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Classification of Infinite–Dimensional Simple Linearly Compact Lie Superalgebras

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Victor G. Kac*

Dedicated to the memory of my friend
Boris Weisfeiler
a remarkable man and mathematician.

Introduction

The present paper was motivated by the problem of classification of operator product expansions (OPE) in conformal field theory. This problem was solved in [DK] in the case when the chiral algebra is generated by finitely many bosonic fields such that in their OPE only linear combinations of these fields and their derivatives occur. An axiomatic description of such a system of fields is called a finite conformal algebra [K6]. The classification of finite conformal algebras uses in an essential way Cartan's classification of pseudogroups of transformations of a finite-dimensional manifold, which, in the modern language, is equivalent to the classification, up to formal equivalence, of Lie algebras of vector fields on a finite-dimensional manifold. The problem of classification of OPE when fermionic fields are allowed as well, or, equivalently, of finite conformal superalgebras, requires an extension of Cartan's theory to the case of supermanifolds. Below I explain the problem in more detail.

Elie Cartan published a solution to the problem (posed by Sophus Lie) of classification of simple infinite-dimensional Lie algebras of vector fields on a finite-dimensional manifold in 1909 [C]. This work had been virtually

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forgotten until the sixties. A resurgence of interest in this area began with the work of Singer and Sternberg [SS] and of Guillemin and Sternberg [GS], which developed an adequate language and machinery of filtered and graded Lie algebras.

The basic problem of the theory is to classify, up to formal equivalence, infinite-dimensional Lie algebras of vector fields acting transitively in a neighborhood of a point x of a complex manifold X. Let L be such a Lie algebra and let L_k ($k \in \mathbb{Z}_+$) denote the subalgebra of L consisting of vector fields that vanish at x up to k-th order. This defines a filtration of L by subspaces of finite codimension, which is *transitive* in the sense that $\dim L/L_0 = \dim X$, or, equivalently, that L_0 contains no non-zero ideals of L. One defines a topology on L by taking $\{L_k\}_{k\in\mathbb{Z}_+}$ to be a fundamental system of neighborhoods of 0. Let $\overline{L}$ be the completion of L in this topology. Two transitive Lie algebras, L and L', of vector fields are called *formally equivalent* if their completions $\overline{L}$ and $\overline{L}'$ are isomorphic topological Lie algebras.

One thus arrives at a problem of classification, up to a continuous isomorphism, of infinite-dimensional *linearly compact* Lie algebras $\overline{L}$, i.e., complete topological Lie algebras that admit a fundamental system of neighborhoods of zero consisting of subspaces of finite codimension, which possess a *fundamental subalgebra* $\overline{L}_0$, i.e., an open subalgebra (of finite codimension) that has no non-zero ideals of $\overline{L}$, cf. [G1].

In [C], Cartan purports to give a classification of simple infinite-dimensional linearly compact Lie algebras. His main idea is the notion of a *primitive* Lie algebra. This is a linearly compact Lie algebra $\overline{L}$ with a maximal fundamental subalgebra $\overline{L}_0$. (Geometrically primitivity means that L does not leave invariant a non-trivial completely integrable differential system.) Every simple $\overline{L}$ can be made primitive by taking any maximal subalgebra containing a fundamental subalgebra. Cartan's list of primitive linearly compact Lie algebras consists of (a) four well-known series: $\overline{W}_m$, $\overline{S}_m$, $\overline{H}_m$ and $\overline{K}_m$ of simple ones, which are respectively the Lie algebra of all formal vector fields in m indeterminates (= all continuous derivations of the algebra of formal power series $\mathbb{C}[[x_1, \dots, x_m]]$) and its subalgebras consisting of divergence zero vector fields, of vector fields annihilating a symplectic form (for m even), and of vector fields multiplying a contact form by a function (for m odd), and (b) two series of non-simple ones which contain $\overline{S}_m$ and $\overline{H}_m$ as ideals of codimension 1.

Here and further the overbar stands for the formal completion of the corresponding Lie (super)algebra of polynomial vector fields, e.g., W_m denotes the Lie algebra of all derivations of the polynomial algebra $\mathbb{C}[x_1, \dots, x_m]$.

The first step of Cartan's paper is the classification of *irreducible* $\overline{L}$, i.e., those for which the representation of $\overline{L}_0/\overline{L}_1$ on $\overline{L}/\overline{L}_0$ is irreducible. (Geometrically irreducibility means that L does not leave invariant any non-trivial differential system, integrable or not.) This result was verified in [SS]. In Cartan's application of this classification to the classification of primitive $\overline{L}$, there seems to be a serious gap (cf. [GS]). In [GQS] the problem was solved by making use of a rather complicated result from analysis.

The first purely algebraic (and very elegant) solution to the problem was found by Weisfeiler [W]. His idea is to choose a minimal $\mathrm{ad}\,\overline{L}_0$-invariant subspace $\overline{L}_{-1}$ of $\overline{L}$ such that $\overline{L}_{-1} \supsetneqq \overline{L}_0$ and construct a new filtration $\overline{L} = \overline{L}_{-d} \supset \overline{L}_{-d+1} \supset \dots \supset \overline{L}_{-1} \supset \overline{L}_0 \supset \overline{L}_1 \supset \dots$ (with new $\overline{L}_1, \dots$). (Geometrically this corresponds to a choice of an invariant irreducible non-integrable differential system.) By considering this *Weisfeiler filtration* one restores the irreducibility of the representation of $\overline{L}_0/\overline{L}_1$ on $\overline{L}_{-1}/\overline{L}_0$ at the expense of the possibility of having the *depth* d of the filtration greater than 1. The associated $\mathbb{Z}$-graded Lie algebra is of the form $\mathfrak{g} = \oplus_{j \geq -d} \mathfrak{g}_j$, and has the following properties:

(G0) $\dim \mathfrak{g}_j < \infty$,

(G1) $\mathfrak{g}_{-j} = \mathfrak{g}_{-1}^j$, for $j \geq 1$,

(G2) if $a \in \mathfrak{g}_j$ with $j \geq 0$ and $[a, \mathfrak{g}_{-1}] = 0$, then $a = 0$,

(G3) the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is irreducible.

Weisfeiler's classification of these $\mathbb{Z}$-graded Lie algebras remained unpublished. In his paper [W] he refers to the paper [K1] where a more general result had been obtained.

Of course, the concluding step after that is to verify that the Lie algebra $\overline{L}$ is uniquely determined by $\mathfrak{g}$, i.e., that the formal completion $\overline{\mathfrak{g}}$ of $\mathfrak{g}$ has no filtered deformations. This can be done by several different techniques developed, in particular, in [SS], [KN], [W] and [K3].

In [G2] Guillemin found a very beautiful new approach to the problem. Using the notion of a characteristic variety, he proved (without the use of classification) that an infinite-dimensional primitive linearly compact Lie algebra has a unique maximal fundamental subalgebra. After that a simple "normalizer trick" almost immediately gives the classification of the $\mathbb{Z}$-graded Lie algebras in question.

At this point it is appropriate to mention an earlier paper, [G1], of Guillemin where he proves Cartan's conjecture on existence in an arbitrary linearly compact Lie algebra with a fundamental subalgebra a finite chain of nested closed ideals such that each of the consecutive quotients is either abelian or of the form $\overline{S}\hat{\otimes}\mathbb{C}[[x_1,\dots,x_n]]$ where $\overline{S}$ is a simple linearly compact Lie algebra.

Let us now turn to superalgebra. At the very end of my paper on classification of finite-dimensional Lie superalgebras [K4], I briefly discussed the problem of classification of simple infinite-dimensional linearly compact Lie superalgebras. This problem has the same geometric origin as in the Lie algebra case with X being a supermanifold. Of course, the first basic example is the Lie superalgebra $\overline{W}(m,n)$ (see Example 4.1 in Section 4) of continuous derivations of the algebra $\mathbb{C}[[x_1,\dots,x_m]]\otimes\Lambda(n)$, where $\Lambda(n)$ stands for the Grassmann algebra in n indeterminates (in other words, $\overline{W}(m,n)$ is the Lie superalgebra of all formal vector fields in m commuting and n anticommuting indeterminates). The remaining three series, $\overline{S}_m$, $\overline{H}_m$ and $\overline{K}_m$ have "super"generalizations as well. They are subalgebras of $\overline{W}(m,n)$, denoted by $\overline{S}(m,n)$, $\overline{H}(m,n)$ (m even) and $\overline{K}(m,n)$ (m odd), which consist respectively of "super"divergence zero vector fields, of vector fields annihilating a "super"symplectic form and of vector fields multiplying a "super"contact form by a function (see Examples 4.2, 4.3 and 4.4 in Section 4). Incidentally, the Lie superalgebras $\overline{W}(0,n)$, $\overline{S}(0,n)$ and $\overline{H}(0,n)$ are finite-dimensional; they form the "non-classical" part of the list of simple finite-dimensional Lie superalgebras (along with a filtered deformation of $\overline{S}(0,n)$). I proposed that, in analogy with Cartan's classification, these four series should give a complete list of simple infinite-dimensional linearly compact Lie superalgebras. Remarkably, the situation turned out to be much more exciting.

It was pointed out by Buttin, Kirillov, Leites and Tulcziev among others (see [L] and references there) that the Schouten bracket makes the space of polyvector fields into a Lie superalgebra. This gives the series, denoted in the present paper by $\overline{HO}(n,n)$ (see Example 4.6), which consists of vector fields from $\overline{W}(n,n)$, annihilating an odd super symplectic form (HO stands for Hamiltonian odd). The next series is $\overline{SHO}(n,n)=\overline{HO}(n,n)\cap\overline{S}(n,n)$ (see Example 4.7). Furthermore, one has the series $\overline{KO}(n,n+1)$ which consists of vector fields from $\overline{W}(n,n+1)$ multiplying an odd super contact form by a function (see Example 4.8). One can take $\overline{KO}(n,n+1)\cap\overline{S}(n,n+1)$ as well, but the situation again is more interesting, as was discovered by Kochetkoff [Ko]. It turns out that for each $\beta \in \mathbb{C}$ one can define the deformed divergence div_β such that $\overline{SKO}(n, n+1; \beta) = \{D \in \overline{KO}(n, n+1) | \ \mathrm{div}_\beta \ D = 0\}$ is a simple superalgebra (see Example 4.9). (One should mention that some of the above Lie superalgebras are not simple, but, apart from small m and n, listed in Examples 4.1-4.4, 4.6-4.9, their derived algebras are simple and have codimension at most 1.)

However, the most surprising discovery was made by Shchepochkina who announced in [S1], the existence of three exceptional simple infinite-dimensional Lie superalgebras. The place of these examples in my classification is discussed in Section 5 (see Remark 5.1). Subsequently she found one more exceptional example (cf. [S2] and Example 4.10 from Section 4 of the present paper). Next, Cheng and I in our work on conformal superalgebras [CK1] and independently Shchepochkina [S2], found another exception (see Example 5.2 in Section 5). Finally, during the work on the present paper one more exception was found (see Example 4.11).

The main result of the present paper is the following theorem (cf. Theorem 6.3 in Section 6).

Theorem 0.1 *Any simple infinite-dimensional linearly compact Lie superalgebra is isomorphic to one of the Lie superalgebras of the following list or its derived subalgebra:*

(a) eight series of completed graded superalgebras: $\overline{W}(m,n)$, $\overline{S}(m,n)$, $\overline{H}(m,n)$, $\overline{K}(m,n)$, $\overline{HO}(n,n)$, $\overline{SHO}(n,n)$, $\overline{KO}(n,n+1)$, $\overline{SKO}(n,n+1;\beta)$,

(b) two series of filtered deformations: $\overline{SHO}(n,n)^\sim$ *(n even),* $\overline{SKO}(n,n+1)^\sim$ *(n odd),*

(c) six exceptional Lie superalgebras: $\overline{E}(1,6)$, $\overline{E}(2,2)$, $\overline{E}(3,6)$, $\overline{E}(3,8)$, $\overline{E}(4,4)$, $\overline{E}(5,10)$.

The major difficulty in the Lie superalgebra case is that, unlike in the Lie algebra case, $\overline{L}$ may contain a lot of maximal fundamental subalgebras. In order to circumvent this difficulty, I introduce the notion of an *even* primitive Lie superalgebra. It is a primitive Lie superalgebra $\overline{L}$ whose maximal fundamental subalgebra $\overline{L}_0$ contains all even ad-exponentiable elements of L; such a subalgebra is called maximally even. Using Guillemin's argument from [G2] I prove that any simple infinite-dimensional linearly compact Lie superalgebra contains a maximally even subalgebra (Corollary 1.1 of Theorem 1.1). (Such a subalgebra is unique in most of the examples, and there are at most two such subalgebras in the remaining examples.) Incidentally, in the finite-dimensional case "maximally even" simply means that $\overline{L}_0$ is a maximal subalgebra containing the even part of $\overline{L}$; the finite-dimensional even primitive Lie superalgebras were classified in [K4].

Another useful notion is that of a quasiprimitive Lie superalgebra: it is a linearly compact Lie superalgebra $\overline{L}$ with a given fundamental subalgebra $\overline{L}_0$ and an $\mathrm{ad}\,\overline{L}_0$-invariant minimal subspace $\overline{L}_{-1}$ containing $\overline{L}_0$ and generating $\overline{L}$ (as an algebra). This weaker property than primitivity, as well as evenness, still remain when one passes to the completion of the associated graded with the Weisfeiler filtration Lie superalgebra (Propositions 2.1 and 2.2).

The main problem which is addressed in the paper is the classification of $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g} = \oplus_{j\geq -d}\mathfrak{g}_j$ which occur as associated graded to the Weisfeiler filtration of an even quasiprimitive infinite-dimensional Lie superalgebra. I use the "normalizer trick" of Guillemin to show that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is *strongly transitive* (Proposition 2.3), meaning that apart from being finite-dimensional (which is (G0) for $j=-1$), faithful (which is (G2) for $j=0$) and irreducible (which is (G3)), it satisfies the property: (G4) if a is a non-zero even element of $\mathfrak{g}_{-1}$, then $[\mathfrak{g}_0, a] = \mathfrak{g}_{-1}$.

It turns out that, in spite of the fact that, unlike in the Lie algebra case, the classification of all faithful irreducible finite-dimensional Lie superalgebra modules is unknown (and probably is impossible), one can give a complete classification of strongly transitive modules V over a finite-dimensional Lie superalgebra $\mathfrak{p}$ (Theorem 3.1). The list consists of two parts:

1. V has a non-zero even element: this part of the list comprises a dozen "classical series" (cases (a)-(j) and (r) of Theorem 3.1) and seven exceptional cases (cases (k)-(q) of Theorem 3.1),

2. all elements of V are odd: then $\mathfrak{p}$ is a Lie algebra, hence a direct sum of simple Lie algebras plus at most 1-dimensional center and V is an arbitrary faithful finite-dimensional irreducible $\mathfrak{p}$-module.

Correspondingly, the classification of the above mentioned $\mathbb{Z}$-graded Lie superalgebras is divided in two parts. The first part, when the $\mathbb{Z}$-gradation of $\mathfrak{g}$ is inconsistent with the $\mathbb{Z}/2\mathbb{Z}$ gradation, is given by Theorem 4.1. The list consists of the above mentioned eight series of Lie superalgebras of polynomial vector fields, excluding $K(1,n)$, with the "principal" or "subprincipal" $\mathbb{Z}$-gradation, two exceptional superalgebras $E(2,2)$ and $E(4,4)$, four "degenerate" series (which are far from being simple) and the extensions by derivations and central elements of these. It more or less corresponds to the list given by Theorem 3.1 (five of the "exceptional" cases of Theorem 3.1 actually correspond to the first members of some series, and the series (r) of Theorem 3.1 do not correspond to any even $\mathbb{Z}$-graded Lie superalgebra).

The second part, when the $\mathbb{Z}$-gradation of $\mathfrak{g}$ is consistent with the $\mathbb{Z}/2\mathbb{Z}$-gradation, is given by Theorem 5.3. The list consists of $K(1,n)$, four exceptional simple superalgebras $E(1,6)$, $E(3,6)$, $E(3,8)$ and $E(5,10)$, nonsimple subalgebras $E'(3,6)$ and $E'(3,8)$, and their extensions by derivations. In this case I use the methods developed in [K4] in order to show that the only possibilities for the $[\mathfrak{g}_0,\mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$ are $so_n(n\neq 2)$, $s\ell_3\boxtimes s\ell_2$ and $\Lambda^2 s\ell_5$.

The final step of the classification of infinite-dimensional simple linearly compact Lie superalgebras is the reconstruction of complete filtered Lie superalgebras from the $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g}=\oplus_j\mathfrak{g}_j$ listed by Theorems 4.1 and 5.3. First, there are the superalgebras $\overline{\mathfrak{g}}=\prod_j\mathfrak{g}_j$ obtained by completion of $\mathfrak{g}$ when $\mathfrak{g}$ is simple. Next, one has to find all simple filtered deformations of $\overline{\mathfrak{g}}$ where $\mathfrak{g}$ is one of the superalgebras listed by Theorems 4.1 and 5.3. For this one can use methods developed in [KN], [K3], [K4] and [CK3]. It turns out (Section 6 and [CK3]) that all non-trivial simple filtered deformations are listed in Theorem 0.1(b) (the first of these deformations was found in [CK3] and the second much earlier in [Ko]).

The notation $X(m,n)$ used here, where $X=W,S,H,K$, etc. means that this is a Lie superalgebra of vector fields on the superspace of dimension (m,n), where the dimension m of the even part is minimal possible and dimension n of the odd part is minimal possible for this m. Note also that in all cases m is equal to the growth (= Gelfand-Kirillov dimension) of $X(m,n)$.

The paper is organized as follows. In the first section I explain the basic properties of linearly compact Lie superalgebras and prove the existence of a maximally even fundamental subalgebra in a simple linearly compact Lie superalgebra (Theorem 1.1 and Corollary 1.1).

In Section 2 the properties (G0)-(G5) of the associated graded of an even quasiprimitive Lie superalgebra are established. In Section 3 I classify strongly transitive finite-dimensional modules (Theorem 3.1).

In Sections 4 and 5 the graded Lie superalgebras with inconsistent and consistent gradation respectively associated to even quasiprimitive Lie superalgebras are classified.

In the last Section 6 filtered deformations of the completions of the above-mentioned graded Lie superalgebras are discussed and the classification of infinite-dimensional simple linearly compact Lie superalgebras is completed (Theorem 6.3). One of the consequences of this result is the classification of simple finite conformal superalgebras announced in [K6], [K7].

Unless otherwise specified, all vector spaces, linear maps and tensor products are considered over the field $\mathbb{C}$ of complex numbers.

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1 Basic properties of primitive Lie superalgebras

In this paper we shall use the superalgebra terminology adapted in [K4]. In particular, a *vector superspace* is a vector space V decomposed in a direct sum of subspaces $V_{\bar{0}}$ and $V_{\bar{1}}$, called the even and odd subspaces respectively. Here and further $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$; if $a \in V_\alpha$, we write $p(a) = \alpha$. By a subspace of a superspace V we mean a subspace U such that $U = (U \cap V_{\bar{0}}) + (U \cap V_{\bar{1}})$. A *superalgebra* is a vector superspace V endowed with a structure of an algebra such that $V_\alpha V_\beta \subset V_{\alpha+\beta}$, $\alpha, \beta \in \{\bar{0}, \bar{1}\}$. A Lie superalgebra is a superalgebra satisfying the super Jacobi and super anti-commutativity axioms, etc.

A topological vector superspace $L = L_{\bar{0}} + L_{\bar{1}}$ is called *linearly compact* if it admits a fundamental system of neighborhoods of zero consisting of subspaces of finite codimension of L such that L is complete in this topology.

Here are some useful properties of a linearly compact superspace L (cf. [G1]). A subspace of L is open iff it is closed and of finite codimension. Another important fact is Chevalley's principle: if $F_1 \supset F_2 \supset \dots$ is a sequence of closed subspaces such that $\cap_j F_j = 0$ and if U is a neighborhood of zero, then $F_j \subset U$ for $j \gg 0$.

A topological Lie superalgebra L is called linearly compact if, as a topological vector superspace, L is linearly compact.

Given subspaces U and V of L, let

$$\begin{aligned} N_U(V) &= \{a \in U \,|\, [a, V] \subset V\}, \\ N^U(V) &= \{a \in V \,|\, [a, U] \subset V\}. \end{aligned}$$

The subspace $N_U(V)$ is the usual normalizer of V in U, whereas $N^U(V)$ is an "inner" normalizer. The proof of the following lemma is straightforward.

Lemma 1.1 *(a) $N_U(V)$ is a subalgebra of the Lie superalgebra L, provided that U is a subalgebra of L.*

(b) $N^U(V)$ is a subalgebra provided that $V \subset U$.

(c) If U and V are open subspaces of L, then $N_U(V)$ and $N^U(V)$ are open as well.

A subalgebra L_0 of L is called *fundamental* if it is proper (i.e., $L_0 \neq L$), open and contains no closed ideals of L.

Fix a fundamental subalgebra L_0 of the linearly compact Lie superalgebra L and choose a subspace L_{-1} of L which generates L as a Lie superalgebra and such that $[L_0, L_{-1}] \subset L_{-1}$. One associates to the triple $L \supset L_{-1} \supset L_0$ the *Weisfeiler filtration* of L [W] by letting inductively for $s \geq 1$:

$$L_{-(s+1)} = [L_{-1}, L_{-s}] + L_{-s},\ L_s = N^{L_{-1}}(L_{s-1})\,.$$

It is straightforward to check that this is indeed a filtration of the form:

$$L = L_{-d} \underset{\neq}{\supset} L_{-d+1} \supset \ldots \supset L_{-1} \supset L_0 \supset L_1 \supset \ldots$$

by open subspaces L_j, in other words:

$$[L_i, L_j] \subset L_{i+j}, \quad \cap_j L_j = 0\,, \quad \dim L/L_j < \infty\,.$$

The number $d \geq 1$ is called the depth of this filtration. By Chevalley's principle, the L_j form a fundamental system of neighborhoods of 0.

An element a of L is called *exponentiable* if the series $\exp(\operatorname{ad} a)$ defines a continuous automorphism of L.

Lemma 1.2 *If L_0 is a fundamental subalgebra of a linearly compact Lie superalgebra L, then any even element a from L_0 is exponentiable.*

Proof Consider the Weisfeiler filtration for the triple $L \supset L \supset L_0$. Since $a \in L_0$, we have $[a, L_j] \subset L_j$ for all j. Since $\dim L/L_j < \infty$, the series $\exp(\operatorname{ad} a)$ converges on L/L_j for each j, hence converges on L to a continuous automorphism. $\square$

Lemma 1.3 (Super Nullstellensatz) *If A is a finitely generated commutative associative superalgebra which contains only one maximal ideal $\mathfrak{m}$, then* $\dim A < \infty$.

Proof Let A be generated by even elements $x_1, \dots, x_m$ and odd elements $\xi_1, \dots, \xi_n$, and denote by $\mathcal{J}$ the ideal generated by $\xi_1, \dots, \xi_n$. Since $A/\mathfrak{m}$ is a field, $\mathcal{J} \subset \mathfrak{m}$. Let $\overline{A} = A/\mathcal{J}$, then $\overline{\mathfrak{m}} = \mathfrak{m}/\mathcal{J}$ is an ideal of $\overline{A}$. If $\overline{\mathfrak{n}}$ is another ideal of $\overline{A}$ and $\mathfrak{n}$ its preimage in A, then $\mathfrak{n} \subset \mathfrak{m}$, hence $\overline{\mathfrak{n}} \subset \overline{\mathfrak{m}}$. Therefore, $\overline{\mathfrak{m}}$ is a unique maximal ideal of $\overline{A}$ and hence, by the ordinary Hilbert's Nullstellensatz, $\dim \overline{A} < \infty$. It follows that $\dim A \leq \dim \overline{A} \cdot 2^n$.

□

Proposition 1.1 *Let L be a linearly compact Lie superalgebra which admits a fundamental subalgebra. Then there exists a proper open subspace H of L which is mapped into itself by every continuous automorphism of L.*

Proof is the same as that of Proposition 3.2 from [G2] using Lemma 1.3 instead of the ordinary Nullstellensatz. □

Theorem 1.1 *(a) Let L be a linearly compact Lie superalgebra with a fundamental subalgebra. Suppose that L has no proper open ideals with a finite-dimensional Lie algebra quotient. Then L admits a proper open subalgebra L_0 which contains all exponentiable elements of L.*

(b) Let L be a simple (i.e., without non-trivial closed ideals) linearly compact Lie superalgebra which is not a finite-dimensional Lie algebra. Then L admits a maximal fundamental subalgebra which contains all exponentiable elements of L.

