

Dedekind Zeta Functions and Quantum Statistical Mechanics

Paula B. Cohen

Vienna, Preprint ESI 617 (1998)

October 13, 1998

Supported by Federal Ministry of Science and Transport, Austria
Available via <http://www.esi.ac.at>

DEDEKIND ZETA FUNCTIONS AND QUANTUM STATISTICAL MECHANICS

Paula B. Cohen

1 Introduction

In 1859 Riemann published an important foundational paper on the Riemann zeta function. Recall that this function is given for $\text{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and that it has a continuation to all the complex plane which is analytic except for a simple pole at $s = 1$. It is straightforward to show that the Riemann zeta function has zeros at the negative even integers and these are called the trivial zeros of the Riemann zeta function. The Riemann hypothesis predicts that the remaining zeros lie on the line $\text{Re}(s) = 1/2$. One knows that the non-trivial zeros of $\zeta(s)$ lie in the band $\text{Re}(s) \in]0, 1[$. The generalisation of this function for a number field is known as the Dedekind zeta function. It encodes much important arithmetical information about the field. One of the major motivations of number theory is to understand more fully the Dedekind zeta function, the most famous challenge being to understand the locus of the zeros of this function and in particular to settle the validity of the Riemann hypothesis. Much powerful work has been done in analytic number theory in the attempt to solve the Riemann hypothesis directly. As the study of the zeros of the Riemann zeta function and its generalisations is so difficult, one may ask how it is possible to recast the problem. For example, Polya and Hilbert proposed that if one can construct a Hilbert space \mathcal{H} and an operator D in \mathcal{H} whose spectrum comprises the zeros of the Riemann zeta function in the band $\text{Re}(s) \in]0, 1[$, then possibly one can settle whether or not $\sqrt{-1}(D - 1/2)$ is self-adjoint or whether $D(1 - D)$ is positive, which would imply the Riemann hypothesis. The point here is that the properties of self-adjointness or of positivity are hopefully easier to check. It is important that the construction of the Hilbert space and the operator should not depend *a priori* on the zeta function, to avoid tautologies. An as yet non-rigorous approach to the Riemann hypothesis initiated by Connes in [C/ras] and being further developed in more recent work of his, includes an interesting and rigorous spectral interpretation of the zeros of the L -functions with Grossencharacter of a global field.

This recent work of Connes derives also from his work with Bost [BC] which was in turn inspired by ideas of Julia [J] and others. The aim is to enrich our knowledge of the Riemann zeta function by creating a dictionary between its

properties and phenomena in statistical mechanics. We discuss in this note a recasting and generalisation of [BC] to the number field case of which fuller details can be found in [Coh]. The starting point of these approaches is the observation that, just as the zeta functions encode arithmetic information, the partition functions of quantum statistical mechanical systems encode their large-scale thermodynamical properties. The first step therefore is to construct a quantum dynamical system with partition function the Riemann zeta function, or the Dedekind zeta function in the general number field case. In order for the quantum dynamical system to reflect the arithmetic of the primes, it must capture also some sort of interaction between them. This last feature translates in the statistical mechanical language into the phenomenon of spontaneous symmetry breaking at a critical temperature with respect to a natural symmetry group. In the region of high temperature, there is a unique equilibrium state as the system is in disorder and symmetric with respect to the action of the symmetry group. In the region of low temperature, a phase transition occurs and the symmetry is broken. This symmetry group acts transitively on a family of possible extremal equilibrium states. The construction of a quantum dynamical system with partition function the Riemann zeta function $\zeta(\beta)$ and spontaneous symmetry breaking or phase transition at its pole $\beta = 1$ with respect to a natural symmetry group was achieved by Bost and Connes in [BC]. A different construction of the basic algebra using crossed products was proposed by Laca and Raeburn and extended to the number field case by them with Arledge in [ALR]. An extension of the work of Bost and Connes to general global fields was done by Harari and Leichtnam in [HL]. The generalisation proposed by Harari and Leichtnam in [HL] fails to capture the Dedekind zeta function as partition function in the case of a number field with class number greater than 1. Their partition function in that case is the Dedekind zeta function with a finite number of non-canonically chosen Euler factors removed. This prompted the author's paper [Coh1] where the full Dedekind zeta function is recovered as partition function. This is achieved by recasting the original construction of Bost and Connes more completely in terms of adèles and ideles.

