representation $U$ does not only realize $\widetilde{\mathcal{P}_+^\uparrow}$-covariance, but even $\mathcal{P}_+^\uparrow$-covariance of the net:*

$$R(2\pi) = \mathrm{id}_{\mathcal{H}}.$$

**Proof:** From Proposition 2.2.1, we get

$$J_{W_1^{\phi/2}} J_{W_1} = R(\tfrac{\phi}{2}) J_{W_1} R(-\tfrac{\phi}{2}) J_{W_1} = R(\tfrac{\phi}{2}) R(\tfrac{\phi}{2}) J_{W_1}^2 = R(\phi).$$

In particular,

$$R(2\pi) = J_{W_1^\pi} J_{W_1} = J_{W_1}^2 = \mathrm{id}_{\mathcal{H}};$$

in the second step we use that the modular conjugations of a standard von Neumann algebra and its commutant coincide. □

The following lemma proves that $\Delta_{\mathfrak{F}(W_1)}$ and $J_{\mathfrak{F}(W_1)}$ are extensions of $J_{W_1}$ and $\Delta_{W_1}$, respectively.

### 2.2.3 Lemma

*For every field $(\mathfrak{H}, \mathfrak{F}, \pi, G)$ over $(\mathcal{H}, \mathfrak{A}, U, \Omega)$, we have*

*(i)* $\Delta_{\mathfrak{F}(W_1)}^{it}\big|_{\mathcal{H}} \, A \, \Delta_{\mathfrak{F}(W_1)}^{-it}\big|_{\mathcal{H}} = \Delta_{W_1}^{it} A \Delta_{W_1}^{-it}$ *for all* $A \in \mathfrak{A}(W_1)''$;

*(ii)* $\Delta_{\mathfrak{F}(W_1)}^{it}\big|_{\mathcal{H}} = \Delta_{W_1}^{it}$;

*(iii)* $\Delta_{\mathfrak{F}(W_1)}^{1/2}\big|_{\mathcal{H} \cap D(\Delta_{\mathfrak{F}(W_1)}^{1/2})} = \Delta_{W_1}^{1/2}$;

*(iv)* $J_{\mathfrak{F}(W_1)}\big|_{\mathcal{H}} = J_{W_1}$;

*(v)* $J_{\mathfrak{F}(W_1)} \, \pi(A) \, J_{\mathfrak{F}(W_1)} = \pi(J_{W_1} A J_{W_1})$ *for all* $A \in \tilde{\mathfrak{A}}$.

**Proof.** It follows from Lemma 2.1.2 that $\Delta_{\mathfrak{F}(W_1)}^{it}$ commutes with the elements of the gauge group $G$ for all $t \in \mathbb{R}$. This implies that every such $\Delta_{\mathfrak{F}(W_1)}^{it}$ maps the $G$-invariant vectors in $\mathfrak{H}$ into $G$-invariant vectors, i.e., $\Delta_{\mathfrak{F}(W_1)}^{it}\mathcal{H} = \mathcal{H}$ because of property (ii) in Definition 2.1.4, and that its adjoint action acts as an automorphism on the commutant of $G$. This – together with the Tomita-Takesaki theorem and the identity $\mathfrak{F}(W_1) \cap G' = \pi(\mathfrak{A}(W_1))''$ following from property (iii) in Definition 2.1.4 – gives that $\mathrm{Ad}(\Delta^{it}_{\mathfrak{F}(W_1)})$ acts as an automorphism on $\pi(\mathfrak{A}(W_1))''$.

Consider now the direct sum decomposition $\pi = \bigoplus_{\iota \in I} \pi_\iota$ of $\pi$ into irreducible representations $\pi_\iota$ in $\Pi_{\mathcal{S}}$. Since the wedge $W_1$ is contained in the spacelike complement of a spacelike cone (take any spacelike cone in $W_1'$), one knows from the definition of $\Pi_{\mathcal{S}}$ that for every $\pi_\iota$, its restriction to the algebra $\mathfrak{A}(W_1)$ is implemented by a unitary operator $U_\iota : \mathcal{H} \to \mathcal{H}_\iota$, so it has a faithful extension $\pi_\iota^{W_1}$ to the algebra $\mathfrak{A}(W_1)''$ which is weakly continuous and which has the property that $\pi_\iota^{W_1}(\mathfrak{A}(W_1)'') = \pi_\iota^{W_1}(\mathfrak{A}(W_1))''$, so the direct sum $\pi = \bigoplus_\iota \pi_\iota$ has a unique weakly continuous faithful extension

$$\mathfrak{A}(W_1)'' \ni A \mapsto \bigoplus_\iota \pi_\iota^{W_1}(A) =: \pi_{W_1}(A).$$

By property (i) in Definition 2.1.4, the inverse of this faithful representation is given by

$$\pi_{W_1}(\mathfrak{A}(W_1)'') \ni B \mapsto \pi_{W_1}^{-1}(B) = B|_{\mathcal{H}} \in \mathfrak{A}(W_1)'',$$

and since this map is weakly continuous, one obtains

$$\pi_{W_1}(\mathfrak{A}(W)'') = \pi_{W_1}(\mathfrak{A}(W_1))''.$$

One may now define a one-parameter group $(\sigma_t)_{t\in\mathbb{R}}$ of automorphisms of $(\mathcal{H}, \mathfrak{A}(W_1)'')$ by

$$\sigma_t(A) := \pi_{W_1}^{-1}\left(\Delta^{it}_{\mathfrak{F}(W_1)}\, \pi_{W_1}(A)\, \Delta^{-it}_{\mathfrak{F}(W_1)}\right), \qquad A \in \mathfrak{A}(W_1)'',$$

and since it has been shown above that the $\Delta^{it}_{\mathfrak{F}(W_1)}$, $t \in \mathbb{R}$, leave the subspace $\mathcal{H}$ invariant, one concludes

$$\sigma_t(A) = \Delta^{it}_{\mathfrak{F}(W_1)}|_{\mathcal{H}} A \Delta^{-it}_{W_1}|_{\mathcal{H}} \qquad \text{for all } A \in \mathfrak{A}(W_1)'', t \in \mathbb{R}.$$

Since $\Delta^{it}_{\mathfrak{F}(W_1)}$ is the modular group of $(\mathfrak{H}, \mathfrak{F}(W_1), \Omega)$, it follows from Theorem 2.1.3 that $\sigma$ satisfies the conditions (i) and (ii) made there. Now Theorem 2.1.3 is applied once more: since $\sigma$ has been shown to be a one-parameter group of automorphisms of the von Neumann algebra $(\mathcal{H}, \mathfrak{A}(W_1)'')$ and since it is satisfies the assumptions (i) and (ii) of Theorem 2.1.3, it follows that $\sigma$ coincides with the modular automorphism group of the standard von Neumann algebra $(\mathcal{H}, \mathfrak{A}(W_1)'', \Omega)$; this proves (i).

(ii) follows from (i): for any $A \in \mathfrak{A}(W_1)''$ and any $t \in \mathbb{R}$, one has

$$\Delta^{it}_{\mathfrak{F}(W_1)}|_{\mathcal{H}} A\Omega = \Delta^{it}_{\mathfrak{F}(W_1)}|_{\mathcal{H}} A \Delta^{-it}_{W_1}|_{\mathcal{H}}\Omega = \Delta^{it}_{W_1} A \Delta^{-it}_{W_1}\Omega = \Delta^{it}_{W_1} A\Omega,$$

so $\Delta^{it}_{\mathfrak{F}(W_1)}|_{\mathcal{H}}$ and $\Delta^{it}_{W_1}$ coincide on a dense subspace of $\mathcal{H}$ and hence – being bounded – on all of $\mathcal{H}$.

(iii) follows from (ii) since the KMS-condition implies, in the sense of quadratic forms:

$$\langle A\Omega, \Delta_{\mathfrak{F}(W_1)}|_{D(\Delta_{\mathfrak{F}(W_1)})\cap\mathcal{H}} B\Omega\rangle = \langle B^*\Omega, A^*\Omega\rangle = \langle A\Omega, \Delta_{W_1} B\Omega\rangle$$
$$\text{for all } A, B \in \mathfrak{A}(W_1)''.$$

Since every positive operator is uniquely determined by its quadratic form, one concludes $\Delta_{\mathfrak{F}(W_1)|_{\mathcal{D}(\Delta_{\mathfrak{F}(W_1)})\cap\mathcal{H}}} = \Delta_{W_1}$, and using the spectral theorem, (iii) follows.

(iv) follows from (iii) since the range of $\Delta^{1/2}_{W_1}$ is dense in $\mathcal{H}$ and

$$J_{\mathfrak{F}(W_1)}\ \Delta^{1/2}_{\mathfrak{F}(W_1)}\Big|_{D(\Delta^{1/2}_{\mathfrak{F}(W_1)})\cap\mathcal{H}} = J_{W_1}\Delta^{1/2}_{W_1}.$$

(v) follows from (iv): because of modular $\mathrm{P}_1$CT-symmetry, $J_{W_1}\widetilde{\mathfrak{A}}J_{W_1} = \widetilde{\mathfrak{A}}$, hence, twice using property (i) in Definition 2.1.4, one obtains for every local observable $A \in \mathfrak{A}_{\mathrm{loc}}$:

$$\begin{aligned} J_{\mathfrak{F}(W_1)}\pi(A)J_{\mathfrak{F}(W_1)}\Omega &= J_{\mathfrak{F}(W_1)}\pi(A)\Omega = J_{\mathfrak{F}(W_1)}\Big|_{\mathcal{H}}\ \pi(A)|_{\mathcal{H}}\,\Omega \\ &= J_{W_1}A\Omega = J_{W_1}AJ_{W_1}\Omega = \pi(J_{W_1}AJ_{W_1})|_{\mathcal{H}}\,\Omega \\ &= \pi(J_{W_1}AJ_{W_1})\Omega. \end{aligned}$$

Since $\Omega$ is separating with respect to $(\mathcal{H}, \mathfrak{A}(W_1)'')$, one easily makes use of translation covariance to prove that it is also separating with respect to $(\mathcal{H}, \mathfrak{A}(W_1 + a)'')$. But since $A$ is a local observable, there is an $a \in \mathbb{R}^{1+s}$ such that $A \in \mathfrak{A}(W_1 + a)''$. This proves the statement for every $A \in \mathfrak{A}_{\mathrm{loc}}$. Since $\mathrm{Ad}J_{\mathfrak{F}(W_1)} : \mathfrak{B}(\mathfrak{H}) \to \mathfrak{B}(\mathfrak{H})$ and $\mathrm{Ad}J_{W_1} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ and $\pi : \widetilde{\mathfrak{A}} \subset \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathfrak{H})$ are continuous with respect to the corresponding norm topologies in $\mathcal{B}(\mathfrak{H})$ and $\mathcal{B}(\mathcal{H})$, the statement extends to $\widetilde{\mathfrak{A}}$. □

**Remark:** In the above argument, the modular groups considered possibly do not implement any symmetry on the net $\mathfrak{A}$. We mention that under the additional assumption that $\Delta^{it}_{W_1}\widetilde{\mathfrak{A}}\Delta^{-it}_{W_1} = \widetilde{\mathfrak{A}}$, $t \in \mathbb{R}$, one can derive

$$\Delta^{it}_{\mathfrak{F}(W_1)}\pi(A)\Delta^{-it}_{W_1} = \pi(\Delta^{it}_{W_1}A\Delta^{-it}_{W_1}) \qquad \text{for all } A \in \widetilde{\mathfrak{A}}, t \in \mathbb{R}$$

from (ii) in the same way as we have obtained (v) from (iv) in the preceding proof.

The next step towards the proof of the spin-statistics theorem is the proof that modular $\mathrm{P}_1$CT-symmetry of $\mathfrak{A}$ implies modular $\mathrm{P}_1$CT-symmetry of the extended field system over $\mathfrak{A}$. This P$_1$CT-theorem holds for all $s \geq 2$ including the even dimensions where a full PCT-theorem is lacking (cf. [65]).

### 2.2.4 Theorem (P$_1$CT-symmetry of the field)

*Let $(\mathfrak{H}, \mathfrak{F}, \pi, G)$ be a $\widetilde{\mathcal{P}_+^\uparrow}$-covariant, normal field over $(\mathcal{H}, \mathfrak{A}, U, \Omega)$.*

*(i)* $J_{\mathfrak{F}(W_1)}\mathfrak{F}(C)J_{\mathfrak{F}(W_1)} = \mathfrak{F}^t(jC)$ *for every* $C \in \mathcal{S}$;

*(ii)* $J_{\mathfrak{F}(W_1)}U_\pi(g)J_{\mathfrak{F}(W_1)} = U_\pi(\widetilde{\mathrm{Ad}}(j)(g))$ *for every* $g \in \widetilde{\mathcal{P}_+^\uparrow}$.

*The antiunitary involution* $\Theta_{W_1} := VJ_{\mathfrak{F}(W_1)} = J_{\mathfrak{F}(W_1)}V^*$ *(with* $V = \frac{1+ik}{1+i}$ *as above) is the P$_1$CT-operator, i.e.*

$$\Theta_{W_1}\mathfrak{F}(C)\Theta_{W_1} = \mathfrak{F}(jC) \qquad \text{for all } C \in \mathcal{S}.$$

**Proof:** Note first that $VJ_{\mathfrak{F}(W_1)} = J_{\mathfrak{F}(W_1)}V^*$ follows from the definition of $V$ by a straightforward computation. From the modular P$_1$CT-symmetry of $\mathfrak{A}$, the preceding lemma, and the fact that the modular objects commute with internal symmetries, it follows that by

$$\begin{aligned}
\mathfrak{H}^J &:= J_{\mathfrak{F}(W_1)}\mathfrak{H} = \mathfrak{H};\\
\mathfrak{F}^J(C) &:= J_{\mathfrak{F}(W_1)}\,\mathfrak{F}(jC)\,J_{\mathfrak{F}(W_1)}, \qquad C \in \mathcal{S};\\
\pi^J(A) &:= J_{\mathfrak{F}(W_1)}\,\pi(J_{W_1}AJ_{W_1})\,J_{\mathfrak{F}(W_1)} = \pi(A), \qquad A \in \tilde{\mathfrak{A}},\\
G^J &:= J_{\mathfrak{F}(W_1)}\,G\,J_{\mathfrak{F}(W_1)} = G
\end{aligned}$$

a second $\widetilde{\mathcal{P}_+^\uparrow}$-covariant normal field $(\mathfrak{H}^J = \mathfrak{H}, \mathfrak{F}^J, \pi^J = \pi, G^J = G)$ over $(\mathcal{H}, \mathfrak{A}, U, \Omega)$ is defined; note that it follows from $\pi^J = \pi$ that $\mathfrak{F}^J$ has the same Bose-Fermi operator as $\mathfrak{F}$ and that $\mathfrak{F}^J$ is covariant under $U_\pi = U_{\pi^J}$. To show that $\mathfrak{F}^J = \mathfrak{F}^t$, let $C_a$ and $C_b$ be two spacelike cones with $C_a \subset W_1$ and $C_b \subset W_1'$. Using the Tomita-Takesaki theorem, one obtains

$$\begin{aligned}
\mathfrak{F}^J(C_a) &= J_{\mathfrak{F}(W_1)}\mathfrak{F}(jC_a)J_{\mathfrak{F}(W_1)} \subset J_{\mathfrak{F}(W_1)}\mathfrak{F}(W_1')J_{\mathfrak{F}(W_1)}\\
&= J_{\mathfrak{F}(W_1)}V^*V\mathfrak{F}(W_1')V^*VJ_{\mathfrak{F}(W_1)} = VJ_{\mathfrak{F}(W_1)}\mathfrak{F}^t(W_1')J_{\mathfrak{F}(W_1)}V^*\\
&= VJ_{\mathfrak{F}(W_1)}\mathfrak{F}(W_1)'J_{\mathfrak{F}(W_1)}V^* = V\mathfrak{F}(W_1)V^* = \mathfrak{F}^t(W_1) = \mathfrak{F}(W_1')'\\
&\subset \mathfrak{F}(C_b)'.
\end{aligned}$$

Since for any spacelike separated spacelike cones $C_a$ and $C_b$, there is a wedge $W$ (which is a Poincaré transform of $W_1$) such that $C_a \subset W$ and $C_b \subset W'$, the net $\mathfrak{F}^J$ is easily shown to be local with respect to the net $\mathfrak{F}$. Twisted duality implies $\mathfrak{F}^J \subset \mathfrak{F}^t$, hence $\Theta_{W_1}\mathfrak{F}(C)\Theta_{W_1} \subset \mathfrak{F}(jC)$ for every

$C \in \mathcal{S}$. Since $\Theta_W$ is an involution, we conclude $\Theta_{W_1}\mathfrak{F}(C)\Theta_{W_1} = \mathfrak{F}(jC)$ and $\mathfrak{F}^J = \mathfrak{F}^t$. From this, the Theorem follows immediately. $\square$

These results given, the proof of the spin-statistics theorem boils down to a simple algebraic computation:

### 2.2.5 Corollary (spin-statistics theorem)

*Let $(\mathfrak{H}, \mathfrak{F}, \pi, G)$ be a covariant, normal field over $(\mathcal{H}, \mathfrak{A}, U, \Omega)$.*

*For every $\phi \in [0, 2\pi]$, denote by $W_1^\phi$ the image of $W_1$ under the rotation of angle $\phi$ in the 1-2-plane, and let $J_{\mathcal{F}(W_1^\phi)}$ be the modular conjugation of $(\mathfrak{H}, \mathfrak{F}(W_1^\phi), \Omega)$. With $r$ as defined in the previous section, define $R_\pi(\phi) := U_\pi(r(\phi))$, $\phi \in \mathbb{R}$.*

*Then one obtains:*

$$R_\pi(\phi) = J_{\mathfrak{F}(W_1^{\phi/2})} J_{\mathfrak{F}(W_1)}.$$

*In particular, $R_\pi(2\pi) = k$, i.e. the spin-statistics connection familiar from 1+3 dimensions holds.*

**Proof:** The first statement immediately follows from the preceding theorem in the same way as Corollary 2.2.2 follows from Proposition 2.2.1.

To obtain the spin-statistics connection, note that $J_{\mathfrak{F}(W_1^\pi)} = V J_{\mathfrak{F}(W_1)} V^*$ follows from wedge duality by Lemma 2.1.2. Since $V J_{\mathfrak{F}(W_1)} = J_{\mathfrak{F}(W_1)} V^*$, one obtains

$$R_\pi(2\pi) = J_{\mathfrak{F}(W_1^\pi)} J_{\mathfrak{F}(W_1)} = V J_{\mathfrak{F}(W_1)} V^* J_{\mathfrak{F}(W_1)} = V^2 J^2_{\mathfrak{F}(W_1)} = V^2 = k.$$

$\square$

## 2.3 Other approaches and open problems

For 1+3 dimensions, the first proof of a spin-statistics theorem for massive particle representations is due to Buchholz and Epstein [33]. It is possible that their proof, at the price of being more involved, yields the more general result, since it does not need the assumption of modular $P_1$CT-symmetry.

On the other hand, it has already been mentioned above that Streater has given an example of a relativistic quantum field which does not exhibit the familiar spin-statistics connection [73], and every proof of the four-dimensional spin-statistics theorem must rely on some assumption which exclude this example. For the above proof, this role is played by Assumption III. Also Buchholz and Epstein make an assumption which rules out Streater's example. Loosely speaking, they assume that the representation of $\widetilde{\mathcal{P}_+^\uparrow}$ under which the massive particle representation of the algebra $\tilde{\mathfrak{A}}$ is covariant is a finite multiple of an irreducible representation of $\widetilde{\mathcal{P}_+^\uparrow}$ when restricted to the one-particle space (which is the image of the representation space under the four momentum's spectral projection associated with the forward mass shell of the particle). It has not been investigated so far how this assumption is related to the above Assumptions II and III.

A proof of the spin-statistics theorem in the same 'algebraic' spirit as the one given above has been found independently by Guido and Longo [49]. In the same paper it has been shown that modular $\mathrm{P}_1$CT-symmetry follows from the assumption that the modular group of the wedge $W_1$ implements the Lorentz boosts, as in the Bisognano-Wichmann setting. As the covariant action of the modular group is used not only to derive modular $\mathrm{P}_1$CT-symmetry, but also for the proof of the spin-statistics theorem itself, the spin-statistics argument given there is less general than the one given above.