Proof We may assume that $\dim L = \infty$. Let H be the subspace of L given by Proposition 1.1 and let $L_0 = N_L(H)$. Then L_0 is an open subalgebra of L. If $L_0 = L$, then H is a proper open ideal of L, a contradiction with the hypothesis of (a). If a is an exponentiable element of L, then by Proposition 1.1, $\exp(t \operatorname{ad} a)H \subset H$ for any $t \in \mathbb{C}$, hence $[a, H] \subset H$ and $a \in L_0$. This proves (a). Statement (b) follows from (a) by taking any maximal subalgebra containing L_0. □

A pair (L, L_0) consisting of a linearly compact Lie superalgebra L and its fundamental subalgebra L_0 is called a *primitive* Lie superalgebra if L_0 is a maximal subalgebra. This primitive Lie superalgebra is called *even* if L_0 contains all exponentiable elements of L.

We have the following corollary of Theorem 1.1.

Corollary 1.1 *If L is a simple linearly compact Lie superalgebra which is not a finite-dimensional Lie algebra, then there exists a subalgebra L_0 of L such that (L, L_0) is an even primitive Lie superalgebra.*

Proposition 1.2 *If L_0' is a proper open subalgebra of L and (L, L_0) is a primitive Lie superalgebra, then either L_0' is a fundamental subalgebra of L, or $L_0' + L_0 = L$. Furthermore any non-zero closed ideal of L is open.*

Proof If L_0' is not a fundamental subalgebra of L, then it contains a non-zero ideal I of L. Since $I \not\subset L_0$, the subalgebra $I + L_0$ must be the whole L due to maximality of L_0. The second claim is proved in the same way as in [G1], Proposition 4.1. $\square$

The following notion is technically more convenient than that of primitivity. A triple (L, L_{-1}, L_0) consisting of a linearly compact superalgebra L, its fundamental subalgebra L_0 and a minimal subspace L_{-1} such that $L_{-1} \supsetneqq L_0$ and $[L_0, L_{-1}] \subset L_{-1}$ is called a *quasiprimitive* Lie superalgebra if L_{-1} generates L (as an algebra). A quasiprimitive Lie superalgebra (L, L_{-1}, L_0) is called *even* if L_0 contains all exponentiable elements of L.

Of course, if (L, L_0) is a primitive Lie superalgebra, choosing a minimal subspace L_{-1} such that $L_{-1} \supsetneqq L_0$ and $[L_0, L_{-1}] \subset L_{-1}$, we obtain a quasiprimitive Lie superalgebra (L, L_{-1}, L_0).

Example 1.1 Any finite-dimensional Lie superalgebra with discrete topology is linearly compact.

Example 1.2 Let $F_m = \mathbb{C}[[x_1, \dots, x_m]]$ be the algebra of formal power series in the indeterminates $x_1, \dots, x_m$ and let $\Lambda(n)$ be the Grassmann superalgebra in the indeterminates $\xi_1, \dots, \xi_n$, Denote by $\overline{\Lambda}(m, n)$ the associative (commutative) superalgebra $F_m \otimes \Lambda(n)$ and by $\overline{\mathcal{J}}$ the ideal of $\overline{\Lambda}(m, n)$ generated by $x_1, \dots, x_m, \xi_1, \dots, \xi_n$. Then $\overline{\Lambda}(m, n)$ is a linearly compact associative superalgebra with topology for which $\left\{\overline{\mathcal{J}}^k\right\}_{k \geq 1}$ form a fundamental system of neighborhoods of 0. Let

$$\overline{W}(m, n) = \operatorname{der} \overline{\Lambda}(m, n)$$

denote the Lie superalgebra of all continuous derivations of the superalgebra $\overline{\Lambda}(m,n)$. It consists of linear operators of the form:

$$\sum_{i=1}^{m} P_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j \frac{\partial}{\partial \xi_j}\,, \text{ where } P_i, Q_j \in \overline{\Lambda}(m,n)\,.$$

Note that $\overline{W}(m,n)$ is a left $\overline{\Lambda}(m,n)$-module and let $\overline{W}(m,n)_k = \overline{\mathcal{J}}^k \overline{W}(m,n)$. Then $\overline{W}(m,n)$ is a linearly compact simple Lie superalgebra with a fundamental system of neighborhoods of 0 consisting of the subalgebras $\overline{W}(m,n)_k$, which form a (Weisfeiler) filtration of $\overline{W}(m,n)$. The pair $(\overline{W}(m,n), \overline{W}(m,n)_0)$ is an even primitive Lie superalgebra (since $\frac{\partial}{\partial x_i}$'s are not exponentiable). Note that, letting $\overline{\mathcal{J}}_i = (x_1, \ldots, x_m, \xi_1, \ldots, \xi_i)$ for $0 \leq i < n$, we obtain primitive Lie superalgebras $(\overline{W}(m,n), \overline{\mathcal{J}}_i \overline{W}(m,n))$. All of them are not even, except for the case $n=1, i=0$.

Example 1.3 Any closed subalgebra L of $\overline{W}(m,n)$ is a linearly compact Lie superalgebra. If no non-trivial closed ideals of $\Lambda(m,n)$ are L-invariant, then $L_0 := L \cap \overline{W}(m,n)_0$ is a fundamental subalgebra of L. Conversely, any linearly compact Lie superalgebra with a fundamental subalgebra L_0 such that $\dim L/L_0 = (m,n)$ is obtained in this way (cf. [B1] and [GS]).

2 Associated graded of even quasiprimitive Lie superalgebras

Let (L, L_{-1}, L_0) be a quasiprimitive Lie superalgebra. As in Section 1, we may associate to this triple the Weisfeiler filtration $L = L_{-d} \supset \ldots \supset L_{-1} \supset L_0 \supset L_1 \supset \ldots$. Let

$$GrL = \oplus_{j \geq -d} \mathfrak{g}_j\,, \; \mathfrak{g}_j = L_j / L_{j+1}\,,$$

be the associated $\mathbb{Z}$-graded Lie superalgebra. It is easy to check that it has the following properties [W]:

(G0) $\dim \mathfrak{g}_j < \infty$ for all j,

(G1) $\mathfrak{g}_{-j} = \mathfrak{g}_{-1}^j$ for $j \geq 1$,

(G2) if $a \in \mathfrak{g}_j$ with $j \geq 0$ and $[a, \mathfrak{g}_{-1}] = 0$, then $a = 0$,

(G3) the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is irreducible.

A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_{j\geq -d}\mathfrak{g}_j$ satisfying (G0)-(G3) is called a *transitive irreducible* graded Lie superalgebra. Properties (G0)-(G2) (resp. (G3)) are usually called the *transitivity* (resp. *irreducibility*) properties respectively. If $\mathfrak{g}_{-d} \neq 0$, the positive integer d is called the *depth* of $\mathfrak{g}$.

In what follows we shall assume, when talking about a $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_j\mathfrak{g}_j$, that (G0) holds. We also shall use the following notation:

$$\mathfrak{g}^- = \oplus_{j<0}\mathfrak{g}_j\ ,\ \mathfrak{g}^+ = \oplus_{j>0}\mathfrak{g}_j\ .$$

The following assertion is clear (cf. [W] and [G2], Lemma 4.1).

Proposition 2.1 *The triple (L, L_{-1}, L_0) is a quasiprimitive Lie superalgebra iff GrL is a transitive irreducible graded Lie superalgebra.*

Lemma 2.1 *If $\mathfrak{g} = \oplus_j\mathfrak{g}_j$ is a transitive irreducible $\mathbb{Z}$-graded Lie superalgebra and I is a $\mathbb{Z}$-graded ideal of $\mathfrak{g}$, then either $I \supset \mathfrak{g}^-$ or $I \subset \mathfrak{g}^-$. If, in addition, $\mathfrak{g}_1 \neq 0$, then $I \cap \mathfrak{g}_{-1} = 0$ in the latter case.*

Proof Let $I_j = I \cap \mathfrak{g}_j$. If $I_j \neq 0$ for some $j \geq 0$, then, by (G2), $I_{-1} \neq 0$, and, by (G3), $I_{-1} \supset \mathfrak{g}_{-1}$, hence, by (G1), $I \supset \mathfrak{g}^-$. If $I \cap \mathfrak{g}_{-1} \neq 0$, then $I \supset \mathfrak{g}_{-1}$ by (G3), hence $I \cap \mathfrak{g}_0 \neq 0$ if $\mathfrak{g}_1 \neq 0$, by (G2). $\square$

Given a $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_{j\geq -d}\mathfrak{g}_j$ of depth $d \geq 1$, consider the associated filtration by subspaces $\mathfrak{g}_{(k)} := \oplus_{j\geq k}\mathfrak{g}_j$ and topology of $\mathfrak{g}$ for which these subspaces form a fundamental system of neighborhoods of zero; let $\overline{\mathfrak{g}}$ (resp. $\overline{\mathfrak{g}}_{(k)}$) be the completion of $\mathfrak{g}$ (resp. $\overline{\mathfrak{g}}_{(k)}$) in this topology. Then we get a filtered linearly compact Lie superalgebra $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}_{(-d)} \supset \overline{\mathfrak{g}}_{(-d+1)} \supset \ldots \supset \overline{\mathfrak{g}}_{(0)} \supset \ldots$. The following lemma follows from Proposition 2.1.

Lemma 2.2 *A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_{j\geq -d}\mathfrak{g}_j$ is transitive irreducible iff the triple $(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}_{(-1)}, \overline{\mathfrak{g}}_{(0)})$ is a quasiprimitive Lie superalgebra.*

Lemma 2.3 *Let L be a linearly compact Lie superalgebra with a fundamental subalgebra L_0 and let $\mathfrak{g} = GrL$ be a graded Lie superalgebra associated to a filtration of L with L_0 as one of its members. Then an even element $a \in L$ is exponentiable iff its image $\overline{a}$ in $\mathfrak{g}$ is exponentiable in $\overline{\mathfrak{g}}$ (where $\overline{a}$ stands for the image of a in L_j/L_{j+1}, j being the minimal index such that $a \notin L_{j+1}$).*

Proof We may assume that L is a subalgebra of $\overline{W}(m,n)$ (cf. Example 1.3) and that $a \notin L_0$ (see Lemma 1.2). Then we may assume that a is of the form:

$$a = \frac{\partial}{\partial x_1} + \sum_i P_i \frac{\partial}{\partial x_i} + \sum_j Q_j \frac{\partial}{\partial \xi_j} + \text{ higher degree terms,}$$

where $\deg P_i = \deg Q_j = k > 0$. Making the change of indeterminates $x_i' = x_i - \widetilde{P}_i$, $\xi_j' = \xi_j - \widetilde{Q}_j$, where $\widetilde{P}_i$ and $\widetilde{Q}_j$ are homogeneous polynomials of degree $k+1$ such that $\frac{\partial \widetilde{P}_i}{\partial x_1} = P_i$, $\frac{\partial \widetilde{Q}_j}{\partial x_1} = Q_j$, we increase k by 1. Thus, we may assume that $a = \frac{\partial}{\partial x_1}$. But then $\exp \lambda(ada)$ (if it converges) acts on L by substitution $x_1 \to x_1 + \lambda$, $\lambda \in \mathbb{C}$, and all other indeterminates unchanged. Hence a is exponentiable in L iff for each monomial p in $x_2, \ldots, x_m, \xi_1, \ldots, \xi_n$ the linear span of all coefficients of p in all coefficients of derivations from L is a finite-dimensional subspace of $\mathbb{C}[x_1]$. This property holds iff it holds in the associated graded of L. $\square$

A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ is called *even* if any even homogeneous exponentiable in $\overline{\mathfrak{g}}$ element of $\mathfrak{g}$ is contained in $\mathfrak{g}_{(0)}$. Lemma 2.3 implies

Proposition 2.2 *A quasiprimitive Lie superalgebra (L, L_{-1}, L_0) is even iff GrL is an even transitive irreducible graded Lie superalgebra.*

Lemma 2.4 *Let $\mathfrak{g} = \oplus_{j \geq -d} \mathfrak{g}_j$ be an even $\mathbb{Z}$-graded Lie superalgebra and let $\mathfrak{b}$ be the maximal graded ideal of $\mathfrak{g}$ contained in $\mathfrak{g}^-$. Then $\mathfrak{b}_{\bar{0}} = 0$.*

Proof An even element from $\mathfrak{b}$ is exponentiable. $\square$

The following is a key lemma. The main idea of its proof is borrowed from [G2].

Lemma 2.5 *Let $\mathfrak{g} = \oplus_{j \geq -d} \mathfrak{g}_j$ be an even transitive irreducible $\mathbb{Z}$-graded Lie superalgebra. Then $\mathfrak{g}$ has the following property*

(G4) *if a is an even element of $\mathfrak{g}_{-1}$, then $[\mathfrak{g}_0, a] = \mathfrak{g}_{-1}$.*

Proof We may assume that $\mathfrak{g}_1 \neq 0$ (since otherwise a is exponentiable). Then $\mathfrak{b}$ is contained in $\oplus_{j \leq -2} \mathfrak{g}_j$. Due to Lemma 2.4, we may assume without loss of generality that $\mathfrak{b} = 0$ (by replacing $\mathfrak{g}$ by $\mathfrak{g}/\mathfrak{b}$; this does not affect exponentiability since $\dim \mathfrak{b} < \infty$).

Consider the subspace $\mathfrak{h} = \oplus_j \mathfrak{h}_j$ of $\mathfrak{g}$ defined by:

$$\mathfrak{h}_j = \mathfrak{g}_j \text{ if } j \neq -1,\ \mathfrak{h}_{-1} = [\mathfrak{g}_0, a]\,,$$

and let $\ell_0 = N_{\mathfrak{g}}(\mathfrak{h})$. Note that ℓ_0 is a graded subalgebra of $\mathfrak{g}$ containing a and all $\mathfrak{g}_j$ for $j \gg 0$.

Suppose that $[\mathfrak{g}_0, a] \neq \mathfrak{g}_{-1}$. Then $\mathfrak{g}_{-1} \not\subset \ell_0$ (since $[\mathfrak{g}_0, \mathfrak{g}_{-1}] = \mathfrak{g}_{-1}$), hence, by Lemma 2.1, any ideal of $\mathfrak{g}$ contained in ℓ_0 must lie in $\oplus_{j \leq -2}\mathfrak{g}_j$, hence is zero. Thus, $\overline{\ell}_0$ is a fundamental subalgebra of $\overline{\mathfrak{g}}$ and therefore, by Lemma 1.2, any even element of $\overline{\ell}_0$, in particular the element a, is exponentiable. This contradicts the hypothesis of the lemma.

$\square$

We shall call a finite-dimensional module V over a Lie superalgebra $\mathfrak{g}$ *strongly transitive* if it is faithful, irreducible and for any non-zero even element v of V one has: $\mathfrak{g} \cdot v = V$. Combining Lemmas 2.3 and 2.5, we obtain the following severe restriction on the associated graded of an even quasiprimitive Lie superalgebra.

Proposition 2.3 *Let (L_1, L_{-1}, L_0) be an even quasiprimitive Lie superalgebra, and let $\mathfrak{g} = \oplus_{j \geq -d}\mathfrak{g}_j$ be the associated graded Lie superalgebra. Then the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is strongly transitive.*

The next lemma provides further restrictions.

Lemma 2.6 *Let $\mathfrak{g} = \oplus_{j \geq -d}\mathfrak{g}_j$ be an even transitive irreducible $\mathbb{Z}$-graded Lie superalgebra. Let $\mathfrak{b} = \oplus_j \mathfrak{b}_j$ be the maximal graded ideal of $\mathfrak{g}$ contained in $\mathfrak{g}^-$. Then*

(G5) *if $r \leq 0$, $s < 0$ and a is a non-zero even element of $\mathfrak{g}_s$, then $[\mathfrak{g}_r, a] + \mathfrak{b}_{r+s} = \mathfrak{g}_{r+s}$.*

Proof We argue in the same way as in the proof of Lemma 2.5. Replace $\mathfrak{g}$ by $\mathfrak{g}/\mathfrak{b}$. Let $\mathfrak{h} = \oplus_j \mathfrak{h}_j$, be a subspace of $\mathfrak{g}$ defined by:

$$\mathfrak{h}_j = \mathfrak{g}_j \text{ if } j \neq r+s \text{ and } \mathfrak{h}_{r+s} = [\mathfrak{g}_r, a]\,,$$

and let $\ell_0 = N_{\mathfrak{g}}(\mathfrak{h})$. Then $a \in \ell_0$. Suppose that $[\mathfrak{g}_r, a] \neq \mathfrak{g}_{r+s}$. Then $\mathfrak{g}^- \not\subset \ell_0$, hence, by Lemma 2.1, any ideal of $\mathfrak{g}$ contained in ℓ_0 must lie in $\mathfrak{g}^-$ and therefore is zero. It follows that $\overline{\ell}_0$ is a fundamental subalgebra of $\overline{\mathfrak{g}}$, hence a is exponentiable in $\overline{\mathfrak{g}}$, a contradiction.

$\square$

The following simple proposition is useful for checking primitivity and simplicity.

Proposition 2.4 *Let $\mathfrak{g} = \oplus_{j \geq -d} \mathfrak{g}_j$ be a $\mathbb{Z}$-graded Lie superalgebra satisfying conditions (G0)-(G3) and, in addition, the following two conditions:*

(G6) $\mathfrak{g}_1$ *generates* $\mathfrak{g}^+$,

(G7) $\mathfrak{g}^-$ *contains no non-zero graded ideals of* $\mathfrak{g}$.

Then

(a) $(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}_{(0)})$ *is a primitive Lie superalgebra.*

(b) *The Lie superalgebra* $\mathfrak{g}$ *(hence* $\overline{\mathfrak{g}}$*) is simple iff the following two conditions hold:*

$$[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0 \, , \quad [\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1 \, .$$

Proof is straightforward. (See [K1] or [G2].)

□

3 Classification of strongly transitive modules

Let $V = V_{\bar{0}} + V_{\bar{1}}$ be a superspace of dimension (m, n), i.e., $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$. Sometimes $m + n$ will also be called the dimension of V. We assume that $m + n > 0$. Let $g\ell(m, n)$ be the Lie superalgebra of all endomorphisms of the superspace V [K4]. If V is a faithful module over a Lie superalgebra $\mathfrak{g}$, we may identify $\mathfrak{g}$ with a subalgebra of $g\ell(m, n)$. We shall describe below examples of strongly transitive modules on V as subalgebras of $g\ell(m, n)$. These subalgebras will be called strongly transitive.

Example 3.1 Any subalgebra $\mathfrak{g}$ of $g\ell(0, n)$, $n \geq 1$, acting irreducibly on V is strongly transitive (since V contains no non-zero even elements). Note that $\mathfrak{g}$ is an ordinary Lie algebra, hence, by the so called Cartan-Jacobson theorem (see e.g., [S] or [OV]) is isomorphic to a direct sum of simple Lie algebras and at most 1-dimensional abelian Lie algebra.

Example 3.2 The Lie superalgebra $g\ell(m,n)$ is strongly transitive for all $m,n \geq 0, m+n > 0$. Its subalgebra [K4]

$$s\ell(m,n) = \{a \in g\ell(m,n) | stra = 0\}$$

is strongly transitive iff $(m,n) \neq (1,0)$.

Example 3.3 Consider a non-degenerate skew-supersymmetric bilinear form $f : V \times V \to \mathbb{C}^{1|0}$, where $\mathbb{C}^{1|0}$ is the $(1,0)$-dimensional superspace (f is an even element of $\Lambda^2 V^*$). We have: $f(V_{\bar{0}}, V_{\bar{1}}) = 0\,, \quad f|_{V_{\bar{0}} \times V_{\bar{0}}}$ is non-degenerate skew-symmetric, so that m is even, and $f|_{V_{\bar{1}} \times V_{\bar{1}}}$ is non-degenerative symmetric. Let (cf. [K4] where a supersymmetric f was considered instead, hence the notation osp there)

$$spo(m,n) = \left\{a \in g\ell(m,n) | f(au,v) + (-1)^{p(a)p(u)} f(u,av) = 0\,, \quad u,v \in V\right\}\,.$$

This is a strongly transitive subalgebra of $g\ell(m,n)$. The subalgebra

$$cspo(m,n) = \mathbb{C}I + spo(m,n)\,,$$

where I is the identity operator on V, is strongly transitive as well.

Example 3.4 Let $\mathfrak{p}$ be the subalgebra $s\ell_m$ or sp_m (m even) of the Lie algebra $g\ell(m,0)$, $m \geq 2$. It acts strongly transitively on the space U of dimension $(m,0)$. Denote by $\mathfrak{p}[\xi]$ the Lie superalgebra $\mathfrak{p} + \mathfrak{p}\xi$, where ξ is an odd element, $\xi^2 = 0$. This Lie superalgebra can be included in $g\ell(m,m)$ by letting it act on $U[\xi] = U + U\xi$ in the obvious way. Consider the following realization of $g\ell(1,1)$:

$$g\ell(1,1) = \mathbb{C}\frac{d}{d\xi} + \mathbb{C}\xi + \mathbb{C}\xi\frac{d}{d\xi} + \mathbb{C}I$$

and introduce the following subalgebra of $g\ell(m,m)$ containing $\mathfrak{p}[\xi]$:

$$\widetilde{\mathfrak{p}}[\xi] = \mathfrak{p}[\xi] + g\ell(1,1)\,.$$

Let $\mathfrak{a}$ be a subalgebra of the Lie superalgebra $g\ell(1,1)$. The subalgebra $\mathfrak{g} = \mathfrak{p}[\xi] + \mathfrak{a}$ of $\widetilde{\mathfrak{p}}[\xi]$ is strongly transitive (on $U[\xi]$) iff the projection of $\mathfrak{g}$ on $\mathbb{C}\frac{d}{d\xi}$ is non-zero. It is easy to see that, up to rescaling of ξ, there are the following possibilities for $\mathfrak{a}$:

(a) $\mathbb{C}\frac{d}{d\xi}$,

(b) $\mathbb{C}(\frac{d}{d\xi}+\xi)+\mathbb{C}I$,

(c) $\mathbb{C}\frac{d}{d\xi}+\mathbb{C}(\alpha\xi\frac{d}{d\xi}+\beta I)$ where $\alpha,\beta\in\mathbb{C}$ and one of them is non-zero,

(d) $\mathbb{C}\frac{d}{d\xi}+\mathbb{C}\xi\frac{d}{d\xi}+\mathbb{C}I$,

(e) $\mathbb{C}\frac{d}{d\xi}+\mathbb{C}\xi+\mathbb{C}I$,

(f) $\mathbb{C}\frac{d}{d\xi}+\mathbb{C}\xi+\mathbb{C}I+\mathbb{C}\xi\frac{d}{d\xi}$.

Example 3.5 Let V be a superspace of dimension (n,n) and let V^* be its dual. Let f be an odd non-degenerate element of Λ^2V^* (= odd skew-supersymmetric bilinear form). Let (cf. [K4])

$$\begin{aligned}\widetilde{p}(n) &= \{a\in g\ell(n,n)|a\cdot f=0\}\ ,\\ c\widetilde{p}(n) &= \mathbb{C}I+\widetilde{p}(n)\,,\\ p(n) &= \{a\in\widetilde{p}(n)|str\,a=0\}\ .\end{aligned}$$

These are strongly transitive subalgebras of $g\ell(n,n)$ iff $n\geq 2$. Recall that in some basis of V, $\widetilde{p}(n)$ consists of matrices of the form $\begin{pmatrix} a & b\\ c & -a\end{pmatrix}$, where a,b,c are $n\times n$ matrices, $c={}^tc$, $b=-{}^tb$, and for $p(n)$, $tr\,a=0$ (the Lie superalgebra $p(n)$ is denoted by $P(n-1)$ in [K4]). Let F be the operator that is identity on $V_{\bar{0}}$ and $-$(identity) on $V_{\bar{1}}$. Given a complex number β, let

$$\widetilde{p}(n\,;\,\beta)=\mathbb{C}(I+\beta F)+p(n)\,.$$

This is again a strongly transitive subalgebra of $g\ell(n,n)$ provided that $n\geq 2$. Note that for $n=2$ these are some of the subalgebras of $\widetilde{s\ell}_2[\xi]$ described in Example 3.4.

Example 3.6 Denote by $\hat{p}(4)$ the subalgebra of $s\ell(4,4)$ consisting of matrices of the form [S2]

$$\begin{pmatrix} a & b\\ c-b^* & -{}^ta\end{pmatrix}+\lambda I\,,$$

where $\lambda\in\mathbb{C}$, $c={}^tc$, $b=-{}^tb$ and b^* stands for the Hodge dual of the skewsymmetric matrix b. This is a strongly transitive subalgebra.