The symmetry group of the system constructed by Bost and Connes is a Galois group, in fact the Galois group over the rational number field of its maximal abelian extension. Using the Artin isomorphism, which says that this symmetry group is also the unit group of the finite ideles, Bost and Connes recover the actual Galois action on the elements of the maximal abelian extension via its action on the equilibrium states of the system. In the general number field case, the symmetry group is again the unit group of the finite ideles, but this group does not in general have a Galois interpretation. See [HL] for a discussion of this point.

2 The main goal

Before stating the problem solved by Bost and Connes in [BC] and its analogue for number fields, we recall a few basic notions from the C^* -algebraic formulation of quantum statistical mechanics. For the background, see [C]. Recall that a C^* -algebra A is an algebra over the complex numbers \mathbb{C} with an adjoint $x \mapsto x^*$, $x \in A$, that is, an anti-linear map with $x^{**} = x$, $(xy)^* = y^*x^*$, $x, y \in A$, and a norm $\|\cdot\|$ with respect to which A is complete and addition and multiplication are continuous operations. One requires in addition that $\|xx^*\| = \|x\|^2$ for all $x \in A$. All our C^* -algebras will be assumed unital. The most basic example of a non-commutative C^* -algebra is $A = M_N(\mathbb{C})$ for $N \geq 2$ an integer. The C^* -algebra plays the role of the “space” on which the system evolves, the evolution itself being described by a 1-parameter group of C^* -automorphisms $\sigma : \mathbb{R} \mapsto \text{Aut}(A)$. The quantum dynamical system is therefore the pair (A, σ_t) . It is customary to use the inverse temperature $\beta = 1/kT$ rather than the temperature T , where k is Boltzmann’s constant. Then, one has the definition of Kubo-Martin-Schwinger (KMS) of an equilibrium state at inverse temperature β . Recall that a state φ on a C^* -algebra A is a positive linear functional on A satisfying $\varphi(1) = 1$. It is the generalisation of a probability distribution.

Definition 1: *Let (A, σ_t) be a dynamical system, and φ a state on A . Then φ is an equilibrium state at inverse temperature β , or KMS_β -state, if for each $x, y \in A$ there is a function $F_{x,y}(z)$, bounded and holomorphic in the band $0 < \text{Im}(z) < \beta$ and continuous on its closure, such that for all $t \in \mathbb{R}$,*

$$F_{x,y}(t) = \varphi(x\sigma_t(y)), \quad F_{x,y}(t + \sqrt{-1}\beta) = \varphi(\sigma_t(y)x). \quad (1)$$

In the case where $A = M_N(\mathbb{C})$, every 1-parameter group σ_t of automorphisms of A can be written in the form,

$$\sigma_t(x) = e^{itH} x e^{-itH}, \quad x \in A, \quad t \in \mathbb{R},$$

for a self-adjoint matrix $H = H^*$. Then for $H \geq 0$ and for all $\beta > 0$, there is a unique KMS_β equilibrium state for (A, σ_t) given by

$$\phi_\beta(x) = \text{Trace}(x e^{-\beta H}) / \text{Trace}(e^{-\beta H}), \quad x \in M_N(\mathbb{C}). \quad (2)$$