In 1+2 (and less) dimensions, there are massive particle representations which violate the Bose-Fermi alternative. Their statistics is described not by the permutation group, but by the braid group. Such representations were named **anyons** by Wilczek [84] (since their statistics parameter may be *any* complex number of modulus 1), for anyons with parastatistics the term **plektons** (derived from the greek expression for 'braid') has been suggested in [45]. These representations are not covered by the Doplicher-Roberts field system with its compact gauge group. For theories with a finite number of sectors, Schomerus has constructed a field system with a non-associative algebra of quantum symmetries [70]; this system includes the anyon representations. It is an open question whether a proof of the spin-statistics theorem in the spirit of the above one can be given in this setting.

In the past years, Guido and Longo have been working on a spin-statistics theorem for anyons. The first step in their strategy was a proof of the spin-statistics theorem for anyonic representations occuring in conformal theories on the circle [50].

Longo has suggested a way how to boil the case of 1+2-dimensional topological charges down to the chiral case [63]: he considers the spacelike cones which are intersections of two images of $W_1$ under rotations. Call this class of spacelike cones $\mathcal{S}_{W_1}$. Every $C \in \mathcal{S}_{W_1}$ has a nonempty intersection $\mathcal{I}$

with the unit circle in the time-zero plane, and every connected subset of some half of this unit circle generates a unique spacelike cone in $\mathcal{S}_{W_1}$. This is a one-to-one correspondence $I \mapsto C_I$ between sufficiently small intervals (call their class $\mathcal{I}$) in $S^1$ and $\mathcal{S}_{W_1}$. Longo now considers the family $(\mathcal{B}(I) := \mathfrak{A}(C_I)'')_{I \in \mathcal{I}}$, which inherits locality and rotational covariance from $\mathfrak{A}$. If $\mathfrak{A}$ satisfies modular reflection covariance and strong additivity, i.e., if given a double cone $\mathcal{O}$ and any finite covering $(\mathcal{O}_\nu)_{\nu \in \mathbb{N}}$ of $\mathcal{O}$ by double cones, one has $\mathfrak{A}(\mathcal{O}) \subset (\bigcup_{\nu \in \mathbb{N}} \mathfrak{A}(\mathcal{O}_\nu))''$, Longo concludes that the spin-statistics theorem follows by mimicking the argument for conformal theories on the circle.

The first proof of a spin-statistics theorem which uses the structures established by the Bisognano-Wichmann theorem has been given by Fröhlich and Marchetti [46]. Their argument includes the anyonic representations mentioned, but it relies on the assumption of the *full* Bisognano-Wichmann structure not only for the net of local observables, but for the whole reduced field bundle. The reduced field bundle is a generalization of the extended field system discussed above. In contrast to the latter, it does not consist of algebras, so the Tomita-Takesaki analysis does not apply in its traditional form (although the corresponding operators are well-defined).

# Chapter 3

# Generalizing Landau's theorem; localization regions for algebras and observables

In the preceding chapter the assumption of modular $P_1CT$-symmetry was crucial. The search for sufficient conditions for this symmetry property and for the corresponding behaviour of the modular group will be the scope of Chapter 4, where two uniqueness theorems for modular symmetries will be proved.

As known from Chapter 2, a local net of observables associates a C$^*$-algebra $\mathfrak{A}(\mathcal{O})$ with every double cone $\mathcal{O}$ in Minkowski space. If $P \supset \mathcal{O}$ is another double cone, it is assumed that $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(P)$, and for an arbitrary open region $R \subset \mathbb{R}^{1+s}$, the algebra $\mathfrak{A}(R)$ is defined to be the C$^*$-algebra generated by all algebras $\mathfrak{A}(\mathcal{O})$ with $\mathcal{O} \in \mathcal{K}$ and $\mathcal{O} \subset R$. A local net associates algebras with regions.

In the sequel it will be discussed how to associate a localization region with a given algebra and even with a single local observable. These two localization prescriptions will be used in the discussion of the two uniqueness theorems in Chapter 4.

The analysis is based on a theorem due to Landau [60]. In order to localize single observables, a new generalization of Landau's theorem will be used. It will be stated and proved below.

This chapter is structured as follows: in Section 3.1, the standard assumptions for this chapter and some additional notation will be introduced, Section 3.2 deals with the different notions of duality in algebraic quantum field theory and their physical relevance, and Section 3.3 gives a couple of mathematical preliminaries. In section 3.4, the theorem of Landau and its consequences for the localization of algebras will be discussed, and the mentioned generalization will be proved. This generalization will be the basis for the analysis of localization regions for single local observables, which is presented in Section 3.5.

## 3.1 Notation and assumptions

As in the preceding chapter, $\mathfrak{A}$ will denote a local net of observables in a Hilbert space $\mathcal{H}$. The following assumptions will be made:

> **Translation covariance and spectrum condition:** $\mathfrak{A}$ is covariant under a strongly continuous, unitary representation $U$ of the translation group; and the spectrum of the four-momentum generating $U$ is contained in the closed forward light cone;
>
> **Existence and uniqueness of a cyclic vacuum vector:** There is an up to a complex phase unique unit vector $\Omega$ in $\mathcal{H}$ which is invariant under $U$ and **cyclic** with respect to the algebra $\widetilde{\mathfrak{A}}$, i.e., $\overline{\widetilde{\mathfrak{A}}\Omega} = \mathcal{H}$; $\Omega$ will be called the **vacuum vector**.
>
> **Weak additivity:** For every double cone $\mathcal{O} \in \mathcal{K}$, one has
>
> $$\left( \bigcup_{a \in \mathbb{R}^{1+s}} \mathfrak{A}(\mathcal{O}+a) \right)'' = \widetilde{\mathfrak{A}}''.$$

Recall that the first two of these assumptions imply that the identical representation $(\mathcal{H}, \mathrm{id}_{\widetilde{\mathfrak{A}}})$ of $\widetilde{\mathfrak{A}}$ is irreducible, which means that the vacuum state is pure. They have also been made in the preceding chapter. In contrast to the Assumptions I in Chapter 2, Poincaré covariance is not assumed unless stated otherwise, and $\mathcal{H}$ does not need to be separable. Assumptions II and III are also obsolete for the sequel. The only assumption which is not familiar from Chapter 2 is weak additivity; it is a sufficient condition for the **Reeh-Schlieder property** that the vacuum is cyclic with respect to every local algebra $\mathfrak{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ [68, 13]. It is typically satisfied by nets arising from Wightman fields, and it is not expected to mean a serious restriction[1].

[1] Actually, Lemma 2.6 in [77] indicates that the Reeh-Schlieder property may be sufficient as an assumption for the subsequent analysis. This will not be worked out here.

In the sequel, $\mathcal{C}$ will denote the class of convex regions which are causally complete proper subsets of $\mathbb{R}^{1+s}$. All the classes of regions specified in the preceding chapter are subclasses of $\mathcal{C}$. The wedges in $\mathcal{W}$ are maximal elements of $\mathcal{C}$ in the sense that for every wedge $W \in \mathcal{W}$, every element $R \in \mathcal{C}$ with $R \supset W$ is a wedge. Every element $R$ of $\mathcal{C}$ is an intersection of wedges ([76], Thm. 3.2). The class of all wedges which contain a region $R \in \mathcal{C}$ will be denoted by $\mathcal{W}_R$.

In general, the causal complement of a convex region is not convex. $\mathcal{C}'$ will denote the class of regions which are causal complements of regions in $\mathcal{C}$. Every region $R$ in $\mathcal{C}'$ is a union of wedges ([76], Thm. 3.2); $\mathcal{W}^R$ will denote the class of all wedges that are subsets of $R$, and $\mathcal{W}^R \cap \mathcal{W}_\mathcal{O} =: \mathcal{W}_\mathcal{O}^R$. Note that $\mathcal{W} = \mathcal{C} \cap \mathcal{C}'$. If two regions $\mathcal{O}$ and $P$ in $\mathcal{C}$ are spacelike with respect to each other, there is a wedge $W \in \mathcal{W}_\mathcal{O}^{P'}$, i.e., a wedge such that $W \in \mathcal{W}_\mathcal{O}$ and $W' \in \mathcal{W}_P$ ([76], Prop. 3.1).

$\mathcal{B}$ will denote the bounded elements of the class $\mathcal{C}$. Clearly, the double cones are in $\mathcal{B}$. Every element of $\mathcal{B}$ is contained in some double cone, and it is precisely the intersection of all such double cones ([76], Prop. 3.8). The class of all double cones which contain a region $\mathcal{O} \in \mathcal{B}$ will be denoted by $\mathcal{K}_\mathcal{O}$, while the class of all double cones which are contained in an arbitrary region $R$ will be called $\mathcal{K}^R$.

## 3.2 Duality, PCT-symmetry, and Borchers classes

Given the net of observables $\mathfrak{A}$, the **dual net** $\mathfrak{A}^d$ is $(\mathfrak{A}^d(\mathcal{O}) := \mathfrak{A}(\mathcal{O}')')_{\mathcal{O}\in\mathcal{K}}$. By locality, $\mathfrak{A}^d$ is an extension of $\mathfrak{A}$. But this extension does not need to satisfy locality.

If it does satisfy locality, it coincides with its own dual net. It then follows that every other local net $(\mathcal{B}(\mathcal{O}))_{\mathcal{O}\in\mathcal{K}}$ of observables which is local with respect to $\mathfrak{A}^d$ is a subnet of $\mathfrak{A}^d$. Given locality of $\mathfrak{A}^d$, any two subnets of $\mathfrak{A}^d$ are local with respect to one another.

For the Wightman framework it has been shown by Borchers that mutually local fields have the same PCT-operator and the same scattering matrix [11]. Borchers also showed for PCT-covariant Wightman fields that mutual locality between irreducible local Wightman fields is not only a reflexive and symmetric, but even a transitive relation, so that mutually local irreducible fields form equivalence classes, called **Borchers classes**. In the algebraic setting the property corresponding to this behaviour is locality of the dual net; for a local net with this property, the dual net contains all nets which are local with respect to $\mathfrak{A}$, i.e., all nets in the 'Borchers class' of

$\mathfrak{A}$.

A local net of observables is said to satisfy **essential duality** if its dual net satisfies locality; it is said to satisfy **Haag duality** if $\mathfrak{A} = \mathfrak{A}^d$ or if $\mathfrak{A}(\mathcal{O})'' = \mathfrak{A}(\mathcal{O}')'$. Clearly, if a net satisfies essential duality, its dual net satisfies Haag duality. Theories with broken symmetries violate Haag duality, while essential duality can still hold [69]. For more examples which violate Haag duality, while satisfying essential duality, see [61]. A brief discussion of these examples will be given at the end of this chapter.

$\mathfrak{A}$ is said to satisfy **wedge duality** if the isotonous family $(\mathfrak{A}(W')')_{W\in\mathcal{W}}$ satisfies locality, which is equivalent to $\mathfrak{A}(W)'' = \mathfrak{A}(W')'$. It follows from the Bisognano-Wichmann results quoted before [8, 9] that all nets arising from finite-component Wightman fields satisfy wedge duality.

Finally, the net $\mathfrak{A}$ is said to satisfy **essential spacelike cone duality** if the isotonous family $(\mathfrak{A}(S')')_{S\in\mathcal{S}}$ satisfies locality. This assumption is needed for the Buchholz-Fredenhagen analysis of massive one-particle representations.

One checks (cf. Lemma 3.5.2 below) that wedge duality implies essential spacelike cone duality and that essential spacelike cone duality implies essential (Haag) duality, since for any two spacelike separated spacelike cones, one can find a wedge which contains one of the two, whereas its spacelike complement contains the other one, and since for any two spacelike separated double cones one can find spacelike separated spacelike cones each containing one of the two double cones. Using this, one easily concludes the stated implications (cf. also Lemma 3.5.2 below).

## 3.3 Commutator functions and wave equation techniques

It is a classical result of the Wightman approach to quantum field theory that one can reconstruct a Wightman field from its vacuum expectation values [72, 54]. For the algebraic approach to quantum field theory, such a result is not known. The following lemma, however, shows how one can reconstruct commutation relations of a net of observables from the behaviour of its vacuum expectation values. Since these have some convenient properties, this will faciliate the subsequent investigations.

### 3.3.1 Lemma

*For an arbitrary double cone $\mathcal{O} \in \mathcal{K}$, let $A$ be an element of $\mathfrak{A}(\mathcal{O}')'$.*

*(i) Assume there is a region $R \subset \mathbb{R}^{1+s}$ which contains some spacelike cone and which has the property that $\langle \Omega, AB\Omega \rangle = \langle \Omega, BA\Omega \rangle$ for all $B \in \mathfrak{A}(R)$. Then it follows that $A \in \mathfrak{A}(R)'$.*

*(ii) Assume there is a double cone $P \in \mathcal{K}$ with the property that $\langle \Omega, AB\Omega \rangle = \langle \Omega, BA\Omega \rangle$ for all $B \in \mathfrak{A}(P)$, and assume there is a double cone $Q \subset P$ with the property that $A \in \mathfrak{A}(Q)'$. Then it follows that $A \in \mathfrak{A}(P)'$.*

*(iii) Assume that there is a double cone $P \in \mathcal{K}$ with the property that $\langle \Omega, AB, \Omega \rangle = \langle \Omega, BA\Omega \rangle$ for all $B \in \mathfrak{A}(P + a)$, $a \in \mathbb{R}^{1+s}$. Then $A \in \mathbb{C}\mathrm{id}_{\mathcal{H}}$.*

**Proof.** (i): If $S$ is a spacelike cone contained in $R$, there is a translation $a \in \mathbb{R}^{1+s}$ such that $S + a \subset R \cap \mathcal{O}'$. Choose $C$ and $D$ in $\mathfrak{A}(S + a)$ and $B$ in $\mathfrak{A}(R)$. Since $A \in \mathfrak{A}(\mathcal{O}')'$, the operators $A$ and $C^*$ commute:

$$\langle C\Omega, ABD\Omega \rangle = \langle \Omega, C^* A\, BD\Omega \rangle = \langle \Omega, AC^* BD\Omega \rangle.$$

Since $C^* BD$ is in $\mathfrak{A}(R)$, the assumption implies

$$\langle \Omega, AC^* BD\Omega \rangle = \langle \Omega, C^* BDA\Omega \rangle,$$

and since $D$ and $A$, in turn, commute because of $A \in \mathfrak{A}(\mathcal{O}')'$, one concludes

$$\langle C\Omega, ABD\Omega \rangle = \langle \Omega, C^* BDA\Omega \rangle = \langle C\Omega, BAD\Omega \rangle.$$

But since $C$ and $D$ are arbitrary elements of $\mathfrak{A}(S + a)$, which is a cyclic algebra with respect to $\Omega$ [2], it follows that $AB = BA$; since $B \in \mathfrak{A}(R)$ was arbitrary, one obtains $A \in \mathfrak{A}(R)'$, which is (i).

(ii) Choose $C$ and $D$ in $\mathfrak{A}(Q)$ and $B$ in $\mathfrak{A}(P)$. Since $A$ has been assumed to be in $\mathfrak{A}(Q)'$, it commutes with $C^*$, so

$$\langle C\Omega, ABD\Omega \rangle = \langle \Omega, C^* A\, BD\Omega \rangle = \langle \Omega, AC^* BD\Omega \rangle.$$

Since $C^* BD$ is in $\mathfrak{A}(P)$, the assumption implies

$$\langle \Omega, AC^* BD\Omega \rangle = \langle \Omega, C^* BDA\Omega \rangle,$$

[2] In contrast to the Reeh-Schlieder property for double cones, this can be shown even without any additivity assumption ([27], p. 279). Statement (i) can, therefore, be shown without assuming weak additivity or the Reeh-Schlieder property for double cones.

and since $D$ and $A$ commute by the assumption that $A \in \mathfrak{A}(Q)'$, one concludes

$$\langle C\Omega, ABD\Omega\rangle = \langle \Omega, C^* BDA\Omega\rangle = \langle C\Omega, BAD\Omega\rangle.$$

But since $C$ and $D$ are arbitrary elements of $\mathfrak{A}(Q)$, which, because of the Reeh-Schlieder property, is a cyclic algebra with respect to $\Omega$ [3] it follows that $AB = BA$; since $B \in \mathfrak{A}(P)$ was arbitrary, one obtains $A \in \mathfrak{A}(P)'$, which is (ii).

(iii) There is a translation $a \in \mathbb{R}^{1+s}$ such that $P + a \subset \mathcal{O}'$, so that $A \in \mathfrak{A}(\mathcal{O}')' \subset \mathfrak{A}(P+a)'$. Now choose a $b \in \mathbb{R}^{1+s}$ such that $P + b$ intersects $P + a$, and let $Q$ be a double cone contained in $(P+b)\cap(P+a)$. Isotony implies that $A \in \mathfrak{A}(Q)'$. Since by assumption, $\langle \Omega, AB\Omega\rangle = \langle \Omega, BA\Omega\rangle$ for all $B \in \mathfrak{A}(P+b)$, (ii) implies that $A \in \mathfrak{A}(P+b)'$. Now one can iterate this procedure: choose an arbitrary $c \in \mathbb{R}^{1+s}$ such that $(P+c)\cap(P+b)$ is nonempty, choose a new double cone $Q$ in this intersection, and conclude from (ii) that $A \in \mathfrak{A}(P+c)'$. Note that only the double cone $P + a$ chosen in the first step needs to be spacelike separated from $\mathcal{O}$, since each step uses the result of the preceding one, so one finds that for *every* $a \in \mathbb{R}^{1+s}$, one proves that $A \in \mathfrak{A}(P+a)'$ with a finite number of steps. The statement now follows from weak additivity. $\square$

Given any two local observables $A, B \in \mathfrak{A}_{\mathrm{loc}}$, the commutator function $f_{A,B}$ will henceforth be defined by

$$\mathbb{R}^{1+s} \ni x \mapsto \langle \Omega, [A, U(x)BU(-x)]\Omega\rangle =: f_{A,B}(x).$$

Due to Lemma 3.3.1, the analysis of the support of this function yields information on the structure of the net. Crucial for this analysis is the fact that $f_{A,B}$ is a boundary value of a solution of the wave equation, and a well-known lemma due to Asgeirsson concerning such solutions immediately implies the following lemma, which, for this reason, will be referred to as Asgeirsson's Lemma. Another important consequence of the 'wave nature' of the function $f_{A,B}$ is a theorem due to Jost, Lehmann and Dyson [55, 42], which will also be recalled for the reader's convenience.

[3]Note that at this stage, it would be sufficient to assume the Reeh-Schlieder property for local algebras; weak additivity is a sufficient condition [13], whereas the attempt to prove the converse implication [15] has turned out to be erronous.