Example 3.7 It is well known that the standard representation of the Lie superalgebra $W(0,n)$ on the Grassmann algebra $\Lambda(n)$ can be deformed (to representation in λ-densities) by letting for a fixed $\lambda \in \mathbb{C}$:

$$a \mapsto a + \lambda \operatorname{div} a\,, \quad a \in W(0,n)$$

(the definition of div and $S(0,n)$ are given in Example 4.2 below). This $W(0,n)$-module is irreducible iff $\lambda \neq 0,1$. For $n=2$, $\lambda \neq 0,1$, it defines a strongly transitive subalgebra of $g\ell(2,2)$, denoted by $w(0,2;\lambda)$ if we reverse the parity of $\Lambda(2)$. (Note that $w(0,2;\frac{1}{2}) = spo(2,2)$ and that $w(0,2;\lambda)$ is isomorphic to $s\ell(2,1)$ as an abstract superalgebra.) Of course, $cw(0,2;\lambda) := w(0,2;\lambda) + \mathbb{C}I$ is strongly transitive as well.

The standard $W(0,2)$-module $\Lambda(2)$ with reversed parity, extended in an obvious way to the semidirect sum $W(0,2) + \Lambda(2)$, is again a strongly transitive subalgebra of $g\ell(2,2)$, which we denote by $\widetilde{w}(0,2)$. Furthermore, $S(0,2) + \Lambda(2)$ and $S(0,2) + (\mathbb{C} + \mathbb{C}\xi_1 + \mathbb{C}\xi_2)$ are still strongly transitive. We denote them by $\widetilde{s}(0,2)$ and $\widetilde{s}^{\circ}(0,2)$ respectively.

Example 3.8 Let $m=n$ and let J be an operator on V such that $J^2 = I$ and $J(V_{\bar{0}}) = V_{\bar{1}}$. Let

$$\widetilde{q}(n) = \{a \in g\ell(n,n) |\, aJ = Ja\}\ .$$

This is a strongly transitive subalgebra of $g\ell(n,n)$ for $n \geq 1$. It contains a subalgebra

$$q(n) = \left\{ \left(\begin{array}{cc} a & b \\ b & a \end{array} \right) \in \widetilde{q}(n) |\, trb = 0 \right\}\ .$$

This is a strongly transitive subalgebra (denoted by $\widetilde{Q}(n-1)$ in [K4]), provided that $n \geq 2$.

Example 3.9 Let H be an odd superspace of dimension n with a non-degenerate symmetric bilinear form $(\ \ ,\ \ )$. The Heisenberg superalgebra $\mathcal{H}_n$ is the superspace $\mathcal{H}_n = \mathbb{C}c + H$, where c is an even central element, with the bracket $[p,q] = (p,q)c$, $p,q \in H$. Note that $U(\mathcal{H}_n)/(c-1)$ is the usual Clifford superalgebra associated to H. (As usual, $U(\mathfrak{g})$ stands for the universal enveloping superalgebra of a Lie superalgebra $\mathfrak{g}$.) Hence $\mathcal{H}_n$ has a unique irreducible module, denoted by S (= spinor module), for which $c = I$.

Its dimension is $(2^{[\frac{n-1}{2}]}, 2^{[\frac{n-1}{2}]})$. The representation of $\mathcal{H}_n$ in S extends to a representation, which we denote by σ_n, of the semidirect sum $so_n + \mathcal{H}_n$ by the following well known formulas, where $\{e_i\}$ is an orthonormal basis of H:

$$\sigma_n((a_{ij})) = \sum_{i,j=1}^{n} a_{ij}\sigma_n(e_i)\sigma_n(e_j)\,, \text{ where } a_{ji} = -a_{ij}\,.$$

This representation of $so_n + \mathcal{H}_n$ gives rise to a subalgebra of $g\ell(2^{[\frac{n-1}{2}]}, 2^{[\frac{n-1}{2}]})$ which we denote by $\widetilde{\mathrm{spin}}_n$. Note that $\widetilde{\mathrm{spin}}_1 = \widetilde{q}(1)$, $\widetilde{\mathrm{spin}}_2 = g\ell(1,1)$ and $\widetilde{\mathrm{spin}}_3 = q(2)$ are strongly transitive subalgebras of $g\ell(1,1)$. It is easy to see that $\widetilde{\mathrm{spin}}_4$ is a strongly transitive subalgebra of $g\ell(2,2)$ but $\widetilde{\mathrm{spin}}_n$ for $n \geq 5$ is not strongly transitive. (Note that $\hat{p}(4)$ contains $\widetilde{\mathrm{spin}}_6$.)

Furthermore, we have: $so_4 \simeq \mathfrak{a}_1 \oplus \mathfrak{a}_2$, where $\mathfrak{a}_i \simeq s\ell_2$ and $\sigma_4(\mathfrak{a}_1)$ (resp. $\sigma_4(\mathfrak{a}_2)$) acts via the standard (resp. trivial) representation of $s\ell_2$ on $S_{\bar{0}}$ and a trivial (resp. standard) representation of $s\ell_2$ on $S_{\bar{1}}$. Then $\mathfrak{a}_1 + \mathcal{H}_4$ still acts strongly transitively on S. We denote the corresponding subalgebra of $g\ell(2,2)$ by spin_4°. Adding to it an arbitrary subalgebra $\mathfrak{a}$ of $\mathfrak{a}_2$ again gives a strongly transitive subalgebra of $g\ell(2,2)$, which we denote by $\mathrm{spin}_4^\circ + \mathfrak{a}$.

Theorem 3.1 *All strongly transitive subalgebras of $g\ell(m,n)$ where $m \geq 1$, $n \geq 0$, are listed in Examples 3.2–3.9. Namely, they are:*

(a) $g\ell(m,n)$,

(b) $s\ell(m,n)$ *for* $m+n \geq 2$,

(c) $spo(m,n)$ *for* $m \geq 2$, m *even,*

(d) $cspo(m,n)$ *for* $m \geq 2$, m *even,*

(e) one of the subalgebras of $\widetilde{s\ell}_m[\xi] \subset g\ell(m,m)$, $m > 2$, *containing* $s\ell_m[\xi]$ *and having a non-zero projection on* $\mathbb{C}\frac{d}{d\xi}$,

(f) one of the subalgebras of $\widetilde{sp}_m[\xi] \subset g\ell(m,m)$, $m \geq 2$, m *even, containing* $sp_m[\xi]$ *and having a non-zero projection on* $\mathbb{C}\frac{d}{d\xi}$,

(g) $\widetilde{\mathrm{p}}(n)$ *for* $n \geq 2$,

(h) $\widetilde{\mathrm{p}}(n\,;\,\beta)$ *for* $n \geq 2$, $\beta \in \mathbb{C}$,

(i) $c\widetilde{p}(n)$ *for* $n \geq 2$*,*

(j) $p(n)$ *for* $n \geq 2$*,*

(k) $w(0,2\,;\,\lambda)$ *for* $\lambda \neq 0,1$*,*

(l) $cw(0,2\,;\,\lambda)$ *for* $\lambda \neq 0,1$*,*

(m) $\widetilde{w}(0,2)$*,*

(n) $\widetilde{s}(0,2)$*,*

(o) $\widetilde{s}^{\circ}(0,2)$*,*

(p) $\mathrm{spin}_4^{\circ} + \mathfrak{a}\,, \quad \mathfrak{a} \subset s\ell_2$*,*

(q) $\widehat{p}(4)$*,*

(r) $\widetilde{q}(n)$ *for* $n \geq 1$*,* $q(n)$ *for* $n \geq 2$*.*

The proof of this theorem is based on several lemmas.

Lemma 3.1 *Let* $\mathfrak{g} \subset g\ell_n$ *be a strongly transitive subalgebra. Then either* $\mathfrak{g} = g\ell_n$*,* $n \geq 1$*, or* $\mathfrak{g} = s\ell_n$*,* $n \geq 2$*, or* $\mathfrak{g} = csp_n$*,* $n \geq 2$*, or* $\mathfrak{g} = sp_n$*,* $n \geq 2$*.*

Proof (cf. [G2]). If v_Λ is the highest weight vector and v_M is the lowest weight vector of the representation of $\mathfrak{g}$ in $\mathbb{C}^n$, then, by strong transitivity, $av_\Lambda = v_M$ for some $a \in \mathfrak{g}$. We may assume that a is a root vector $e_{-\alpha}$, where α is a positive root. Hence $\Lambda + (-M) = \alpha$ and $[\mathfrak{g},\mathfrak{g}]$ is simple (by faithfulness). But both summands on the left are non-zero dominant, hence α is a dominant root with $\sum_i \alpha(H_i) \geq 2$, where H_i are all simple coroots. It follows from the well-known list of dominant roots, that $[\mathfrak{g},\mathfrak{g}]$ is either $s\ell_n$ or sp_n, with α the highest root.

$\square$

Corollary 3.1 *If* $V = V_{\bar{0}} + V_{\bar{1}}$ *is a strongly transitive module over a Lie superalgebra* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$*, then the* $\mathfrak{g}_{\bar{0}}$*-module* $V_{\bar{0}}$ *is irreducible; moreover, the image of* $\mathfrak{g}_{\bar{0}}$ *in* $EndV_{\bar{0}}$ *is one of the linear Lie algebras listed by Lemma 3.1.*

Lemma 3.2 *Let* V *be a strongly transitive module over a Lie superalgebra* $\mathfrak{g}$ *and let* $\mathfrak{a}$ *be a non-zero abelian ideal of* $\mathfrak{g}$*. Then there are two possibilities:*

(a) $\mathfrak{g}$ *is one of the following subalgebras of* $g\ell(2,2)$*:* $\widetilde{w}(0,2)$*,* $\widetilde{s}^\circ(0,2)$*,* $\widetilde{s}(0,2)$*,*

(b) $\mathfrak{a}$ *is an even central subalgebra acting on* V *by scalar operators.*

Proof There exists $\lambda \in \mathfrak{a}^*$ such that the associated weight space $V^\lambda = \{v \in V | a(v) = \lambda(a)v, a \in \mathfrak{a}\}$ is non-zero. Let $\mathfrak{g}^\lambda = \{g \in \mathfrak{g} | \lambda([g,\mathfrak{a}]) = 0\}$ be the stabilizer of λ, and let $U = U(\mathfrak{g}^\lambda)v_\lambda$, where $v_\lambda \in V^\lambda$ is a non-zero (even or odd) vector. Then U is a direct sum of isomorphic 1-dimensional $\mathfrak{g}^\lambda$-modules, hence by Blattner's theorem ([B1], [Ch]) we obtain:

$$V = Ind_{\mathfrak{g}^\lambda}^{\mathfrak{g}} U \,. \tag{3.1}$$

Since $\dim V < \infty$, we conclude that

$$\mathfrak{g}^\lambda \supset \mathfrak{g}_{\bar{0}} \,. \tag{3.2}$$

Going over to the Zariski closure, we may assume that $\mathfrak{g}_{\bar{0}}$ is an algebraic Lie algebra; let $\mathfrak{s}$ be a maximal reductive subalgebra of $\mathfrak{g}_{\bar{0}}$. Due to Corollary 3.1, $\mathfrak{s}$ acts irreducibly on $V_{\bar{0}}$ (since the nil-radical of $\mathfrak{g}_{\bar{0}}$ acts trivially). With respect to the adjoint representation of $\mathfrak{s}$ on $\mathfrak{g}$, we have a decomposition as $\mathfrak{s}$-modules:

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}^\lambda \,, \tag{3.3}$$

where, due to (3.2), we have:

$$\mathfrak{g}' \subset \mathfrak{g}_{\bar{1}} \,,\; \mathfrak{s} \subset \mathfrak{g}^\lambda \,. \tag{3.4}$$

Due to (3.1)-(3.4), we conclude that, as an $\mathfrak{s}$-module, $V_{\bar{0}}$ is a direct sum of at least $[\frac{1}{2}(1+\dim\mathfrak{g}')]$ modules. It follows from Corollary 3.1 that

$$\dim \mathfrak{g}' \leq 2 \,. \tag{3.5}$$

If $\dim \mathfrak{g}' = 2$, we see that, as an $\mathfrak{s}$-module:

$$V = U + \mathfrak{g}' \otimes U + \Lambda^2 \mathfrak{g}' \otimes U \,.$$

It follows from Corollary 3.1 that $\dim U = (0,1)$ (otherwise $V_{\bar{0}}$ is not irreducible as an $\mathfrak{s}$-module), hence $\dim V = (2,2)$ and the $\mathfrak{s}$-module $V_{\bar{0}}$ is either $s\ell_2$ or $g\ell_2$, while the $[\mathfrak{s},\mathfrak{s}]$-module $V_{\bar{1}}$ is trivial. It is easy to see now that $\mathfrak{g}$

lies in the subalgebra $\widetilde{w}(0,2)$ of $g\ell(2,2)$ (defined in Example 3.7). It follows that $\mathfrak{g}$ is one of the strongly transitive subalgebras listed in Example 3.7.

If $\mathfrak{g} = \mathfrak{g}^\lambda$, then V^λ is a $\mathfrak{g}$-submodule of V, hence $V = V^\lambda$ and lemma is proved.

Thus, the following situation remains:

$$\mathfrak{g} = \mathbb{C}b + \mathfrak{g}^\lambda ,$$

where b is a non-zero odd element such that $[\mathfrak{s}, b] \subset \mathbb{C}b$, and

$$V = U \oplus bU .$$

Again, by Corollary 3.1, it follows that $U = V_{\bar{0}}$, hence $\mathfrak{g}^\lambda = \mathfrak{g}_{\bar{0}}$ and $\mathfrak{a}_{\bar{0}}$ acts by scalar operators on $V_{\bar{0}}$. If $\mathfrak{a} \neq \mathfrak{a}_{\bar{0}}$, then $b \in \mathfrak{a}$, hence $[b, \mathfrak{a}] = 0$ and therefore $\mathfrak{a}_{\bar{0}}$ acts by scalar operators on V and is a central subalgebra of $\mathfrak{g}$; then $[\mathfrak{g}_{\bar{0}}, b] \subset \mathbb{C}b$ and since $bV^\lambda = 0$ we conclude that $\mathfrak{g}V^\lambda \subset V^\lambda$, hence $V = V^\lambda$ and lemma is proved.

Thus, $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}_{\bar{0}}$ acting by scalar operators on $V_{\bar{0}}$. Since $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, we have $[b, \mathfrak{a}] \subset \mathfrak{a}$, and since b is an odd element, we conclude that $[b, \mathfrak{a}] = 0$. Hence $\mathfrak{a}$ acts by scalar operators on V.

□

Lemma 3.3 *Let V be a strongly transitive module over a Lie superalgebra $\mathfrak{g}$ and let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. Then provided that $\mathfrak{r} \neq 0$, there are three possibilities:*

- *(a) $\mathfrak{g}$ is one of the following subalgebras of $g\ell(1,1)$: $g\ell(1,1)$, $s\ell(1,1)$, $\widetilde{q}(1)$ (and $\mathfrak{g} = \mathfrak{r}$),*
- *(b) $\mathfrak{g}$ is one of the following subalgebras of $g\ell(2,2)$: $\widetilde{w}(0,2)$, $\widetilde{\mathrm{spin}_4}$, $\widetilde{s}(0,2)$, $\widetilde{s}^\circ(0,2)$, $q(2)$, $\widetilde{q}(2)$, $\mathrm{spin}_4^\circ + \mathfrak{a}$ (where $0 \subseteq \mathfrak{a} \subsetneqq s\ell_2$) (and $(\mathfrak{g}/\mathfrak{r}) \simeq s\ell(2,1)$, so_4, $s\ell_2$ in the remaining cases, respectively),*
- *(c) $\mathfrak{r}$ is an even 1-dimensional central subalgebra of $\mathfrak{g}$ acting on V by scalar operators.*

Proof Let $\mathfrak{r} = \mathfrak{r}^{(0)} \supset \mathfrak{r}^{(1)} \supset \ldots \supset \mathfrak{r}^{(k-1)} \supset \mathfrak{r}^{(k)} \supset 0$ be the derived series of $\mathfrak{r}$ with $\mathfrak{r}^{(k)} \neq 0$. By Lemma 3.2, we may assume that: $\mathfrak{r}^{(k)} = \mathbb{C}c$, where c is a central element of $\mathfrak{g}$ acting as identity on V. We may also assume that

$\dim V \neq (1,1)$ and that $k \geq 1$; let $\mathfrak{p} = \mathfrak{r}^{(k-1)}$ for short. We have: $[x, \mathfrak{p}] = \mathbb{C}c$ for any non=zero $x \in \mathfrak{p}$, since otherwise x generates an abelian ideal of $\mathfrak{g}$, which contradicts Lemma 3.2. Since $\dim V < \infty$, the superspace $\mathfrak{p}/\mathbb{C}c$ is purely odd. Therefore $\mathfrak{p} = H + \mathbb{C}c$, where H is a non-zero odd subspace of $\mathfrak{g}$, c is an even central element represented in V by I and $[p,q] = (p,q)c$ for $p, q \in H$, where $(\ \ ,\ \ )$ is a non-degenerate symmetric bilinear form on H, which is invariant under the adjoint action of $\mathfrak{g}_{\bar{0}}$; we also have: $[\mathfrak{g}_{\bar{1}}, H] \subset \mathbb{C}c$. Note that $\mathfrak{p}$ is the Heisenberg superalgebra considered in Example 3.9.

Recall (see Example 3.9) that $\mathfrak{p}$ has a unique irreducible representation σ in a vector space S such that $\sigma(c) = I$. Let $\mathfrak{p}^-$ be a maximal isotropic subspace of H and let U be the subspace of V consisting of vectors annihilated by $\mathfrak{p}^-$ (vacuum subspace). Then, as a $\mathfrak{p}$-module, V is isomorphic to $U \otimes S$, where $\mathfrak{p}$ acts via $1 \otimes \sigma$. We shall identify V with $U \otimes S$. The representation σ in S extends from $\mathfrak{p}$ to the whole $\mathfrak{g}$ by the following formulas (cf. Example 3.9):

$$\sigma(a) = \sum_{i,j} a_{ij}\sigma(e_i)\sigma(e_j) \text{ if } a \in \mathfrak{g}_{\bar{0}}\,,\ [a, e_i] = \sum_j a_{ij} e_j\,,$$

$$\sigma(b) = \sum_i b_i \sigma(e_i) \text{ if } b \in \mathfrak{g}_{-1}\,,\ [b, e_i] = b_i c\,.$$

This representation of $\mathfrak{g}$ in S extends to $V = U \otimes S$ via $1 \otimes \sigma$.

We let $\mu(g) = g - (1 \otimes \sigma)g\,,\quad g \in \mathfrak{g}$. It is clear that $\mu(\mathfrak{p}) = 0$ and hence μ is a representation of $\mathfrak{g}$ in V commuting with $\mathfrak{p}$. Therefore, the subspace U of V is $\mu(\mathfrak{g})$-invariant and is a $\mathfrak{g}/\mathfrak{p}$-module via μ. Thus, the action of $g \in \mathfrak{g}$ in the $\mathfrak{g}$-module $V = U \otimes S$ looks as follows:

$$g = (\mu \otimes 1)g + (1 \otimes \sigma)g\,.$$

Since V is irreducible, the $\mathfrak{g}$-module U must be irreducible too.

Let $\mathfrak{s}$ be a maximal reductive subalgebra of $\mathfrak{g}_{\bar{0}}$ (cf. Lemma 3.2). If $\mathfrak{s}$ is abelian, then $\mathfrak{g}_{\bar{0}}$ is solvable, hence $\mathfrak{g}$ is solvable ([K4], Proposition 1.3.3), hence either $\dim V = 1$ or $\dim V_{\bar{0}} = \dim V_{\bar{1}}$ ([K4], Proposition 5.2.3); by Corollary 3.1, it follows that in the latter case $\dim V_{\bar{0}} = \dim V_{\bar{1}} = 1$.

Thus, we may assume that $\mathfrak{s}$ is not abelian and, as an $[\mathfrak{s},\mathfrak{s}]$- module, $V_{\bar{0}} = U_{\bar{0}} \otimes S_{\bar{0}} + U_{\bar{1}} \otimes S_{\bar{1}}$. It follows that $\dim U = 1$ and hence $\mathfrak{g}$ is a subalgebra of $\widetilde{\mathrm{spin}}_n$ if n is even and of $\widetilde{\mathrm{spin}}_n + \mathbb{C}d$ if n is odd, where $n = \dim H$ and d is an odd endomorphism of the $\mathfrak{p}$-module S (cf. Example 3.9). It follows that $n = 3$ or 4. Then it is straightforward to see that $\mathfrak{g}$ is $q(2)$ or $\widetilde{q}(2)$ if $n = 3$ and $\mathfrak{g}$ is one of the subalgebras of $\widetilde{\mathrm{spin}}_4$ containing spin_4° if $n = 4$.

$\square$

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra of type $X = A, B, C, D, F, G, P$ or Q, and let $\mathrm{Der}\,\mathfrak{g}$ denote the Lie superalgebra of derivations of $\mathfrak{g}$ (it is described by [K4], Proposition 5.1.2). A central extension by even center of a subalgebra of $\mathrm{Der}\,\mathfrak{g}$ containing $\mathfrak{g}$ will be called an almost simple Lie superalgebra of classical type X.

Lemma 3.4 *Let $\mathfrak{g}$ be an almost simple Lie superalgebra of classical type and let V be a strongly transitive module over $\mathfrak{g}$ of dimension (m, n), where $m, n \geq 1$. Then the corresponding strongly transitive subalgebra of $g\ell(m, n)$ is one of the following:*

(a) $g\ell(m, n)$ and $s\ell(m, n)$ for $m + n \geq 3$ or $(m, n) = (2, 0)$,

(b) $spo(m, n)$ and $cspo(m, n)$ for $m \geq 2$,

(c) $w(0, 2; \lambda)$ and $cw(0, 2; \lambda)$ for $\lambda \neq 0, 1$,

(d) $\widetilde{p}(n; \beta), c\widetilde{p}(n), p(n)$ and $\widetilde{p}(n)$ for $n \geq 3$ and $\hat{p}(4)$,

(e) $\widetilde{q}(n)$ and $q(n)$ for $n \geq 3$.

Proof is similar to that of Lemma 3.1. Let Λ be the highest weight of the $\mathfrak{g}_{\bar{0}}$-module $V_{\bar{0}}$ and M the lowest weight of an irreducible component of the $\mathfrak{g}_{\bar{0}}$-module $V_{\bar{1}}$. By strong transitivity, $e_{-\alpha} v_\Lambda = v_M$ for an odd positive root α of $\mathfrak{g}$, hence $\alpha = \Lambda + (-M)$ is a dominant odd root of $\mathfrak{g}$. Moreover, due to Corollary 3.1, if $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$ has at least 2 simple components, we may assume that the restriction of M to at least one simple component of $\mathfrak{g}_{\bar{0}}$ is non-zero. Hence

$$\sum_i \alpha(H_i) \geq 2 \tag{3.6}$$

if $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$ has at least 2 simple components, where H_i are the simple coroots of $\mathfrak{g}_{\bar{0}}$. If $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$ has only one simple component and (3.6) does not hold, then by the above remarks, $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_0]$ must act trivially on $V_{\bar{1}}$.

A quick inspection of cases shows that α must be the highest weight of an irreducible submodule of the adjoint representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$.

It easily follows from the above discussion that if $\mathfrak{g}$ is of a type A, B, C or D, then only the possibilities (a), (b) and (c) of the lemma occur. Since in our situation the number k of [K4], Theorem 8 is always 1, the three exceptional superalgebras are ruled out. Finally, if $\mathfrak{g}$ is of type P (resp. Q), only (c) (resp. (d)) are possible.

$\square$

Lemma 3.5 *Let $\mathfrak{g}$ be an almost simple Lie superalgebra of Cartan type (but not of classical type). Then $\mathfrak{g}$ has no strongly transitive modules.*

Proof Recall [K4] that $\mathfrak{g}$ admits a filtration by subalgebras $\mathfrak{g} \supset \mathfrak{g}_{(0)} \supset \mathfrak{g}_{(1)} \supset \dots$ where $\mathfrak{g}_{(1)}$ is the radical of $\mathfrak{g}_{(0)}$ and $\mathfrak{s} := \mathfrak{g}_{(0)}/\mathfrak{g}_{(1)}$ is one of the Lie algebras $g\ell_n$, $s\ell_n$, so_n, cso_n. Let V be a strongly transitive $\mathfrak{g}$-module. Then, by Corollary 3.1, V is a quotient of the module induced from the even irreducible $\mathfrak{g}_{(0)}$-module U on which $\mathfrak{g}_{(1)}$ acts trivially and $\mathfrak{s}$ acts via the standard representation of $g\ell_n$, $s\ell_n$, sp_n or csp_n.