This has the familiar form of a Gibbs state and is easily seen to satisfy the KMS_β condition of Definition 1. The KMS_β states can therefore be seen as generalisations of Gibbs states. The normalisation constant $\text{Trace}(e^{-\beta H})$ is known as the partition function of the system. A symmetry group G of the dynamical system (A, σ_t) is a subgroup of $\text{Aut}(A)$ commuting with σ :

$$g \circ \sigma_t = \sigma_t \circ g, \quad g \in G, t \in \mathbb{R}.$$

Consider now a system (A, σ_t) with interaction. Then, guided by quantum statistical mechanics, we expect to see the following features. When the temperature is high, so that β is small, the system is in disorder, there is no interaction between its constituents and the state of the system does not see the action of the symmetry group G : the KMS_β -state is unique. As the temperature is lowered, the constituents of the system begin to interact. At a critical temperature β_0 a phase transition occurs and the symmetry is broken. The symmetry group G then permutes transitively a family of extremal KMS_β -states generating the possible states of the system after phase transition: the KMS_β -state is no longer unique. This phase transition phenomenon is known as spontaneous symmetry breaking at the critical inverse temperature β_0 . The partition function should have a pole at β_0 . For a fuller explanation, see [BC]. The problem solved by Bost and Connes was the following.

Problem 1: *Construct a dynamical system (A, σ_t) with partition function the zeta function $\zeta(\beta)$ of Riemann, where $\beta > 0$ is the inverse temperature, having spontaneous symmetry breaking at the pole $\beta = 1$ of the zeta function with respect to a natural symmetry group.*

As mentioned in the introduction, the symmetry group is the unit group of the ideles, given by $W = \prod_p Z_p^*$ where the product is over the primes p and $Z_p^* = \{u_p \in \mathbb{Q}_p : |u_p|_p = 1\}$. We use here the normalisation $|p|_p = p^{-1}$. This is the same as the Galois group $\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$. Here \mathbb{Q}^{ab} is the maximal abelian extension of the rational number field \mathbb{Q} , which in turn is isomorphic to its maximal cyclotomic extension, that is the extension obtained by adjoining to \mathbb{Q} all the roots of unity. The interaction detected in the phase transition comes about from the interaction between the primes coming from considering at once all the embeddings of the non-zero rational numbers \mathbb{Q}^* into the completions \mathbb{Q}_p of \mathbb{Q} with respect to the prime valuations $|\cdot|_p$.

The natural generalisation of this problem to the number field case was solved in [Coh1] and is the following.

Problem 2: *Given a number field K , construct a dynamical system (A, σ_t) with partition function the Dedekind zeta function $\zeta_K(\beta)$, where $\beta > 0$ is the inverse temperature, having spontaneous symmetry breaking at the pole $\beta = 1$ of the Dedekind function with respect to a natural symmetry group.*

Recall that the Dedekind zeta function is given by

$$\zeta_K(s) = \sum_{\mathcal{C} \in \mathcal{O}} \frac{1}{N(\mathcal{C})^s}, \quad \text{Re}(s) > 1. \quad (3)$$

Here \mathcal{O} is the ring of integers of K and the summation is over the ideals \mathcal{C} of K contained in \mathcal{O} . The symmetry group is the unit group of the finite ideles of K .

For the natural generalisation to the function field case see [HL]. For the sake of exposition, we restrict ourselves in the sequel to the case of the rational numbers, that is to a discussion of Problem 1.

3 Construction of the C^* -algebra

We give a different construction of the C^* -algebra of [BC] to that found in their original paper. It is essentially equivalent to the construction of [ALR], except that we work with adeles and ideles. In the generalisation to the number field case, this makes quite a difference. Let \mathcal{A} denote the finite adeles of \mathbb{Q} , that is the restricted product of \mathbb{Q}_p with respect to \mathbb{Z}_p . Recall that this restricted product consists of the infinite vectors $(a_p)_p$, indexed by the primes p , such that $a_p \in \mathbb{Q}_p$ with $a_p \in \mathbb{Z}_p$ for almost all primes p . The (finite) adeles form a ring under componentwise addition and multiplication. The (finite) ideles \mathcal{J} are the invertible elements of the adeles. They form a group under componentwise multiplication. Let Z_p^* be those elements of $u_p \in \mathbb{Z}_p$ with $|u_p|_p = 1$. Notice that an idele $(u_p)_p$ has $u_p \in \mathbb{Q}_p^*$ with $u_p \in Z_p^*$ for almost all primes p . Let