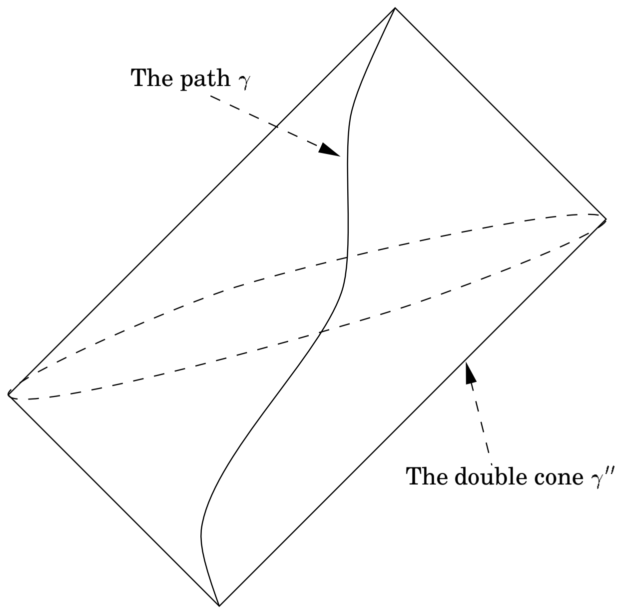


**Figure 3.1: Asgeirsson's double cone lemma**

### 3.3.2 Lemma (Asgeirsson)

*(i) If the commutator function $f_{A,B}$ and all its partial derivatives are zero along a timelike line segment $\gamma$, $f_{A,B}$ vanishes in the entire double cone $\gamma''$.*

*(ii) If $f_{A,B}$ vanishes in an open neighbourhood of a timelike curve segment $\gamma$, it also vanishes in the double cone $\gamma''$.*

**Proof.** The Fourier transform of the operator valued function $\mathbb{R}^{1+s} \ni x \mapsto U(x)$ is the spectral measure of the four-momentum operator. It follows that the Fourier transform $\hat{f}_{A,B}$ of the function $f_{A,B}$ is a finite (signed) measure, and by the spectrum condition, one has $\mathrm{supp}\,\hat{\mathrm{f}}_{\mathrm{A,B}} \subset \overline{\mathrm{V}}$. It follows that the function

$$F(x,\sigma) := \frac{1}{(2\pi)^2} \int \cos(\sigma\sqrt{k^2})\, e^{ikx}\, d\hat{f}_{A,B}(k)$$

is a continuous function with $F(x,0) = f_{A,B}(x)$ for all $x \in \mathbb{R}^{1+s}$. This $F$ is a solution of the 1+(s+1)-dimensional wave equation. (i) is a well known property of solutions of the wave equation (see, e.g., [10], sect. 4.4.D (p. 183 f)). (ii), which is also well known, is an easy consequence of (i). □

Both above formulations are used in the applications. Formulation (i) is appropriate for extensions of the vanishing locus of $f_{A,B}$ (the region where

$f_{A,B}$ vanishes) within $\mathbb{R}^{1+s}$, since the set where $f_{A,B}$ is known to vanish typically is the closure of its open kernel. In the proof of Theorem 3.3.4, however, the Lemma will be used in order to investigate the vanishing locus of the wave $F$. Since locality, as it stands, only implies that $F$ is zero in some subset of $\mathbb{R}^{1+s}$, which is a null set in $\mathbb{R}^{1+(s+1)}$, one makes use of the fact that $F$ has been constructed in such a way that all its partial derivatives, including the one in the $\sigma$-direction, are zero at all points where $F = 0$ due to locality; one may then conclude that the region where $F = 0$ also extends into the $\sigma$-direction.

### 3.3.3 Definition

Let $R$ be any region in Minkowski space.

(i) $R$ will be called **Asgeirsson complete** if for every timelike curve $\gamma \subset R$, the double cone $\gamma''$ is also a subset of in $R$. The smallest Asgeirsson complete extension of $R$ will be called the **Asgeirsson hull** of $R$

(ii) $R$ will be called **double cone complete** if it contains as subsets all double cones with tips in $R$, i.e., if $R = (R + V_+) \cap (R - V_+)$.

(iii) $R$ will be called a **Jost-Lehmann-Dyson region** if it is double cone complete and if every maximal timelike curve in $\mathbb{R}^{1+s}$ intersects $R \cup R'$.

It is clear from the definition that every double cone complete region is Asgeirsson complete. Furthermore, every causally complete region is double cone complete. The causal complement of a Jost-Lehmann-Dyson region is a Jost-Lehmann-Dyson region, too. All causally complete and convex regions are Jost-Lehmann-Dyson regions.

The union of two disjoint causally complete regions is still Asgeirsson complete, but it does not need to be double cone complete, not even causally complete. As an example, consider two disjoint double cones or wedges which are not spacelike with respect to each other. In particular, the union of two Jost-Lehmann-Dyson regions does not need to be a Jost-lehmann-Dyson region.

An example of a double cone complete region which is neither causally complete nor a Jost-Lehmann-Dyson region is the region

$$R := \{x \in \mathbb{R}^{1+s} : 1 < x^2 < 2, x_0 > 0\},$$

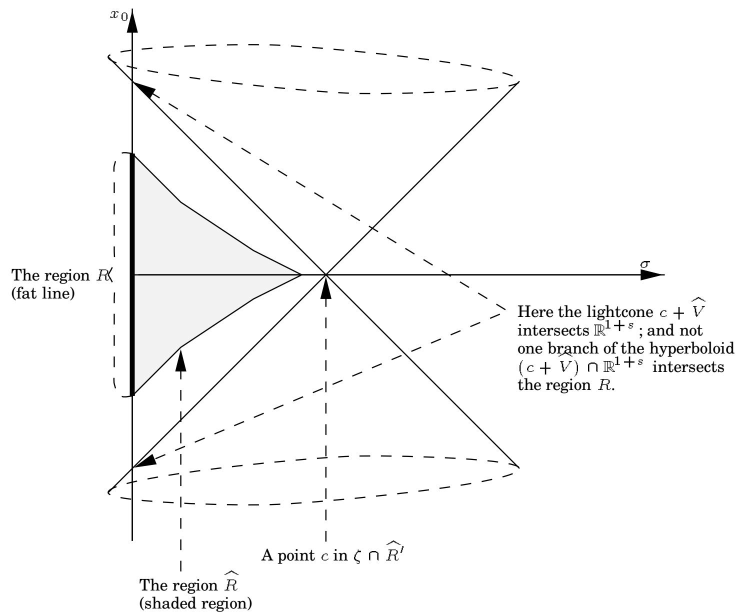


**Figure 3.2: The proof of the Jost-Lehmann-Dyson theorem.**
*For simplicity it has been assumed that the Cauchy surface $\zeta$ mentioned in the proof of Theorem 3.3.4 is the time zero plane, which does not need to be a possible choice (but which is possible as soon as $R$ is invariant under time reflection). The scenario is symmetric under a $\sigma$-reflection, since the function $F$ is even in $\sigma$. Only the $\sigma > 0$-half plane has been sketched.*

since there are timelike curves which do not intersect $R$, e.g., the curve $\mathbb{R} \ni t \mapsto (t, \sqrt{t^2+1}, 0, \ldots, 0)$.

Finally, the time slice region $\{x \in \mathbb{R}^{1+s} : \; 0 < x_0 < 1\}$ is a Jost-Lehmann-Dyson region, but not causally complete.

### 3.3.4 Theorem (Jost, Lehmann, Dyson)

*Let $A$ and $B$ be local observables, and assume that the commutator function $f_{A,B}$ vanishes in a Jost-Lehmann-Dyson region $R$. Then the support of $f_{A,B}$ is contained not only in the complement of $R$, but even in the (in general, smaller) union of all* **admissible mass hyperboloids***, i.e., the mass hyperboloids*

$$H_{a,\sigma} := \{x \in \mathbb{R}^{1+s} : \; (x-a)^2 = \sigma^2\}, \qquad a \in \mathbb{R}^{1+s}, \, \sigma \in \mathbb{R},$$

*which do not intersect $R$.*

**Sketch of proof.** Define $F$ as in the proof of Lemma 3.3.2. Since $F$ is a solution of the wave equation, it is well-known that for every Cauchy surface $\zeta$ in $\mathbb{R}^{1+(s+1)}$, there exists a distribution $F_\zeta$ with support in $\zeta$ such that $F = F_\zeta * D_{1+(s+1)}$, where $D_{1+(s+1)}$ denotes a fundamental solution of the 1+(s+1)-dimensional wave equation (see, e.g., [10], pp. 175-184). The support of $D_{1+(s+1)}$ is contained in the closed light cone $\overline{\widehat{V}}$ of $\mathbb{R}^{1+(s+1)}$ [4]. Since $R$ is a Jost-Lehmann-Dyson region in $\mathbb{R}^{1+s}$, its 1+(s+1)-dimensional Asgeirsson hull $\widehat{R}$ is easily seen to be a Jost-Lehmann-Dyson region in $\mathbb{R}^{1+(s+1)}$. Provided this region is 'well-behaved', there is a Cauchy surface $\zeta$ in $\widehat{R} \cup \widehat{R}'$. This Cauchy surface has the property that for every point $c \in \zeta$, either both the forward and the backward part of $\overline{\widehat{V}} + c$ or neither of them intersects $R$. The former case occurs if and only if $c \in \zeta \cap \widehat{R}$. The latter case occurs if and only if $c \in \zeta \cap \widehat{R}'$, the Asgeirsson hull $\widehat{R}$ of $R$ and the spacelike complement being taken in the spacetime $\mathbb{R}^{1+(s+1)}$. But since all partial derivatives of $F$ can be checked to vanish in all points in $R$, one obtains from Lemma 3.3.2 that $F$ vanishes in $\widehat{R}$, the support of $F_\zeta$ contains only points of the second kind, i.e., it is contained in $\widehat{R}' \cap \zeta$. This implies that the support of $F$ is contained in $(\widehat{R} \cap \zeta) + \overline{\widehat{V}}$.

Since $f_{A,B}$ is a boundary value of $F$ and since the intersection of $\overline{\widehat{V}} + c$ with $\mathbb{R}^{1+s}$ is the convex hull of a shifted mass hyperboloid, the support of the boundary value $f_{A,B}$ of the function $F$ is contained in the union of admissible mass hyperboloids, as stated. $\square$

In the theory of analytic functions in several complex variables, domains of analyticity can be extended in the way found in the Jost-Lehmann-Dyson theorem [26].

## 3.4 How to localize observables: a generalization of Landau's theorem

Using the above wave equation techniques, Landau [60] proved the following:

### 3.4.1 Theorem (Landau)

*If the closures of two double cones $\mathcal{O}$ and $P$ are disjoint, then*

$$\mathfrak{A}(\mathcal{O}')' \cap \mathfrak{A}(P')' = \mathbb{C}\mathrm{id}_{\mathcal{H}}.$$

This immediately implies the following corollary (cf. [6]).

[4]This notation is consistent since $\widehat{V}$ is, indeed, the 1+(s+1)-dimensional Asgeirsson hull of $V$. Note that $\widehat{\overline{V}} \neq \overline{\widehat{V}}$.

### 3.4.2 Corollary

*For every bounded, causally complete and convex region $\mathcal{O}$ and every arbitrary open region $M \subset \mathbb{R}^{1+s}$, the algebra $\mathfrak{A}(\mathcal{O}')'$ contains $\mathfrak{A}(M)$ as a subalgebra if and only if $\overline{\mathcal{O}} \supset \overline{M}$.*

**Proof.** By isotony and locality, the condition is sufficient. To prove that it is necessary, assume $\overline{M}$ not to be contained in $\overline{\mathcal{O}}$ as a subset. Then, since $\mathcal{K}$ is a topological base and since the region $M\backslash\overline{\mathcal{O}}$ has a nonempty interior, $M\backslash\overline{\mathcal{O}}$ contains a double cone $P \in \mathcal{K}$ whose closure is disjoint from $\overline{\mathcal{O}}$. It follows that a wedge $W$ can be found whose closure is disjoint from $\overline{\mathcal{P}}$ and whose interior contains $\overline{\mathcal{O}}$. But by Proposition 3.8 (b) in [76], on can now conclude that there is a double cone $Q$ with $Q \subset W$ and $Q \supset \mathcal{O}$ (note that $\mathcal{O}$ itself does not need to be a double cone). Landau's theorem now implies that $\mathfrak{A}(P) \cap \mathfrak{A}(Q')' = \mathbb{C}\mathrm{id}_{\mathcal{H}}$. It follows from the Reeh-Schlieder property that $\mathfrak{A}(P) \not\subset \mathbb{C}\mathrm{id}_{\mathcal{H}}$, so $\mathfrak{A}(P) \not\subset \mathfrak{A}(Q')'$. Since $\mathfrak{A}(P) \subset \mathfrak{A}(M)$ follows from isotony, $\mathfrak{A}(M)$ cannot be a subset of $\mathfrak{A}(Q')'$, and since $\mathcal{O} \subset Q$, it cannot be a subset of $\mathfrak{A}(\mathcal{O}')'$. □

This already implies that for an $\mathcal{O}$ satisfying the assumptions of the corollary, the region

$$L(\mathfrak{A}(\mathcal{O}')') := \bigcup\{P \in \mathcal{K} : \mathfrak{A}(P) \subset \mathfrak{A}(\mathcal{O}')'\},$$

which will be called the **localization region** of the algebra $\mathfrak{A}(\mathcal{O}')'$, coincides with $\mathcal{O}$. The proof of Corollary 3.4.2 can be made shorter as soon as one knows that Landau's theorem still works if one of the two double cones is replaced by a wedge. That this, indeed, is possible, has been shown in the context of the proof of the $\mathrm{P}_1$CT-part of the first uniqueness theorem for modular symmetries (Theorem. 2.1 in [59]).

### 3.4.3 Theorem

*If the closures of a double cone $\mathcal{O}$ and a wedge $W$ are disjoint, then*

$$\mathfrak{A}(\mathcal{O}')' \cap \mathfrak{A}(W')' = \mathbb{C}\mathrm{id}_{\mathcal{H}}.$$

Following precisely the same, even a simpler line of argument as above, while using this generalized version of Landau's theorem, one concludes that in Lemma 3.4.2, the assumption that $\mathcal{O}$ is bounded may be omitted.

### 3.4.4 Corollary

*For every causally complete convex region $R \subset \mathbb{R}^{1+s}$ and every arbitrary open region $M \subset \mathbb{R}^{1+s}$, the algebra $\mathfrak{A}(R')'$ contains $\mathfrak{A}(M)$ as a subalgebra if and only if $\overline{R} \supset \overline{M}$.*

**Proof.** By isotony and locality, the condition is sufficient. To prove that it is necessary, assume $\overline{M}$ not to be contained in $\overline{R}$ as a subset. Then, since $\mathcal{K}$ is a topological base and since the region $M \backslash \overline{R}$ has a nonempty interior, $M \backslash \overline{R}$ contains a double cone $\mathcal{O} \in \mathcal{K}$ whose closure is disjoint from $\overline{R}$. It follows that a wedge $W$ can be found whose closure is disjoint from $\overline{\mathcal{O}}$ and whose interior contains $\overline{R}$. Landau's theorem now implies that $\mathfrak{A}(\mathcal{O}) \cap \mathfrak{A}(W')' = \mathbb{C}\mathrm{id}_{\mathcal{H}}$. It follows from the Reeh-Schlieder property that $\mathfrak{A}(\mathcal{O}) \not\subset \mathbb{C}\mathrm{id}_{\mathcal{H}}$, so $\mathfrak{A}(\mathcal{O}) \not\subset \mathfrak{A}(W')'$. Since $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(M)$ follows from isotony, $\mathfrak{A}(M)$ cannot be a subset of $\mathfrak{A}(W')'$, and since $R \subset W$, it cannot be a subset of $\mathfrak{A}(R')'$. □

For every causally complete convex region $R$ it follows that $L(\mathfrak{A}(R')') = L(\mathfrak{A}(R)) = R$.

In order to investigate the localization behaviour of a single local observable, a further generalization of Landau's theorem will be used. It is the main result of this section. It also includes the statement of Theorem 2.1 in [59]. $\mathfrak{A}^d_{\mathrm{loc}}$ will denote the algebra of local observables of the dual net $\mathfrak{A}^d$.

### 3.4.5 Theorem (empty-intersection theorem)

*Let $(W_\nu)_{1\leq\nu\leq n}$ be a family of $n$ wedges in $\mathcal{W}$. If $\bigcap_\nu \overline{W}_\nu = \emptyset$, then*

$$\mathfrak{A}^d_{\mathrm{loc}} \cap \bigcap_\nu \mathfrak{A}(W'_\nu)' = \mathbb{C}\mathrm{id}_{\mathcal{H}}\,.$$

**Proof.** Choose an $A \in \mathfrak{A}^d_{\mathrm{loc}} \cap \bigcap_\nu \mathfrak{A}(W'_\nu)'$, and let $\mathcal{O}$ be a double cone with $A \in \mathfrak{A}(\mathcal{O}')'$.

Since $n$ is finite and since the closures of the wedges have an empty common intersection, there is a double cone $P$ which contains the origin and which is so small that the wedges $\tilde{W}_\nu := \overline{(W_\nu - P)''}$, $\nu \leq n$, still have an empty common intersection. Choose $B \in \mathfrak{A}(P)$, and define $\tilde{\mathcal{O}} := \overline{(\mathcal{O} - P)''}$; then the commutator function $f_{A,B}$ vanishes in the region $R := \tilde{\mathcal{O}}' \cup \bigcup_\nu \tilde{W}'_\nu$.

There is no admissible mass hyperboloid for this region, not even for the smaller region $\bigcup_\nu \tilde{W}'_\nu$. To see this, note that if a (shifted) mass hyperboloid is disjoint from a set-theoretic union of open wedges, so is the unique shift of the closed light cone $\overline{V}$ which contains the hyperboloid. Now choose $x \in \mathbb{R}^{1+s}$ such that $x + \overline{V}$ is disjoint from all $\tilde{W}'_\nu$, $\nu \leq n$. This is equivalent to $\{x\}' \supset \bigcup_\nu \tilde{W}'_\nu$, i.e.,

$$x \in \bigcap_\nu \tilde{W}''_\nu = \bigcap_\nu \tilde{W}_\nu = \emptyset.$$

Hence, there is no admissible mass hyperboloid for $\bigcup_\nu \tilde{W}'_\nu$, so there is, a fortiori, no admissible mass hyperboloid for the larger region $R$.

If $R$ is a Jost-Lehmann-Dyson region, it follows from Theorem 3.3.4 that $f_{A,B}(x)$ vanishes for all $x \in \mathbb{R}^{1+s}$ and all $B \in \mathfrak{A}(P)$, so using part (iii) of Lemma 3.3.1, one concludes that $A \in \mathbb{C}\mathrm{id}_{\mathcal{H}}$, and the proof is complete.

But since $R$ does not need to be a Jost-Lehmann-Dyson region, Asgeirsson's lemma will be used to show that the function $f_{A,B}$ even vanishes in the 1+s-dimensional Asgeirsson hull $\hat{R}$ of $R$. Since there is no admissible hyperboloid for $R$, there is, a fortiori, no admissible hyperboloid for $\hat{R}$, so the proof will be complete as soon as $\hat{R}$ has been shown to be a Jost-Lehmann-Dyson region in $\mathbb{R}^{1+s}$.

To this end, choose coordinates such that $\tilde{\mathcal{O}}$ is a symmetric double cone centered at the origin, and let $\rho_0$ be its radius[5]. For every $\rho > 0$, let $Z_\rho = \{x = (x_0, \vec{x}) \in \mathbb{R}^{1+s} : \|\vec{x}\| = \rho\}$ be the boundary of the cylinder of radius $\rho$ around the time axis in $\mathbb{R}^{1+s}$, and define

$$\begin{aligned} R_{\rho,0} &:= \tilde{\mathcal{O}}' \cap Z_\rho, \\ R_{\rho,\nu} &:= \tilde{W}'_\nu \cap Z_\rho, \qquad \nu \leq n. \end{aligned}$$

Due to the above choice of coordinates, the region $R_{\rho,0}$ is a strip:

$$R_{\rho,0} = \{x \in Z_\rho : |x_0| \leq \rho - \rho_0\}$$

(which is empty if $\rho < \rho_0$). Note that with respect to the spacetime structure $Z_\rho$ inherits from Minkowski space, this is a Jost-Lehmann-Dyson region in $Z_\rho$.

The wedges $\tilde{W}'_\nu$ are Jost-Lehmann-Dyson regions in $\mathbb{R}^{1+s}$. With respect to the inherited spacetime structure of $Z_\rho$, the region $R_{\rho,\nu}$ is also a Jost-Lehmann-Dyson region in $Z_\rho$.

Since $R$ does not need to be a Jost-Lehmann-Dyson region in $\mathbb{R}^{1+s}$, the region

$$R_\rho := R \cap Z_\rho = \bigcup_{0 \leq \mu \leq n} R_{\rho,\mu}$$

[5]To construct this double cone, consider the $\rho_0$-ball $\vec{B}_{\rho_0}(0)$ around the origin in the time-zero plane; then $\tilde{\mathcal{O}}$ is the causal completion of this region: $\mathcal{O} = \vec{B}_{\rho_0}(0)''$. The radius of a double cone has also a Poincaré invariant geometrical significance: it is half the geodesic distance between the tips of the double cone.

does not need to be a Jost-Lehmann-Dyson region in $Z_\rho$.