If $\mathfrak{g}$ is of type $W(n)$, $n \geq 3$, $\widetilde{S}(n)$, $n \geq 4$, or $S(n)$, $n \geq 4$, then V is isomorphic to $\Lambda(n)/\mathbb{C}1$ (resp. a submodule of codimension 1 in it) with reversed parity if $\mathfrak{g}$ is of type $W(n)$ or $\widetilde{S}(n)$ (resp. $S(n)$). In all these cases, however, $V_{\bar{0}}$ is spanned by $\xi_i, \xi_i\xi_j\xi_k, \dots$, hence is not an irreducible $\mathfrak{s}$-module.

The case of $\mathfrak{g}$ of type $H(n)$ with $n > 6$ is ruled out since then $\mathfrak{s}$ is not of type A_n or C_n. If $n = 5$ (resp. $n = 6$), then the only possibility for the $\mathfrak{s}$-module $V_{\bar{0}}$ is sp_4 (resp. $s\ell_4$). One checks directly that $V_{\bar{0}}$ cannot be irreducible in these cases, which rules out types $H(5)$ and $H(6)$ as well.

□

Recall that a semisimple finite-dimensional Lie superalgebra $\mathfrak{g}$ contains an ideal of the form $S = \oplus_{i=1}^{k} \Lambda(n_i) \otimes S_i$, called the *socle* of $\mathfrak{g}$, where S_i are simple Lie superalgebras, n_i are non-negative integers, and $\mathfrak{g}$ is contained in the Lie superalgebra of derivations of S (see [K4], [C]). We shall call the number $k \geq 1$ the length of $\mathfrak{g}$. We let $S_i(n) = \Lambda(n) \otimes S_i$ for short.

Lemma 3.6 *Let $\widetilde{\mathfrak{g}}$ be a central extension of a semisimple Lie superalgebra $\mathfrak{g}$ by an even center, and let V be a strongly transitive module over $\widetilde{\mathfrak{g}}$. Then*

(a) The length k of $\mathfrak{g}$ is 1.

(b) Either the socle S of $\mathfrak{g}$ is a simple Lie superalgebra, or $S = \Lambda(1) \otimes S_1$, where S_1 is a simple Lie algebra and $V = V_{\bar{0}} + \xi_1 V_{\bar{0}}$ as an S-module.

Proof By Corollary 3.1, we may assume that $(S_1)_{\bar{0}}$ acts irreducibly on $V_{\bar{0}}$. Let Λ be the highest weight of this module. Suppose that $k \geq 2$. Then $(S_2)_{\bar{0}} V_{\bar{0}} = 0$. Let M be a non-zero lowest weight of $(S_2)_{\bar{0}}$ in $V_{\bar{1}}$. Arguing as in the proof of Lemma 3.4, we see that $\Lambda - M$ is a root of $\oplus_i S_i$ whose restriction to S_1 and S_2 is non-zero, a contradiction proving (a).

Thus $S = \Lambda(m) \otimes S_1$ where $\Lambda(m)$ is a Grassmann algebra in the indeterminates $\xi_1, \dots, \xi_m$ and S_1 is a simple Lie superalgebra.

Suppose that S_1 is not a Lie algebra. Let U be a non-trivial irreducible S_1-submodule of V. Then, by Corollary 3.1, $U_{\bar{0}} = V_{\bar{0}}$, $(\xi_1(S_1)_{\bar{1}})V_{\bar{0}} = 0$, hence $(\xi_1(S_1)_{\bar{0}})V_{\bar{0}} = 0$ if $m \geq 1$. But then $(\xi_1(S_1)_{\bar{0}})V_{\bar{1}}$ is a non-zero submodule of $V_{\bar{0}}$, a contradiction.

Hence either S is a simple Lie superalgebra, or S_1 is a simple Lie algebra. In the latter case, by Corollary 3.1, the S_1-module $V_{\bar{0}}$ is isomorphic to $s\ell_n$ or sp_n, $n \geq 2$, and $V = V_{\bar{0}} + \sum_i \xi_i V_{\bar{0}} + \sum_{i<j} \xi_i \xi_j V_{\bar{0}} + \dots$. It follows that $m = 1$, proving (b).

$\square$

Proof of Theorem 3.1. Let $\mathfrak{g} \subset g\ell(m,n)$, $m \geq 1$, $n \geq 0$, be a strongly transitive subalgebra. If $n = 0$, then, by Lemma 3.1, $\mathfrak{g}$ is one of the linear Lie algebras (a)-(d) with $n = 0$. Thus, we may assume that $n \geq 1$. If $m = n = 1$, then it is easy to see that only $g\ell(1,1)$, $s\ell(1,1)$ and $\widetilde{q}(1)$ are possible. Hence, by Lemma 3.3, either $\mathfrak{g}$ is one of the linear Lie superalgebras $(m)-(p)$ listed by the theorem, or we may assume that the radical of $\mathfrak{g}$ is an even central subalgebra acting on V by scalars. If $\mathfrak{g}$ is an almost simple Lie superalgebra, then, by Lemmas 3.4 and 3.5 we may have only cases (a)-(d), (g)-(l) and (q)-(r) of the theorem. Finally, if $\mathfrak{g}$ is a central extension of a semisimple Lie superalgebra by an even center and $\mathfrak{g}$ is not almost simple, by Lemma 3.6 we may have only cases (e) and (f) of the theorem.

$\square$

4 Classification of even transitive irreducible graded Lie superalgebras: the case of inconsistent gradation

Before stating the main theorem of this section, we consider some examples. The basic definitions may be found in [K4]. The proofs of the statements in these examples (that are not entirely obvious) may be found in [S2] and [CK2].

Example 4.1 (general superalgebras of vector fields) Let $\Lambda(m,n)$ be the associative (commutative) superalgebra $\mathbb{C}[x_1, \dots, x_m] \otimes \Lambda(n)$, $m, n \geq 0$, $m + n > 0$ (it is a dense subalgebra of the linearly compact algebra $\overline{\Lambda}(m,n)$). Let $W(m,n)$ denote the Lie superalgebra of all derivations of the superalgebra $\Lambda(m,n)$. It consists of linear operators of the form:

$$\sum_{i=1}^{m} P_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j \frac{\partial}{\partial \xi_j}\,, \quad \text{where } P_i\,, Q_j \in \Lambda(m,n)\,, \tag{4.1}$$

hence it is a dense subalgebra of $\overline{W}(m,n)$. It is a simple Lie superalgebra if $(m,n) \neq (0,1)$.

Let $(a_1,\dots,a_m|b_1,\dots,b_n)$ be an $(m+n)$-tuple of integers such that all the a_i are positive. Then, letting

$$\deg x_i = a_i = -\deg \frac{\partial}{\partial x_i}\,,\ \deg \xi_j = b_j = -\deg \frac{\partial}{\partial \xi_j}$$

defines a $\mathbb{Z}$-gradation of the Lie superalgebra $W(m,n)$. (All $\mathbb{Z}$-gradations of $W(m,n)$ satisfying (G0) are obtained, up to automorphism, in this way, cf. [K5].) This is called a *$\mathbb{Z}$-gradation of type* $(a_1,\dots,a_m|b_1,\dots,b_n)$. The $\mathbb{Z}$-gradation of type $(1,\dots,1|1,\dots,1)$ (resp. $(1,\dots,1|0,\dots,0)$) is called *principal* (resp. *subprincipal*) gradation of $W(m,n)$.

The Lie superalgebra $W(m,n)$ with one of the above gradations is an even graded Lie superalgebra in the following two cases:

(a) Arbitrary $W(m,n)$ with the principal gradation. (It is the associated graded for the even primitive Lie superalgebra $(\overline{W}(m,n), \overline{W}(m,n)_0)$.) It is an even irreducible transitive $\mathbb{Z}$-graded Lie superalgebra of depth 1.

(b) The Lie superalgebras $W(m,1)$, $m \geq 1$, with the *subprincipal* gradation. They are even irreducible transitive $\mathbb{Z}$-graded Lie superalgebras of depth 1.

Remark 4.1 The associated graded to the primitive Lie superalgebra $(\overline{W}(m,n), \overline{J}_i W(m,n))$ considered in Example 1.2 is $W(m,n)$ with $\mathbb{Z}$-gradation of type $(1,\dots,1|1,\dots,1,0,\dots,0)$ with $n-i$ zeros.

Example 4.2 (special superalgebras of vector fields) Denote by $S'(m,n)$ the subalgebra of $W(m,n)$ which consists of operators D of the form (4.1) with zero *divergence* defined by:

$$\operatorname{div} D = \sum_i \frac{\partial P_i}{\partial x_i} + \sum_j (-1)^{p(Q_j)} \frac{\partial Q_j}{\partial \xi_j}\,,$$

and let $S(m,n) = [S'(m,n), S'(m,n)]$. Then $S(m,n)$ coincides with $S'(m,n)$ if $m > 1$ or $m = 0$, $n \geq 3$, and then it is simple. One has:

$$S'(1,n) = S(1,n) + \mathbb{C}\xi_1 \dots \xi_n \frac{\partial}{\partial x_1} ,$$

and $S(1,n)$ is simple iff $n \geq 2$.

All the $\mathbb{Z}$-gradations of $W(m,n)$ defined in Example 4.1 induce $\mathbb{Z}$-gradations of $S(m,n)$ and $S'(m,n)$. The *principal* and *subprincipal* gradations of these Lie superalgebras are defined as those induced by principal and subprincipal gradations of $W(m,n)$.

We have the following two cases of even $\mathbb{Z}$-graded Lie superalgebras, both of depth 1:

(a) $S(m,n)$ and $S'(1,n)$ with the principal gradation. They are irreducible transitive $\mathbb{Z}$-graded Lie superalgebras, except for $S(1,1)$ which is not irreducible.

(b) $S(m,1)$ with the subprincipal gradation. They are transitive irreducible $\mathbb{Z}$-graded Lie superalgebras iff $m \geq 2$ (for $m = 1$ transitivity fails).

Example 4.3 (Hamiltonian superalgebras) Let $m = 2s$ be even and consider the following *even Hamiltonian* differential form:

$$h_{m,n} = 2\sum_{i=1}^{s} dx_i \wedge dx_{s+i} + \sum_{j=1}^{n} d\xi_j \, d\xi_{n-j+1} .$$

Denote by $H'(m,n)$ the subalgebra of $W(m,n)$ consisting of operators that annihilate $h_{m,n}$, and let $H(m,n) = [H'(m,n), H'(m,n)]$. The Lie superalgebra $H'(m,n)$ consists of operators of the form:

$$\begin{aligned} H_f &= \sum_{i=1}^{s} \left(\frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_{s+i}} - \frac{\partial f}{\partial x_{s+i}} \frac{\partial}{\partial x_i} \right) \\ &- (-1)^{p(f)} \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \xi_{n-j+1}} + \frac{\partial f}{\partial \xi_{n-j+1}} \frac{\partial}{\partial \xi_j} \right) , \end{aligned}$$

where $f \in \Lambda(m,n)$. One has: $H(m,n)$ coincides with $H'(m,n)$ if $m \geq 2$, and then it is simple. Furthermore:

$$H'(0,n) = H(0,n) + \mathbb{C}H_{\xi_1 \dots \xi_n} ,$$

and $H(0,n)$ is simple iff $n \geq 5$.

The gradation of type $(a_1, \dots | \, b_1, \dots)$ of $W(m,n)$ induces a gradation on $H'(m,n)$ (and $H(m,n)$) iff the differential form $h_{m,n}$ is homogeneous in this gradation [K5] (if we put $\deg dx_i = a_i$ and $\deg d\xi_j = b_j$). In particular, the principal gradation of $W(m,n)$ and the gradation of type $(1, \dots, 1| \, 2, 0)$ of $W(m,2)$ induce gradations on the Hamiltonian superalgebras, called the *principal* and *subprincipal* gradations respectively of these superalgebras. Again we have two cases of even transitive irreducible $\mathbb{Z}$-graded Lie superalgebras:

(a) $H'(0,n)$, $n \geq 3$ and $H(m,n)$, $m \geq 2$, with principal gradation, which have depth 1,

(b) $H(m,2)$, $m \geq 2$, with subprincipal gradation, which have depth 2 ($\dim \mathfrak{g}_{-2} = (0,1)$).

Example 4.4 (contact superalgebras) Let $m = 2s+1$ be odd and consider the following *even contact* differential form:

$$k_{m,n} = dx_m + \sum_{i=1}^{s} (x_i \, dx_{s+i} - x_{s+i} \, dx_i) + \sum_{j=1}^{n} \xi_j \, d\xi_{n-j+1} \, .$$

Let $K(m,n) = \{D \in W(m,n) | Dk_{m,n} = fk_{m,n}$ for some $f \in \Lambda(m,n)\}$. The Lie superalgebra $K(m,n)$ is always simple. Since $K(1,2)$ is isomorphic to $W(1,1)$, we shall always assume when talking about $K(m,n)$ that $(m,n) \neq (1,2)$.

As before, a $\mathbb{Z}$-gradation of $W(m,n)$ induces one on $K(m,n)$ if the differential form $k_{m,n}$ is homogeneous. In particular, the $\mathbb{Z}$-gradation of type $(1, \dots, 1, 2| \, 1, \dots, 1)$ (resp. $(1, \dots, 1, 2|2, 0)$) of $W(m,n)$ (resp. $W(m,2)$) induces a $\mathbb{Z}$-gradation of $K(m,n)$ (resp. $K(m,2)$), called the *principal* (resp. *subprincipal*) gradation of these Lie superalgebras. Again, we have two cases of even transitive irreducible $\mathbb{Z}$-graded Lie superalgebras:

(a) $K(m,n)$ with principal gradation, which have depth 2 ($\dim \mathfrak{g}_{-2} = (1,0)$).

(b) $K(m,2)$ with subprincipal gradation, which have depth 2 ($\dim \mathfrak{g}_{-2} = (1,1)$).

Example 4.5 Let $P = \oplus_j P_j$ be one of the following Lie algebras: $W(m,0)$, $S(m,0)$, $H(m,0)$ or $K(m,0)$ with the principal gradation. As in Example 3.4, consider the Lie superalgebra $P[\xi]$ with the gradation, called principal, defined by letting $\deg \xi = 0$. As in Example 3.4, consider the semi-direct sum $\widetilde{P}[\xi] = P[\xi] + g\ell(1,1)$ with $\deg g\ell(1,1) = 0$. Let $\mathfrak{a}$ be a subalgebra of $g\ell(1,1)$ with a non-zero projection on $\mathbb{C}\frac{d}{d\xi}$. Then $P[\xi] + \mathfrak{a}$ is an even transitive irreducible $\mathbb{Z}$-graded Lie superalgebra.

Example 4.6 (odd Hamiltonian superalgebras) Consider the following odd Hamiltonian differential form:

$$ho_n = \sum_{i=1}^{n} dx_i \, d\xi_i \, .$$

Denote by $HO(n,n)$ the subalgebra of $W(n,n)$ consisting of operators that annihilate ho_n. This Lie superalgebra consists of operators of the form:

$$HO_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i} \right) , \quad f \in \Lambda(m,n) \, .$$

It is simple iff $n \geq 2$. The principal gradation of $W(m,n)$ induces a $\mathbb{Z}$-gradation, called again *principal*, of $HO(m,n)$. This is an even irreducible transitive $\mathbb{Z}$-graded Lie superalgebra of depth 1 if $n \geq 2$ (for $n = 1$ it is not irreducible).

We shall need the following explicit formula:

$$\begin{aligned} [HO_f, HO_g] &= HO_{\{f,g\}_{ho}} \, , \text{where} \\ \{f,g\}_{ho} &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} \right) . \end{aligned}$$

Note that $HO(2,2)$ with gradation of type $(1,1|0,0)$ is isomorphic to $S(2,1)$ with the principal gradation (since $W(0,2) \simeq s\ell(2,1)$). Therefore, we shall always consider $HO(n,n)$ with $n \geq 3$.

Example 4.7 (special odd Hamiltonian superalgebras). Denote by $SHO'(n,n)$ the subalgebra of divergence zero operators of $HO(n,n)$ and let $SHO(n,n)$ denote its derived subalgebra. Then one has:

$$SHO'(n,n) = SHO(n,n) + \mathbb{C} HO_{\xi_1 \dots \xi_n} \, .$$

The Lie superalgebra $SHO(n,n)$ is simple iff $n \geq 3$. The principal gradation is defined as in Example 4.6. The $\mathbb{Z}$-graded Lie superalgebras $SHO'(n,n)$ with $n \geq 2$ and $SHO(n,n)$ with $n \geq 3$ are even irreducible transitive of depth 1 ($SHO(n,n)$ for $n = 1,2$ and $SHO'(1,1)$ are not irreducible.)

Example 4.8 (odd contact superalgebras). Consider the following odd contact differential form:

$$ko_n = d\xi_{n+1} + \sum_{i=1}^{n} (x_i \, d\xi_i + \xi_i \, dx_i) \, .$$

Let $KO(n, n+1) = \{D \in W(n, n+1) | D(ko_n) = f(ko_n)$ for some $f \in \Lambda(n, n+1)\}$. This Lie superalgebra is simple for all $n \geq 1$. Here we have two cases of even transitive irreducible $\mathbb{Z}$-graded Lie superalgebras:

(a) The $\mathbb{Z}$-gradation of $W(n, n+1)$ of type $(1, \dots, 1|1, \dots, 1, 2)$ induces a $\mathbb{Z}$-gradation, called *principal*, of $KO(n, n+1)$. This is an even irreducible transitive $\mathbb{Z}$-graded Lie superalgebra of depth 2 (and $\dim \mathfrak{g}_{-2} = (0,1)$), provided that $n \geq 2$ (for $n = 1$ it is not irreducible).

(b) The gradation of type $(1, \dots, 1| 0, \dots, 0, 1)$ of $W(n, n+1)$ induces a $\mathbb{Z}$-gradation of $KO(n, n+1)$, called its *subprincipal* gradation. These $\mathbb{Z}$-graded superalgebras are transitive irreducible of depth 1 for $n \geq 2$, but only $KO(2,3)$ with subprincipal gradation is even.

In the description of the next example we shall use that $KO(n, n+1)$ consists of operators of the form ($f \in \Lambda(n, n+1)$):

$$\begin{aligned} KO_f &= HO_f + (E(f) - 2f)\frac{\partial}{\partial \xi_{n+1}} + (-1)^{p(f)}\frac{\partial f}{\partial \xi_{n+1}} E \, , \\ \text{where } E &= \sum_{i=1}^{n} (x_i \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial \xi_i}) \, . \end{aligned}$$

One has: $[KO_f, KO_g] = KO_{\{f,g\}_{ko}}$, where

$$\{f,g\}_{ko} = \{f,g\}_{ho} + (E-2)f\frac{\partial g}{\partial \xi_{n+1}} + (-1)^{p(f)}\frac{\partial f}{\partial \xi_{n+1}}(E-2)g \, .$$

Remark 4.2 Let $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial \xi_i}$ be the odd Laplacian. Then

$$\begin{aligned} \operatorname{div} HO_f &= 2(-1)^{p(f)}\Delta(f)\,, \\ \operatorname{div} KO_f &= 2(-1)^{p(f)}(\Delta(f) + (E-1)\frac{\partial f}{\partial \xi_{n+1}})\,. \end{aligned}$$

Remark 4.3 The Lie superalgebras $W(1,1)$, $K(1,2)$ and $KO(1,2)$ are isomorphic. However, the principal gradation of $W(1,1)$ only is irreducible.

Example 4.9 (special odd contact superalgebras) Given $\beta \in \mathbb{C}$, introduce the deformed divergence (cf. Remark 4.2):

$$\operatorname{div}_\beta f = 2(-1)^{p(f)}(\Delta(f) + (E - n\beta)\frac{\partial f}{\partial \xi_{n+1}})\,,$$

and define the following subalgebra of $KO(n, n+1)$ (cf. [Ko]):

$$SKO'(n, n+1; \beta) = \{KO_f \,|\, \operatorname{div}_\beta f = 0\}\ .$$

As before, we denote by $SKO(n, n+1; \beta)$ its derived algebra. It is simple iff $n \geq 2$. One has: $SKO'(n, n+1; \beta) = SKO(n, n+1; \beta)$ for all $n \geq 2$ and all β with the following two exceptions (cf. [Ko]):

$$\begin{aligned} SKO'(n, n+1; 1) &= SKO(n, n+1; 1) + \mathbb{C}KO_{\xi_1 \xi_2 \ldots \xi_{n+1}}\,, \\ SKO'(n, n+1; \frac{n-2}{n}) &= SKO(n, n+1; \frac{n-2}{n}) + \mathbb{C}KO_{\xi_1 \ldots \xi_n}\,. \end{aligned}$$

Again we have two cases of even transitive irreducible Lie superalgebras:

(a) The principal gradation of $KO(n, n+1)$ induces a $\mathbb{Z}$-gradation, called *principal*, of all these Lie superalgebras. All of them are even irreducible transitive $\mathbb{Z}$-graded Lie superalgebras of depth 2 (and $\dim \mathfrak{g}_{-2} = (0,1)$), provided that $n \geq 2$ (for $n = 1$ irreducibility fails).

(b) The subprincipal gradation of $KO(n, n+1)$ induces a gradation of $SKO'(n, n+1; \beta)$ and $SKO(n, n+1; \beta)$, called their *subprincipal* gradation. These $\mathbb{Z}$-graded Lie superalgebras are transitive irreducible of depth 1 for $n \geq 2$, but they are even only for $n = 2$ (and arbitrary β).

Example 4.10 (exceptional Lie superalgebra $E(4,4)$ [S2]) There exists a simple $\mathbb{Z}$-graded Lie superalgebra of depth 1: $E(4,4) = \oplus_{j \geq -1} \mathfrak{g}_j$ such that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ gives rise to the linear Lie superalgebra $\hat{p}(4)$. It is an even irreducible transitive $\mathbb{Z}$-graded Lie superalgebra.

Example 4.11 (exceptional Lie superalgebra $E(2,2)$ [CK2]). There exists a simple $\mathbb{Z}$-graded Lie superalgebra of depth 1: $E(2,2)=\oplus_{j\geq -1}\mathfrak{g}_j$, such that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ gives rise to the linear Lie superalgebra spin_4°. One has:

$$\mathrm{Der}\ E(2,2)=E(2,2)+s\ell_2\,,$$

where $\deg s\ell_2=0$ and the $(\mathfrak{g}_0+s\ell_2)$-module $\mathfrak{g}_{-1}$ gives rise to the linear Lie superalgebra $\widetilde{\mathrm{spin}}_4$. Thus, for any subalgebra $\mathfrak{a}$ of $s\ell_2$ we get an even irreducible transitive $\mathbb{Z}$-graded Lie superalgebra $E(2,2)+\mathfrak{a}$.

The gradations in Examples 4.10 and 4.11 are called *principal*.

The following two propositions are used in the sequel. Their proofs are straightforward (though a bit messy) and may be found in [CK2]. Here and further we use the following notations. Let $\mathfrak{g}=\oplus_{j\geq -d}\mathfrak{g}_j$ be a $\mathbb{Z}$-graded Lie superalgebra. Then $\mathbb{C}_0+\mathfrak{g}$ denotes the semidirect sum of $\mathbb{C}z$ with $\mathfrak{g}$, where z is a derivation of $\mathfrak{g}$ such that $z|_{\mathfrak{g}_j}=jI$ and $\deg z=0$. If $\mathfrak{g}$ is transitive, then $\mathbb{C}_0+\mathfrak{g}$ is transitive iff $\mathfrak{g}_0$ contains no non-zero elements acting as scalars on $\mathfrak{g}_{-1}$. Also, if $\mathfrak{g}_0=p(m)$ or $cp(m)$, one can form the semidirect sum of $\mathbb{C}F$ with $\mathfrak{g}$; this will be denoted by $\mathbb{C}_0'+\mathfrak{g}$. Furthermore, if $d=1$ and $\mathfrak{g}_{-1}$ has an even (resp. odd) skew-supersymmetric bilinear form $(\ ,\ )$ which is invariant with respect to $[\mathfrak{g}_0,\mathfrak{g}_0]$, then $\mathfrak{g}$ has a central extension, denoted by $\mathbb{C}_{-2}+\mathfrak{g}$, constructed by adding to $\mathfrak{g}$ the even (resp. odd) central 1-dimensional subspace $\mathbb{C}c$ in degree -2 and defining a new bracket on $\mathfrak{g}_{-1}$ by:

$$[x,y]\widehat{\ }=[x,y]+(x,y)c,\quad x,y\in\mathfrak{g}_{-1}\,.$$

Proposition 4.1 *All even transitive irreducible $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g}=\oplus_{j\geq -1}\mathfrak{g}_j$ of depth 1 with the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ giving rise to the indicated below subalgebra of $g\ell(m,n)$, where $m\geq 1$, are as follows (with restrictions on m and n given in Examples 4.1-4.9):*

I. Superalgebras with the principal gradation:

(a) $g\ell(m,n)$: $W(m,n)$, $\mathbb{C}_0+S(m,n)$, $\mathbb{C}_0+S'(m,n)$,

(b) $s\ell(m,n)$: $S(m,n)$, $S'(m,n)$,

(c) $spo(m,n)$: $H(m,n)$,

(d) $cspo(m,n)$: $\mathbb{C}_0+H(m,n)$,

(e) $s\ell_m[\xi]+\mathfrak{a}$: $W(m,0)[\xi]+\mathfrak{a}$, $S(m,0)[\xi]+\mathfrak{a}$,

(f) $sp_m[\xi]+\mathfrak{a}$: $H(m,0)[\xi]+\mathfrak{a}$,

(g) $\widetilde{p}(m)$: $HO(m,m)$, $\mathbb{C}'_0+SHO'(m,m)$, $\mathbb{C}'_0+SHO'(m,m)$,

(h) $c\widetilde{p}(m)$: $\mathbb{C}_0+HO(m,m)$, $\mathbb{C}_0+\mathbb{C}'_0+SHO(m,m)$, $\mathbb{C}_0+\mathbb{C}'_0+SHO'(m,m)$,

(i) $p(m)$: $SHO(m,m)$, $SHO'(m,m)$,

(j) $cp(m)$: $\mathbb{C}_0+SHO(m,m)$, $\mathbb{C}_0+SHO'(m,m)$,

(k) $\hat{p}(4)$: $E(4,4)$,

(l) $\mathrm{spin}_4^\circ+\mathfrak{a}$: $E(2,2)+\mathfrak{a}$.