$$\mathcal{R} = \prod_p \mathbb{Z}_p, \quad I = \mathcal{J} \cap \mathcal{R}, \quad W = \prod_p Z_p^*.$$

Further, let \mathcal{I} denote the semigroup of integral ideals of \mathbb{Z} . It is the semigroup of \mathbb{Z} -modules of the form $m\mathbb{Z}$ where $m \in \mathbb{Z}$. Notice that I as above is also a semigroup. We have a natural short exact sequence,

$$1 \rightarrow W \rightarrow I \rightarrow \mathcal{I} \rightarrow 1. \quad (4)$$

The map $I \rightarrow \mathcal{I}$ in this short exact sequence is given as follows. To $(u_p)_p \in I$ associate the ideal $\prod_p p^{\text{ord}_p(u_p)}$ where $\text{ord}_p(u_p)$ is determined by the formula $|u_p|_p = p^{-\text{ord}_p(u_p)}$. It is clear that this map is surjective with kernel W , that is that the above sequence is indeed short exact. By the Strong Approximation Theorem we have

$$\mathbb{Q}/\mathbb{Z} \simeq \mathcal{A}/\mathcal{R} \simeq \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p \quad (5)$$

and we have therefore a natural action of I on \mathbb{Q}/\mathbb{Z} by multiplication in \mathcal{A}/\mathcal{R} and transport of structure. We use here that $I\mathcal{R} \subset \mathcal{R}$. Mostly we shall work in \mathcal{A}/\mathcal{R} rather than \mathbb{Q}/\mathbb{Z} . We have the following straightforward Lemma (see [Coh1]).

Lemma 1: For $a = (a_p)_p \in I$ and $y \in \mathcal{A}/\mathcal{R}$, the equation

$$ax = y$$

has $n(a) =: \prod_p p^{\text{ord}_p(a_p)}$ solutions in $x \in \mathcal{A}/\mathcal{R}$. Denote these solutions by $[x : ax = y]$.

In the above lemma it is important to bear in mind that we are computing modulo \mathcal{R} . Now, let $C[\mathcal{A}/\mathcal{R}] =: \text{span}\{\delta_x : x \in \mathcal{A}/\mathcal{R}\}$ be the group algebra of \mathcal{A}/\mathcal{R} over \mathbb{C} , so that $\delta_x \delta_{x'} = \delta_{x+x'}$ for $x, x' \in \mathcal{A}/\mathcal{R}$. We have (see for comparison [ALR]),

Lemma 2: *The formula*

$$\alpha_a(\delta_y) = \frac{1}{n(a)} \sum_{[x:ax=y]} \delta_x$$

for $a \in I$ defines an action of I by endomorphisms of $C^*(\mathcal{A}/\mathcal{R})$.

The endomorphism α_a for $a \in I$ is a one-sided inverse of the map $\delta_x \mapsto \delta_{ax}$ for $x \in \mathcal{A}/\mathcal{R}$, so it is like a semigroup “division”. The C^* -algebra can be thought of as the operator norm closure of $C[\mathcal{A}/\mathcal{R}]$ in its natural left regular representation in $l^2(\mathcal{A}/\mathcal{R})$. We now appeal to the notion of semigroup crossed product developed by Laca and Raeburn and used in [ALR], applying it to our situation. A covariant representation of $(C^*(\mathcal{A}/\mathcal{R}), I, \alpha)$ is a pair (π, V) where

$$\pi : C^*(\mathcal{A}/\mathcal{R}) \rightarrow B(\mathcal{H})$$

is a unital representation and

$$V : I \rightarrow B(\mathcal{H})$$

is an isometric representation in the bounded operators in a Hilbert space \mathcal{H} . The pair (π, V) is required to satisfy,

$$\pi(\alpha_a(f)) = V_a \pi(f) V_a^*, \quad a \in I, \quad f \in C^*(\mathcal{A}/\mathcal{R}).$$