But for every $\nu \leq n$, there is a $\rho_\nu > \rho_0$ such that $R_{\rho,0} \cup R_{\rho,\nu}$ is a Jost-Lehmann-Dyson region in $Z_\rho$ for every $\rho > \rho_\nu$: the edge $E_\nu$ of the wedge $\tilde{W}'_\nu$ is a spacelike flat (s-1)-dimensional submanifold of $\mathbb{R}^{1+s}$. Therefore it is contained in the spacelike complement of $\tilde{\mathcal{O}}$ up to a bounded set (which may be empty). If one chooses $\rho_\nu$ so large that this bounded set is surrounded by the manifold $Z_{\rho_\nu}$, it follows for every $\rho \geq \rho_\nu$ that $E_\nu \cap Z_\rho \subset R_{\rho,0}$. Now one can check by some elementary geometric considerations that the region $R_{\rho,0} \cup R_{\rho,\nu}$ is a double cone complete extension of the Jost-Lehmann-Dyson region $R_{\rho,0}$. Such a region, in turn, is a Jost-Lehmann-Dyson region.

For every $\rho > \max_{\nu \leq n} \rho_\nu =: \hat{\rho}$, this implies that the region $R_\rho$ is a finite union of Jost-Lehmann-Dyson regions each of which contains the Jost-Lehmann-Dyson region $R_{\rho,0}$. But such a union is, again, a Jost-Lehmann-Dyson region.

It follows that for every $\rho > \hat{\rho}$, the 1+s-dimensional Asgeirsson hull $\hat{R}_\rho$ of $R_\rho$ is a Jost-Lehmann-Dyson region in $\mathbb{R}^{1+s}$. On the other hand, for every $\rho > \hat{\rho}$ and for $0 \leq \nu \leq n$, the part of $\tilde{\mathcal{O}}'$ or $\tilde{W}'_\nu$, respectively, which is surrounded by $Z_\rho$ is a subset of the $\mathbb{R}^{1+s}$-Asgeirsson hull $\hat{R}_{\rho,\nu}$ of $R_{\rho,\nu}$. It follows that

$$R \subset \bigcup_{\rho > \hat{\rho}} \bigcup_{\nu \leq n} \hat{R}_{\rho,\nu} = \bigcup_{\rho > \hat{\rho}} \hat{R}_\rho,$$

and one checks that this region is the $\mathbb{R}^{1+s}$-Asgeirsson hull $\hat{R}$ of $R$. Since the Jost-Lehmann-Dyson region $\hat{R}_\rho$ increases with $\rho$, one concludes that $\hat{R}$ is a Jost-Lehmann-Dyson region. This is what remained to be shown, so the proof is complete. $\square$

## 3.5 The localization region of a single local observable

Theorem 3.4.5 prepares the definition of a localization region for local observables. The existence of a nonempty localization region for every local observable is established by the following proposition. In the sequel, the notation introduced in Section 3.1 will be used: $\mathcal{C}$ will denote the class of all causally complete and convex regions in $\mathbb{R}^{1+s}$, and $\mathcal{B}$ will denote the regions belonging to $\mathcal{C}$ which are bounded. As before, $\mathcal{K} \subset \mathcal{B}$ will denote the class of all double cones, and $\mathcal{W} \subset \mathcal{C}$ will denote the class of all wedges.

### 3.5.1 Proposition

*Let $\mathcal{X}$ be any of the classes $\mathcal{K}$, $\mathcal{B}$, $\mathcal{W}$ and $\mathcal{C}$. For every $A \in \mathfrak{A}_{\mathrm{loc}}$ which is not a multiple of the identity, the* **localization regions**

$$\begin{aligned} L^{\mathcal{X}}(A) &:= \bigcap\{\overline{\mathcal{O}} : \mathcal{O} \in \mathcal{X} : A \in \mathfrak{A}(\mathcal{O})''\} \\ L_d^{\mathcal{X}}(A) &:= \bigcap\{\overline{\mathcal{O}} : \mathcal{O} \in \mathcal{X} : A \in \mathfrak{A}(\mathcal{O}')'\} \end{aligned}$$

*are nonempty regions in $\mathcal{B}$. Between them, one has the following equalities and inclusions:*

$$\begin{array}{ccccccc} L^{\mathcal{B}}(A) & = & L^{\mathcal{K}}(A) & \supset & L_d^{\mathcal{K}}(A) & = & L_d^{\mathcal{B}}(A) \\ & & \cup & & \cup & & \\ L^{\mathcal{C}}(A) & = & L^{\mathcal{W}}(A) & \supset & L_d^{\mathcal{W}}(A) & = & L_d^{\mathcal{C}}(A) \end{array}$$

**Proof.** We start with the proof of the equalities and inclusions. The equalities immediately follow from the definitions, since on the one hand, $\mathcal{K} \subset \mathcal{B}$ and $\mathcal{W} \subset \mathcal{C}$, while on the other hand, every region in $\mathcal{B}$ is an intersection of double cones in $\mathcal{K}$ and every region in $\mathcal{C}$ is an intersection of wedges in $\mathcal{W}$ (see Section 2.1). The inclusions in the upper and the lower row of the diagram immediately follow from locality. The inclusions in the two columns follow from the fact that every double cone is an intersection of wedges and that, by isotony, an observable contained in the algebra associated with a given double cone is contained in all algebras associated with wedges containing this double cone.

By these inclusions, it is sufficient to prove that $L_d^{\mathcal{W}}(A)$ is nonempty if $A \notin \mathbb{C}\mathrm{id}$. It already follows from Theorem 3.4.5 that the intersection of every finite family of wedges whose algebras contain $A$ is nonempty. But the family of all wedges whose algebras contain $A$ is never finite.

Since $A$ is a local observable, there is a double cone $\mathcal{O}$ with $A \in \mathfrak{A}(\mathcal{O})$, and it follows from isotony and locality that $L_d^{\mathcal{W}}(A) \subset \mathcal{O}$. But this implies that

$$L_d^{\mathcal{W}}(A) = \bigcap\{\overline{\mathcal{O} \cap W} : W \in \mathcal{W}, A \in \mathfrak{A}(W')'\},$$

which is an intersection of subsets of the compact set $\overline{\mathcal{O}}$. But if for a class of closed subsets of a compact space, every finite subclass has a nonempty intersection, it follows that the whole class has a nonempty intersection. This is the finite intersection property, which is just the Heine-Borel property of compact topological spaces formulated in terms of closed sets instead of open sets ([67], p. 98). □

In the sequel the maps $\mathfrak{A}_{\mathrm{loc}} \ni A \mapsto L^{\mathcal{X}}(A) \in \mathcal{B}$ and $\mathfrak{A}_{\mathrm{loc}} \ni A \mapsto L_d^{\mathcal{X}}(A) \in \mathcal{B}$ will be referred to as **localization prescriptions**. Two problems arise if one wants to interprete the above definitions:

> **Problem 1:** There are several of them, and others may easily be defined. One may ask whether there is one 'physical' localization prescription or whether several distinct localization prescriptions play different roles.
>
> **Problem 2:** Not one of the above localization prescriptions is known to satisfy **locality** in the sense that two local observables commute if their localization regions are spacelike with respect to each other.

The localization prescription $L_d^{\mathcal{W}}$ is the one which — compared with the other prescriptions under consideration — associates the smallest localization region with a local observable. Evidently, this is a first partial answer to Problem 1.

But from a physical viewpoint, it is not necessarily the strongest localization prescription which can be regarded as the 'best' one, but one may prefer to look for the strongest localization prescription which satisfies locality (if such a prescription exists). It might happen that the localization prescriptions $L^{\mathcal{W}}$ and $L_d^{\mathcal{K}}$ both satisfy locality, while $L_d^{\mathcal{W}}$ does not[6]. Since

[6]On the other hand it cannot happen that $L^{\mathcal{K}}$ violates locality if the weaker localization prescription

$$\mathfrak{A}_{\mathrm{loc}} \ni A \mapsto \bigcap\{\overline{\mathcal{O}} : \mathcal{O} \in \mathcal{K}, A \in \mathfrak{A}(\mathcal{O})\},$$

(where the local C*-algebras themselves are tested instead of their weak closures) does satisfy locality. This is why this type of localization prescription is not discussed at this stage.

there is, in general, no inclusion relation between $L^{\mathcal{W}}$ and $L_d^{\mathcal{K}}$, it might happen in this case that there are two distinct 'strongest' localization prescriptions satisfying locality.

Clearly, the localization prescriptions $L^{\mathcal{K}}$ and $L_d^{\mathcal{K}}$ coincide if the net satisfies Haag duality, and the prescriptions $L^{\mathcal{W}}$ and $L_d^{\mathcal{W}}$ coincide if the net satisfies wedge duality. Furthermore, wedge duality also makes $L_d^{\mathcal{W}}$ coincide with $L_d^{\mathcal{K}}$ by the following lemma (cf. also [21], Lemma 4.1).

### 3.5.2 Lemma

*Assume the net $\mathfrak{A}$ to satisfy wedge duality. For every region $R \in \mathcal{C}$, one has*

$$\mathfrak{A}(R')' = \bigcap_{W \in \mathcal{W}_R} \mathfrak{A}(W)'' =: \mathcal{M}(R),$$

*and the net $\mathcal{M}$ satisfies locality.*

**Proof.** We first show that the net $(\mathfrak{A}(R')')_{R\in\mathcal{C}}$ satisfies locality. This immediately follows from the fact remarked above that if $R$ and $S$ are spacelike separated regions in $\mathcal{C}$, there is a wedge $W \in \mathcal{W}$ with $R \subset W$ and $S \subset W'$. For such a constellation, one has

$$\mathfrak{A}(R')' \subset \mathfrak{A}(W')' = \mathfrak{A}(W)'' \subset \mathfrak{A}(S')'',$$

which is the stated locality for the net $(\mathfrak{A}(R')')_{R\in\mathcal{C}}$.

One proves in the same way that the net $\mathcal{M}$ satisfies locality with respect to $\mathfrak{A}$, i.e., $\mathcal{M}(R) \subset \mathfrak{A}(R')'$ for all $R \in \mathcal{C}$. On the other hand,

$$\mathfrak{A}(R')' \subset \bigcap_{W \in \mathcal{W}_R} \mathfrak{A}(W')' = \bigcap_{W \in \mathcal{W}_R} \mathfrak{A}(W)'' = \mathcal{M}(R) \qquad \text{for all } R \in \mathcal{C},$$

and this completes the proof. □

In the following lemma and in the discussion of the second uniqueness theorem for modular symmetries, wedge duality will be assumed. Henceforth, the localization region $L_d^{\mathcal{W}}(A) = L^{\mathcal{W}}(A) = L_d^{\mathcal{K}}(A)$ will simply be denoted by $L(A)$ for every $A \in \mathfrak{A}_{\mathrm{loc}}^d$, where $\mathfrak{A}_{\mathrm{loc}}^d$ is, as above, the algebra of local observables of the dual net $\mathfrak{A}^d$. Given wedge duality, it easily follows from Theorem 3.4.5 that $L$ associates a nonempty localization region with every $A$ not only in $\mathfrak{A}_{\mathrm{loc}}$, but even in $\mathfrak{A}_{\mathrm{loc}}^d$.

### 3.5.3 Theorem (nonempty-intersection theorem)

*Assume $\mathfrak{A}$ to satisfy wedge duality.*

*(i) The localization prescription $L : \mathfrak{A}_{\mathrm{loc}} \to \mathcal{B}$ satisfies locality if and only if for every finite family $(W_\nu)_{1\le\nu\le n}$ of wedges, one has*

$$\mathfrak{A}_{\mathrm{loc}} \cap \bigcap_\nu \mathfrak{A}(W_\nu)'' = \mathfrak{A}_{\mathrm{loc}} \cap \mathfrak{A}^d(\bigcap_\nu W_\nu).$$

*(ii) The extension $L : \mathfrak{A}^d_{\mathrm{loc}} \to \mathcal{B}$ satisfies locality if and only if for every finite family $(W_\nu)_{1\le\nu\le n}$ of wedges, one has*

$$\mathfrak{A}^d_{\mathrm{loc}} \cap \bigcap_\nu \mathfrak{A}(W_\nu)'' = \mathfrak{A}^d_{\mathrm{loc}} \cap \mathfrak{A}^d\left(\bigcap_\nu W_\nu\right).$$

**Proof.** We prove the first statement; the second is easily obtained from the first by replacing the local net $\mathfrak{A}$ by the local net $\mathfrak{A}^d$.

To see that the condition is sufficient, let $A$ be a local observable in $\mathfrak{A}_{\mathrm{loc}}$, and define

$$\mathcal{W}'_A := \{W \in \mathcal{W} : \ A \in \mathfrak{A}(W)',\ W \,\text{open}\}.$$

Since $\bigcup \mathcal{W}'_A = L(A)'$, it is clear that for every local observable $B \in \mathfrak{A}_{\mathrm{loc}}$ with $L(B) \subset L(A)'$, the class $\mathcal{W}'_A$ is an open covering of $L(B)$. Since $L(B)$ is compact, there is a finite subcovering $\breve{\mathcal{W}}'_A$, and since $A$ is a local observable, this finite covering can be chosen such that the region

$$\tilde{L}(A) := \bigcap_{W\in\breve{\mathcal{W}}'_A} W'$$

is compact. Now the condition implies that

$$A \in \mathfrak{A}_{\mathrm{loc}} \cap \bigcap_{W\in\breve{\mathcal{W}}'_A} \mathfrak{A}(W')' = \mathfrak{A}_{\mathrm{loc}} \cap \mathfrak{A}^d\left(\bigcap_{W\in\breve{\mathcal{W}}'_A} W'\right) = \mathfrak{A}_{\mathrm{loc}} \cap \mathfrak{A}^d(\tilde{L}(A)).$$

The closure of the region $\tilde{L}(A)$ is still spacelike separated from $L(B)$. Conversely, one proves in the same way that there is a region $\tilde{L}(B)$ with $B \in \mathfrak{A}^d(\tilde{L}(B))''$ and $\tilde{L}(B) \subset \tilde{L}(A)'$. But now it follows from the locality of $\mathfrak{A}^d$ that $A$ and $B$ commute. This proves that the condition is sufficient.

To prove that the condition is necessary, let $(W_\nu)_{1\le\nu\le n}$ be a finite family of wedges. Whenever $A \in \mathfrak{A}_{\mathrm{loc}} \cap \bigcap_\nu \mathfrak{A}(W_\nu)''$ and $B \in \mathfrak{A}_{\mathrm{loc}} \cap \mathfrak{A}(X)''$ for any $X \in \mathcal{W}^{(\bigcap_\nu W_\nu)'} =: \mathcal{W}^{(\cap W)'}$, locality of $L$ implies that $AB = BA$, and one concludes that

$$\begin{aligned} A \in \bigcap_{X\in\mathcal{W}^{(\cap W)'}} (\mathfrak{A}_{\mathrm{loc}} \cap \mathfrak{A}(X)'')' &= \bigcap_{X\in\mathcal{W}^{(\cap W)'}} \mathfrak{A}(X)' = \bigcap_{X\in\mathcal{W}^{(\cap W)'}} \mathfrak{A}(X')'' \\ &= \bigcap_{X\in\mathcal{W}_{\cap W}} \mathfrak{A}(X)'' = \mathfrak{A}^d\left(\bigcap_\nu W_\nu\right). \end{aligned}$$

The last step applies Lemma 3.5.2. This proves that $\mathfrak{A}_{\mathrm{loc}} \cap \bigcap_\nu \mathfrak{A}(W_\nu)'' \subset \mathfrak{A}_{\mathrm{loc}} \cap \mathfrak{A}^d\left(\bigcap_\nu W_\nu\right)$. Since the converse inclusion follows from isotony and wedge duality, the proof is complete. □

### 3.5.4 Proposition

*Assume $\mathfrak{A}$ to satisfy wedge duality, and suppose that the dual net of $\mathfrak{A}$ satisfies* **strong additivity for wedges**, *i.e., for every wedge $W \in \mathcal{W}$ and every double cone $\mathcal{O}$ with $W \subset W + \mathcal{O}$, one has*

$$\mathfrak{A}(W)'' \subset \left( \bigcup_{a \in W} \mathfrak{A}^d(\mathcal{O} + a) \right)'' .$$

*Then the localization prescription $L$ satisfies locality.*

**Proof.** It is sufficient to derive the following from the assumptions: if $A$ is a local observable and if $X$ is a wedge whose closure is spacelike separated from $L(A)$, then $A \in \mathfrak{A}(X)'$.

So let $A$ and $X$ be chosen in this way. For any neighbourhood $\mathcal{O}$ of the origin, choose $B \in \mathfrak{A}^d(\mathcal{O})$, and let $\mathcal{W}_A$ be the class of all wedges $W \in \mathcal{W}$ with $A \in \mathfrak{A}(W)''$. Due to wedge duality, the commutator function $f_{A,B}$ defined above vanishes in the region $R := \bigcup_{W \in \mathcal{W}_A} (W - \mathcal{O})' \subset (L(A) - \mathcal{O})'$, which is a set-theoretic union of wedges. As already stated in the proof of Theorem 3.4.5, it follows from this that a mass hyperboloid $H$ is admissible with respect to $R$ if and only if the whole unique shift of the lightcone which contains $H$ is disjoint from $R$. Since $R$ is easily seen to be a Jost-Lehmann-Dyson region, one concludes from Theorem 3.3.4 that $f_{A,B}$ vanishes in $R'' = (L(A) - \mathcal{O})'$.

Since $B \in \mathfrak{A}^d(\mathcal{O})$ is arbitrary, it follows that $A \in \mathfrak{A}^d(\mathcal{O} + x)'$ for every $x \in R''$. But since the closures of $X$ and $L(A)$ have been assumed to be spacelike separated, $\mathcal{O}$ can been chosen so small that $X \subset R''$, hence, strong additivity for wedges implies that $A \in \mathfrak{A}(X)'$, as stated. □

The assumption of strong additivity for wedges has been used extensively by Thomas and Wichmann in [76]. The authors have obtained results in the spirit of Proposition 3.5.4 and Theorem 3.4.5, but the latter are not implied by the former results.

To illustrate a situation where the condition of Theorem 3.5.3 is violated, consider some regular triangle in the time zero plane, and let $\mathcal{O}$, $P$ and $Q$ be the double cones in $\mathcal{K}$ which are generated by the $r$-balls centered

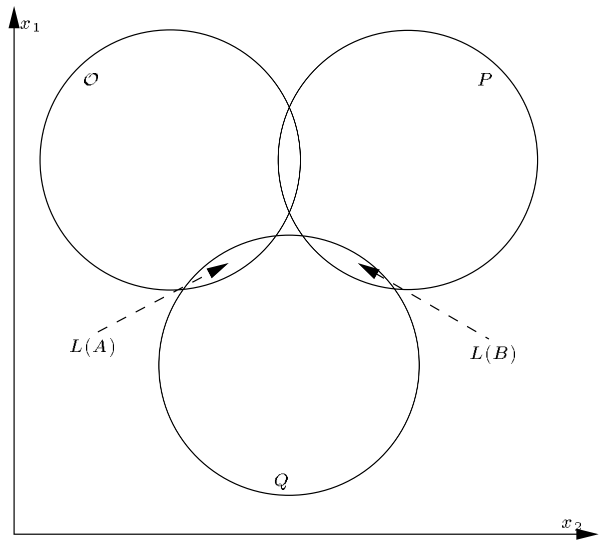


**Figure 3.3: The 'Mickey Mouse Problem'**

*If $A \in \mathfrak{A}(\mathcal{O}) \cap \mathfrak{A}(Q)$, while $B \in \mathfrak{A}(P) \cap \mathfrak{A}(Q)$, the localization regions $L(A)$ and $L(B)$ are spacelike separated; but it cannot yet be concluded that $A$ and $B$ commute.*

at the corners of this triangle. Assume a local observable $A$ to be contained in $\mathfrak{A}(\mathcal{O}')'$ and in $\mathfrak{A}(Q')'$, while another local observable $B$ is contained in $\mathfrak{A}(P')'$ and $\mathfrak{A}(Q')'$. If the radius of the three double cones is bigger than half the side length and smaller than the distance between the triangle's centre of mass and its corners, then the three double cones have pairwise nonempty intersections, while their common intersection is empty. In this case, the localization regions $L(A)$ and $L(B)$ are spacelike with respect to one another, but without strong additivity for wedges or something similar, there is no reason why $A$ and $B$ should commute, since not any two of the double cones $\mathcal{O}$, $P$ and $Q$ are spacelike separated.