II. Superalgebras with the subprincipal gradation:

(a) $\widetilde{s}\ell_m[\xi]$: $W(m,1)$, $S(m,1)+\mathbb{C}_0+\mathbb{C}\xi$,

(b) $s\ell_m[\xi]+\mathbb{C}\frac{d}{d\xi}+\mathbb{C}((m-1)\xi\frac{d}{d\xi}+I)$: $S(m,1)$,

(c) $s\ell_m[\xi]+\mathbb{C}\xi\frac{d}{d\xi}+\mathbb{C}I$: $\mathbb{C}_0+S(m,1)$,

(d) $w(0,2;\lambda)$: $SKO(2,3;1-\frac{1}{\lambda})$,

(e) $cw(0,2;\lambda)$: $\mathbb{C}_0+SKO(2,3;1-\frac{1}{\lambda})$,

(f) $\widetilde{w}(0,2)$: $KO(2,3)$,

(g) $\widetilde{s}^\circ(0,2)$: $SKO(2,3;1)$,

(h) $\widetilde{s}(0,2)$: $SKO'(2,3;1)$.

III. (a) $\widetilde{q}(n)$ *or* $q(n)$: *none.*

Proposition 4.2 *All even transitive irreducible $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g}=\oplus_{j\geq -d}\mathfrak{g}_j$ of depth $d\geq 2$ with the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ giving rise to the indicated below subalgebra of $g\ell(m,n)$, where $m\geq 1$, have depth 2 and are as follows:*

I. Superalgebras with the principal gradation and 1-dimensional even $\mathfrak{g}_{-2}$:

(a) $spo(m,n)$: $\mathbb{C}_{-2}+H(m,n)$,

(b) $cspo(m,n)$: $\mathbb{C}_{-2}+\mathbb{C}_0+H(m,n)$, $K(m,n)$.

II. Superalgebras with the principal gradation and 1-dimensional odd $\mathfrak{g}_{-2}$:

(a) $\widetilde{p}(m)$: $\mathbb{C}_{-2}+HO(m,m)$, $\mathbb{C}_{-2}+\mathbb{C}_0'+SHO(m,m)$, $\mathbb{C}_{-2}+\mathbb{C}_0'+SHO'(m,m)$,

(b) $c\widetilde{p}(m)$: $KO(m,m+1)$, $\mathbb{C}_{-2}+\mathbb{C}_0+HO(m,m)$, $\mathbb{C}_{-2}+\mathbb{C}_0+\mathbb{C}_0'+SHO(m,m)$, $\mathbb{C}_{-2}+\mathbb{C}_0+\mathbb{C}_0'+SHO'(m,m)$, $\mathbb{C}_0+SKO(m,m+1;\beta)$,

(c) $p(m)$: $\mathbb{C}_{-2}+SHO(m,m), \mathbb{C}_{-2}+SHO'(m,m)$,

(d) $cp(m)$: $\mathbb{C}_{-2}+\mathbb{C}_0+SHO(m,m+1), \mathbb{C}_{-2}+\mathbb{C}_0+SHO'(m,m)$,

(e) $\widetilde{p}(m,\beta)$: $SKO(m,m+1;\beta), SKO'(m,m+1;\beta)$,

(f) $sp_m[\xi]+\mathbb{C}\xi\frac{d}{d\xi}+\mathbb{C}\frac{d}{d\xi}:((H(m,0)+\mathbb{C}_{-2})[\xi]+\mathbb{C}\xi\frac{d}{d\xi}+\mathbb{C}\frac{d}{d\xi})/\mathbb{C}_{-2}$.

III. *Superalgebras with the subprincipal gradation and* 1*-dimensional odd* $\mathfrak{g}_{-2}$*:*

(a) $sp_m[\xi]+\mathbb{C}\xi\frac{d}{d\xi}+\mathbb{C}\frac{d}{d\xi}$: $H(m,2)$.

IV. *Superalgebras with the principal gradation and* $\dim\mathfrak{g}_{-2}=(1,1)$*:*

(a) $csp_m[\xi]+\mathfrak{a}$: $K(m+1,0)[\xi]+\mathfrak{a}$, $(\mathbb{C}_{-2}+H(m,0))[\xi]+\mathfrak{a}$ *with a non-trivial projection on* $\mathbb{C}\xi$.

V. *Superalgebras with the subprincipal gradation and* $\dim\mathfrak{g}_{-2}=(1,1)$*:*

(a) $\widetilde{sp}_m[\xi]$: $K(m+1,2)$,

(b) $csp_m[\xi]+\mathfrak{a}:\mathbb{C}_{-2}+H(m,2)+\mathfrak{a}$.

Now we can state and prove the main theorem of this section.

Theorem 4.1 *Let* $\mathfrak{g}=\oplus_{j\geq -d}\mathfrak{g}_j$ *be a* $\mathbb{Z}$*-graded even transitive irreducible Lie superalgebra. Suppose that the gradation of* $\mathfrak{g}$ *is not consistent (i.e., that* $\mathfrak{g}_{-1}$ *contains a non-zero even element). Then* $\mathfrak{g}$ *is one of the* $\mathbb{Z}$*-graded Lie superalgebras listed below.*

I. *The following Lie superalgebras with the principal* $\mathbb{Z}$*-gradation:*

(a) $W(m,n)$ *with* $m\geq 1$,

(b) $S(m,n)$ *with* $m\geq 2, S(1,n)$ *and* $S'(1,n)$ *with* $n\geq 2$,

(c) $H(m,n)$ *with* $m\geq 2$ *even,*

(d) $K(m,n)$ *with* $m \geq 3$ *odd,*

(e) $SHO(m,m)$ *with* $m \geq 3$*, and* $SHO'(m,m)$ *with* $m \geq 2$*,*

(f) $HO(m,m)$ *with* $m \geq 2$*,*

(g) $SKO(m,m+1;\beta)$ *and* $SKO'(m,m+1;\beta)$ *for* $m \geq 2$*,*

(h) $KO(m,m+1)$ *for* $m \geq 2$*,*

(i) $E(4,4)$*,*

(j) $E(2,2)+\mathfrak{a}$*, where* $\mathfrak{a}$ *is a subalgebra of* $s\ell_2$*,*

(k) $P[\xi]+\mathfrak{a}$*, where* P *is one of the Lie algebras* $W(m,0)$*,* $S(m,0)$*, or* $H(m,0)$ *and* $\mathfrak{a} \subset g\ell(1,1)$ *has a non-trivial projection on* $\mathbb{C}\frac{\partial}{\partial\xi}$*,*

(l) $(\mathbb{C}_{-2}+H(m,0))[\xi]+\mathfrak{a}$ *and* $K(m,0)[\xi]+\mathfrak{a}$*, where* $\mathfrak{a}$ *is as in* (k)*,*

(m) $((H(m,0)+\mathbb{C}_{-2})[\xi]+\mathbb{C}\xi\frac{d}{d\xi}+\mathbb{C}\frac{d}{d\xi})/\mathbb{C}_{-2}$.

II. *The following Lie superalgebras with the subprincipal gradation:*

(a) $W(m,1)$ *with* $m \geq 1$*,*

(b) $S(m,1)$ *and* $S(m,1)+\mathbb{C}_0+\mathbb{C}\xi$ *with* $m \geq 2$*,*

(c) $H(m,2)$ *with* $m \geq 2$ *even,*

(d) $K(m,2)$ *with* $m \geq 3$ *odd,*

(e) $KO(2,3)$*,*

(f) $SKO(2,3;\beta)$ *and* $SKO'(2,3;\beta)$.

III. (a) $\mathbb{C}_0'+\mathfrak{g}$*, where* $\mathfrak{g}$ *is of type Ie,*

(b) $\mathbb{C}_{-2}+\mathfrak{g}$*, where* $\mathfrak{g}$ *is of types Ie, If, IIIa,*

(c) $\mathbb{C}_{-2}+H(m,0)[\xi]+\mathfrak{a}$*, where* $\mathfrak{a}=\mathbb{C}\frac{d}{d\xi}$ *or* $\mathfrak{a}=\mathbb{C}\frac{d}{d\xi}+\mathbb{C}I$.

IV. $\mathbb{C}_0+\mathfrak{g}$*, where* $\mathfrak{g}$ *is one of the above* $\mathbb{Z}$*-graded superalgebras for which* $\mathfrak{g}_0$ *has a trivial center.*

Proof By Proposition 2.1, the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ gives rise to a strongly transitive subalgebra of $g\ell(m,n)$, where (m,n) is the dimension of $\mathfrak{g}_{-1}$, $m \geq 1$. All such subalgebras are listed by Theorem 3.1.

Let $\mathfrak{b}=\oplus_{j\leq -2}\mathfrak{b}_j$ be the maximal ideal of $\mathfrak{g}$ contained in $\mathfrak{g}^-$ and let $\mathfrak{g}'=\mathfrak{g}/\mathfrak{b}=\oplus_j\mathfrak{g}_j'$. Recall that, by Lemma 2.4, $\mathfrak{b}$ is odd, hence it is abelian and $[(\mathfrak{g}_0)_{\bar{1}},\mathfrak{b}]=0$.

Due to Lemma 2.6, $\mathfrak{g}'_{-2} = [a, \mathfrak{g}_{-1}]$, where a is a non-zero even element of $\mathfrak{g}_{-1}$. Hence $\dim \mathfrak{g}'_{-2} < \dim \mathfrak{g}_{-1}$. But in all cases listed in Theorem 3.1 except for (e), (f), (k) and (l), $\mathfrak{g}_{-1}$ is the irreducible non 1-dimensional module over $\mathfrak{g}_0$ of minimal dimension (with the non-zero action of the center of $\mathfrak{g}_0$ in case (p)); in cases (e) and (f) the only other possibility for an irreducible $\mathfrak{g}_0$-module of smaller dimension is the $(1,1)$-dimensional module which is trivial on $\mathfrak{p}[\xi]$. In cases (k) and (l), the condition $\mathfrak{g}'_{-2} = [a, \mathfrak{g}_{-1}]$ for an even weight vector rules out the possibility $\dim \mathfrak{g}'_{-2} > 1$ by looking at $s\ell_2$-weights.

Thus, all irreducible subquotients of the $\mathfrak{g}_0$-module $\mathfrak{g}'_{-2}$ are either $(1,0)$-dimensional, or $(0,1)$-dimensional, or $(1,1)$-dimensional. Since the $\mathfrak{g}_0$-module $\mathfrak{g}'_{-2}$ is a quotient of $\Lambda^2 \mathfrak{g}_{-1}$, it is easy to see by inspection of the list given by Theorem 3.1 that one has four possibilities:

1) $\dim \mathfrak{g}'_{-2} = 0$,

2) $\dim \mathfrak{g}'_{-2} = (1,0)$ and the $[\mathfrak{g}_0, \mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$ has an even invariant skew-supersymmetric bilinear form,

3) $\dim \mathfrak{g}'_{-2} = (0,1)$ and the $[\mathfrak{g}_0, \mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$ has an odd such form,

4) $\dim \mathfrak{g}'_{-2} = (1,1)$ and the $[\mathfrak{g}_0, \mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$ has both even and odd such forms.

In Case 1) depth $\mathfrak{g}' = 1$, then either $1'$) depth $\mathfrak{g} = 1$ or $1''$) $\dim \mathfrak{g}_{-2} = (0,1)$ which is possible if the $[\mathfrak{g}_0, \mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$ has an odd skew-supersymmetric invariant bilinear form. In Case $1'$) we use Proposition 4.1 which shows that only the following cases of Theorem 4.1 occur: I (a)-(c), (e), (f), (i)-(k), II (a)-(c), (e), (f), III (a) and IV. In Case $1''$) $\mathfrak{g}_{-2}$ is central, hence $\mathfrak{g}_{-3} = 0$ and depth $\mathfrak{g} = 2$; by Proposition 4.2, we get only Cases III (b) and (c) of Theorem 4.1.

In Cases 2)-4) the depth of $\mathfrak{g}' = 2$ since otherwise, as above, $\mathfrak{g}'_{-3} = [\mathfrak{g}'_{-2}, a]$ for some even $a \in \mathfrak{g}_{-1}$, hence $\dim \mathfrak{g}'_{-3} \leq \dim \mathfrak{g}'_{-2}$. But taking bracket of $\mathfrak{g}_{-1}$ with an element from $\mathfrak{g}'_{-2}$ (resp. $\mathfrak{g}_{-2}$) establishes a homomorphism of $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules $\mathfrak{g}_{-1} \to \mathfrak{g}'_{-3}$ (resp. $\mathfrak{g}_{-3}$). Hence $\mathfrak{g}'_{-3} = 0$ and $\mathfrak{g}_{-3} = 0$ (since $\mathfrak{b}_3$ is odd). Thus, in Cases 2)-4) the depth of $\mathfrak{g}$ is 2. Since the $\mathfrak{g}_0$-module $\mathfrak{g}_{-2}$ is a quotient of $\Lambda^2 \mathfrak{g}_{-1}$, we see, as above, that the only possibilities for $\mathfrak{g}_{-2}$ are: $2'$) $\dim \mathfrak{g}_{-2} = (1,0)$ and $2''$) $\dim \mathfrak{g}_{-2} = (1,1)$ in Case 2); and 3) $\dim \mathfrak{g}_{-2} = (0,1)$ (resp. 4) $\dim \mathfrak{g}_{-2} = (1,1)$) in Cases 3) (resp. 4)). Using Proposition 4.2, we see that in Case $2'$) we have only Case I (d) of Theorem 4.1, in Case $2''$) we have only Case III (b) of Theorem 4.1, in Case 3) we have only Cases I (g), (h), (m) and IV of Theorem 4.1, and in Case 4) we have only cases I(l) and II(d) of Theorem 4.1.

□

5 Classification of even transitive irreducible graded Lie superalgebras: the case of consistent gradation

A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ is called *consistent* if $\mathfrak{g}_{\bar{0}} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_{2j}$ and $\mathfrak{g}_{\bar{1}} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_{2j+1}$. Note that a consistent transitive $\mathbb{Z}$-graded Lie superalgebra of depth 1 is finite-dimensional (since it can be embedded in $W(0, \dim\mathfrak{g}_{-1})$). A consistent $\mathbb{Z}$-graded Lie superalgebra of depth ≥ 2 is infinite-dimensional (since otherwise all even elements are exponentiable). We shall treat these cases separately in Theorems 5.1 and 5.3.

In the statement of the following theorem and further on we shall use the following notation: $s\ell_n$, sp_n and so_n denote the standard modules of these Lie algebras, $s\ell_n^*$ denotes the contragredient module, spin_7 denotes the 8-dimensional spinor representation of so_7, 1 stands for the trivial 1-dimensional representation. The sign $\boxtimes$ stands for the outer tensor product of $\mathfrak{g}_1$-module V_1 and $\mathfrak{g}_2$-module V_2, i.e., the $\mathfrak{g}_1\oplus\mathfrak{g}_2$-module $V_1\otimes V_2$, and $\otimes$ stands for the usual tensor product of $\mathfrak{g}$-modules. As usual, S^kV and Λ^kV denote the k-th symmetric and exterior powers of the $\mathfrak{g}$-module V (in the "super" sense). The 1-dimensional module over $\mathbb{C}$ for which $1\mapsto a$ is denoted by $\mathbb{C}(a)$. Similar notation is used for Lie superalgebras.

Theorem 5.1 *([K4], Theorem 4) The following is a complete list of transitive irreducible consistent $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g} = \oplus_{j=-1}^{k}\mathfrak{g}_j$ of depth 1 and $k\geq 1$:*

I. The $\mathfrak{g}_0$-modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are contragredient and $k=1$:

- *(a) $s\ell(m,n)$, $m\neq n$, $m,n\geq 1$ ($\mathfrak{g}_0$-module $\mathfrak{g}_{-1} = g\ell_m\boxtimes s\ell_n$),*
- *(b) $s\ell(n,n)$, $n\geq 2$ ($\mathfrak{g}_0$-module $\mathfrak{g}_{-1} = s\ell_n\boxtimes s\ell_n$),*
- *(c) $spo(m,2)$, m even ≥ 2 ($\mathfrak{g}_0$-module $\mathfrak{g}_{-1} = csp_m$),*
- *(d) $\mathbb{C}_0+\mathfrak{g}$, where $\mathfrak{g}$ is of type I(b).*

II. The $\mathfrak{g}_0$-modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are not contragredient and $k=1$:

(a) $p(n)$, $n \geq 3$ *($\mathfrak{g}_0$-module $\mathfrak{g}_{\mp 1}$ (resp. $\mathfrak{g}_{\pm 1}$) $= S^2 s\ell_n$ (resp. $\Lambda^2 s\ell_n^*$)),*

(b) $\mathfrak{p}[\xi] + \mathbb{C}\frac{d}{d\xi}$*, where* $\mathfrak{p}$ *is a simple Lie algebra ($\mathfrak{p}_0$-module $\mathfrak{p}_{-1} = ad\,\mathfrak{p}$ and $\mathfrak{p}_1 = 1$),*

(c) $\mathbb{C}_0 + \mathfrak{g}$*, where* $\mathfrak{g}$ *is of types II (a), (b).*

III. $k > 1$:

(a) $W(0,n)$, $n \geq 3$*, with principal gradation ($\mathfrak{g}_0$-module $\mathfrak{g}_{-1} = g\ell_n$),*

(b) $S(0,n)$, $n \geq 4$*, with principal gradation ($\mathfrak{g}_0$-module $\mathfrak{g}_{-1} = s\ell_n$),*

(c) $H(0,n)$, $n \geq 5$*, with principal gradation ($\mathfrak{g}_0$-module $\mathfrak{g}_{-1} = so_n$),*

(d) $H'(0,n)$, $n \geq 4$*, with principal gradation ($\mathfrak{g}_0$-module $\mathfrak{g}_{-1} = so_n$),*

(e) $\mathbb{C}_0 + \mathfrak{g}$*, where* $\mathfrak{g}$ *is of types III (b)-(d).*

A transitive $\mathbb{Z}$-graded Lie superalgebra is called *bitransitive* if, in addition to properties (G0), (G1), (G2), it satisfies the following two properties:

(G8) if $a \in \mathfrak{g}_j$ with $j < -1$ and $[a, \mathfrak{g}_1] = 0$, then $a = 0$,

(G9) $\mathfrak{g}_j = \mathfrak{g}_1^j$ for each $j \geq 1$.

Recall (see [K1], [K4]) that for any transitive local Lie superalgebra $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, there exists a unique bitransitive $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$.

Theorem 5.2 *The following is a complete list of bitransitive irreducible consistent $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g} = \oplus_{j \geq -d} \mathfrak{g}_j$ of depth $d \geq 2$ ($d = 2$ unless otherwise stated), such that the $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are contragredient (in parenthesis the $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules $\mathfrak{g}_j$ for $-1 \geq j \geq -d$ are described; the $\mathfrak{g}_0$-modules $\mathfrak{g}_j$ and $\mathfrak{g}_{-j}$ are contragredient):*

(a) $spo(2m,n)$, $m \geq 1$, $n \geq 1$, $n \neq 2$ $\quad$ $(\mathfrak{g}_{-1} = s\ell_m \boxtimes so_n,\ \mathfrak{g}_{-2} = S^2 s\ell_m \boxtimes 1)$,

(b) $spo(2m,2n)$, $m \geq 1$, $n \geq 2$ $\quad$ $(\mathfrak{g}_{-1} = s\ell_n \boxtimes sp_{2m},\ \mathfrak{g}_{-2} = \Lambda^2 s\ell_n \boxtimes 1)$,

(c) $D(2,1;\alpha)$ $\quad$ $(\mathfrak{g}_{-1} = so_4,\ \mathfrak{g}_{-2} = 1)$,

(d) $F(4)$ $\quad$ $(\mathfrak{g}_{-1} = \mathrm{spin}_7,\ \mathfrak{g}_{-2} = 1)$,

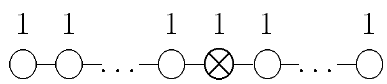
(e) $F(4)$ $\quad$ $(\mathfrak{g}_{-1} = s\ell_2 \boxtimes sp_4,\ \mathfrak{g}_{-2} = 1 \boxtimes so_5)$,

(f) $F(4)$ $\quad$ *($\mathfrak{g}_{-1} = s\ell_3 \boxtimes s\ell_2$, $\mathfrak{g}_{-2} = s\ell_3^* \boxtimes 1, \mathfrak{g}_{-3} = 1 \boxtimes s\ell_2$, $\mathfrak{g}_{-4} = s\ell_3 \boxtimes 1$),*

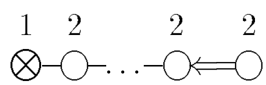
(g) *G(3)* $\quad$ *($\mathfrak{g}_{-1} = 7$-dimensional irreducible G_2-module, $\mathfrak{g}_{-2} = 1$).*

Proof The Lie algebra $\mathfrak{g}_0$ is a direct sum of a semisimple Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ and at most 1-dimensional center $\mathbb{C}c$. Let e_i, f_i, h_i, $i = 1, \dots, r$, be the Chevalley generators of $[\mathfrak{g}_0, \mathfrak{g}_0]$, let e_0 (resp. f_0) be the lowest (resp. highest) weight vector of the $[\mathfrak{g}_0, \mathfrak{g}_0]$-module $\mathfrak{g}_1$ (resp. $\mathfrak{g}_{-1}$), and let $h_0 = [e_0, f_0]$. Then the elements e_i, f_i, h_i, $i = 0, 1, \dots, r$, generate a finite-dimensional contragredient Lie superalgebra $\mathfrak{g}'$ [K4] such that $\mathfrak{g} = \mathfrak{g}' + \mathbb{C}c$. The contragredient Lie superalgebra $\mathfrak{g}'$ satisfies the conditions of Theorem 3 of [K4] and also has a unique odd simple root. Thus, we must select among all diagrams given by Proposition 2.5.6 of [K4] those with a unique non-white node. A complete list of these diagrams, along with the coefficients of the highest root, is as follows (unfortunately some of the diagrams for $F(4)$ and $G(3)$ were inadvertently omitted in [K4]); here $m, n \geq 1$ and m is even for $spo(m, n)$:

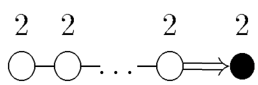
$s\ell(m, n)$



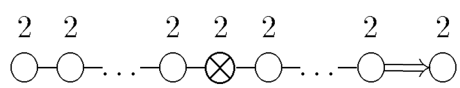
$spo(m, 2), m > 2$



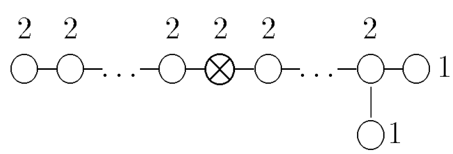
$spo(m, 1)$



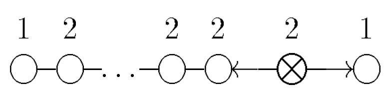
$spo(m, n)$, $n > 1$ odd



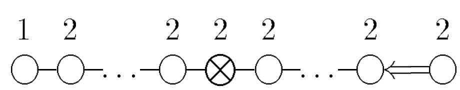
$spo(m, n)$, n even



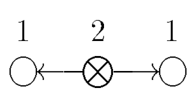
$spo(2, n), n$ even



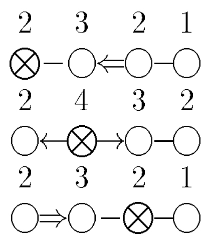
$spo(m, n), m > 2, n$ even



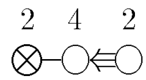
$D(2, 1; \alpha)$

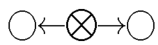


$F(4)$



$G(3)$



Here the diagram  corresponds to the matrix $\begin{pmatrix} 2 & -1 & 0 \\ \alpha & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, where α is arbitrary in the case $D(2,1;\alpha)$, $\alpha = -1$ in the case $spo(2,n)$ and $\alpha = 3/2$ in the case $F(4)$.