Notice that the V_a are not in general unitary. Such a representation is given by (λ, L) on $l^2(\mathcal{A}/\mathcal{R})$ with orthonormal basis $\{e_x : x \in \mathcal{A}/\mathcal{R}\}$ where λ is the left regular representation of $C^*(\mathcal{A}/\mathcal{R})$ on $l^2(\mathcal{A}/\mathcal{R})$ and

$$L_a e_y = \frac{1}{\sqrt{n(a)}} \sum_{[x:ax=y]} e_x.$$

The universal covariant representation, through which all other covariant representations factor, is called the (semigroup) crossed product $C^*(\mathcal{A}/\mathcal{R}) \times_\alpha I$. This algebra is the universal C^* -algebra generated by the symbols $\{e(x) : x \in \mathcal{A}/\mathcal{R}\}$ and $\{\mu_a : a \in I\}$ subject to the relations

$$\mu_a^* \mu_a = 1, \quad \mu_a \mu_b = \mu_{ab}, \quad a, b \in I, \quad (6)$$

$$e(0) = 1, \quad e(x)^* = e(-x), \quad e(x)e(y) = e(x+y), \quad x, y \in \mathcal{A}/\mathcal{R}, \quad (7)$$

$$\frac{1}{n(a)} \sum_{[x:ax=y]} e(x) = \mu_a e(y) \mu_a^*, \quad a \in I, y \in \mathcal{A}/\mathcal{R}. \quad (8)$$

The relations in (6) reflect a multiplicative structure, those in (7) an additive structure and those in (8) how these multiplicative and additive structures are related via the crossed product action. Julia [J] observed that by using only the

multiplicative structure of the integers one cannot hope to capture an interaction between the different primes. When $u \in W$ then μ_u is a unitary, so that $\mu_u^* \mu_u = \mu_u \mu_u^* = 1$ and we have for all $x \in \mathcal{A}/\mathcal{R}$,

$$\mu_u e(x) \mu_u^* = e(u^{-1}x), \quad \mu_u^* e(x) \mu_u = e(ux). \quad (9)$$

Therefore we have a natural action of W as inner automorphisms of $C^*(\mathcal{A}/\mathcal{R}) \times_\alpha I$ using (9).

To recover the C^* -algebra of [BC] we must split the short exact sequence (4). The ideals in \mathcal{I} are all of the form mZ for some $m \in Z$. This generator m is determined up to sign. Consider the image of $|m|$ in I under the diagonal embedding $q \mapsto (q)_p$ of Q^* into I , where the p -th component of $(q)_p$ is the image of q in Q_p^* under the natural embedding of Q^* in Q_p^* . The map

$$+ : mZ \mapsto (|m|)_p \quad (10)$$

defines a splitting of (4). Let I_+ denote the image and define A to be the semigroup crossed product $C^*(\mathcal{A}/\mathcal{R}) \times_\alpha I_+$ with the restricted action α from I to I_+ . By transport of structure using (5), this algebra is easily seen to be isomorphic to a semigroup crossed product of $C^*(Q/Z)$ by N_+ , where N_+ denotes the positive natural numbers. This is the algebra constructed in [BC] (see also [ALR]). From now on, we use the symbols $\{e(x) : x \in Q/Z\}$ and $\{\mu_a : a \in N_+\}$. It is essential to split the short exact sequence in this way in order to obtain the symmetry breaking phenomenon.