To avoid these problems, it is not necessary to assume that $\mathfrak{A}(X) \cap \mathfrak{A}(Y) = \mathfrak{A}(X \cap Y)$ for all $X, Y \in \mathcal{C}$. Clearly, this would imply the condition in Theorem 3.5.3. It is popular in algebraic quantum field theory to consider the algebra $\mathfrak{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$, as the algebra of all observables that can be measured in $\mathcal{O}$. If an observable can be measured in $\mathcal{O}$ and in $P$, one could ask whether it should be measurable in $\mathcal{O} \cap P$, too. But there is no reason why this should be the case. Considering generalized free fields, Landau has given examples of local nets which do not satisfy Haag duality, while they do satisfy essential duality [61] (and they apparently satisfy wedge duality as well as weak additivity). The examples yield local nets that satisfy the assumptions of Theorem 3.5.3, while, in general, $\mathfrak{A}(\mathcal{O}) \cap \mathfrak{A}(P) \neq \mathfrak{A}(\mathcal{O} \cap P)$.

To illustrate the geometrical trick of Landau's example, start from some local net $\mathfrak{B}$ of observables in 1+(s+1) dimensions, and with every double cone $\mathcal{O} = (a + V_+) \cap (b + V_-)$ in $\mathbb{R}^{1+s}$, associate the algebra

$$\mathfrak{B}_0(\mathcal{O}) := \mathfrak{B}((a + \widehat{V}_+) \cap (b + \widehat{V}_-)) =: \mathfrak{B}(\widehat{\mathcal{O}}),$$

where, as before, $\widehat{V}_+$ and $\widehat{V}_-$ denote the 1+(s+1)-dimensional forward and backward light cone, respectively.

One easily checks that $\mathfrak{B}_0(\mathcal{O}) \cap \mathfrak{B}_0(P)$ might not coincide with $\mathfrak{B}_0(\mathcal{O} \cap P)$, since the intersection of the 1+(s+1)-dimensional Asgeirsson hulls of $\mathcal{O}$ and $P$ differs from the 1+(s+1)-dimensional Asgeirsson hull of the intersection $\mathcal{O} \cap P$, i.e., $\widehat{\mathcal{O}} \cap \widehat{P} \neq \widehat{\mathcal{O} \cap P}$. Indeed, Landau has given examples for theories where the corresponding algebras differ. In particular, they differ if the 'large' net $\mathfrak{B}$ has the intersection property, i.e., if $\mathfrak{B}(\mathcal{O}) \cap \mathfrak{B}(P) = \mathfrak{B}(\mathcal{O} \cap P)$ for all $\mathcal{O}, P \in \mathcal{B}$. This proves that the intersection property cannot be a general property of all local nets of observables.

On the other hand, Landau has shown that all his $\mathfrak{B}_0$ satisfy essential duality. This implies that for every $\mathcal{O} \in \mathcal{K}$, one has $\mathfrak{B}_0(\mathcal{O}')' = \mathfrak{B}(\underline{\mathcal{O}}')'$, where $\underline{\mathcal{O}}$ denotes the region $\bigcup_{\sigma \in \mathbb{R}} \mathcal{O} + \mathbb{R} e_{s+1}$, i.e., the double cone $\mathcal{O}$ smeared out in the s+1st spacelike direction. In Landau's examples, this yields the net associated with some generalized free field, and this net is known to satisfy all the conditions of Theorem 3.5.3 and Proposition 3.5.4.

We conclude this chapter with the remark that the question what the intersection of two algebras of local observables contains has arised earlier, as, e.g., the remarks in Section III.4.2 of Haag's monograph [51] show. It appears that Haag's 'Tentative Postulate' that the map $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ be a homomorphism from the orthocomplemented lattice of all causally complete regions (which, in general, are neither bounded nor convex) of Minkowski space into the orthocomplemented lattice of von Neumann algebras on a Hilbert space is far from being proved as it stands (cf. also Haag's heuristic remarks which illustrate the physical limits of the postulate). But if a net satisfies wedge duality and strong additivity for wedges, the above results, indeed, lead to a partial proof of Haag's 'Tentative Posulate': for arbitrary finite families of wedges, one obtains relations in the spirit of (III.4.7) through (III.4.11) in [51] for the dual net. One should, however, note that the above results are far from establishing the lattice homomorphism discussed by Haag.

# Chapter 4

# The two uniqueness theorems for modular symmetries

The assumption of modular $P_1CT$-symmetry made in chapter 2 makes sense for the net of observables as well as for the extended field system constructed from the massive single particle representations. But as soon as modular symmetry holds for the net of observables, it has been shown there to hold for this field system, too.

The same argument also works for the modular group: if the modular group of $\mathfrak{A}(W_1)''$ implements boosts, so does the modular group of $\mathfrak{F}(W_1)$. Furthermore, it has been shown by Guido and Longo that if $\Delta_{W_1}^{it}$ implements the 1-boosts in all Lorentz frames, one can conclude modular $P_1CT$-symmetry [49], and Brunetti, Guido, Longo and Borchers have shown that the commutation relations between the modular groups of different wedges are the same as the commutation relations of the corresponding subgroups of the Poincaré group: they generate a representation of the restricted Poincaré group [25, 17]. On the other hand, Buchholz, Florig, Dreyer and Summers have found that also modular $P_1CT$-symmetry implies Poincaré symmetry [32].

Therefore this chapter will, on the one hand, be dealing both with modular $P_1CT$-symmetry and modular Lorentz-symmetry, and on the other hand, the analysis will be confined to the properties of the modular objects of the net of observables only.

The spin-statistics theorem is not the only application of modular symmetries. For conformal theories of local observables, the Bisognano-Wichmann modular symmetries have been established by different groups in different ways [24, 47, 44]. Conversely, chiral theories in 1+1 dimensions can even be *constructed* from two algebras whose modular objects satisfy certain conditions [81]. This result may also be generalized to higher dimensions [82, 83].

As well as the subsequent analysis, these results mainly depend on a theorem recently found by Borchers (Theorem II.9 in [16]). The main implication of this theorem is that in (the vacuum sector of) a theory satisfying translation covariance and the spectrum condition, the Bisognano-Wichmann modular objects commute with the translation operators in the same way Lorentz boosts and $\mathrm{P_1CT}$-operators would commute with these operators. The corresponding relations will be referred to as **Borchers' commutation relations**. In 1+1 dimensions, Borchers concluded that each local net of observables satisfying translation covariance and the spectrum condition may be enlarged to a local net of observables satisfying Poincaré covariance and the Bisognano-Wichmann modular symmetry principles. The situation in higher dimensions, however, remained an open problem.

In this chapter, two uniqueness results will be derived from Borchers' commutation relations, again for the case of at least 1+2 space-time dimensions: the first one proves that the only symmetries which *can* be implemented by $J_{W_1}$ and $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, are the $\mathrm{P_1CT}$-symmetry and the 1-boosts, respectively. As a symmetry, every unitary or antiunitary operator is admitted under whose adjoint action every algebra of local observables is mapped onto some algebra associated with some open region in Minkowski space. Using a trick proposed by Trebels [78], a last translational degree of freedom which was left open in [59] will be eliminated. The result will be stated and discussed in Section 4.1, while the proof will be given in the Sections 4.3 and 4.4.

The second uniqueness theorem assumes that the localization region of a local observable develops continuously under the action of the modular group $\Delta_{W_1}^{it}$. It is shown that in this case, the localization region develops like under the action of a boost. The result is stated and discussed in Section 4.2, the proof is given in Section 4.4.

## 4.1 Statement and discussion of the first uniqueness theorem for modular symmetries

As before, $\mathfrak{A}$ will denote local net of observables in a Hilbert space $\mathcal{H}$. The standard assumptions will be the same as in Chapter 3: translation covariance, spectrum condition, existence and uniqueness of a pure vacuum state, and weak additivity. The representation $U$ of the translation group and the vacuum vector $\Omega$ are denoted as before.

The wedge $W_1$, the Tomita operator $S_{W_1}$ of the standard von Neumann algebra $(\mathcal{H}, \mathfrak{A}(W_1)'', \Omega)$, its modular operator $\Delta_{W_1}$ and its modular conjugation $J_{W_1}$ are defined as in Chapter 2.

Crucial for the sequel is the following theorem due to Borchers (Theorem II.9 in [16]):

### 4.1.1 Theorem (Borchers)

*Let $\mathfrak{M}$ be a subalgebra of $\mathfrak{B}(\mathcal{H})$ such that $(\mathcal{H}, \mathfrak{M}, \Omega)$ is a standard von Neumann algebra, and let $J_{\mathfrak{M}}$ and $\Delta_{\mathfrak{M}}$ be its modular conjugation and its modular operator, respectively. Let $(T(r))_{r\in\mathbb{R}}$ be a strongly continuous one-parameter group of unitaries which has a positive generator and which for each $r \geq 0$ satisfies the conditions*

$$\begin{array}{lrcl} (a) & T(r)\Omega & = & \Omega; \\ (b) & T(r)\mathfrak{M}T(-r) & \subset & \mathfrak{M}. \end{array}$$

*Then for each $r \in \mathbb{R}$, the following relations, which will be referred to as* **Borchers' commutation relations***, hold:*

$$\begin{array}{lrcll} (i) & J_{\mathfrak{M}}T(r)J_{\mathfrak{M}} & = & T(-r); & \\ (ii) & \Delta_{\mathfrak{M}}^{it}T(r)\Delta_{\mathfrak{M}}^{-it} & = & T(e^{-2\pi t}r) & \text{for all } t \in \mathbb{R}. \end{array}$$

Now introduce light cone coordinates as follows:

$$\mathbb{R}^{1+s} \ni a \mapsto (a_+ := a_0 + a_1,\ \ a_- := a_0 - a_1,\ \ a_2, \ldots, a_s).$$

Applying Theorem 4.1.1 to the coordinates $a_+$ and $a_-$ (inserted for $r$), and applying Lemma 2.1.2 to the other coordinates, one obtains, in the present setting, for each $a \in \mathbb{R}^{1+s}$:

$$\begin{array}{lrcll} (i) & J_{W_1}U(a)J_{W_1} & = & U(ja), & \\ (ii) & \Delta_{W_1}^{it}U(a)\Delta_{W_1}^{-it} & = & U(V_1(-2\pi t)a) & \text{for all } t \in \mathbb{R}, \end{array}$$

where $V_1(2\pi t)$ denotes the Lorentz boost by $(2\pi t)$ in the 1-direction. So $J_{W_1}$ and $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, have the same commutation relations with the translations as a $\mathrm{P}_1$CT-operator and the group of Lorentz boosts in the 0-1-direction would have, respectively. For 1+1 dimensions, it easily follows that the net of observables may be enlarged to a local net which generates the same wedge algebras (and, hence, the same corresponding modular operator and conjugation) as the original one and which has the property that $J_{W_1}$ is a PCT-operator **(modular PCT-symmetry)**, whereas $\Delta_{W_1}^{it}$ implements the Lorentz boost by $-2\pi t$ for each $t \in \mathbb{R}$ **(modular Lorentz symmetry)**.

The first uniqueness theorem for modular symmetries states that in 1+2 and more dimensions, $J_{W_1}$ or $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, can be shown to be a $\mathrm{P}_1$CT-operator or a 0-1-Lorentz boost, respectively, as soon as $J_{W_1}$ or $\Delta_{W_1}^{it}$ is *any* symmetry in the following sense:

### 4.1.2 Definition

A unitary or an antiunitary operator $K$ in $\mathcal{H}$ is called a **symmetry** of $\mathfrak{A}$ if for each $\mathcal{O} \in \mathcal{K}$, there are open sets $M_{\mathcal{O}}, N_{\mathcal{O}} \subset \mathbb{R}^{1+s}$ with

$$K\mathfrak{A}(\mathcal{O})K^* = \mathfrak{A}(M_{\mathcal{O}}), \qquad K^*\mathfrak{A}(\mathcal{O})K = \mathfrak{A}(N_{\mathcal{O}}).$$

Before discussing this definition, we state the first main results of this chapter.

### 4.1.3 Theorem

*Let $K$ be a symmetry of $\mathfrak{A}$, and let $\kappa$ be a causal automorphism*[1] *of $\mathbb{R}^{1+s}$ such that*

$$KU(a)K^* = U(\kappa a) \qquad \textit{for all } a \in \mathbb{R}^{1+s}.$$

*Then there is a unique $\xi \in \mathbb{R}^{1+s}$ such that*

$$K\mathfrak{A}(\mathcal{O})K^* = \mathfrak{A}(\kappa\mathcal{O} + \xi) \qquad \text{for all } \mathcal{O} \in \mathcal{K}.$$

[1]Recall that a **causal automorphism** of $\mathbb{R}^{1+s}$ is a bijection $f : \mathbb{R}^{1+s} \to \mathbb{R}^{1+s}$ which preserves the causal structure of $\mathbb{R}^{1+s}$, i.e., $f(x)$ and $f(y)$ are timelike with respect to eacht other if and only if $x$ and $y$ are timelike with respect to each other. Without assuming linearity or continuity, one can show that the group of all causal automorphisms of $\mathbb{R}^{1+s}$ is generated by the elements of the Poincaré group and the dilatations [1, 3, 2, 86, 20]. Since the transformations implemented on the translations by Borchers' commutation relations happen to be causal transformations in all applications to be discussed below, this assumption means no loss of generality.

From Theorems 4.1.1 and 4.1.3, the following will be concluded:

### 4.1.4 Proposition (first uniqueness theorem for modular symmetries)

*(i) If $J_{W_1}$ is a symmetry, then*

$$J_{W_1}\mathfrak{A}(\mathcal{O})J_{W_1} = \mathfrak{A}(j\mathcal{O}) \qquad \textit{for all } \mathcal{O} \in \mathcal{K}.$$

*(ii) If the unitaries $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, are symmetries, then*

$$\Delta_{W_1}^{it}\mathfrak{A}(\mathcal{O})\Delta_{W_1}^{-it} = \mathfrak{A}(V_1(-2\pi t)\mathcal{O}) \qquad \textit{for all } \mathcal{O} \in \mathcal{K}.$$

Another application of Theorem 4.1.3 is the following:

### 4.1.5 Proposition (uniqueness theorem '1a' for modular symmetries)

*Assume $\mathfrak{A}$ to be Poincaré covariant, and assume that the vacuum vector $\Omega$ is not only cyclic, but also separating with respect to the algebra $\mathfrak{A}(V_+)''$.*

*(i) If the modular conjugation $J_+$ of $(\mathcal{H}, \mathfrak{A}(V_+)'', \Omega)$ is a symmetry, then*

$$J_+\mathfrak{A}(\mathcal{O})J_+ = \mathfrak{A}(-\mathcal{O}) \qquad \textit{for all } \mathcal{O} \in \mathcal{K}.$$

*(ii) If for some $t \in \mathbb{R}$, the modular unitary $\Delta_+^{it}$ of $(\mathcal{H}, \mathfrak{A}(V_+)'', \Omega)$ is a symmetry, then*

$$\Delta_+^{it}\mathfrak{A}(\mathcal{O})\Delta_+^{-it} = \mathfrak{A}(e^{-2\pi t}\mathcal{O}).$$

Since massive theories cannot be dilation invariant unless their mass spectrum is dilation invariant (cf., e.g., [64]), the interesting models satisfying condition (ii) are massless theories. But it follows from the scattering theory for massless fermions and bosons in 1+3 or 1+1 dimensions (see [27, 28, 29]) that each of the conditions (i) and (ii) of Proposition 4.1.5 implies a massless theory to be free (i.e., its S-matrix is trivial) (see [28, 30, 34]).

Note that all modular symmetries considered in the Propositions 4.1.4 and 4.1.5 have been found in [16] for the 1+1-dimensional case.

The above notion of a symmetry has been designed (almost) as wide as appeared possible for the preceding results. It will turn out that due to the strong impact of Borchers' commutation relations, it is, a priori, not necessary to assume that a symmetry is induced by a point transformation, i.e., that there is a bijection $f : \mathbb{R}^{1+s} \to \mathbb{R}^{1+s}$ such that $M_{\mathcal{O}} = f(\mathcal{O})$ and $N_{\mathcal{O}} = f^{-1}(\mathcal{O})$ for every $\mathcal{O} \in \mathcal{K}$. It is not even necessary to assume that the symmetries form a group. What is essential in the definition is that the image of any local algebra under the adjoint action of $K$ is generated by the local algebras it contains as subsets. If one associates with every algebra $\mathfrak{M} \subset \mathfrak{B}(\mathcal{H})$ its **localization region** $L(\mathfrak{M})$ **with respect to the net** $\mathfrak{A}$ by

$$L(\mathfrak{M}) := \bigcup \{\mathcal{O} \in \mathcal{K} : \mathfrak{A}(\mathcal{O}) \subset \mathfrak{M}\},$$

then one knows from Chapter 3 that for every region $\mathcal{O} \in \mathcal{B}$, one has $L(\mathfrak{A}(\mathcal{O})) = \mathcal{O}$. In general, $L(\mathfrak{A}(M)) \supset M$.

Definition 4.1.2 could also refer to other prescriptions to extend the net $(\mathfrak{A}(\mathcal{O}))_{\mathcal{O}\in\mathcal{K}}$ to the index set of all open sets in $\mathbb{R}^{1+s}$. Indeed, there are situations where this makes sense. If, for example, the local algebras (associated with double cones) are von Neumann algebras, not all algebras of the form $\mathfrak{A}(M)$, $M \subset \mathbb{R}^{1+s}\,\mathrm{open}$, need to be von Neumann algebras. Since the adjoint action of any unitary operator maps von Neumann algebras onto von Neumann algebras, the above definition may exclude a large class of open regions.

In this case, one may prefer to consider the net

$$(\mathcal{R}(M) := \mathfrak{A}(M)'')_{M\subset\mathbb{R}^{1+s}\,\mathrm{open}}.$$

The class of symmetries with respect to the net $\mathcal{R}$ may be larger than the class of symmetries of the net $(\mathfrak{A}(M))_{M\subset\mathbb{R}^{1+s}\,\mathrm{open}}$, and the above results would still apply if one replaced the net $\mathfrak{A}$ by the net $\mathcal{R}$. For this reason, a more general definition of symmetry was given in [59]. But since the relevance of this generalization appears to be rather a technical one, this definition will not be used in the sequel in order to faciliate reading. The proof of the new result would be precisely the same if the more general definition of a symmetry were used.

Definition 4.1.2 includes the spacetime symmetries and internal symmetries which are familiar from high energy physics. The internal symmetries — these include the quantum symmetries [70] already mentioned in the introduction of chapter 2, and they include local gauge symmetries since they do not need to be translation invariant — are trivially included, since they leave the local algebras invariant. On the other hand, special conformal transformations may map bounded onto unbounded regions, so they would be excluded a priori if the regions $M_{\mathcal{O}}$ and $N_{\mathcal{O}}$ were assumed to be bounded regions or even double cones.

Clearly, Definition 4.1.2 does not include those supersymmetries that interpolate between bosonic and fermionic sectors. This does not imply any loss of generality, since already the Tomita-Takesaki Theorem guarantees that neither the modular group nor the modular conjugation can turn an observable into a fermionic field operator. Furthermore, symmetries in the sense of Definition 4.1.2 do not need to leave the vacuum vector invariant; but since the modular objects which are considered have this property by construction, this does not mean any gain of generality for the present context.

The first uniqueness theorem for modular symmetries is similar to the results of Keyl [56] and Araki [5]: in both papers, the notion of a symmetry is more restrictive than the one used here, whereas both authors can avoid the use of the spectrum condition. Araki assumes, in addition, that algebras of local observables which are localized in *timelike* separated regions are not contained in each other's commutants; this property is violated by the massless free field in any even space-time dimension [27, 28, 29, 30].

## 4.2 Statement and discussion of the second uniqueness theorem for modular symmetries

In the first uniqueness theorem it is assumed that the adjoint actions of $J_{W_1}$ and $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, map every local algebra $\mathfrak{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$, *onto* the algebra $\mathfrak{A}(M_{\mathcal{O}})$ associated with some open region $M_{\mathcal{O}}$ in Minkowski space. This means that, in a slightly weakened sense, the net structure has to be preserved. This is the restrictive aspect of the assumption. On the other hand, the shape of the region $M_{\mathcal{O}}$ is left completely arbitrary, the map $\mathcal{K} \ni \mathcal{O} \mapsto M_{\mathcal{O}}$ is not even assumed to be induced by a point transformation. All symmetries familiar from high energy physics are admitted a priori, and a lot of 'pathological' maps are admitted as well. In this aspect, the assumptions of the first uniqueness theorem are rather weak.