The depth of $\mathfrak{g}$ is then equal to the label of the non-white node, the Dynkin diagram of $[\mathfrak{g}_0, \mathfrak{g}_0]$ is obtained by removing the non-white node, and non-zero labels of the highest weight of the $[\mathfrak{g}_0, \mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$ equal 1 for the nodes connected to the non-white node. The $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules $\mathfrak{g}_j$ for $-d \leq j \leq -2$ are computed directly.

$\square$

Corollary 5.1 *Let $\mathfrak{g} = \oplus_{j \geq -d} \mathfrak{g}_j$ be an even transitive consistent $\mathbb{Z}$-graded Lie superalgebra such that the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ contains a submodule $\mathfrak{g}'_1$ contragredient to $\mathfrak{g}_{-1}$, and denote by $\mathfrak{g}' = \oplus_j \mathfrak{g}'_j$ the bitransitive $\mathbb{Z}$-graded Lie superalgebra with local part $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}'_1$. Then the $\mathfrak{g}_0$-modules $\mathfrak{g}'_j$ and $\mathfrak{g}'_{-j}$ are contragredient, $\dim \mathfrak{g}' < \infty$ and there are the following possibilities for $\mathfrak{g}'$ and the $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$:*

I. $\mathfrak{g}' = \mathfrak{g}'_{-1} + \mathfrak{g}_0 + \mathfrak{g}'_1$*:*

- *(a)* $\mathfrak{g}' = s\ell(m,n)$, $m+n \geq 0$, $m+n > 0, \mathfrak{g}_{-1} = s\ell_m \boxtimes s\ell_n$,
- *(b)* $\mathfrak{g}' = spo(m,2)$, *m even* ≥ 2, $\mathfrak{g}_{-1} = sp_m$.

II. $\mathfrak{g}' = \oplus_{j=-2}^{2} \mathfrak{g}'_j$ *and the center of* $\mathfrak{g}_0$ *is 1-dimensional:*

- *(a)* $spo(2,n)$, $n \geq 1$, $n \neq 2$, $\mathfrak{g}_{-1} = so_n$, $\mathfrak{g}_{-2} = 1$,
- *(b)* $spo(2m,4)$, $m \geq 2$, $\mathfrak{g}_{-1} = s\ell_2 \boxtimes sp_{2m}$, $\mathfrak{g}_{-2} = 1$,

(c) $spo(2m,6)$, $m \geq 1$, $\mathfrak{g}_{-1} = s\ell_3 \boxtimes sp_{2m}$, $\mathfrak{g}_{-2} = s\ell_3^* \boxtimes 1$,

(d) $D(2,1;\alpha)$, $\mathfrak{g}_{-1} = so_4$, $\mathfrak{g}_{-2} = 1$,

(e) $F(4)$, $\mathfrak{g}_{-1} = \mathrm{spin}_7$, $\mathfrak{g}_{-2} = 1$,

(f) $G(3)$, $\mathfrak{g}_{-1} =$ *7-dimensional irreducible G_2-module*, $\mathfrak{g}_{-2} = 1$.

Proof The local part of the Lie superalgebra $\mathfrak{g}'$ admits a Cartan involution which induces the Cartan involution on $\mathfrak{g}_0$ and exchanges $\mathfrak{g}_{-1}$ and $\mathfrak{g}'_1$. It induces a Cartan involution of $\mathfrak{g}'$ which exchanges $\mathfrak{g}'_{-j}$ and $\mathfrak{g}'_j$, hence $\mathfrak{g}'_{-j}$ and $\mathfrak{g}'_j$ are contragredient and $\dim \mathfrak{g}' < \infty$.

In the case when depth $\mathfrak{g}' = 1$ we may apply Theorem 5.1, Case I.

In the case when depth $\mathfrak{g}' \geq 2$ we use Theorem 5.2. We apply Lemma 2.6 in order to eliminate some cases of Theorem 5.2. Since $S^2 s\ell_m$ is strongly transitive only for $m = 1$, case (a) is possible only if $m = 1$, which gives case II(a) of the corollary. Similarly, since $\Lambda^2 s\ell_n$ is strongly transitive only for $n = 2$ and 3, we get cases II(b), (c) of the corollary. Case (e) of Theorem 5.2 is ruled out since so_5 is not a strongly transitive module. Finally, case (f) of Theorem 5.2 is ruled out since $\mathfrak{g}'_{-4} = [a, \mathfrak{g}'_{-2}]$ for some non-zero $a \in \mathfrak{g}'_{-2}$ (by Lemma 2.6), hence $\dim \mathfrak{g}'_{-4} < \dim \mathfrak{g}'_{-2}$.

$\square$

Before stating the main theorem of this section (Theorem 5.3) we introduce some more examples. The proofs of the statements in these examples may be found in [CK2].

Example 5.1 The contact superalgebra $K(1,n)$ with the principal $\mathbb{Z}$-gradation (see Example 4.4) is an even transitive irreducible consistent Lie superalgebra of depth 2 for $n = 1$ and $n \geq 3$ (for $n = 2$ irreducibility fails).

Example 5.2 (exceptional Lie superalgebra $E(1,6)$ [CK1], [S2]) Consider the Lie superalgebra $K(1,6) = \oplus_{j \geq -2} \mathfrak{g}_j$ with the principal gradation. The $[\mathfrak{g}_0, \mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$ gives rise to the linear Lie algebra $so_6 \simeq \Lambda^2 s\ell_4$ and the $[\mathfrak{g}_0, \mathfrak{g}_0]$-module $\mathfrak{g}_1$ is isomorphic to $\mathfrak{g}^*_{-1} \oplus \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$, where $\mathfrak{g}_1^+$ and $\mathfrak{g}_1^-$ are $[\mathfrak{g}_0, \mathfrak{g}_0]$-submodules of $\mathfrak{g}_1$ isomorphic to $S^2 s\ell_4$ and $S^2 s\ell_4^*$ respectively. Denote by $E(1,6)$ the graded subalgebra of $K(1,6)$ generated by $\mathfrak{g}_{-1} + \mathfrak{g}_0 + (\mathfrak{g}^*_{-1} + \mathfrak{g}_1^+)$. This is a simple even transitive irreducible consistent $\mathbb{Z}$-graded Lie superalgebra of depth 2. (Note that taking $\mathfrak{g}_1^-$ in place of $\mathfrak{g}_1^+$ gives an isomorphic superalgebra. Also, taking $\mathfrak{g}_1^+ + \mathfrak{g}_1^-$ gives $\mathbb{C}_{-2} + H(0,n)$, taking $\mathfrak{g}_1^+$ or $\mathfrak{g}_1^-$ gives $\widehat{p}(4)$ and taking $\mathfrak{g}^*_{-1}$ gives $spo(2,6)$, all of which are finite-dimensional graded Lie superalgebras of depth ≥ 2, hence not even.)

Example 5.3 (exceptional Lie superalgebra $E(5,10)$) There exists a unique bitransitive $\mathbb{Z}$-graded Lie superalgebra, denoted by $E(5,10)$, such that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ (resp. $\mathfrak{g}_1$) gives rise to the linear Lie algebra $\Lambda^2 s\ell_5$ (resp. highest component of $s\ell_5 \otimes \Lambda^2 s\ell_5$). It has depth 2 with the $\mathfrak{g}_0$-module $\mathfrak{g}_{-2}$ being $s\ell_5^*$ and it is simple. Its explicit construction is very easy (see [CK2] for details). Let $E(5,10)_{\bar{0}}$ be the Lie algebra $S(5,0)$ of divergence 0 polynomial vector fields on $\mathbb{C}^5$, i.e., polynomial vector fields annihilating the volume form $v = dx_1 \wedge \ldots \wedge dx_5$. Let $E(5,10)_{\bar{1}}$ be the space of closed polynomial differential 2-forms on $\mathbb{C}^5$. The bracket of $E(5,10)_{\bar{0}}$ with $E(5,10)_{\bar{1}}$ is defined by the usual action of vector fields on differential forms. In order to define bracket of two elements from $E(5,10)_{\bar{1}}$, note that vector fields may be identified with differential 4-forms by contracting a vector field with the volume form v. Hence we may define $[w_1, w_2] = w_1 \wedge w_2$ for $w_1, w_2 \in E(5,10)_{\bar{1}}$. The $\mathbb{Z}$-gradation of $E(5,10)$ is defined by letting $\deg x_i = -\deg \partial/\partial x_i = 2$, $\deg dx_i = -\frac{1}{2}$.

Example 5.4 (exceptional Lie superalgebras $E(3,6)$ and $E(3,8)$) There exist two bitransitive $\mathbb{Z}$-graded Lie superalgebras, denoted by $E(3,6)$ and $E(3,8)$, such that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ (resp. $\mathfrak{g}_1$) is isomorphic to $s\ell_3 \boxtimes s\ell_2 \boxtimes \mathbb{C}(-1)$ (resp. $(S^2 s\ell_3 \boxtimes s\ell_2 \boxtimes \mathbb{C}(1)) + s\ell_3^* \boxtimes s\ell_2 \boxtimes \mathbb{C}(1))$. The graded superalgebra $E(3,6)$ has depth 2 with the $\mathfrak{g}_0$-module $\mathfrak{g}_{-2}$ isomorphic to $s\ell_3 \boxtimes 1 \boxtimes \mathbb{C}(-2)$. The graded superalgebra $E(3,8)$ has depth 3 with the $\mathfrak{g}_0$-module $\mathfrak{g}_{-2}$ isomorphic to $s\ell_3 \boxtimes 1 \boxtimes \mathbb{C}(-2)$ and the $\mathfrak{g}_0$-module $\mathfrak{g}_{-3}$ isomorphic to $1 \boxtimes s\ell_2 \boxtimes \mathbb{C}(-3)$. Both superalgebras are simple. Their explicit construction is slightly more complicated than that of $E(5,10)$ and may be found in [CK2].

Remark 5.1 [CK2] The Lie superalgebras $E(3,6)$, $E(3,8)$ and $E(5,10)$ have a $\mathbb{Z}$-gradation of depth 1, 2 and 2 respectively (with $\dim \mathfrak{g}_{-2} = (1,0)$ in the second and third cases) such that $\mathfrak{g}_0$ is isomorphic to $W(0,3)$, $W(0,3) + \mathbb{C}$ and $S(0,4) + \mathbb{C}$ respectively, and the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is the space $\Lambda(3)/\mathbb{C}$ with reversed parity, the space of half-densities in 3 odd indeterminates and the space $\Lambda(4)/(\mathbb{C} + \mathbb{C}\xi_1\xi_2\xi_3\xi_4)$ with reversed parity respectively. The existence of such $\mathbb{Z}$-graded Lie superalgebras was announced in [S1], and a proof was given in [S2], but the proof in the second case is incorrect.

Example 5.5 (exceptional Lie superalgebras $E'(3,6)$ and $E'(3,8)$) We shall denote by $E'(3,6)$ and $E'(3,8)$ the subalgebras of $E(3,6)$ and $E(3,8)$ respectively generated by the subspaces $\mathfrak{g}_{-1} + [\mathfrak{g}_0, \mathfrak{g}_0] + \mathfrak{g}'_1$, where $\mathfrak{g}'$ is the

$[\mathfrak{g}_0,\mathfrak{g}_0]$-submodule of $\mathfrak{g}_1$ isomorphic to $S^2 s\ell_3 \boxtimes s\ell_2$. It is easy to see that $E'(3,6)$ is isomorphic to the semidirect sum of $s\ell_2$ which is put in degree 0 and $SHO(3,3)$ with the gradation of type $(2,2,2\,|\,3,3,3)$, and $E'(3,8)$ is isomorphic to the semidirect sum of $s\ell_2$ (put in degree 0) and the central extension of $SHO(3,3)$ (with this gradation) by odd 2-dimensional center put in degree -3.

Example 5.6 ($s\ell_2 + S(1,2)$ [P]) The Lie algebra $s\ell_2$ acts by outer derivations on the Lie superalgebra $S(1,2)$ preserving the gradation of type $(2|1,1)$. We denote the resulting $\mathbb{Z}$-graded semi-direct sum by $s\ell_2 + S(1,2)$. It is isomorphic to $\mathrm{Der}\, S(1,2)$.

Proposition 5.1 ([CK2]) *All even transitive irreducible consistent $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g} = \oplus_{j\geq -d}\mathfrak{g}_j$ with the non-zero $\mathfrak{g}_0$-modules $\mathfrak{g}_j$ for $-1 \geq j \geq -d$ indicated below are as follows:*

(a) $\mathfrak{g}_{-1} = so_n \boxtimes \mathbb{C}(-1)\,,\ n \geq 1\,,\ n \neq 2\,,\ \mathfrak{g}_{-2} = \mathbb{C}(-2)\ :\ K(1,n),\ s\ell_2 + S(1,2)$ *and* $E(1,6)$,

(b) $\mathfrak{g}_{-1} = \Lambda^2 s\ell_5\,,\ \mathfrak{g}_{-2} = s\ell_5^*\ :\ E(5,10)$,

(c) $\mathfrak{g}_{-1} = \Lambda^2 s\ell_5 \boxtimes \mathbb{C}(-1)\,,\ \mathfrak{g}_{-2} = s\ell_5^* \boxtimes \mathbb{C}(-2)\ :\ \mathbb{C}_0 + E(5,10)$,

(d) $\mathfrak{g}_{-1} = s\ell_3 \boxtimes s\ell_2 \boxtimes \mathbb{C}(-1)\,,\ \mathfrak{g}_{-2} = s\ell_3^* \boxtimes 1 \boxtimes \mathbb{C}(-2)\ :\ E(3,6)$ *and* $\mathbb{C}_0 + E'(3,6)$,

(e) $\mathfrak{g}_{-1} = s\ell_3 \boxtimes s\ell_2 \boxtimes \mathbb{C}(-1)\,,\ \mathfrak{g}_{-2} = s\ell_3^* \boxtimes 1 \boxtimes \mathbb{C}(-2)\,,\ \mathfrak{g}_{-3} = 1 \boxtimes s\ell_2 \boxtimes \mathbb{C}(-3)\ :\ E(3,8)$ *and* $\mathbb{C}_0 + E'(3,8)$,

(f) $\mathfrak{g}_{-1} = s\ell_3 \boxtimes s\ell_2\,,\ \mathfrak{g}_{-2} = s\ell_3^* \boxtimes 1\ :\ E'(3,6)$,

(g) $\mathfrak{g}_{-1} = s\ell_3 \boxtimes s\ell_2\,,\ \mathfrak{g}_{-2} = s\ell_3^* \boxtimes 1\,,\ \mathfrak{g}_{-3} = 1 \boxtimes s\ell_2\ :\ E'(3,8)$.

Now we can state the main theorem of this section.

Theorem 5.3 *An even transitive irreducible consistent $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_{j\geq -d}\mathfrak{g}_j$ of depth $d \geq 2$ is isomorphic to one of the following $\mathbb{Z}$-graded Lie superalgebras:*

(a) $K(1,n)\,,\ n \geq 1\,,\ n \neq 2$,

(b) $E(1,6)$ *and* $s\ell_2 + S(1,2)$*,*

(c) $E(3,6)$*,*

(d) $E(3,8)$*,*

(e) $E(5,10)$*,*

(f) $E'(3,6)$ *and* $E'(3,8)$*,*

(g) $\mathbb{C}_0 + \mathfrak{g}$*, where* $\mathfrak{g}$ *is of type (e) or (f).*

Consider the decomposition of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ in a direct sum of irreducible submodules:

$$\mathfrak{g}_1 = \oplus_{s=1}^{t} \mathfrak{g}_1^{[s]} \,. \tag{5.1}$$

Let F be a highest weight vector of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ and E_{M_s} be a lowest weight vector of the $\mathfrak{g}_0$-module $\mathfrak{g}_1^{[s]}$. Lemmas that follow put various restrictions on the decomposition (5.1) and the weights Λ and M_s.

Lemma 5.1 *At most one of the modules* $\mathfrak{g}_1^{[s]}$ *is contragredient to* $\mathfrak{g}_{-1}$*, unless the* $\mathfrak{g}_0$*-module* $\mathfrak{g}_{-1}$ *is isomorphic to* cso_4*.*

Proof Suppose that there are two such modules, say $\mathfrak{g}_1^{[1]}$ and $\mathfrak{g}_1^{[2]}$. Let $E'_{-\Lambda}$ and $E''_{-\Lambda}$ be their lowest weight vectors, and let $h' = [E'_{-\Lambda}\,,\, F_\Lambda]\,,\, h'' = [E''_{-\Lambda}\,,\, F_\Lambda]$. If $h' = ch''$ for some $c \in \mathbb{C}$, then $[E'_{-\Lambda} - cE''_{-\Lambda}\,,\, F_\Lambda] = 0$, hence $[E'_{-\Lambda} - cE''_{-\Lambda}\,,\, \mathfrak{g}_{-1}] = 0$, which contradicts transitivity of $\mathfrak{g}$. Hence $\mathfrak{g}_{-1} + [\mathfrak{g}_{-1}, \mathfrak{g}_1^{[i]}] + \mathfrak{g}_1^{[i]}$, $i = 1, 2$, must be local parts of non-isomorphic bi-transitive $\mathbb{Z}$-graded Lie superalgebras, which we shall denote by $\mathfrak{p}'$ and $\mathfrak{p}''$. Corollary 5.1 shows that this may happen only in two cases:

(a) $(\mathfrak{p}', \mathfrak{p}'') = (D(2,1;\alpha)\,,\, D(2,1;\beta))\,, \quad \alpha, \beta \in \mathbb{C}$,

(b) $(\mathfrak{p}', \mathfrak{p}'') = (s\ell(3,2)\,,\, spo(2,6))$.

(In case (a), $D(2,1;\alpha)$ may degenerate in $\mathbb{C}_0 + s\ell(2,2)$.) Thus, we have to rule out case (b).

Recall that the diagram of $spo(2,6)$ at hand is $\bigcirc\!\!-\!\!\bigcirc\!\leftarrow\!\otimes\!\rightarrow\!\bigcirc$ with the highest root $\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$, and $\theta - \alpha_j$ is a root iff $j = 2$ (α_j denote simple roots). The lowest root vector is $e_{-\theta} = [f_1 f_2 f_3 f_4 f_3 f_2]$. Here

e_i, f_i, h_i, $i = 1, 2, 4$, are Chevalley generators of $\mathfrak{p}_0''$ and $f_3 = F_\Lambda$, $e_3 = E''_{-\Lambda}$; we let $e_3' = E'_{-\Lambda}$. Then $[e_{-\theta}, e_3'] = 0$ since $\theta - \alpha_3$ is not a root, hence $[[e_{-\theta}, f_3], e_3'] = \theta(h')e_{-\theta} \neq 0$ and $[e_{-\theta}, f_3] \neq 0$. Next, $[[e_{-\theta}, f_3], f_4] \neq 0$ since its bracket with e_4 is non-zero (because $\theta - \alpha_4$ is not a root and $\alpha_3(h_4) \neq 0$). Finally, $[[[e_{-\theta}, f_3], f_4], f_3] \neq 0$ since its bracket with e_3 is non-zero (because $\theta - \alpha_3$ is not a root and $\alpha_4(h_3) \neq 0$). Thus, $\mathfrak{g}_{-4} \neq 0$. But this contradicts the following remark (which will be used again later).

Remark 5.2 If, under the assumptions of Theorem 5.3, the $[\mathfrak{g}_0, \mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$ is isomorphic to $s\ell_3 \boxtimes s\ell_2$, and $\mathfrak{g}_{-2} \neq 0$, then only the following possibilities for the $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules $\mathfrak{g}_j$, $j \leq -2$, may occur: (a) $\mathfrak{g}_{-2} = s\ell_3^* \boxtimes 1$, $\mathfrak{g}_{-3} = 0$, (b) $\mathfrak{g}_{-2} = s\ell_3^* \boxtimes 1$, $\mathfrak{g}_{-3} = 1 \boxtimes s\ell_2$, $\mathfrak{g}_{-4} = 0$. As above, proof is immediate by Lemma 2.6.

$\square$

Now let $M = M_s$ be the lowest weight of a $\mathfrak{g}_0$-module $\mathfrak{g}_1^{[s]}$, which is not contragredient to $\mathfrak{g}_{-1}$, i.e., $\Lambda + M \neq 0$. Then we clearly have

$$[F_\Lambda\,,\, E_M] = e_{-\alpha}\,,\ \Lambda + M = -\alpha\,, \tag{5.2}$$

where $e_{-\alpha}$ is the root vector of $[\mathfrak{g}_0, \mathfrak{g}_0]$ attached to the root $-\alpha$. Let $\mathfrak{g}_0^{[s]}$ denote the simple component of $[\mathfrak{g}_0, \mathfrak{g}_0]$ containing $e_{-\alpha}$, let $g_{-1}^{[s]}$ and $\mathfrak{g}_1^{[s]}$ be the irreducible $\mathfrak{g}_0^{[s]}$-submodules of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ containing F_Λ and E_M respectively. Let $\mathfrak{g}^{[s]}$ be the bitransitive Lie superalgebra with local part $\mathfrak{g}_{-1}^{[s]} + \mathfrak{g}_0^{[s]} + \mathfrak{g}_1^{[s]}$.

The same argument as in the beginning of the proof of Lemma 5.1 gives the next lemma.

Lemma 5.2 *All the $\mathfrak{g}_0^{[s]}$-modules $\mathfrak{g}_1^{[s]}$ for distinct s, such that $\Lambda + M_s \neq 0$, are inequivalent.*

The following lemma puts severe restrictions on possible Λ and α. It follows from [K4], Lemmas 4.1.4 and 4.1.5 and their proofs.

Lemma 5.3 *Let $\mathfrak{g} = \oplus_{j=-d}^{k} \mathfrak{g}_j$ be a consistent $\mathbb{Z}$-graded bitransitive Lie superalgebra such that $\mathfrak{g}_0$ is a simple Lie algebra, $\mathfrak{g}_{\pm 1}$ are irreducible $\mathfrak{g}_0$-modules and $d, k \in \{1, 2, \ldots, \infty\}$. Let Λ (resp. M) be the highest (resp. lowest) weight of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ (resp. $\mathfrak{g}_1$), and suppose that $\Lambda + M = \alpha$, where α is a positive root of $\mathfrak{g}_0$.*

I. Provided that either d or k is finite, one has:

(a) $(\Lambda, \alpha) = 0$.

(b) If β is a positive root of $\mathfrak{g}_0$ such that $\alpha + \beta$ is a root and $\alpha - \beta$ is not, then $2(\Lambda, \beta)/(\beta, \beta) = 1$ and $2(\alpha, \beta)/(\beta, \beta) = -1$.

II. Provided that d is finite, one has:

If β and γ are positive roots of $\mathfrak{g}_0$ such that $\alpha + \beta$ is a root, but $\alpha - \beta$, $\alpha - \gamma$ and $\beta - \gamma$ are not, and if $(\Lambda, \gamma) \neq 0$, then $2\Lambda - \beta - \gamma$ is a weight of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-2}$.