In particular, this replacement of I by I_+ now means that the group W acts by outer automorphisms. For $x \in A$, one has that $\mu_u^* x \mu_u$ is still in A (computing in the larger algebra $C^*(\mathcal{A}/\mathcal{R}) \times_\alpha I$), but now this defines an outer action of W . This coincides with the definition of W as the symmetry group as in [BC].

4 The time evolution and the KMS_β -states

Using the abstract description of the C^* -algebra A of §3, to define the time evolution σ of our dynamical system (A, σ) it suffices to define it on the symbols $\{e(x) : x \in Q/Z\}$ and $\{\mu_a : a \in N_+\}$. For $t \in \mathbb{R}$, let σ_t be the automorphism of A defined by

$$\sigma_t(\mu_m) = m^{it}, \quad m \in N_+, \quad \sigma_t(e(x)) = e(x), \quad x \in Q/Z. \quad (11)$$

By (6) and (9) we clearly have that the action of W commutes with this 1-parameter group σ_t . Hence W will permute the extremal KMS_β -states of (A, σ_t) . To describe the KMS_β -states for $\beta > 1$, we shall represent (A, σ_t) on a Hilbert space. Namely, following [BC], let \mathcal{H} be the Hilbert space $l^2(N_+)$ with canonical orthonormal basis $\{\varepsilon_m, m \in N_+\}$. For each $u \in W$, one has a representation π_u of A in $B(\mathcal{H})$ given by,

$$\pi_u(\mu_m) \varepsilon_n = \varepsilon_{mn}, \quad m, n \in N_+$$

$$\pi_u(e(x))\varepsilon_n = \exp(2i\pi nu \circ x)\varepsilon_n, \quad n \in \mathbb{N}_+, \quad x \in \mathbb{Q}/\mathbb{Z}. \quad (12)$$

Here $u \circ x$ for $u \in W$ and $x \in \mathbb{Q}/\mathbb{Z}$ is the multiplication induced by transport of structure using (5). One verifies easily that (12) does indeed give a C^* -algebra representation of A . Let H be the unbounded operator in \mathcal{H} whose action on the canonical basis is given by

$$H\varepsilon_n = (\log n)\varepsilon_n, \quad n \in \mathbb{N}_+. \quad (13)$$

Then clearly, for each $u \in W$, we have

$$\pi_u(\sigma_t(x)) = e^{itH}\pi_u(x)e^{-itH}, \quad t \in \mathbb{R}, \quad x \in A.$$

Notice that, for $\beta > 1$,

$$\text{Trace}(e^{-\beta H}) = \sum_{n=1}^{\infty} \langle e^{-\beta H} \varepsilon_n, \varepsilon_n \rangle = \sum_{n=1}^{\infty} n^{-\beta} \langle \varepsilon_n, \varepsilon_n \rangle = \sum_{n=1}^{\infty} n^{-\beta},$$

so that the Riemann zeta function appears as a partition function of Gibbs state type. We can now state the main result of [BC].

Theorem: (Bost-Connes) *The dynamical system (A, σ_t) has symmetry group W . The action of $u \in W$ is given by $[u] \in \text{Aut}(A)$ where*

$$[u] : e(y) \mapsto e(u \circ y), \quad y \in \mathbb{Q}/\mathbb{Z}, \quad [u] : \mu_a \mapsto \mu_a, \quad a \in \mathbb{N}.$$

This action commutes with σ ,

$$[u] \circ \sigma_t = \sigma_t \circ [u], \quad u \in W, \quad t \in \mathbb{R}.$$

Moreover,

(1) *for $0 < \beta \leq 1$, there is a unique KMS_β state. (It is a factor state of Type III_1 with associated factor the Araki-Woods factor R_∞ .)*

(2) *for $\beta > 1$ and $u \in W$, the state*

$$\phi_{\beta,u}(x) = \zeta(\beta)^{-1} \text{Trace}(\pi_u(x)e^{-\beta H}), \quad x \in A$$

is a KMS_β state for (A, σ_t) . (It is a factor state of Type I_∞). The action of W on A induces an action on these KMS_β states which permutes them transitively and the map $u \mapsto \phi_{\beta,u}$ is a homomorphism of the compact group W onto the space \mathcal{E}_β of extremal points of the simplex of KMS_β states for (A, σ_t) .