The second uniqueness theorem differs from the first one in several aspects. The first difference is that wedge duality will be assumed in the second uniqueness theorem, while it occurs as a result of the first one. In addition, it will be assumed that the localization prescription $L$ introduced in Chapter 3 for single local observables satisfies locality. As shown there in Lemma 3.5.3, this follows from standard assumptions which are slightly stronger than those sufficient to find a nonempty localization region, but which are not expected to mean a serious restriction.

The next difference to the first uniqueness theorem is that the action of a modular group on a single local observable $A$ is considered rather than its action on the whole net. It is assumed that for some given local observable $A \in \mathfrak{A}_{\mathrm{loc}}$, the operators $A_t := \Delta_{W_1}^{it} A \Delta_{W_1}^{-it}$, are also local observables for succiently small $t \in \mathbb{R}$ and that the localization region of the observable $A_t$ depends continuously on $t$ for small $t$, i.e., that for every sequence $(t_\nu)_{\nu \in \mathbb{N}}$ which converges to some sufficiently small $t_\infty$, the localization $L(A_{t_\infty})$ consists precisely of all accumulation points of sequences $(x_\nu)_{\nu \in \mathbb{N}}$ with $x_\nu \in L(A_{t_\nu})$.

### 4.2.1 Proposition (second uniqueness theorem for modular symmetries)

*Assume the net $\mathfrak{A}$ to satisfy wedge duality, and assume the localization prescription $L : \mathfrak{A}_{\mathrm{loc}} \to \mathcal{B}$ defined in Chapter 3 to satisfy locality (cf. Lemma 3.5.3). Let $A$ be a local observable, and assume that there exists an $\varepsilon > 0$ such that all $A_t$, $t \in [0, \varepsilon]$, are local observables and such that the function $[0, \varepsilon] \ni t \mapsto L(A_t)$ is continuous in the above sense.*

*Then $L(A_t) = V_1(-2\pi t) L(A)$ for every $t \in [0, \varepsilon]$.*

To illustrate the continuity assumption of this theorem, note that the curve $\mathbb{R} \ni t \mapsto A_t$ is continuous in the strong operator topology. Since the algebras $\mathfrak{A}(\mathcal{O})''$, $\mathcal{O} \in \mathcal{K}$, are von Neumann algebras, hence strongly closed, it follows that whenever one has a sequence $(t_n)_{n \in \mathbb{N}}$ which converges to some $t_\infty \in \mathbb{R}$ and which has the property that all $A_{t_n}$ are contained in a given local algebra $\mathfrak{A}(\mathcal{O})$ associated with some fixed double cone $\mathcal{O} \in \mathcal{K}$, the observable $A_{t_\infty}$ is also contained in $\mathfrak{A}(\mathcal{O})$, which implies $L(A_{t_\infty}) \subset \mathcal{O}$. In this sense, the map $t \mapsto L(A_t)$ is upper continuous. Under the action of a strongly continuous group, the localization region does not 'explode' suddenly.

On the other hand, the localization region may 'collapse', i.e., not every limit of a convergent sequence $x_n$ with $x_n \in L(A_{t_n})$ for all $n \in \mathbb{N}$ needs to be contained in $L(A_{t_\infty})$. In this sense, the map $t \mapsto L(A_t)$ is not known to be lower continuous. If it *is* lower continuous, it is, by upper continuity, actually continuous, i.e., if $(t_n)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ are convergent series with the property that $x_n \in L(A_{t_n})$ for all $n \in \mathbb{N}$, then the limit of the sequence $(x_n)_{n \in \mathbb{N}}$ is in the localization region of $A_{t_\infty}$.

It would be worth while to find sufficient criteria or even a general proof for this assumption. At present, this appears to be rather a hard task, and an idea how to do it has not been found so far.

The net structure is, in contrast to the first uniqueness theorem, not assumed to be preserved under the adjoint action of $\Delta_{W_1}^{it}$, since only a single local observable is considered. It is not assumed that the image of any local algebra under the adjoint action of the modular group contains any local algebra as a subset, let alone that it is, as in the first uniqueness theorem, an algebra of the form $\mathfrak{A}(M_\mathcal{O})$ for some open region $M_\mathcal{O} \subset \mathbb{R}^{1+s}$. Furthermore, not all local observables need to fulfill the assumptions of the theorem, and even if they do, it is not required that $\Delta_{W_1}^{it}\mathfrak{A}(\mathcal{O})\Delta_{W_1}^{-it}$ contains any algebra of the form $\mathfrak{A}(\mathcal{P})$, $\mathcal{P} \in \mathcal{K}$ (unless $t = 0$, of course). The net structure of $\mathfrak{A}$ may be lost completely under the action of the modular group.

On the other hand, the assumption that every local observable $A$ is mapped onto some other local observable under the adjoint action of the modular group prohibits $A$ to be mapped onto an observable localized in an unbounded region. But for every bounded open region $\mathcal{O}$ there are conformal transformations which map $\mathcal{O}$ onto an unbounded region; these transformations are excluded a priori. This is a restrictive assumption which was not necessary in the first uniqueness theorem, but which is obtained there as a result.

The second uniqueness theorem shows that if the modular group $\Delta_{W_1}^{it}$, $t \in \mathbb{R}$, does not boost a given observable, the localization region of the observable develops discontinuously. In so far, it plays a role complementary to the role of the first uniqueness theorem.

## 4.3 Proof of Theorem 4.1.3

In the sequel, $K$ and $\kappa$ are defined as in Theorem 4.1.3. For every spacetime region $M \subset \mathbb{R}^{1+s}$, $\mathcal{K}^M$ denotes (as before) the class of all double cones contained in $M$ (in contrast to $\mathcal{K}_M$, which denotes the double cones containing $M$), and for every algebra $\mathfrak{M} \subset \mathfrak{B}(\mathcal{H})$, $\mathcal{K}^\mathfrak{M}$ will denote the class of all double cones $\mathcal{O} \in \mathcal{K}$ with $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{M}$.

The proof will be subdivided into a couple of lemmas. The first one implies that for every $\mathcal{O} \in \mathcal{K}$, the regions $M_\mathcal{O}$ and $N_\mathcal{O}$ are bounded. It makes use of the observation that a region $M$ is bounded if and only if its difference region $M - M$ is bounded and of the fact that difference sets can be expressed in terms of translations. Since the behaviour of translations under the action of the symmetry $K$ is known by assumption, one can prove the following lemma.

### 4.3.1 Lemma

*For every double cone $\mathcal{O} \in \mathcal{K}$, one has*

$$L(K\mathfrak{A}(\mathcal{O})K^*) - L(K\mathfrak{A}(\mathcal{O})K^*) = \kappa(\mathcal{O} - \mathcal{O}).$$

**Proof:** Using the assumptions of Theorem 4.1.3, one obtains

$$\begin{aligned}
&L(K\mathfrak{A}(\mathcal{O})K^*) - L(K\mathfrak{A}(\mathcal{O})K^*) = L(\mathfrak{A}(M_{\mathcal{O}})) - L(\mathfrak{A}(M_{\mathcal{O}}))\\
&\quad = \{a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^{\mathfrak{A}(M_{\mathcal{O}})} : \mathfrak{A}(P+a) \subset \mathfrak{A}(M_{\mathcal{O}})\}\\
&\quad = \{a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^{\mathfrak{A}(M_{\mathcal{O}})} : \ KU(\kappa^{-1}a)K^*\mathfrak{A}(P)KU(-\kappa^{-1}a)K^* \subset \mathfrak{A}(M_{\mathcal{O}})\}\\
&\quad = \kappa\{a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^{\mathfrak{A}(M_{\mathcal{O}})} : U(a)\underbrace{K^*\mathfrak{A}(P)K}_{=\mathfrak{A}(N_p)}U(a) \subset \underbrace{K^*\mathfrak{A}(M_{\mathcal{O}})K}_{=\mathfrak{A}(\mathcal{O})}\}\\
&\quad \subset \kappa\{a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^{\mathfrak{A}(M_{\mathcal{O}})} : \exists Q \in \mathcal{K}^{\mathfrak{A}(N_P)} : \mathfrak{A}(Q+a) \subset \mathfrak{A}(\mathcal{O})\}.
\end{aligned}$$

Since the definitions and isotony imply

$$\mathcal{K}^{\mathfrak{A}(N_P)} = \mathcal{K}^{K^*\mathfrak{A}(P)K} \subset \mathcal{K}^{K^*\mathfrak{A}(M_{\mathcal{O}})K} = \mathcal{K}^{\mathfrak{A}(\mathcal{O})},$$

and since Corollary 3.4.2 implies $\mathcal{K}^{\mathfrak{A}(\mathcal{O})} = \mathcal{K}^{\mathcal{O}}$, one obtains

$$\begin{aligned}
L(K\mathfrak{A}(\mathcal{O})K^*) - L(K\mathfrak{A}(\mathcal{O})K^*) &\subset \kappa\{a \in \mathbb{R}^{1+s} : \exists Q \in \mathcal{K}^{\mathcal{O}} : \mathfrak{A}(Q+a) \subset \mathfrak{A}(\mathcal{O})\}\\
&= \kappa(\mathcal{O} - \mathcal{O}).
\end{aligned}$$

The last step makes use of Corollary 3.4.2. Conversely,

$$\begin{aligned}
\kappa(\mathcal{O} - \mathcal{O}) &= \kappa\{a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^{\mathcal{O}} : \mathfrak{A}(P+a) \subset \mathfrak{A}(\mathcal{O})\}\\
&= \{a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^{\mathcal{O}} : \mathfrak{A}(P+\kappa^{-1}a) \subset \mathfrak{A}(\mathcal{O})\}\\
&= \{a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^{\mathcal{O}} : \ K^*U(a)K\mathfrak{A}(P)K^*U(-a)K \subset \mathfrak{A}(\mathcal{O})\}\\
&= \{a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^{\mathcal{O}} : \mathfrak{A}(M_P + a) \subset \mathfrak{A}(M_{\mathcal{O}})\}\\
&\subset \{a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^{\mathcal{O}} : \exists Q \in \mathcal{K}^{\mathfrak{A}(M_P)} : \mathfrak{A}(Q+a) \subset \mathfrak{A}(M_{\mathcal{O}})\},
\end{aligned}$$

and since

$$\mathcal{K}^{\mathfrak{A}(M_P)} = \mathcal{K}^{K\mathfrak{A}(P)K^*} \subset \mathcal{K}^{K\mathfrak{A}(\mathcal{O})K^*} = \mathcal{K}^{\mathfrak{A}(M_{\mathcal{O}})},$$

one obtains

$$\begin{aligned}
\kappa(\mathcal{O} - \mathcal{O}) &\subset \{a \in \mathbb{R}^{1+s} : \exists Q \in \mathcal{K}^{\mathfrak{A}(M_{\mathcal{O}})} : \mathfrak{A}(Q+a) \subset \mathfrak{A}(M_{\mathcal{O}})\}\\
&= L(\mathfrak{A}(M_{\mathcal{O}})) - L(\mathfrak{A}(M_{\mathcal{O}})),
\end{aligned}$$

where the last step makes use of Corollary 3.4.2. This completes the proof. □

The next lemma proves that strict inclusions of double cones are preserved under the symmetry $K$. Again, this is boiled down to translating local algebras up and down Minkowski space and using the commutation relations between the symmetry $K$ and the translation operators. One makes use of the observation that $\overline{\mathcal{O}} \subset P$ if and only if $\mathcal{O}$ can be translated within $P$ into all directions.

### 4.3.2 Lemma

*For any two double cones $\mathcal{O}, P \in \mathcal{K}$ with $\overline{\mathcal{O}} \subset P$,*

$$\overline{L(K\mathfrak{A}(\mathcal{O})K^*)} \subset L(K\mathfrak{A}(P)K^*).$$

**Proof:** $\overline{\mathcal{O}} \subset P$ if and only if the set $\{a \in \mathbb{R}^{1+s} : \mathcal{O} + a \subset P\}$ is a neighbourhood of the origin of $\mathbb{R}^{1+s}$. After using Corollary 3.4.2, a couple of elementary transformations yield

$$\begin{aligned}
\{a \in \mathbb{R}^{1+s} : \mathcal{O} + a \subset P\} &= \{a \in \mathbb{R}^{1+s} : \mathfrak{A}(\mathcal{O} + a) \subset \mathfrak{A}(P)\} \\
&= \{a \in \mathbb{R}^{1+s} : K^*U(\kappa a)K\mathfrak{A}(\mathcal{O})K^*U(-\kappa a)K \subset \mathfrak{A}(P)\} \\
&= \{a \in \mathbb{R}^{1+s} : \mathfrak{A}(M_{\mathcal{O}} + \kappa a) \subset \mathfrak{A}(M_P)\} \\
&= \kappa^{-1}\{a \in \mathbb{R}^{1+s} : \mathfrak{A}(M_{\mathcal{O}} + a) \subset \mathfrak{A}(M_P)\} \\
&\subset \kappa^{-1}\{a \in \mathbb{R}^{1+s} : L(\mathfrak{A}(M_{\mathcal{O}})) + a \subset L(\mathfrak{A}(M_P))\},
\end{aligned}$$

and since $\kappa$ is a linear automorphism of $\mathbb{R}^{1+s}$, it follows that $\overline{\mathcal{O}}$ can be a subset of $P$ only if

$$\{a \in \mathbb{R}^{1+s} : L(\mathfrak{A}(M_{\mathcal{O}})) + a \subset L(\mathfrak{A}(M_P))\}$$

is a neighbourhood of the origin. This implies the statement. □

The next lemma proves that $K$ and $K^*$ implement a homeomorphism of $\mathbb{R}^{1+s}$ onto itself.

### 4.3.3 Lemma

*Let $x \in \mathbb{R}^{1+s}$ be arbitrary, and let $(\mathcal{O}_\nu)_{\nu\in\mathbb{N}}$ be a neighbourhood base of $x$ consisting of double cones $\mathcal{O}_\nu \in \mathcal{K}$.*

*Then* $(L(K\mathfrak{A}(\mathcal{O}_\nu)K^*))_{\nu\in\mathbb{N}}$ *is a neighbourhood base of a (naturally, unique) point* $\tilde{\kappa}(x) \in \mathbb{R}^{1+s}$*, and* $(L(K^*\mathfrak{A}(\mathcal{O}_\nu)K))_{\nu\in\mathbb{N}}$ *is a neighbourhood base of a point* $\hat{\kappa}(x) \in \mathbb{R}^{1+s}$*. The functions* $x \mapsto \tilde{\kappa}(x)$ *and* $x \mapsto \hat{\kappa}(x) = \tilde{\kappa}^{-1}(x)$ *are continuous.*

**Proof:** Without loss of generality, one may assume that $\overline{\mathcal{O}}_{\nu+1} \subset \mathcal{O}_\nu$ for all $\nu \in \mathbb{N}$. It follows from $L(\mathfrak{A}(\mathcal{O})) = \mathcal{O}$ for all $\mathcal{O} \in \mathcal{K}$ and Lemma 4.3.1 that all $L(K\mathfrak{A}(\mathcal{O}_\nu)K^*)$, $\nu \in \mathbb{N}$, are bounded sets, and it follows from Lemma 4.3.2 that

$$\overline{L(K\mathfrak{A}(\mathcal{O}_{\nu+1})K^*)} \subset L(K\mathfrak{A}(\mathcal{O}_\nu)K^*).$$

Therefore, the intersection of this family is nonempty, and Lemma 4.3.1 implies that the diameter of $L(K\mathfrak{A}(\mathcal{O}_\nu)K^*)$ tends to zero as $\nu$ tends to infinity. This implies that the intersection contains precisely one point $\tilde{\kappa}(x)$, as stated. The corresponding statements for $K^*$ are proved analogously.

This proves that $x \mapsto \tilde{\kappa}(x)$ is a bijective point transformation. Let $(x_\nu)_{\nu\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{1+s}$ which converges to a point $x_\infty$. Then there is a neighbourhood base $(\mathcal{O}_\nu)_{\nu\in\mathbb{N}}$ of $x_\infty$ with $x_\nu \in \mathcal{O}_\nu$ for all $\nu \in \mathbb{N}$. But since $\tilde{\kappa}(x_\nu) \in \tilde{\kappa}(\mathcal{O}_\nu)$ for all $\nu \in \mathbb{N}$, and since $\tilde{\kappa}(\mathcal{O}_\nu)$ is a neighbourhood base of $\tilde{\kappa}(x_\infty)$, it follows that $\tilde{\kappa}(x_\nu)$ tends to $\tilde{\kappa}(x_\infty)$ as $\nu \to \infty$. Since this line of argument applies to $\hat{\kappa} = \tilde{\kappa}^{-1}$ as well, it follows that $\kappa^{-1}$ and $\tilde{\kappa}$ are continuous, as stated. □

The next lemma determines the function $\tilde{\kappa}$ up to a constant translation.

### 4.3.4 Lemma

*For every* $x \in \mathbb{R}^{1+s}$*, one has*

$$\tilde{\kappa}(x) = \tilde{\kappa}(0) + \kappa x.$$

**Proof:** Let $(\mathcal{O}_\nu)_{\nu\in\mathbb{N}}$ be a neighbourhood base of $o$. Then $(\mathcal{O}_\nu + x)_{\nu\in\mathbb{N}}$ is a neighbourhood base of $x$, and

$$\bigcap_{\nu\in\mathbb{N}} L(K\mathfrak{A}(\mathcal{O}_\nu + x)K^*) = \bigcap_{\nu\in\mathbb{N}} \tilde{\kappa}(\mathcal{O}_\nu + x) = \{\tilde{\kappa}(x)\}.$$

On the other hand,

$$\begin{aligned}
\bigcap_{\nu\in\mathbb{N}} L(K\mathfrak{A}(\mathcal{O}_\nu + x)K^*) &= \bigcap_{\nu\in\mathbb{N}} L(U(\kappa x)K\mathfrak{A}(\mathcal{O}_\nu)K^*U(-\kappa x)) \\
&= \kappa x + \bigcap_{\nu\in\mathbb{N}} \tilde{\kappa}(\mathcal{O}_\nu) \\
&= \kappa x + \{\tilde{\kappa}(0)\}.
\end{aligned}$$

□

Theorem 4.1.3 may now be proved as follows. By Lemma 4.3.4, $\tilde{\kappa}(\mathcal{O})$ is a double cone for every $\mathcal{O} \in \mathcal{K}$. Since $K$ has been assumed to be a symmetry, one concludes that for every $\mathcal{O} \in \mathcal{K}$, one has $K\mathfrak{A}(\mathcal{O})K^* \subset \mathfrak{A}(\tilde{\kappa}(\mathcal{O}))$ and $K^*\mathfrak{A}(\mathcal{O})K \subset \mathfrak{A}(\tilde{\kappa}^{-1}(\mathcal{O}))$, hence

$$\mathfrak{A}(\tilde{\kappa}(\mathcal{O}) = KK^*\mathfrak{A}(\tilde{\kappa}(\mathcal{O}))KK^* \subset K\mathfrak{A}(\tilde{\kappa}^{-1}(\mathcal{O}))K^* = \mathcal{K}\mathfrak{A}(\mathcal{O})K^*,$$

which completes the proof of Theorem 4.1.3. □

## 4.4 Proof of the two uniqueness theorems

### 4.4.1 Proof of Proposition 4.1.4 (i) ($\mathrm{P}_1$CT-part of the first uniqueness theorem)

It follows from Theorem 4.1.3 that there is a function $\tilde{\jmath}\mathbb{R}^{1+s} \to \mathbb{R}^{1+s}$ such that

$$J\mathfrak{A}(\mathcal{O})J = \mathfrak{A}(\tilde{\jmath}(\mathcal{O})) \qquad \text{for all } \mathcal{O} \in \mathcal{K}.$$

This $\tilde{\jmath}$ satisfies the relation

$$\tilde{\jmath}(x) = \tilde{\jmath}(0) + jx =: \iota + jx \qquad \text{for all } x \in \mathbb{R}^{1+s}.$$

It remains to be shown that $\iota = 0$. Since $J$ is an involution, so is $\tilde{\jmath}$. This implies

$$x = \tilde{\jmath}(\tilde{\jmath}(x)) = \tilde{\jmath}(\iota + jx) = \iota + j\iota + x \qquad \text{for all } x \in \mathbb{R}^{1+s},$$

which gives $\iota = -j\iota$, hence $\iota_2 = \cdots = \iota_s = 0$. That $\iota_0 = \iota_1 = 0$, is equivalent to $\tilde{\jmath}(W_1) = W_1'$, but this is easy to see since $W_1' \subset \tilde{\jmath}(W_1)$ follows from locality and the Tomita-Takesaki theorem, while it is also the Tomita-Takesaki theorem which implies $\mathfrak{A}(\tilde{\jmath}(W_1))'' \subset \mathfrak{A}(W_1)'$, whence one concludes that $\tilde{\jmath}(W_1) \subset W_1'$ by using Corollary 3.4.2. This completes the proof. □

In the sequel, a well-known generalization of Asgeirsson's Lemma will be used repeatedly. It is called the **double cone theorem** and has been found by Borchers and Vladimirov [79, 12, 80, 18]. Below, it will be applied together with the **edge of the wedge theorem** due to Bogoliubov (cf., e.g., [72, 80, 18]). For the reader's convenience, both theorems are recalled here. For every $\varepsilon > 0$, $B_\varepsilon$ will denote the $\varepsilon$-ball in $\mathbb{R}^2$, and $n$ will denote some natural number.