Lemma 5.4 *If $\mathfrak{g}^{[s]} = \oplus_j \mathfrak{g}_j^{[s]}$ has depth at least 2 and $\Lambda + M_s \neq 0$, then the $\mathfrak{g}_0^{[s]}$-module $\mathfrak{g}_{-1}^{[s]}$ is isomorphic to $\Lambda^2 s\ell_5$, the $\mathfrak{g}_0^{[s]}$-module $\mathfrak{g}_{-2}^{[2]}$ is isomorphic to $s\ell_5^*$ and $\mathfrak{g}_{-3}^{[s]} = 0$. Moreover, $\mathfrak{g}_1^{[s]}$ is isomorphic to the highest component of $s\ell_5 \otimes \Lambda^2 s\ell_5$.*

Proof We shall skip the superscript $[s]$ when it shall cause no confusion. If $\mathfrak{g}_{-2}$ is a trivial $\mathfrak{g}_0$-module, then taking bracket with it establishes an isomorphism of $\mathfrak{g}_0$-modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, and also $\mathfrak{g}_{-1}$ is then a selfcontragredient module. This is impossible since $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are not contragredient. Thus, due to Lemma 2.6, the $\mathfrak{g}_0$-module $\mathfrak{g}_{-2}$ is $s\ell_n$ or $s\ell_n^*$ or sp_n. But $\mathfrak{g}_{-2} \subset S^2\mathfrak{g}_{-1}$ which rules out sp_n and $s\ell_{2n}$ (since the corresponding groups contain $-I$).*

Thus, $\mathfrak{g}_0 = s\ell_n$ with $n \geq 3$ odd. The non-zero labels of Λ break the Dynkin diagram of $s\ell_n$ in connected components to which Λ restricts trivially. By Lemma 3.5 I (a) and (b), α is the sum of all simple roots of one of these connected components, all non-zero labels of Λ equal 1 and their number is at most 2, i.e., either $\Lambda = \omega_i$ or $\Lambda = \omega_i + \omega_j$ with $i < j$, where ω_i denote fundamental weights. In the latter case, $\alpha = \alpha_{i+1} + \ldots + \alpha_{j-1}$, so that taking $\beta = \alpha_i$ and $\gamma = \alpha_j$, we deduce from Lemma 5.3 II that $2\omega_i + 2\omega_j - \alpha_i - \alpha_j$ is a weight of $s\ell_n$ (the standard module), a contradiction.

Thus, $\Lambda = \omega_i$, and, up to passing to the contragredient module, we may assume that $i \leq n-2$ and $\alpha = \alpha_{i+1} + \ldots + \alpha_{n-1}$. Since $s\ell_n$ is not contained in $S^2 s\ell_n$,one has: $i \geq 2$. We let $\beta = \alpha_i$ and $\gamma = \alpha_1 + \alpha_2 + \ldots + \alpha_{i+1}$. Then Lemma 5.3 II gives us that $2\omega_i - \beta - \gamma = \omega_{i-1} - \omega_1 + \omega_{i+2}$ is a weight of $s\ell_n$. This is possible in the following two cases:

*Here and further in this section S^2 and Λ^2 mean the ordinary symmetric and exterior square, i.e., we disregard the parity.

(a) $i = n-2$, $\alpha = \alpha_{n-1}$,

(b) $i = 2$, $n = 5$, $\alpha = \alpha_3 + \alpha_4$.

In case (a) $\Lambda = \omega_{n-2}$, $M = -\omega_{n-2} - \alpha_{n-1} = -2\omega_{n-1}$, hence $\mathfrak{g}^{[s]}$ is isomorphic to $p(n)$ which has depth 1. In case (b) the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is $\Lambda^2 s\ell_5$ and $\mathfrak{g}_{-2} \subset S^2\mathfrak{g}_{-1}$ can be only $s\ell_5^*$. Also, $M = -\omega_2 - \alpha_3 - \alpha_4 = -\omega_3 - \omega_4$. It remains to show that $\mathfrak{g}_{-3} = 0$.

First, $\mathfrak{g}_{-4} = [a, \mathfrak{g}_{-2}]$ for any non-zero $a \in \mathfrak{g}_{-2}$ (by Lemma 2.6), hence $\dim \mathfrak{g}_{-4} < \dim \mathfrak{g}_{-2}$ and therefore, since $\mathfrak{g}_{-4} \subset \mathfrak{g}_{-2} \otimes \mathfrak{g}_{-2}$, we see that $\mathfrak{g}_{-4} = 0$. Similarly, $\mathfrak{g}_{-3} = [a, \mathfrak{g}_{-1}]$ for any non-zero $a \in \mathfrak{g}_{-2}$, hence $\dim \mathfrak{g}_{-3} \leq \dim \mathfrak{g}_{-1}$. Since $\mathfrak{g}_{-3} \subset \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}$ as $\mathfrak{g}_0$-modules, and $\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}$, apart from the highest components, whose dimension is greater than that of $\mathfrak{g}_{-1}$, contains $s\ell_5$ with multiplicity 1 (see [OV]), the only possibility for the $\mathfrak{g}_0$-module $\mathfrak{g}_{-3}$ is $s\ell_5$. Suppose that this possibility does occur. Then consider the vector $F'' = [f_{123} f_2 F\ , F]$. Here and further we use notation $f_{123} = [[f_1, f_2], f_3]$, $f_2 F = [f_2, F]$, etc. Using (5.2), we see that $[F'', E] \neq 0$, hence F'' is a weight vector of $\mathfrak{g}_{-2}$, and since its weight is ω_4, it is a highest weight vector. Next, it is easy to see that $F''' = [f_{234} F\ , F'']$ is a highest weight vector of $\mathfrak{g}_{-3}$ (of weight ω_1). One checks directly, using (5.2), that $[F'''\ , E] = 0$, hence $[\mathfrak{g}_{-3}, \mathfrak{g}_1] = 0$. But $[\mathfrak{g}_{-3}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-4} = 0$ as well, hence $[\mathfrak{g}_{-3}, \mathfrak{g}_0] = [\mathfrak{g}_{-3}, [\mathfrak{g}_{-1}, \mathfrak{g}_1]] = 0$, a contradiction.

$\square$

The following corollary is immediate by Theorem 5.1 and Lemma 5.4.

Corollary 5.2 *If the $\mathfrak{g}_0^{[s]}$-modules $\mathfrak{g}_{-1}^{[s]}$ and $\mathfrak{g}_1^{[s]}$ are not contragredient, there are only the following possibilities for the $\mathfrak{g}_0^{[s]}$-modules $\mathfrak{g}_{-1}^{[s]}$ and $\mathfrak{g}_1^{[s]}$:*

(a) $\mathfrak{g}_{-1}^{[s]} = S^2 s\ell_n$, $\mathfrak{g}_1^{[s]} = \Lambda^2 s\ell_n$, $n \geq 3$,

(b) $\mathfrak{g}_{-1}^{[s]} = \Lambda^2 s\ell_n$, $\mathfrak{g}_1^{[s]} = S^2 s\ell_n$, $n \geq 3$,

(c) $\mathfrak{g}_{-1}^{[s]} = ad\, \mathfrak{p}$, *where* $\mathfrak{p}$ *is a simple Lie algebra,* $\mathfrak{g}_1^{[s]} = 1$,

(d) $\mathfrak{g}_{-1}^{[s]} = s\ell_n$, $\mathfrak{g}_1^{[s]} =$ *highest component of* $s\ell_n \otimes \Lambda^2 s\ell_n^*$, $n \geq 4$,

(e) $\mathfrak{g}_{-1}^{[s]} = so_n$, $\mathfrak{g}_1^{[s]} = \Lambda^3 so_n$, $n \geq 5$, $n \neq 6$,

(f) $\mathfrak{g}_{-1}^{[s]} = \Lambda^2 s\ell_5$, $\mathfrak{g}_{-2}^{[s]} = s\ell_5^*$, $\mathfrak{g}_1^{[s]} =$ *highest component of* $s\ell_5 \otimes \Lambda^2 s\ell_5$.

In all cases except for (f), $\mathfrak{g}^{[s]}_{-2} = 0$, and in case (f), $\mathfrak{g}^{[s]}_{-3} = 0$.

Lemma 5.5 *Suppose that, under the assumptions of Theorem 5.3, there are at least two submodules of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ and one of them, say $\mathfrak{g}^{[s]}_1$ is contragredient to $\mathfrak{g}_{-1}$. Let $\overline{\mathfrak{g}} = \mathfrak{g}/\mathfrak{b} = \oplus_{j\geq -d}\overline{\mathfrak{g}}_j$, where $\mathfrak{b}$ is the maximal ideal of $\mathfrak{g}$. Then only the following possibilities for $\mathfrak{g}^{[s]}$ and the $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules $\overline{\mathfrak{g}}_j$, $-1 \geq j \geq -d$, may occur:*

(a) $spo(2,n)$, $n \geq 1$, $n \neq 2$, $\mathfrak{g}_{-1} = so_n$, $\overline{\mathfrak{g}}_{-2} = 1$,

(b) $s\ell_n(3,2)$, $\mathfrak{g}_{-1} = s\ell_3 \boxtimes s\ell_2$, $\overline{\mathfrak{g}}_{-2} = s\ell_3^* \boxtimes 1$, $\overline{\mathfrak{g}}_{-3} = 1 \boxtimes s\ell_2$,

(c) $spo(2,6)$, $\mathfrak{g}_{-1} = s\ell_3 \boxtimes s\ell_2$, $\overline{\mathfrak{g}}_{-2} = s\ell_3^* \otimes 1$,

(d) $D(2,1\,;\,\alpha)$, $\mathfrak{g}_{-1} = so_4$, $\overline{\mathfrak{g}}_{-2} = 1$.

Proof Due to Lemma 5.1, either we have case (d) or, apart from the $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$ contragredient to $\mathfrak{g}_{-1}$, which we denote by $\mathfrak{g}'_1$, there is another one, which is not contragredient to $\mathfrak{g}_{-1}$, which we denote by $\mathfrak{g}''_1$. Let $\mathfrak{g}'$ and $\mathfrak{g}''$ denote the corresponding bitransitive Lie superalgebras. We have: $[\mathfrak{g}_0, \mathfrak{g}_0] = [\mathfrak{g}'_0, \mathfrak{g}'_0]$, and $\mathfrak{g}_{-1}$ and $\mathfrak{g}'_{-1}$ are isomorphic modules. One of the simple components of $\mathfrak{g}_0$ must be $\mathfrak{g}''_0$ and the $\mathfrak{g}''_0$-irreducible submodule of $\mathfrak{g}_{-1}$ containing F_Λ must be isomorphic to the $\mathfrak{g}''_0$-module $\mathfrak{g}''_{-1}$.

Thus, we need to compare the list of modules given by Corollary 5.1, which gives all possibilities for the $[\mathfrak{g}'_0, \mathfrak{g}'_0]$-module $\mathfrak{g}'_{-1}$ with the list of modules given by Corollary 5.2, which gives all possibilities for the $\mathfrak{g}''_0$-module $\mathfrak{g}''_{-1}$.

All possible pairs are given by the following table.

	$[\mathfrak{g}'_0, \mathfrak{g}'_0]$-module $\mathfrak{g}'_{-1}$	$\mathfrak{g}'$	$\mathfrak{g}''_0$-module $\mathfrak{g}''_{-1}$		$\mathfrak{g}''$
(a)	$s\ell_3 \boxtimes s\ell_n$	$s\ell(3,n)$	$s\ell_3$		$p(3)$
(b)	$s\ell_3 \boxtimes sp_{2m}$	$spo(2m,6)$	$s\ell_3$		$p(3)$
(c)	$s\ell_m \boxtimes s\ell_n$	$s\ell(m,n)$	$s\ell_n$	$(n \geq 4)$	$S(0,n)$
(d)	so_3	$spo(2,3)$	so_3		$so_3[\xi]$
(e)	so_n	$spo(2,n)$	so_n	$(n \geq 5)$	$H(0,5)$

Recall now, that, due to Lemma 2.6, either the depth of $\mathfrak{g}$ is 1 or the $[\mathfrak{g}_0, \mathfrak{g}_0]$-module $S^2\mathfrak{g}_{-1}$ must give rise to the linear Lie algebra $s\ell_n$ or sp_n.

This permits only $n = 2$ in case (a) and $m = 1$ in case (b), and we may apply Remark 5.1.

$\square$

Lemma 5.6 *Suppose that, under the assumptions of Theorem 5.3 all the* $\mathfrak{g}_0$*-submodules* $\mathfrak{g}_1^{[s]}$ *of* $\mathfrak{g}_1$ *are not contragredient to* $\mathfrak{g}_{-1}$*. Then (see notation of Lemma 5.5) only the following possibilities for the* $[\mathfrak{g}_0, \mathfrak{g}_0]$*-modules* $\overline{\mathfrak{g}}_j$*,* $-1 \geq j \geq -d$*, and* $\mathfrak{g}_1$ *may occur:*

(a) $\mathfrak{g}_{-1} = so_6$, $\overline{\mathfrak{g}}_{-2} = 1$, $\mathfrak{g}_1 = \Lambda^3 so_6$,

(b) $\mathfrak{g}_{-1} = s\ell_3 \boxtimes s\ell_2$, $\overline{\mathfrak{g}}_{-2} = s\ell_3^* \boxtimes 1$, $\mathfrak{g}_1 = S^2 s\ell_3 \boxtimes s\ell_2$,

(c) $\mathfrak{g}_{-1} = s\ell_3 \boxtimes s\ell_2$, $\overline{\mathfrak{g}}_{-2} = s\ell_3^* \boxtimes 1$, $\overline{\mathfrak{g}}_{-3} = 1 \boxtimes s\ell_2$, $\mathfrak{g}_1 = S^2 s\ell_3 \boxtimes s\ell_2$,

(d) $\mathfrak{g}_{-1} = \Lambda^2 s\ell_5$, $\overline{\mathfrak{g}}_{-2} = s\ell_5^*$, $\mathfrak{g}_1 =$ *highest component of* $sl_5 \otimes \Lambda^2 s\ell_5$.

Proof Decompose $[\mathfrak{g}_0, \mathfrak{g}_0]$ in a direct sum of simple Lie algebras and consider the corresponding decompositions of weights Λ and $M^{[s]}$:

$$[\mathfrak{g}_0, \mathfrak{g}_0] = \oplus_{j=1}^r \mathfrak{g}_{0j}\,,\ \Lambda = \sum_{j=1}^r \Lambda_j\,,\ M^{[s]} = \sum_{j=1}^r M_j^{[s]}\,.$$

If $\Lambda_j + M_j^{[s]} = 0$ (resp. $\neq 0$) for some s, we shall refer the $\mathfrak{g}_{0j}$-module corresponding to Λ_j as to the contragredient case (resp. non-contragredient case). Due to Corollaries 5.1 and 5.2, the following is a complete list of possibilities in each case (we list in parenthesis the highest weight of the non-contragredient partner from $\mathfrak{g}_1$; $\mathfrak{p}$ stands for a simple Lie algebra):

	contragredient		non-contragredient	
(a)	$s\ell_n$, $n \geq 2$	(a)	$s\ell_n$, $n \geq 4$	$(\omega_1 + \omega_{n-2})$
(b)	sp_n, $n \geq 4$	(b)	so_n, $n > 6$	(ω_3)
(c)	so_n, $n \geq 3$, $n \neq 4$	(c)	so_5	(ω_2)
(d)	spin_7	(d)	$\Lambda^2 s\ell_n$, $n \geq 3$	$(2\omega_{n-1})$
(e)	G_2	(e)	$\Lambda^2 s\ell_4$	$(2\omega_1$ or $2\omega_3)$
		(f)	$S^2 s\ell_n$, $n \geq 3$	(ω_{n-2})
		(g)	$ad\,\mathfrak{p}$	(0)
		(h)	$\Lambda^2 s\ell_5$	$(\omega_1 + \omega_2)$

First, let us study the case $r=1$, i.e., $[\mathfrak{g}_0,\mathfrak{g}_0]$ simple. If $t=1$ (see (5.1)), we have case (d) of the lemma (by Lemma 5.4), so we may assume that $t\geq 2$. Then, due to Lemma 5.2, we have to pick out from the right column of the table those linear Lie algebras which appear at least twice. These are $\Lambda^2 s\ell_4$ and $\Lambda^2 s\ell_5$. In the first case the $[\mathfrak{g}_0,\mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$ (resp. $\mathfrak{g}_1$) is so_6 (resp. $\Lambda^3 so_6$), which gives possibility (a) of the lemma. In the second case, the $[\mathfrak{g}_0,\mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$ is $\Lambda^2 s\ell_5$, we denote its highest weights vector by F, and the $[\mathfrak{g}_0,\mathfrak{g}_0]$-module $\mathfrak{g}_1$ is a direct sum of the $s\ell_5$-modules with highest weights $\omega_1+\omega_2$ and $2\omega_4$, we denote their lowest weight vectors by E and E'. We have:

$$[F,E]=[f_3,f_4]\,,\ [F,E']=f_1\,. \tag{5.3}$$

Consider the vectors F'' and F''' introduced in the proof of Lemma 5.4. Using (5.3), we obtain: $[F'',E']=0$, hence $[F''',E']=[[f_{234}F,F''],E']=[f_{234},f_1]F''\neq 0$. This contradicts Lemma 5.4.

Note that the condition $S^2\mathfrak{g}_{-1}\supset\mathfrak{g}_{-2}$ implies that the tensor square of the modules from the right column of the table must contain $s\ell_n$ or sp_n, $n\geq 1$. This immediately excludes cases (a), (e) and (f).

Let now $r\geq 2$ and assume that case (g) of the right column does not occur. Then we may assume that $\Lambda_1+M_1^{[1]}=-\alpha$ where α is a positive root of $\mathfrak{g}_{01}$ (since cases (f) and (g) are excluded) and $\Lambda_j+M_j^{[1]}=0$ for $j\geq 2$. We employ Lemma 5.3. Let β be a positive root of $\mathfrak{g}_{01}$ such that $(\Lambda,\beta)\neq 0$ and $\alpha+\beta$ is a root but $\alpha-\beta$ is not (such β exists by Lemma 5.3 I(a)). Let γ be a positive root of $\mathfrak{g}_{02}$ such that $(\Lambda,\gamma)\neq 0$. It follows from Lemma 5.3 II that $2\Lambda-\beta-\gamma$ is a weight of the $[\mathfrak{g}_0,\mathfrak{g}_0]$-module $\mathfrak{g}_{-2}$. Since (by Lemma 2.6) the corresponding linear Lie algebra is $s\ell_n$ or sp_n, $n\geq 1$, we conclude that the $[\mathfrak{g}_0,\mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$ is isomorphic to $s\ell_3\otimes s\ell_2$, while the $[\mathfrak{g}_0,\mathfrak{g}_0]$-module $\mathfrak{g}_{-2}$ is isomorphic to $s\ell_3^*\otimes\mathbb{C}$. Due to Remark 5.1, this is case (b) or (c) of the lemma.

Finally, consider the case (g) when $\Lambda_1=\theta$, the highest root of $\mathfrak{p}=\mathfrak{g}_{01}$, and $M^{[1]}=0$, so that $[F,E]=e_\theta$. Then $r\geq 2$, otherwise $\dim\mathfrak{g}<\infty$. Let β be a positive root of $\mathfrak{g}_{02}$ such that $(\Lambda,\beta)\neq 0$. Then we have:

$$[[e_{-\theta}F,e_{-\beta}F],E]=[e_\theta,e_{-\theta}]e_{-\beta}F\neq 0\,.$$

Hence $2\Lambda-\theta-\beta$ is a weight of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-2}$. When, restricted to $\mathfrak{g}_{01}$, it is θ, which is impossible by Lemma 2.6.

$\square$

Lemma 5.7 *Let $\mathfrak{g} = \oplus_{j\geq -d}\mathfrak{g}_j$, $d \geq 2$, be a transitive irreducible consistent Lie superalgebra for which $\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ is a local part of one of $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g}'$ listed in Corollary 5.1 and suppose that* $\dim \mathfrak{g}_{-1} > 1$. *Then $\mathfrak{g}_j = \mathfrak{g}_1^j$, hence* $\dim \mathfrak{g} < \infty$.

Proof We use notation of the proof of Theorem 5.2 and an argument from [K2], Lemma 4.2. Suppose the contrary, and take the minimal $j \geq 2$ such that $\mathfrak{g}_j \neq \mathfrak{g}_1^j$. Take a weight vector E_β of the $\mathfrak{g}_0$-module $\mathfrak{g}_j$ outside of $\mathfrak{g}_1^j$. Then we have:

$$[E_\beta, f_0] = e_{\beta-\alpha_0} \in \mathfrak{g}_1^{j-1}\,,$$

where $\beta - \alpha_0$ is a positive root of $\mathfrak{g}'$. Taking bracket of both sides with the root vector $e_{-\beta+\alpha_0}$, we obtain:

$$\pm[[E_\beta, e_{-\beta+\alpha_0}], f_0] + [E_\beta, [f_0, e_{-\beta+\alpha_0}]] = h_{\alpha_0-\beta}\,. \qquad (5.4)$$

But the first summand on the left is a multiple of $\mathfrak{h}_{\alpha_0} = [e_0, f_0]$. If β is not a root of $\mathfrak{g}'$, then the second summand in (5.4) is zero, hence β is a multiple of α_0. If β is a root of $\mathfrak{g}'$, then adding to E_β a root vector e_β we can add to the left-hand side of (5.4) an arbitrary multiple of h_β, which again shows that β is a multiple of α_0. Since $\beta - \alpha_0$ is a positive root of $\mathfrak{g}'$, we obtain that $\beta = 2\alpha_0$. Thus, any weight of the $\mathfrak{g}_0$-module $\mathfrak{g}_j/\mathfrak{g}_1^j$ is $2\alpha_0$, which is impossible if $\dim \mathfrak{g}_{-1} > 1$.

$\square$

Lemma 5.8 *Let $\mathfrak{g} = \oplus_{j\geq -d}\mathfrak{g}_j$ be an even transitive consistent $\mathbb{Z}$-graded Lie superalgebra of depth $d \geq 2$. Then $\mathfrak{g}^-$ contains no ideals of $\mathfrak{g}$ and one has the following possibilities for the $[\mathfrak{g}_0, \mathfrak{g}_0]$-module $\mathfrak{g}_j$, $-1 \geq j \geq -d$:*

(a) $\mathfrak{g}_{-1} = so_n$, $n \geq 1$, $n \neq 2$, $\mathfrak{g}_{-2} = 1$,

(b) $\mathfrak{g}_{-1} = s\ell_3 \boxtimes s\ell_2$, $\mathfrak{g}_{-2} = s\ell_3^* \boxtimes 1$,

(c) $\mathfrak{g}_{-1} = s\ell_3 \boxtimes s\ell_2$, $\mathfrak{g}_{-2} = s\ell_3^* \boxtimes 1$, $\mathfrak{g}_{-3} = 1 \boxtimes s\ell_2$,

(d) $\mathfrak{g}_{-1} = \Lambda^2 s\ell_5$, $\mathfrak{g}_{-2} = s\ell_5^*$.

Proof Let $\mathfrak{b} = \oplus_{j\leq -2}\mathfrak{b}_j$ be the graded maximal ideal of $\mathfrak{g}$ contained in $\mathfrak{g}^-$. Recall, that, by Lemma 2.4, $\mathfrak{b}_{-2j} = 0$, $j \geq 1$.

It follows from Lemmas 5.1, 5.5, 5.6, 5.7 and Remark 5.1 that all possibilities for the $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules $\overline{\mathfrak{g}}_{-j}$, $-1 \geq j \geq -d$, are listed by the lemma. It remains to prove that $\mathfrak{b} = 0$.

Since $\mathfrak{b}_{-2}$ and $\mathfrak{b}_{-4}$ are zero, we see that $[\mathfrak{b}_{-3}, \mathfrak{g}_{\pm 1}] = 0$. Hence $[\mathfrak{b}_{-3}, [\mathfrak{g}_0, \mathfrak{g}_0]] = 0$ since in all cases $[\mathfrak{g}_0, \mathfrak{g}_0] \subset [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$. Hence $\mathfrak{b}_{-3}$ is a trivial $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules contained in $\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}$, hence $\mathfrak{b}_{-3} = 0$.

$\square$

End of Proof of Theorem 5.3. It follows from Lemma 5.8 and Proposition 5.1.

Remark 5.3 Notation $X(m,n)$, where $X = W, S, H, K, HO$, etc., for the $\mathbb{Z}$-graded Lie superalgebra $\oplus_{j \geq -d} \mathfrak{g}_j$ carries the following information:

$$\dim \oplus_{j<0} \mathfrak{g}_j = (m,n)\,.$$

In other words, $X(m,n)$ acts transitively on a supermanifold of dimension (m,n). It is easy to show that the growth (= Gelfand–Kirillov dimension) of $X(m,n)$ is equal to m in all cases.

6 Classification of infinite-dimensional simple linearly compact Lie superalgebras

An immediate consequence of Theorems 4.1 and 5.3 is the following theorem.

Theorem 6.1 *An infinite-dimensional even transitive irreducible $\mathbb{Z}$-graded Lie superalgebra is isomorphic to one of the $\mathbb{Z}$-graded Lie superalgebras listed by Theorems 4.1 and 5.3.*

Due to Proposition 2.1 and Lemma 2.3, we get the following corollary.