(3) *the ζ function of Riemann is the partition function of (A, σ_t) .*

Part (1) of the above theorem is difficult and the reader is referred to [BC] for complete details, as for a full proof of (2). That for $\beta > 1$ the KMS_β -states given in part (2) fulfil Definition 1 of §2 is a straightforward exercise. Notice that they have the form of Gibbs equilibrium states.

5 Concluding remarks

The Theorem of the preceding section solves Problem 1 of §2. More information is contained in its proof however. As mentioned in the Introduction, given the existence of the Artin isomorphism in class field theory for the rationals, one can recover the Galois action of W explicitly. It is still an open problem to exhibit this Galois action in terms of an analogue of (A, σ_t) in a satisfactory way for general number fields.

Another exciting feature occurs in the analysis of the proof of part (1) of the Theorem of the preceding section. If one treats also the infinite places, working with the full adèles A and ideles (the restricted product now extending over the archimedean places as well as over the non-archimedean places (primes)), Connes has observed that the von-Neumann algebra of Type III_∞ in the region $0 < \beta \leq 1$ has in its continuous decomposition the Type II_∞ factor given by the crossed product of $L^\infty(A)$ by the action of Q^* by multiplication. This can be interpreted as the von-Neumann algebra associated to the orbit space A/Q^* and it is this space which plays a fundamental role in Connes' proposed approach to the Riemann hypothesis in [C/ras].

References

- [ALR] J. Arledge, M. Laca, I. Raeburn, *Semigroup crossed products and Hecke algebras arising from number fields*, Doc. Mathematica **2** (1997) 115–138.
- [BC] J-B. Bost, A. Connes, *Hecke Algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series), **1**, 411–457 (1995).
- [Coh1] P.B. Cohen, *A C^* -dynamical system with Dedekind zeta partition function and spontaneous symmetry breaking*, soumis aux Actes des Journées Arithmétiques de Limoges, 1997. Preprint de l'IRMA de l'UST de Lille.
- [Coh2] P.B. Cohen, *Sur la mécanique statistique d'après les travaux de Bost-Connes*, soumis aux Actes des Journées Académiques de Lille, 1998.
- [C] A. Connes, *Noncommutative Geometry*, Academic Press, 1994. (Version française: Géométrie non commutative, InterEditions, Paris, 1990.)
- [C/ras] A. Connes, *Formule de trace en géométrie non commutative et hypothèse de Riemann*, C.R. Acad. Sci. Paris, **t.323**, Série 1 (Analyse), 1231–1236 (1996) .
- [D] J. Dixmier, *Les C^* -algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.

- [**HL**] D. Harari, E. Leichtnam, *Extension du phénomène de brisure spontanée de symétrie de Bost-Connes au cas des corps globaux quelconques*, *Selecta Mathematica*, New Series 3 (1997), 205-243.
- [**J**] B. Julia, *Statistical Theory of Numbers*, in *Number Theory and Physics*, Springer Proceedings in Physics, Vol. 47, 1990.
- [**Pa**] K.R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, *Monographs in Mathematics* Vol. 85, Birkhäuser, 1992.
- [**R**] B. Riemann, *Monatsberichte der Berliner Akad.* (1859)
- [**T**] E.C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Second Edition revised by D.R. Heath-Brown, Oxford University Press, New York, 1988.

URA AGAT CNRS, Mathématiques Pures et Appliquées
Université des Sciences et Technologies de Lille
F-59655 Villeneuve d'Ascq Cedex, France
pcohen@gat.univ-lille1.fr