### 4.4.2 Theorem (edge of the wedge theorem)

*Let $C$ be a nonempty, open and convex cone in $\mathbb{R}^n$. For some $\varepsilon > 0$, assume that $g_+$ is a function analytic in the tube $\mathbb{R}^n + i(C \cap B_\varepsilon)$, whereas $g_-$ is a function analytic in the tube $\mathbb{R}^n - i(C \cap B_\varepsilon)$. If there is an open region $\gamma \subset \mathbb{R}^n$ where $g_+$ and $g_-$ have a common boundary value in the sense of distributions, then $g_+$ and $g_-$ are branches of a function $g$ whose domain of analyticity contains a complex neighbourhood $\Gamma$ of $\gamma$.*

### 4.4.3 Theorem (double cone theorem)

*Within the setting and notation of Theorem 4.4.2, let $c$ be any smooth curve in $\gamma$ which has all its tangent vectors in $C$. Then the domain of analyticity of $g$ contains a complex neighbourhood of the double cone $(c + C) \cap (c - C)$.*

### 4.4.4 Proof of Proposition 4.1.4 (ii) (Lorentz part of the first uniqueness theorem)

For every $t \in \mathbb{R}$, Theorem 4.1.3 implies the existence of a unique $\xi(t) \in \mathbb{R}^{1+s}$ with

$$\Delta^{it}\mathfrak{A}(\mathcal{O})\Delta^{-it} = \mathfrak{A}(V_1(-2\pi t)\mathcal{O} + \xi(t)) \qquad \text{for all } \mathcal{O} \in \mathcal{K}.$$

By Lemma 2.1.2 it is clear that $\xi(t) + W_1 = W_1$. Since $\mathcal{O} \in \mathcal{K}$ implies $V_1(-2\pi t)\mathcal{O}+\xi(t) \in \mathcal{K}$, and since $(\Delta^{it})_{t\in\mathbb{R}}$ is a one-parameter group, it follows that $(\xi(t))_{t\in\mathbb{R}}$ is a one-parameter group of translations with $\xi(0) = 0$. This implies that there is a $\xi \in \mathbb{R}^{1+s}$ such that $\xi(t) = t\xi$ for all $t \in \mathbb{R}$.

It remains to be shown that $\xi = 0$. To this end, choose any $x \in W_1$, and consider the curve $c(t) := V_1(-2\pi t)x + t\xi$, $t \in \mathbb{R}$, for some vector $\xi$ pointing in the 2-direction. $c$ is a causal curve if and only if $\xi = 0$, if $\xi \neq 0$, it becomes spacelike for large positive or negative $t$. This observation and the proof that motions along spacelike curves cannot be implemented by the modular group under consideration are due to Trebels [78]. The following argument differs from his formulation, but the crucial idea to apply the double cone theorem is due to him.

If $\xi \neq 0$, there is, for every $\varepsilon > 0$, a double cone $\mathcal{O} \subset W_1$ with the property that $V_1(-2\pi\varepsilon)\mathcal{O} + \varepsilon\xi$ is spacelike with respect to $\mathcal{O}$. It follows that there are an $a \in \mathbb{R}^{1+s}$ and a $\delta > 0$ such that

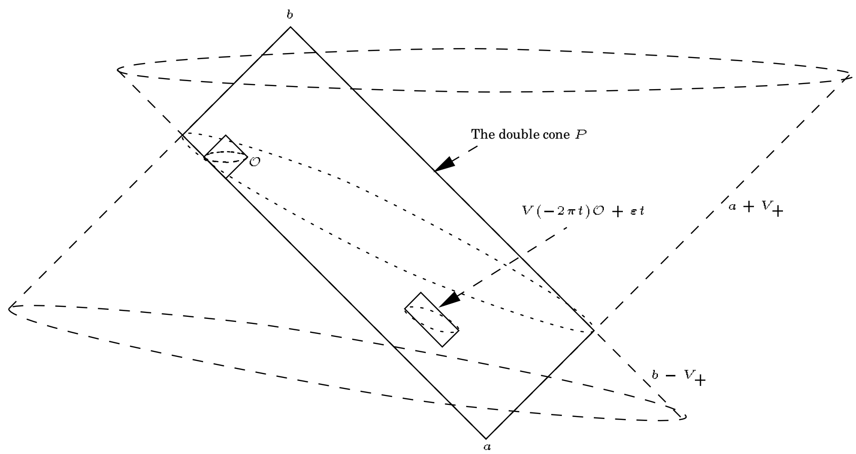


**Figure 4.1: First uniqueness theorem: the double cone** $P$

(i) $V_1(-2\pi t)\mathcal{O} + t\xi - \delta t e_0 \subset a + V_+$ for all $t \in [0, \varepsilon]$;

(ii) $\overline{\mathcal{O}} \not\subset a + V_+$.

On the other hand, there is a $b \in \mathbb{R}^{1+s}$ such that

(iii) $V_1(-2\pi t)\mathcal{O} + t\xi \subset b - V_+$ for all $t \in [0, \varepsilon]$.

See Figure 4.4.4. Now denote $P := (a + V_+) \cap (b - V_+)$, choose $A \in \mathfrak{A}(\mathcal{O})$ and $B \in \mathfrak{A}(P')$, denote by $e_0$ the unit vector in the time direction, and consider the function $g_{A,B}$ defined by

$$\begin{aligned}
\mathbb{R}^2 \ni (t,s) \mapsto g_{A,B}(t,s) &:= \left\langle \Omega, [B, U(se_0)\Delta^{it} A \Delta^{-it} U(-se_0)]\Omega \right\rangle \\
&= \left\langle \Omega, BU(se_0)\Delta^{it} A\Omega \right\rangle - \left\langle \Omega, A\Delta^{-it} U(-se_0) B\Omega \right\rangle \\
&= \left\langle \Omega, BU(se_0)\Delta^{it} A\Omega \right\rangle - \overline{\langle \Omega, B^* U(se_0)\Delta^{it} A^*\Omega \rangle} \\
&=: g_+(t,s) - g_-(t,s).
\end{aligned}$$

By conditions (i) and (iii), the function $g_{A,B}$ vanishes in the closure of the open triangle $\gamma$ with corners $(0,0)$, $(\varepsilon, 0)$ and $(\varepsilon, -\delta\varepsilon)$. Clearly, $\gamma$ contains a smooth curve which joins $(0,0)$ to $(\varepsilon, -\delta\varepsilon)$ and which has tangent vectors in the cone $C := \{(t,s) \in \mathbb{R}^2 : \ t > 0,\ s < 0\}$. It will be shown that by the double cone theorem, $g_{A,B}$ vanishes in the whole open rectangle $]0, \varepsilon[ \times ] - \delta\varepsilon, 0[$. Since $g_{A,B}$ is continuous, it follows that it even vanishes in the closed rectangle $[0, \varepsilon] \times [-\delta\varepsilon, 0]$. Since $B \in \mathfrak{A}(P')$ and $A \in \mathfrak{A}(\mathcal{O})$ are arbitrary, Lemma 3.3.1 implies that $\mathfrak{A}(\mathcal{O} - \delta\varepsilon e_0) \subset \mathfrak{A}(P')'$. But since by condition (ii), the double cone $\mathcal{O} - \delta\varepsilon e_0$ cannot be contained in $P$ no matter how small $\delta\varepsilon$ is, this is in conflict with Corollary 3.4.2, so it follows that $\xi = 0$, which implies the statement.

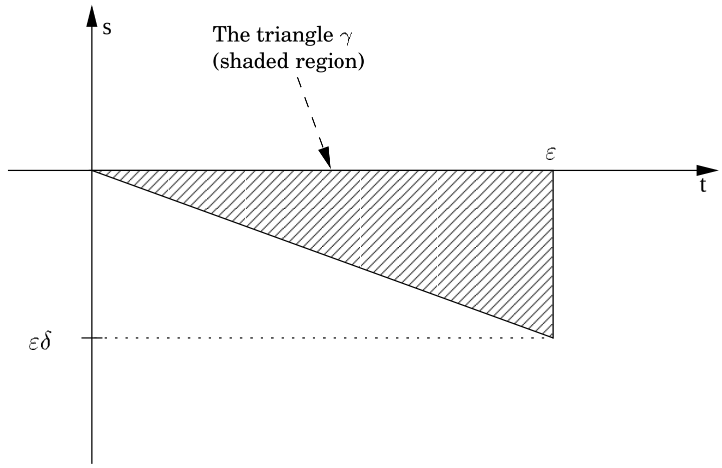


**Figure 4.2: First uniqueness theorem: the triangle $\gamma$**

It remains to be shown that the function $g_{A,B}$ fulfills the assumptions of the double cone theorem. Using elementary arguments from spectral theory it can be shown that given any $\rho > 0$, any vector $\phi$ in the domain of $\Delta^\rho$ and any $\psi \in \mathcal{H}$, the function $\mathbb{R} \ni t \mapsto \langle \psi, \Delta^{it}\phi\rangle$ has an extension to a function which is continuous on the strip $\{t \in \mathbb{C} : -\rho \leq \operatorname{Im} t \leq 0\}$ and analytic on the interior of this strip (cf. [62], Lemma 8.1.10 (p. 351)).

Since the vectors $A\Omega$ and $A^*\Omega$ are in the domain of $\Delta^{\frac{1}{2}}$, it follows that for every $\psi \in \mathcal{H}$, the functions $\mathbb{R} \ni t \mapsto \langle \psi, \Delta^{it} A\Omega\rangle$ and $\mathbb{R} \ni t \mapsto \overline{\langle \psi, \Delta^{it} A^*\Omega\rangle}$ have extensions which are continuous in the strips $\{t \in \mathbb{C} : -\frac{1}{2} \leq \operatorname{Im} t \leq 0\}$ and $\{t \in \mathbb{C} : 0 \leq \operatorname{Im} \leq \frac{1}{2}\}$, respectively, and which are analytic in the interior of these strips.

On the other hand, it follows from the spectrum condition that for any two vectors $\phi, \psi \in \mathcal{H}$, the functions $\mathbb{R} \ni s \mapsto \langle \psi, U(se_0)\phi\rangle$ and $\mathbb{R} \ni s \mapsto \overline{\langle \psi, U(se_0)\phi\rangle}$ have extensions which are continuous in the (complex) closed upper and lower half plane, respectively, and analytic in the interior of these half planes.

This proves that the function $g_+$ has a continuous extension to the tube $\mathbf{T}_+ := \{(t,s) \in \mathbb{C}^2 : -1/2 \leq \operatorname{Im} t \leq 0, \operatorname{Im} s \geq 0\}$ and that at each interior point of this strip, this extension is analytic separately in $t$ and in $s$. Using Hartogs' fundamental theorem which states that a function of several complex variables is holomorphic if and only if it is holomorphic separately in each of these variables [53, 80], it follows that $g_+$ is, as a function in two complex variables, analytic in the interior of $\mathbf{T}_+$. It follows in the same way that $g_-$ has the corresponding properties for the tube $-\mathbf{T}_+ =: \mathbf{T}_-$. The tubes $\mathrm{T}_+$ and $\mathrm{T}_-$ contain the smaller tubes $\mathbb{R}^2 - i\overline{C \cap B_{\frac{1}{2}}}$ and $\mathbb{R}^2 + i\overline{C \cap B_{\frac{1}{2}}}$.

Since $g_+$ and $g_-$ coincide as continuous functions in the closure of $\gamma$, they coincide as distributions in the open region $\gamma$, and it follows from the edge of the wedge theorem that they are branches of a function $g$ which is analytic in a complex neighbourhood $\Gamma$ of $\gamma$. But since $\gamma$ contains a smooth curve joining the points $(0,0)$ and $(\varepsilon, -\delta\varepsilon)$ with tangent vectors in $C$, it follows from the double cone theorem that the function $g$ is analytic in the region

$$((0,0) + C) \cap ((\varepsilon, -\delta\varepsilon) - C) = ]0, \varepsilon[ \times ] - \delta\varepsilon, 0[.$$

This implies that $g_{A,B}$ vanishes in this region, which is all that remained to be shown, so the proof is complete. □

### 4.4.5 Proof of Proposition 4.1.5 ('uniqueness theorem 1a')

For every $a \in \mathbb{R}^{1+s}$, Theorem 4.1.1 implies the commutation relations

$$\begin{aligned} J_+ U(a) J_+ &= U(-a); \\ \Delta_+^{it} U(a) \Delta_+^{-it} &= U(e^{-2\pi t} a) \qquad \text{for all } t \in \mathbb{R}. \end{aligned}$$

If, respectively, $J_+$ or $\Delta_+^{it}$ is a symmetry, Theorem 4.1.3 implies that it can differ from the stated symmetry at most by a translation. Since $V_+$ is Lorentz-invariant, $J_+$ and $\Delta_+^{it}$, $t \in \mathbb{R}$, commute with all $U(g)$, $g \in L_+^\uparrow$. However, there are no nontrivial translations which commute with all $g \in L_+^\uparrow$; this proves Proposition 4.1.5. □

### 4.4.6 Proof of Proposition 4.2.1 (second uniqueness theorem)

From the Tomita-Takesaki Theorem it follows that the modular group under consideration leaves the algebras $\mathfrak{A}(W_1)''$ and $\mathfrak{A}(W_1)'$ invariant. By wedge duality, it also leaves the algebra $\mathfrak{A}(W_1')''$ invariant. Borchers' commutation relations now imply that the algebras associated with the images of $W_1$ under arbitrary translations transform under the adjoint action of $\Delta_{W_1}^{it}$ as under the Lorentz boost $V_1(-2\pi t)$. It follows that for every $A \in \mathfrak{A}_{\text{loc}}$, one already has

$$(L(A_t) + W_1) \cap (L(A_t) + W_1') \subset V_1(-2\pi t)\left((L(A) + W_1) \cap (L(A) + W_1')\right),$$

where $A_t := \Delta_{W_1}^{it} A \Delta_{W_1}^{-it}$, as above. Therefore it is sufficient to control the extension of $L(A_t)$ in the 2- and the 3-direction. Furthermore, Borchers' commutation relations imply that it is sufficient to prove the theorem for the case that $L(A) \subset W_1$, since every localization region may be shifted into $W_1$ by some translation.

For some local observable $A$ localized in $W_1$, we discuss the negative 2-direction; the other directions behave the same way. To this end, define the continuous function

$$\mathbb{R} \ni t \mapsto \Sigma(t) := \min\{x_2 : x \in L(A_t))\},$$

and choose coordinates such that $\Sigma(0) = 0$. It will be shown that for any $\varepsilon > 0$, the assumption $\Sigma(\varepsilon) > 0$ leads to a contradiction; if $\Sigma(\varepsilon) < 0$, one may turn the spacetime upside down (along the time axis) in order to obtain an analogous argument.

If $\Sigma(\varepsilon) > 0$, there is an $x \in W_1$ such that

$$(V_1(-2\pi\varepsilon)x - x)^2 - \Sigma(\varepsilon)^2 < 0,$$

and since $L(A_t)$ has been assumed to depend continuously on $t$, it follows that there are an $a \in \mathbb{R}^{1+s}$, a $\sigma^\sharp \geq 0$ and a $\delta > 0$ such that for $A^x := U(x)AU(-x)$, one has

(i) $L(A^x_t) + (\sigma^\sharp - \delta t)e_0 \subset a + \overline{V_+}$ for all $t \in [0, \varepsilon]$.

(ii) $L(A^x) \not\subset a + V_+$.

On the other hand, there exists a $b \in \mathbb{R}^{1+s}$ such that

(iii) $L(A^x_t) + \sigma^\sharp e_0 \subset b - \overline{V}_+$ for all $t \in [0, \varepsilon]$,

since $L(A^x_t) = L(A_t) + x$ depends continuously on $t$.

Now define $P := (a + V_+) \cap (b - V_+)$, and for any $B \in \mathfrak{A}(P')$, consider – as in the proof of Proposition 4.1.4 – the function $g_{A^x,B}$ defined by

$$\mathbb{R}^2 \ni (t, s) \mapsto g_{A^x,B}(t, s) := \langle\Omega, BU(se_0)\Delta^{it}_{W_1} A^x\Omega\rangle - \langle\Omega, A^x\Delta^{-it}_{W_1} U(-se_0)B\Omega\rangle.$$

Since $L(A^x_t) = L(A_t) + x$ depends continuously on $t$, locality implies that there is a region with continuous boundaries in which this function vanishes for all choices of $B$. By Conditions (i) and (iii), this region contains the triangle $\gamma$ with the corners $(0, \sigma^\sharp)$, $(\varepsilon, \sigma^\sharp)$ and $(\varepsilon, \sigma^\sharp - \delta\varepsilon)$ (Figure 4.4.6). As in the proof of Proposition 4.1.4, it can be shown that $g_{A^x,B}$ also vanishes in the closed rectangle $[0, \varepsilon] \times [\sigma^\sharp - \delta\varepsilon, \sigma^\sharp]$.

Since $B \in \mathfrak{A}(P')$ is arbitrary, this implies, by Lemma 3.3.1, that $L(A^x) + (\sigma^\sharp - \delta\varepsilon)e_0 \subset P$. This is in conflict with Corollary 3.4.5 if $\sigma^\sharp < \delta\varepsilon$. But this does not need to be the case.