Corollary 6.1 *If (L, L_{-1}, L_0) is an infinite-dimensional even quasiprimitive linearly compact Lie superalgebra, then the $\mathbb{Z}$-graded Lie superalgebra associated with the Weisfeiler filtration of L is isomorphic to one of the $\mathbb{Z}$-graded Lie superalgebras listed by Theorems 4.1 and 5.3.*

Remark 6.1 If $\mathfrak{g} = \oplus_{j \geq -d} \mathfrak{g}_j$ is one of the $\mathbb{Z}$-graded Lie superalgebras listed by Theorems 4.1 and 5.3, then $(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}_{(-1)}, \overline{\mathfrak{g}}_{(0)})$ is an even quasiprimitive Lie superalgebra (recall that $\overline{\mathfrak{g}}$ is the formal completion of $\mathfrak{g}$, $\overline{\mathfrak{g}}_{(-1)} = \prod_{j \geq -1} \mathfrak{g}_j$ and

$\overline{\mathfrak{g}}_{(0)} = \prod_{j\geq 0} \mathfrak{g}_j$). Almost all pairs $(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}_{(0)})$ are primitive. The only exceptions are III(a), (b) from Theorem 4.1 and examples $\overline{E}'(3,8)$ and $\mathbb{C}_0 + \overline{E}'(3,8)$ from Theorem 5.3 (when $\mathfrak{g}_{-d} + \overline{\mathfrak{g}}_{(0)}$ is a subalgebra).

Remark 6.2 One can show that a primitive Lie superalgebra is semisimple (i.e., has no closed abelian ideals). Semisimple linearly compact Lie superalgebras that admit a fundamental subalgebra can be described in terms of simple linearly compact Lie superalgebras in the same way as in the finite-dimensional case [K4], [Ch]. It is a super analog of a more precise version of the Cartan–Guillemin theorem [G1]. Using this, one can describe primitive Lie superalgebras in terms of primitive simple ones.

Now we can turn to the discussion of classification of infinite-dimensional simple linearly compact Lie superalgebras. Let L be such a Lie superalgebra. By Theorem 1.1, L has a subalgebra L_0 such that (L, L_0) is an even primitive Lie superalgebra. Consider an irreducible Weisfeiler filtration $L = L_{-d} \supset L_{-d+1} \supset \ldots \supset L_{-1} \supset L_0 \supset L_1 \supset \ldots$ and let $\mathfrak{g} = \oplus_{j\geq -d}\mathfrak{g}_j$ be the associated graded Lie superalgebra. By Corollary 6.1, $\mathfrak{g}$ is one of the $\mathbb{Z}$-graded Lie superalgebras listed by Theorems 4.1 and 5.3.

A linearly compact filtered Lie superalgebra L whose associated graded is $\mathfrak{g}$ is called a *filtered deformation of* $\overline{\mathfrak{g}}$. Of course, $\overline{\mathfrak{g}}$ is a filtered deformation of $\overline{\mathfrak{g}}$, called the *trivial* filtered deformation; note that $\overline{\mathfrak{g}}$ is simple iff $\mathfrak{g}$ is. If L is simple, it is called a *simple* filtered deformation of $\mathfrak{g}$. If $\overline{\mathfrak{g}}$ is the only (resp. the only simple) filtered deformation of $\overline{\mathfrak{g}}$, we shall say that $\overline{\mathfrak{g}}$ has no filtered (resp. simple filtered) deformations.

Thus, the classification of infinite-dimensional simple linearly compact Lie superalgebras is reduced to find all (up to isomorphism) simple filtered deformations of all $\mathbb{Z}$-graded Lie superalgebras listed by Theorems 4.1 and 5.3.

Let $\mathfrak{s}$ be a maximal reductive subalgebra in the even part of $\mathfrak{g}_0 = L_0/L_1$. Using Levi theorem, we may find linear maps $\varphi_k : \mathfrak{s} \to L_0$ for each $k \geq 1$ such that the induced map $\overline{\varphi}_k : \mathfrak{s} \to L_0/L_k$ is an injective homomorphism and $\varphi_k(s) - \varphi_{k+1}(s) \in L_{k+1}$ for all $s \in \mathfrak{s}$. Taking limit as $k \to \infty$, we get an injective homomorphism φ: $\mathfrak{s} \hookrightarrow L_0$. We shall identify $\varphi(\mathfrak{s})$ with $\mathfrak{s}$. Note that $[\mathfrak{s}, L_n] \subset L_n$ for each n and that $\mathfrak{s}$-modules L_n/L_{n+1} and $\mathfrak{g}_n$ are isomorphic.

By the complete reducibility theorem, the $\mathfrak{s}$-modules L_n/L_{n+k} are completely reducible for all $k \geq 1$, so that we may find a complementary $\mathfrak{s}$-

submodule $\mathfrak{g}_n(k)$ to L_{n+1}/L_{n+k} in L_n/L_{n+k}. Again, as $k \to \infty$, we get an $\mathfrak{s}$-module decomposition $L_n = \mathfrak{m}_n \oplus L_{n+1}$ where $\mathfrak{m}_n = \mathfrak{g}_n(\infty)$ and $\mathfrak{g}_n$ are isomorphic $\mathfrak{s}$-modules.

Thus, we have obtained a decomposition

$$L = \prod_{j \geq -d} \mathfrak{m}_j \tag{6.1}$$

as $\mathfrak{s}$-modules, where $\mathfrak{s} \subset \mathfrak{m}_0$ and $L_n = \mathfrak{m}_n \oplus L_{n+1}$ for each n.

Lemma 6.1 *(a) If $\mathfrak{s}$ contains a non-zero central element c then $\overline{\mathfrak{g}}$ has no filtered deformations.*

(b) If $\mathfrak{g}$ is one of the types I(j), (k) or II(a), (b),or IV of Theorem 4.1 or (f) and (g) of Theorem 5.3, then L cannot be simple, hence $\overline{\mathfrak{g}}$ has no simple filtered deformations.

Proof In case (a), if c is normalized such that $c|_{\mathfrak{m}_{-1}} = -1$, then $c|_{\mathfrak{m}_j} = j$, hence $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j}$ in (6.1) and $L \simeq \overline{\mathfrak{g}}$. In case (b) it is easy to show that $\mathfrak{m}_0 \not\subset [L, L]$. For example, if $d = 1$ and $\mathfrak{m}_0 \subset [L, L]$, then $\mathfrak{b} = \mathfrak{m}_{-1} + \mathfrak{m}_0$ is a semisimple subalgebra of L but there are no such semisimple Lie superalgebras.

□

Lemma 6.2 *Suppose that decomposition (6.1) has the following properties:*

$$\mathfrak{m}_{-1}^k \subset \mathfrak{m}_{-k} \textit{ for all } k \geq 2 (\textit{hence } \mathfrak{m}_{-1}^k = \mathfrak{m}_{-k}), \tag{6.2}$$

$$\mathfrak{m}_0 \textit{ is a subalgebra of } L, \tag{6.3}$$

$$[\mathfrak{m}_0, \mathfrak{m}_{-1}] \subset \mathfrak{m}_{-1}, \tag{6.4}$$

$$[\mathfrak{m}_1, \mathfrak{m}_{-1}] \subset \mathfrak{m}_0. \tag{6.5}$$

Then, provided that $\mathfrak{g}$ satisfies (G6) (from Section 2), $\overline{\mathfrak{g}}$ has no filtered deformations.

Proof First, we prove that

$$[\mathfrak{m}_0, \mathfrak{m}_1] \subset \mathfrak{m}_1. \tag{6.6}$$

Indeed, for any $x \in \mathfrak{g}_{-1}$ we have:

$$[x, [\mathfrak{m}_0, \mathfrak{m}_1]] = [[x, \mathfrak{m}_0], \mathfrak{m}_1] + [\mathfrak{m}_0, [x, \mathfrak{m}_1]] \subset \mathfrak{m}_0$$

by (6.3), (6.4) and (6.5), hence (6.6) follows by transitivity of $\mathfrak{g}$. Next we prove

$$\mathfrak{m}_1^n \subset \mathfrak{m}_n \text{ (hence } \mathfrak{m}_1^n = \mathfrak{m}_n) \text{ for } n \geq 1\,. \tag{6.7}$$

Using (6.5) and (6.6), we have by induction on n: $[x, \mathfrak{m}_1^{n+1}] \subset \mathfrak{m}_1^n$, and we again use transitivity of $\mathfrak{g}$. Next:

$$[\mathfrak{m}_{-1}, \mathfrak{m}_n] \subset \mathfrak{m}_{n-1} \tag{6.8}$$

since $[\mathfrak{m}_{-1}, \mathfrak{m}_1^n] \subset \mathfrak{m}_1^{n-1}$ by (6.5) and (6.6). By (6.2)-(6.4), (6.6) and (6.7) we have:

$$[\mathfrak{m}_0, \mathfrak{m}_n] \subset \mathfrak{m}_n \text{ for all } n\,.$$

This along with (6.2), (6.7) and (6.8) shows that $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j}$.

□

Remark 6.3 If $\mathfrak{g}$ satisfies the property:

$$[x, \mathfrak{g}_1] = 0 \text{ for } x \in \mathfrak{g}_j\,,\ j \geq -d\,, \text{ implies } x = 0\,,$$

then (6.2) follows from $[\mathfrak{m}_{-1}, \mathfrak{m}_{-1}] \subset \mathfrak{m}_{-2}$. Indeed for $x \in \mathfrak{m}_1$ we have by induction on $k \geq 2$: $[x, \mathfrak{m}_{-1}^k] \subset \mathfrak{m}_{-1}^{k-1}$, hence $\mathfrak{m}_{-1}^k \subset \mathfrak{m}_{-k}$ for all $k \geq 2$.

Lemma 6.3 *$\overline{E}(5,10)$ has no filtered deformations.*

Proof In this case $\mathfrak{s} = \mathfrak{m}_0$, hence (6.3) and (6.4) hold automatically, and the $\mathfrak{m}_0$-module $\mathfrak{m}_{2j}$ (resp. $\mathfrak{m}_{2j+1}$) is isomorphic to the highest component of $S^{j+1}(s\ell_5^*) \otimes s\ell_5$ (resp. $S^{j+1}(s\ell_5) \otimes \Lambda^2 s\ell_5$), see Example 5.3. Hence, $S^2\mathfrak{m}_{-1}$ does not contain a submodule isomorphic to $\mathfrak{m}_j$ for $j \neq -2$, hence $[\mathfrak{m}_{-1}, \mathfrak{m}_{-1}] = \mathfrak{m}_{-2}$ and, by Remark 6.3, (6.2) holds. Finally, in order to check (6.5) we have to show that $\mathfrak{m}_{-1} \otimes \mathfrak{m}_1$ does not contain a submodule isomorphic to $\mathfrak{m}_{2j}$ for $j \geq 1$, which is straightforward (see [OV]). Hence, by Lemma 6.2, $\overline{E}(5,10)$ has no filtered deformations.

□

Lemma 6.4 *$\overline{S}(m,n)$ and $\overline{S}'(m,n)$ with principal gradation have no filtered deformations if $m, n \geq 1$.*

Proof Let L be a filtered deformation of $\overline{S}(m,n)$ or $\overline{S}'(m,n)$ and let $\mathfrak{s} \subset L$ be the reductive subalgebra constructed above. Let us embed L in $\overline{W}(m,n) = \prod_{j\geq -1} W(m,n)_j$ (principal gradation), see Example 1.3. By Levi-Maltsev theorem (see [OV] or [S]), we may assume that $\mathfrak{s} \subset \overline{W}(m,n)_0 = g\ell(m,n)$. Hence $\mathfrak{s} = s\ell(m,n)_{\overline{0}}$. In particular, $\mathfrak{s}$ contains all operators $h_{ij} = x_i\frac{\partial}{\partial x_i} + \xi_j\frac{\partial}{\partial \xi_j}$, which span a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$. It is easy to see that the weights of $\mathfrak{h}$ that occur in $\overline{W}(m,n)_{-1}$ (= linear span of $\frac{\partial}{\partial x_i}$'s and $\frac{\partial}{\partial \xi_j}$'s.) do not occur in $\overline{W}(m,n)_j$ with $j \geq 0$. It follows that $\mathfrak{m}_{-1}$ from decomposition (6.1) coincides with $\overline{W}(m,n)_{-1}$. Hence, by transitivity, $\mathfrak{m}_j \subset \overline{W}(m,n)_j$ for all j, proving the claim.

□

Remark 6.4 It is well-known (and easy to show using Lemma 6.2) that $\overline{S}(m,0)$ has no filtered deformations. On the other hand, $S(0,n)$ for n even does have them [K4].

Lemma 6.5 $\overline{S}(m,1)$ *with subprincipal gradation has no filtered deformations if* $m \geq 2$.

Proof Let L be a filtered deformation. Then $\mathfrak{s}$ is isomorphic to $s\ell_m \oplus \mathbb{C}c$, where ad c acts as a scalar k on $(\mathfrak{m}_k)_{\overline{0}}$. It follows that $\overline{S}(m,1)_{\overline{0}} = \prod_{k\geq -1}(\mathfrak{m}_k)_{\overline{0}}$ is a completed graded Lie algebra isomorphic to $\overline{W}(m,0)$ with principal gradation. Furthermore, $(\mathfrak{m}_{-1})_{\overline{1}}$ is the standard $s\ell_m$-module with the eigenvalue of c equal $m-1$ and $(\mathfrak{m}_k)_{\overline{1}}$ for $k \geq 0$ is a direct sum of $\mathfrak{s}$-modules: $(\mathfrak{m}_k)_{\overline{1}} = V_k + V_k'$, where $V_k \simeq$ highest component of $s\ell_m \otimes S^{k+1}s\ell_m^*$ with eigenvalue of c equal $m+k$ and $V_k' = S^k s\ell_m$ with eigenvalue of c equal $-m+k$. Moreover, $V_0' = \mathbb{C}b$, where ad b maps $L_{\overline{0}}$ to 0 and is an isomorphism $L_{\overline{1}} \to L_{\overline{0}}$ as $s\ell_m$-modules. This implies that the conditions of Lemma 6.2 are satisfied, hence decomposition (6.1) is a $\mathbb{Z}$-gradation.

□

The *growth* of a linearly compact Lie superalgebra L is defined as the growth (see [K1]) of the graded Lie algebra GrL for any decreasing filtration of L (by Chevalley's principle, it is independent of the choice of filtration). It is clear that growth $L = 0$ iff $\dim L < \infty$, and that growth $L \geq 1$ otherwise.

Theorem 6.2 *A simple linearly compact Lie superalgebra* L *of growth* 1 *is isomorphic to one of the following Lie superalgebras:* $\overline{W}(1,n)$ *for* $n \geq 0$, $\overline{S}(1,n)$ *for* $n \geq 2$, $\overline{K}(1,n)$ *for* $n \geq 1$ *or* $\overline{E}(1,6)$.

Proof Due to Corollary 6.1, L has a filtration for which the associated graded Lie superalgebra $\mathfrak{g}$ is one of those listed by Theorems 4.1 and 5.3. Due to the growth condition, we must select those with $m = 1$ (see Remark 5.2). Here is the list:

I. (inconsistent principal gradation)

   (a) $W(1,n)$ with $n \geq 0$,

   (b) $S(1,n)$ and $S'(1,n)$ with $n \geq 2$,

   (c) $P[\xi] + \mathfrak{a}$ with $P = W(1,0)$,

II. (inconsistent subprincipal gradation) $W(1,1)$,

III. (consistent gradation)

   (a) $K(1,n)$ with $n \geq 1$, $n \neq 2$,

   (b) $E(1,6)$,

   (c) $s\ell_2 + S(1,2)$.

IV. $\mathbb{C}_0 + \mathfrak{g}$, where $\mathfrak{g}$ is one of the examples I-III for which $\mathfrak{g}_0$ has a trivial center.

As has been shown above, the cases I(c), III(c) and IV are ruled out since L is simple. Also, we have just shown that in the remaining cases $\overline{\mathfrak{g}}$ has no filtered deformations. This proves the theorem.

$\square$

Remark 6.5 Using methods developed in [DK], it is easy to derive from Theorem 6.2 the classification of finite simple conformal superalgebras (and hence the finite simple formal distribution Lie superalgebras) announced in [K6], [K7].

A more systematic way of describing filtered deformation is developed in [CK3] based on [KN]. It gives the description of simple filtered deformation in all the remaining cases.

Example 6.1 (filtered deformation of $\mathbb{C}_{-2}+\overline{SHO}(n,n)$, n even [CK3]). The Lie superalgebra $\mathbb{C}_{-2}+\overline{SHO}'(n,n)$ can be identified with the subspace $\overline{\Lambda}(n,n)^{\Delta}$ of $\overline{\Lambda}(n,n)$ consisting of elements f such that $\Delta(f)=0$ (see Remark 4.2) and the bracket $\{\,.\,,\,.\,\}_{ho}$ defined in Example 4.6. Its derived algebra is the subspace $\overline{\Lambda}(n,n)^{\Delta}_0$ of $\overline{\Lambda}(n,n)^{\Delta}$ of codimension 1, which consists of elements with zero projection on the monomial $\mathbb{C}\xi_1\dots\xi_n$. We denote by $\overline{CSHO}(n,n)^{\sim}$, n even, the space $\overline{\Lambda}(n,n)^{\Delta}_0$ with the following deformed bracket ($f,g\in\overline{\Lambda}(n,n)^{\Delta}_0$):

$$[f,g]=\{f,g\}_{ho}+\alpha(fg)\,,$$

where $\alpha(b)=\{\xi_1,\dots\xi_n,b\}_{ho}$ if b is a monomial in the x_i, and $\alpha(b)=0$ for all other monomials. The superalgebra $\overline{CSHO}(n,n)^{\sim}$ is simple for $n\geq 2$ (n even). Note that $\mathbb{C}_{-2}+\overline{SHO}'(n,n)$ has a similar deformation, which we denote by $\overline{CSHO}'(n,n)^{\sim}$, but it is not simple since its derived algebra has codimension 1 (and coincides with $\overline{CSHO}(n,n)^{\sim}$).

Example 6.2 (filtered deformation of $SHO'(n,n)$, n even [CK3]). Denote by $\overline{SHO}(n,n)^{\sim}$, n even, the space $\overline{\Lambda}(n,n)^{\Delta}/\mathbb{C}$ (see Examples 6.1) with the following deformed bracket:

$$[f,g]=\{f,g\}_{ho}+\alpha_1(fg)\,,$$

where $\alpha_1(b)=\{\xi_1\dots\xi_n,b\}_{ho}$ if b is a monomial in the x_i or $b=x_i\xi_i$, and $\alpha(b)=0$ for all other monomials. The superalgebra $\overline{SHO}(n,n)^{\sim}$ is simple for $n\geq 2$ (n even).

Example 6.3 (filtered deformation of $SKO(n,n+1;\frac{n+2}{n})$, n odd, cf. [Ko]). Let $\Delta'=\Delta+(E-(n+2))\frac{\partial}{\partial\xi_{n+1}}$. The Lie superalgebra $SKO(n,n+1;\frac{n+2}{n})$ is identified with the subspace $\overline{\Lambda}(n,n+1)^{\Delta'}$ of $\overline{\Lambda}(n,n+1)$ consisting of elements f such that $\Delta'(f)=0$, with the bracket $\{\,.\,,\,.\,\}_{ko}$ defined in Example 4.8 (see Example 4.9). We denote by $\overline{SKO}(n,n+1)^{\sim}$, n odd, the space $\overline{\Lambda}(n,n+1)^{\Delta'}$ with the following deformed bracket:

$$[f,g]=\{f,g\}_{ko}+\alpha_2(fg)\,,$$

where $\alpha_2(b)=\{\xi_1\dots\xi_{n+1},b\}_{ko}-2b\xi_1\dots\xi_n$ if b is a monomial in the x_i and $\alpha_2(b)=0$ for all other monomials. The superalgebra $\overline{SKO}(n,n+1)^{\sim}$ is simple for $n\geq 3$ (n odd).

It is shown in [CK3] that the only non-trivial simple deformations in all cases of Theorems 4.1 and 5.3 are those given by Examples 6.1-6.3. However, the Lie superalgebras $\overline{CSHO}(n,n)^{\sim}$ and $\overline{SHO}(n,n)^{\sim}$ are isomorphic. This proves the main theorem of the paper:

Theorem 6.3 *The following is a complete list of simple infinite-dimensional linearly compact Lie superalgebras* ($m \geq 1$)*:*

(a) $\overline{W}(m,n)$*,*

(b) $\overline{S}(m,n)$ *with* $(m,n) \neq (1,0)$*,* $(1,1)$*,*

(c) $\overline{H}(m,n)$ *with* $m \geq 2$*,* m *even,*

(d) $\overline{K}(m,n)$ *with* $m \geq 1$*,* m *odd,*

(e) $\overline{HO}(m,m)$ *with* $m \geq 2$*,*

(f) $\overline{SHO}(m,m)$ *with* $m \geq 3$*,*

(g) $\overline{SHO}(m,m)^{\sim}$ *with* $m \geq 2$*,* m *even,*

(h) $\overline{KO}(m,m+1)$*,*

(i) $\overline{SKO}(m,m+1;\beta)$ *with* $m \geq 2$*,* $\beta \in \mathbb{C}$*,*

(j) $\overline{SKO}(m,m+1)^{\sim}$ *with* $m \geq 3$*,* m *odd,*

(k) $\overline{E}(1,6)$*,* $\overline{E}(2,2)$*,* $\overline{E}(3,6)$*,* $\overline{E}(3,8)$*,* $\overline{E}(4,4)$*,* $\overline{E}(5,10)$*.*

Remark 6.6 Here are all the isomorphisms between the Lie superalgebras listed in Theorem 6.3:

$$\overline{W}(1,1) \simeq \overline{K}(1,2) \simeq \overline{KO}(1,2)\,,\; \overline{S}(2,1) \simeq \overline{HO}(2,2)\,.$$

In conclusion of the paper we describe all derivations:

Proposition 6.1 *The Lie superalgebra* $\mathrm{Der}\, L$ *of a simple infinite-dimensional linearly compact Lie superalgebra* L *is as follows* ($m \geq 1$)*:*

(a) If L *is one of the Lie superalgebras* $\overline{W}(m,n)$ *,* $\overline{SHO}(m,m)^{\sim}$*,* $\overline{K}(m,n)$*,* $\overline{KO}(m,m+1)$*,* $\overline{SKO}(m,m+1;0)$ *with* $m \geq 3$*,* $\overline{SKO}(m,m+1)^{\sim}$*,* $\overline{E}(4,4)$*,* $\overline{E}(1,6)$*,* $\overline{E}(3,6)$*,* $\overline{E}(3,8)$*, then* $\mathrm{Der}\, L = L$*.*

(b) If L is one of the Lie superalgebras $\overline{S}(m,n)$, $\overline{H}(m,n)$, $\overline{HO}(m,m)$ with $m \geq 2$, $\overline{SKO}(m,m+1;\beta)$ with $m \geq 2$ and $\beta \neq 0$, $\frac{m-2}{m}$, $\overline{SKO}(2,3;\beta)$, $\overline{E}(5,10)$, then $\mathrm{Der}\, L = \mathbb{C}_0 + L$.

(c) If L is one of the Lie superalgebras $\overline{S}(1,n)$ with $n \geq 3$, $\overline{SHO}(m,m)$ with $m \geq 4$, $\overline{SKO}(m,m+1;\frac{m-2}{m})$ with $m \geq 3$, then $\mathrm{Der}\, L$ *is* $\mathbb{C}_0 + \overline{S}'(1,n)$, $\mathbb{C}_0 + \overline{SHO}'(m,m)$, *and* $\overline{SKO}'(m,m+1;\frac{m-2}{m})$ *respectively.*

(d) If $L = \overline{SHO}(3,3)$, $\overline{E}(2,2)$ or $\overline{S}(1,2)$, then $\mathrm{Der}\, L = s\ell_2 + L$.

Proof is the same as that of Proposition 5.1.2 from [K4]. $\square$

Postscript. This paper (to appear in October 1998 issue of Advances in Math.) was presented at the ESI in September 1998. In this talk, after giving the list of maximal compact subgroups K of the groups of inner automorphisms of the six infinite-dimensional simple Lie superalgebras, I suggested that one of the exceptional Lie superalgebras $E(3,6)$ or $E(3,8)$ might be the algebra of supersymmetries of the Standard Model, since the group $SU_3 \times SU_2 \times U_1$ is the group of symmetries of this model and it is the group K for both $E(3,6)$ and $E(3,8)$. David Broadhurst commented that similarly the exceptional superalgebra $E(5,10)$ might be the algebra of supersymmetries of the hypothetical Grand Unified Model since $K = SU_5$ for $E(5,10)$. He also asked whether $E(3,6)$ or $E(3,8)$ can be embeded in $E(5,10)$. I replied that $E(3,8)$ cannot be embeded since $E(5,10)$ has no consistent gradations of debth 3, but that $E(3,6)$ can, since its non-positive part can be embeded in that of $E(5,10)$.

It is thus natural to conjecture that the Standard Model can be extended to the Grand Unified Model in such a way that the algebra $E(3,6)$ of supersymmetries of the Standard Model is embeded in the algebra $E(5,10)$ of supersymmetries of the Grand Unified Model.

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