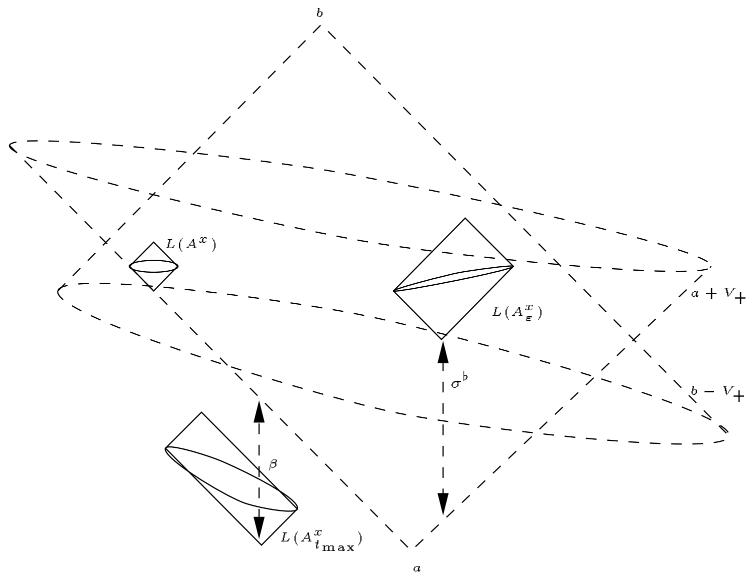


**Figure 4.3: Second uniqueness theorem: construction of $P$**

*The three localization regions have been depicted as double cones (which they do not need to be in general). $t_{\max}$ is the parameter which leads to the double cone which is the 'lowest' with respect to the lower boundary of the light cone $a + V_+$. Such a $t_{\max}$ does exist since $L(A^x_t)$ depends continuously on $t$.*

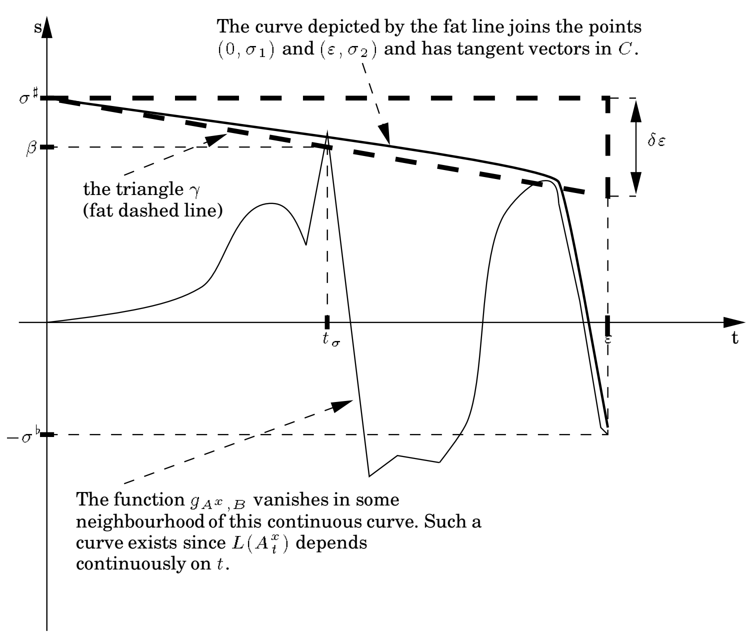


**Figure 4.4: Second uniqueness theorem: where $g_{A^x}$ vanishes**

To conclude the proof for the case that $\sigma^\sharp \geq \delta\varepsilon$, note that it follows from property (iii) of the double cone $P$ that there is a $\sigma^\flat > 0$ such that $g_{A^x,B}(\varepsilon, -\sigma^\flat) = 0$, and since $L(A_t^x) = L(A_t) + x$ depends continuously on $t$, one concludes that $\sigma^\flat$ can be chosen such that some point in the triangle $\gamma$ can be joined to the point $(\varepsilon, -\sigma^\flat)$ by a smooth curve with tangent vectors in $C$ (i.e., which is the graph of a monotonously decreasing function) around which the function $g_{A^x,B}$ vanishes (cf. Figure 4.4.6). This implies that a smooth curve with tangent vectors in $C$ can be found which joins the points $(0, \sigma^\sharp)$ and $(\varepsilon, -\sigma^\flat)$ and around which the function $g_{A^x,B}$ vanishes. But now the double cone theorem implies that $g_{A^x,B}$ vanishes in the open rectangle $]0, \varepsilon[ \times ] -\sigma^\flat, \sigma^\sharp[$, continuity of $g_{A^x,B}$ implies that this function also vanishes in the closure of this rectangle, and this, as above, leads to a contradiction with Corollary 3.4.2 and completes the proof. $\square$

# Chapter 5

# Conclusion and outlook

As already mentioned in Chapter 2, the counterexample due to Streater [73] makes that every proof of the spin-statistics theorem in 1+3 dimensions must be based on assumptions which rule this example out. Such assumptions have been made in Chapter 2 (compactness of the group of internal symmetries) as well as in an older proof of the spin-statistics theorem due to Buchholz and Epstein [33] (at most finite degeneracy of particles), and it is an open question how the different assumptions are related.

On the other hand, the Bisognano-Wichmann theorem is a theorem about *finite-component* Wightman fields. This resctriction excludes Streater's example, and it clearly implies the Buchholz-Epstein assumption that every particle is at most finite degenerate. An open question is whether the Buchholz-Epstein assumption could be sufficient to prove the Bisognano-Wichmann theorem in the algebraic setting.

The above uniqueness theorems on modular symmetries are not the only attempt to find sufficient conditions for the modular symmetry properties investigated in this thesis. Recently, Borchers published a simple analytic continuation trick which derives the Bisognano-Wichmann modular symmetries from a so-called 'reality condition', which is of a rather technical nature and, therefore, not recalled here [18]. By now, a derivation of this condition from physical principles is not known.

The proof of the spin-statistics theorem in 1+2 dimensions due to Guido and Longo appears not to rely on any assumptions which rule out Streater's example. This does not seem to be a problem, since Streater's example does not work in 1+2 dimensions (it relies on the fact that in 1+3 dimensions, a spin-one-half representation of the Poincaré group can be turned into its adjoint representation by means of a unitary intertwiner, which is impossible in 1+2 dimensions). It is, however, an indication that the spin-statistics theorem in 1+2 dimensions might be a more universal property of relativistic quantum fields than it is in 1+3 dimensions, and it may be that the assumption of modular $P_1$CT-symmetry can, in principle, be avoided.

Chapter 3 was intended to give the preliminaries for the analysis of modular symmetries. Conversely, Thomas and Wichmann have also investigated the implications of modular Lorentz symmetry for the localization behaviour of a local observable. Assuming modular Lorentz symmetry, strong additivity for wedges and an intersection property[1], they found that the localization region of an observable $A$ with respect to a minimal Poincaré covariant local net generated by $A$ is the smallest region $\mathcal{O}_A$ in $\mathcal{B}$ with the property that for any $(a, \Lambda) \in \mathcal{P}_+^\uparrow$, one has $(a, \Lambda)\mathcal{O}_A \subset \mathcal{O}_A'$ if and only if $[A, U(a, \Lambda)AU(a, \lambda)^*] = 0$, which is unique up to a translation [77]. This definition of a localization region does no longer refer to any other observables of the net, it shows that a localization region of a local observable can be defined such that it is a property of the observable itself without referring to any other observables. This is an interesting consequence of modular Lorentz symmetry, which, in a way, looks 'dual' with respect to the above results.

[1]This assumption makes the construction of a nonempty localization region straightforward, but it is not a standard assumption and has been avoided above

## Acknowledgements

This work was partly supported by the Deutsche Forschungsgemeinschaft. Essential progress was made during a stay at the Erwin Schrödinger Institute in Vienna, where I had the occasion to join the project 'Local Quantum Physics' in September/October 1997. I would like to thank the organizers of this project for their kind invitation, the Erwin Schrödinger Institute for warm hospitality, and the other participants, namely H.-J. Borchers, D. Guido, S. Trebels and E. Wichmann, for their inspiring and encouraging cooperation and interest.

During a visit to Göttingen, I obtained fruitful hints M. Lutz and M. Requardt.

I am indebted to W. Kunhardt who read several preliminary versions extremely carefully and critically and gave important hints at errors and possible improvements. I would also like to thank C. Adler, D. Arlt, K. Bresser, K. Fredenhagen, G. Pinter and D. Prange, who read and improved the manuscript, convinced me that illustrations are worth while and feasible, and helped me to convince the computer of this.

Finally I would like to thank K. Fredenhagen for his patient and encouraging interest and cooperation.

# Bibliography

[1] Alexandrov, A. D.: On Lorentz transformations, Uspekhi Mat. Nauk. **5**, No. 3 (37), p. 187 (1950)

[2] Alexandrov, A. D.: Mappings of Spaces with Families of Cones and Space-Time Transformations, *Annali di matematica* **103**, 229-257 (1975)

[3] Alexandrov, A. D., Ovchinnikova, V. V.: Notes on the foundations of relativity theory, *Vestnik Leningrad Univ.* **14**, p. 95 (1953)

[4] Araki, H.: A Generalization of Borchers' Theorem, *Helv. Phys. Acta* **36**, 132 (1963)

[5] Araki, H.: Symmetries in a Theory of Local Observables and the Choice of the Net of Local Algebras, *Rev. Math. Phys.*, Special Issue, 1-14 (1992)

[6] Bannier, U.: Intrinsic Algebraic Characterization of Space-Time Structure, *Int. J. Theor. Phys.* **33**, 1797-1809 (1994)

[7] Baumgärtel, H., Wollenberg, M.: Causal Nets of Operator Algebras, Akademie-Verlag Berlin, 1992

[8] Bisognano, J. J., Wichmann, E. H.: On the Duality Condition for a Hermitean Scalar Field, *J. Math. Phys.* **16**, 985-1007 (1975)

[9] Bisognano, J. J., Wichmann, E. H.: On the Duality Condition for Quantum Fields, *J. Math. Phys.* **17**, 303 (1976)

[10] Bogoliubov, N. N., Logunov, A. A., Oksak, A. I., Todorov, I. T., General Principles of Quantum Field Theory, Kluwer, Dordrecht 1990

[11] Borchers, H.-J.: Über die Mannigfaltigkeit der interpolierenden Felder zu einer kausalen S-Matrix, *Nuovo Cimento* **15**, 784 (1960)

[12] Borchers, H.-J.: Über die Vollständigkeit lorentzinvarianter Felder in einer zeitartigen Röhre, *Nuovo Cimento* **19**, 787-796 (1961)

[13] Borchers, H.-J.: On the Vacuum State in Quantum Field Theory, II, *Commun. Math. Phys.* **1**, 57 (1965)

[14] Borchers, H.-J.: A Remark on a Theorem of B. Misra, *Commun. Math. Phys.* **4**, 315-323 (1967)

[15] Borchers, H.-J.: On the Converse of the Reeh-Schlieder Theorem, *Commun. Math. Phys.* **10**, 269-273 (1968)

[16] Borchers, H.-J.: The CPT-Theorem in Two-Dimensional Theories of Local Observables, *Commun. Math. Phys.* **143**, 315-332 (1992)

[17] Borchers, H.-J.: Half-sided Modular Inclusion and the Construction of the Poincaré Group, *Commun. Math. Phys.* **179**, 703-723 (1996)

[18] Borchers, H.-J.: Translation Group and Particle Representations in Quantum Field Theory, Springer, Berlin, Heidelberg 1996

[19] Borchers, H.-J.: On Poincaré transformations and the modular group of the algebra associated with a wedge, preprint, 1997, and to be published

[20] Borchers, H.-J., Hegerfeldt, G. C.: The Structure of Space-Time Transformations, *Commun. Math. Phys.* **28**, 259-266 (1972)

[21] Borchers, H.-J., Yngvason, J.: Transitivity of locality and duality in quantum field theory. Some modular aspects, *Rev. Math. Phys.* **6**, 597-619 (1994)

[22] Bratteli, O., Robinson, D.: Operator Algebras and Quantum Statistical Mechanics I, Springer 1979 (Berlin, Heidelberg, New York)

[23] Bredon, G. E.: Topology and Geometry, Springer 1993 (Berlin, Heidelberg, New York)

[24] Brunetti, R., Guido, D., Longo, R.: Modular Structure and Duality in Conformal Quantum Field Theory, *Commun. Math. Phys.* **156**, 201-219 (1993)

[25] Brunetti, R., Guido, D., Longo, R.: Group Cohomology, Modular Theory and Space-Time Symmetries, *Rev. Math. Phys.* **7**, 57-71 (1995)

[26] Bros, J., Messiah, A., Stora, R.: A Problem of Analytic Completion Related to the Jost-Lehmann-Dyson Formula, *J. Math. Phys.* **2**, 639-651 (1961)

[27] Buchholz, D.: Collision Theory for Massless Fermions, *Commun. Math. Phys.* **42**, 269-279 (1975)

[28] Buchholz, D.: Collision Theory for Waves in Two Dimensions and a Characterization of Models with Trivial S-Matrix, *Commun. Math. Phys.* **45**, 1-8 (1975)

[29] Buchholz, D.: Collision Theory of Massless Bosons, *Commun. Math. Phys.* **52**, 147-173 (1977)

[30] Buchholz, D.: On the Structure of Local Quantum Fields with Non-Trivial Interaction, in: Proceedings of the International Conference on Operator Algebras, Ideals and Their Applications in Theoretical Physics, Leipzig 1977, Teubner 1978 (Stuttgart)

[31] Buchholz, D., Doplicher, S., Longo, R., Roberts, J.: A New Look at Goldstone's Theorem, Rev. Math. Phys. (Special Issue, 1992), 47-82

[32] Buchholz, D., Dreyer, O., Florig, M., Summers, S. J.: Geometric Modular Action and spacetime Symmetry Groups, preprint 1998

[33] Buchholz, D., Epstein, H.: Spin and Statistics of Quantum Topological Charges, *Fizika* **17** (3), 329-343 (1985)

[34] Buchholz, D., Fredenhagen, K.: Dilations and interaction, *J. Math. Phys.* **18**, 1107-1111 (1977)

[35] Buchholz, D., Fredenhagen, K.: Locality and the Structure of Particle States, *Commun. Math. Phys.* **84** 1-54 (1982)

[36] Buchholz, D., Wichmann, E. H.: Causal Independence and the Energy-Level Density of States in Local Quantum Field Theory, *Commun. Math. Phys.* **106**, 321-344 (1986)

[37] Doplicher, S., Haag, R., Roberts, J. E.: Fields, Observables and Gauge Transformations, II *Commun. Math. Phys.* **12**, 1-23 (1969)

[38] Doplicher, S., Haag, R., Roberts, J. E.: Local Observables and Particle Statistics I, *Commun. Math. Phys.* **23**, 199-230 (1971)

[39] Doplicher, S., Haag, R., Roberts, J. E.: Local Observables and Particle Statistics II, *Commun. Math. Phys.* **35**, 49-85, (1974)

[40] Doplicher, S. Longo, R.: Standard and split inclusions of von Neumann algebras, *Invent. math.* **75**, 493-536 (1984)

[41] Doplicher, S., Roberts, J. E.: Why There is a Field Algebra with a Compact Gauge Group Describing the Superselection Structure in Particle Physics, *Commun. Math. Phys.* **131**, 51-107 (1990)

[42] Dyson, F. J.: Integral Representations of Causal Commutators, *Phys. Rev* **110**, 1460 (1958)

[43] Fierz, M.: Über die relativistische Theorie kräftefreier Teilchen mit beliebigem Spin, *Helv. Phys. Acta* **12**, 3 (1939)

[44] Fredenhagen, K., Jörß, M.: Conformal Haag-Kastler Nets, Pointlike Localized Fields and the Existence of Operator Product Expansions, *Commun. Math. Phys.* **176**, 541-554 (1996)

[45] Fredenhagen, K., Rehren, K.-H., Schroer, B.: Superselection Sectors with Braid Group Statistics and Exchange Algebras I, *Commun. Math. Phys.* **125**, 201 (1989)

[46] Fröhlich, J., Marchetti, P. A.: Spin-Statistics Theorem and Scattering in Planar Quantum Field Theories with Braid Statistics, *Nucl. Phys. B* **356**, 533-573 (1991)

[47] Gabbiani, F., Fröhlich, J.: Operator Algebras and Conformal Quantum Field Theory, *Commun. Math. Phys.* **155**, 569-640 (1993)

[48] Guido, D., Longo, R.: Relativistic Invariance and Charge Conjugation, *Commun. Math. Phys.* **148**, 521-551 (1992)

[49] Guido, D., Longo, R.: An Algebraic Spin and Statistics Theorem, *Commun. Math. Phys.* **172**, 517-534 (1995)

[50] Guido, D., Longo, R.: The Conformal Spin and Statistics Theorem, *Commun. Math. Phys.* **181**, 11-36 (1996)

[51] Haag, R.: Local Quantum Physics, Springer 1992 (Berlin, Heidelberg, New York)

[52] Haag, R., Kastler, D.: An Algebraic Approach to Quantum Field Theory, *J. Math. Phys.* **5**, 848-861 (1964)

[53] Hartogs, F.: Zur Theorie der Funktionen mehrerer komplexer Veränderlicher, insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten, *Math. Ann.* **62**, 1-88 (1906)

[54] Jost, R.: The General Theory of Quantized Fields, Am. Math. Soc., Providence, Rhode Island, 1965

[55] Jost, R., Lehmann, H.: Integral Representation of Causal Commutators, *Nuovo Cimento* **5**, 1598-1608 (1957)

[56] Keyl, M.: Remarks on the relation between causality and quantum fields, *Class. Quantum Grav.* **10**, 2353-2362 (1993)

[57] Kuckert, B.: PCT- und Poincarésymmetrie als modulare Strukturen; ein algebraisches Spin-Statistik-Theorem, diploma thesis, Hamburg 1994

[58] Kuckert, B.: A New Approach to Spin & Statistics, *Lett. Math. Phys.* **35**, 319-335 (1995)

[59] Kuckert, B.: Borchers' Commutation Relations and Modular Symmetries in Quantum Field Theory, *Lett. Math. Phys.* **41**, 307-320 (1997)

[60] Landau, L. J.: A Note on Extended Locality, *Commun. Math. Phys.* **13**, 246-253 (1969)

[61] Landau, L. J.: On Local Functions of Fields, *Commun. Math.Phys.* **39**, 49-62 (1974)

[62] Li Bing-Ren: Introduction to Operator Algebras, World Scientific, Singapore 1992

[63] Longo, R.: On the spin-statistics relation for topological charges, in: Doplicher, S., Longo, R., Roberts, J. E., Zsido, L. (eds.): Operator Algebras and Quantum Field Theory, proceedings of the conference at the Accedemia Nazionale dei Lincei, Rome 1996 (International Press)

[64] Mack, G., Salam, A.: Finite-Component Field Representations of the Conformal Group, *Ann. Phys.* **53**, 174-202 (1969)

[65] Manoharan, A. C.: Some Considerations on Configuration Space Analyticity in General Quantum Field Theory, in: K. T. Mahanthappa, W. E. Brittin (eds.): Mathematical Methods in Theoretical Physics, Gordon and Breach 1969 (New York) (Conference held in Boulder 1968)

[66] Pauli, W.: On the Connection of Spin and Statistics, *Phys. Rev.* **58** , 716-722 (1940)

[67] Reed, M. Simon, B.: Functional Analysis, Academic Press, New York 1972

[68] Reeh, H., Schlieder, S.: Bemerkungen zur Unitäräquivalenz von lorentzinvarianten Feldern, *Nuovo Cimento* **22**, 1051 (1961)

[69] Roberts, J. E.: Spontaneously Broken Gauge Symmetries and Superselection Rules, Proceedings of the International School of Mathematical Physics at the University of Camerino, 1974

[70] Schomerus, V.: Construction of Field Algebras with Quantum Symmetry from Local Observables, *Commun. Math. Phys.* **169**, 193-236 (1995)

[71] Stone, M. (ed.): Quantum Hall Effect, World Scientific 1992 (Singapore)

[72] Streater, R. F., Wightman, A. S.: PCT, Spin & Statistics, and All That, Benjamin 1964 (New York)

[73] Streater, R. F.: Local Fields with the Wrong Connection Between Spin and Statistics, *Commun. Math. Phys.* **5**, 88-98 (1967)

[74] Takesaki, M.: Tomita's Theory of Modular Hilbert Algebras and Its Applications, Lecture Notes in Mathematics **128**, Springer 1970 (New York)

[75] Takesaki, M.: Theory of Operator Algebras I, Springer 1979 (New York)

[76] Thomas, L. J., Wichmann, E. H.: On the causal Structure of Minkowski Spacetime, *J. Math. Phys.* **38**, 5044-5086

[77] Thomas, L. J., Wichmann, E. H.: Standard forms of local nets in quantum field theory, *J. Math. Phys.* **39**, 2643-2681 (1998)

[78] Trebels, S.: PhD-thesis, Göttingen 1997, to be published

[79] Vladimirov, V. S.: The construction of envelopes of holomorphy for domains of a special type (in Russian), *Doklady Akad. Nauk SSSR* **134**, 251-254 (1960)

[80] Vladimirov, V. S.: Methods of the Theory of Functions of Many Complex Variables, M. I. T. Press, Cambridge (MA), 1966

[81] Wiesbrock, H.-W.: Conformal Quantum Field Theory and Half-sided Modular Inclusions, *Commun. Math. Phys.* **158**, 537-544 (1993)

[82] Wiesbrock, H.-W.: Modular inclusions and intersections of algebras in QFT, in: Doplicher, S., Longo, R., Roberts, J. E., Zsido, L.: Operator algebras and Quantum Field Theory, proceedings of the conference at the Accademia Nazionale dei Lincei, Rome 1996 (International Press)

[83] Wiesbrock, H.-W.: Modular intersections of von Neumann algebras in quantum field theory, FU Berlin, *Commun. Math. Phys.* **193**, 263-285 (1998)

[84] Wilczek, F.: Quantum mechanics of fractional spin particles, *Phys. Rev. Lett.* **49**, 957 (1982)

[85] Yngvason, J.: A Note on Essential Duality, *Lett. Math. Phys.* **31**, 127-141 (1994)

[86] Zeeman, E. C.: Causality Implies the Lorentz Group, *J. Math. Phys.* **5**, 490-493 (1964)